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SIGMA MODELS, MINIMAL SURFACES AND SOME RICCI FLAT PSEUDO-RIEMANNIAN GEOMETRIES

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Abstract. We consider the sigma models where the base metric is proportional to the metric of the configuration space. We show that the corresponding sigma model equation admits a Lax pair. We also show that this type of sigma models in two dimensions are intimately related to the minimal surfaces in a flat pseudo-Riemannian 3-space. We define two dimensional surfaces conformally related to the minimal surfaces in flat three dimensional geometries which enable us to give a construction of the metrics of some even dimensional Ricci flat (pseudo-) Riemannian geometries.

1. Introduction

Let M be a 2-dimensional manifold with local coordinates $x^\mu = (x, y)$ and $\Lambda^{\mu\nu}$ be the components of a tensor field in M. Let P be an 2×2 matrix with a nonvanishing constant determinant. We assume that P is a Hermitian ($P^\dagger = P$) matrix. Then the field equations of the sigma-model we consider is given as follows

$$\frac{\partial}{\partial x^\alpha}\left(\Lambda^{\alpha\beta} P^{-1} \frac{\partial P}{\partial x^\beta}\right) = 0\,. \tag{1.1}$$

The integrability of the above equation has been studied in [1] where the matrix function P and the tensor $\Lambda^{\alpha\beta}$ were considered independent. The sigma model equation given above is integrable provided Λ satisfies the conditions

$$\partial_\alpha\left(\frac{1}{\sigma}\Lambda^{\alpha\beta}\partial_\beta \sigma\right) = 0\,, \qquad \partial_\alpha\left(\frac{1}{\sigma}\Lambda^{\beta\alpha}\partial_\beta \phi\right) = 0\,, \tag{1.2}$$

where σ and ϕ are determinant and antisymmetric part of the tensor $\Lambda^{\alpha\beta}$ respectively.

We have classified in [1] possible forms of the tensor $\Lambda^{\alpha\beta}$ under these conditions of integrability. The case where Λ and P are related has been considered in [2]. As an example, let $P = g$ where g is a 2×2 symmetric matrix. Letting also $\Lambda^{\alpha\beta} = g^{\alpha\beta}$, the inverse components of the metric $g_{\alpha\beta}$, then (1.1) becomes

$$\frac{\partial}{\partial x^{\alpha}}\left(g^{\alpha\beta} g^{-1} \frac{\partial g}{\partial x^{\beta}}\right) = 0 . \tag{1.3}$$

The above sigma model equation is integrable and the Lax equation is simply given by [2]

$$\epsilon^{\alpha\beta} \frac{\partial}{\partial x^{\beta}} \Psi = \frac{1}{k^2+\sigma} (k g^{\alpha\beta} - \sigma \epsilon^{\alpha\beta}) g^{-1} \frac{\partial g}{\partial x^{\beta}} \Psi . \tag{1.4}$$

Integrability conditions are satisfied because $\det g = \sigma$ (a constant) and g is symmetric. Here k is an arbitrary constant (the spectral parameter), $\epsilon^{\alpha\beta}$ is the Levi-Civita tensor with $\epsilon^{12} = 1$.

In the theory of surfaces in $\mathbb{R}^3$ there is a class, the minimal surfaces, which have special importance both in physics and mathematics [3,4]. Let $S = \{(x,y,z) \in \mathbb{R}^3;\ z = h(x,y)\}$ define a surface $S \in \mathbb{R}^3$ which is the graph of a differentiable function $\phi(x,y)$. This surface is called minimal if ϕ satisfies the condition

$$(1+\phi_x^2)\phi_{yy} - 2\phi_x\phi_y\phi_{xy} + (1+\phi_y^2)\phi_{xx} = 0 . \tag{1.5}$$

The Gaussian curvature K of the surface S is given by

$$K = \frac{\phi_{xx}\phi_{yy} - \phi_{xy}^2}{(1+\phi_x^2+\phi_y^2)^2} . \tag{1.6}$$

Here in this work we generalize the above treatment to more general geometries. Instead of $\mathbb{R}^3$ we take a pseudo-Euclidean manifold M_3 and two surfaces with any signature.

Let (S,g) denote a two dimensional geometry where S is a surface in a three dimensional flat manifold M_3 and g is a (pseudo-)Riemannian metric on S with a non vanishing determinant, $\det g$. Furthermore we assume that the metric components $g_{\alpha\beta}$ satisfies the following conditions

$$\partial_{\mu}(g^{\mu\nu} g^{-1} \partial_{\nu} g) = 0 , \tag{1.7}$$

$$R + \frac{1}{4}\, \mathrm{Tr}[g^{\mu\nu} \partial_{\mu} g^{-1} \partial_{\nu} g] = 0 , \tag{1.8}$$

where R is the Ricci scalar of S. We shall see in the following sections that some surfaces which are conformally related to minimal surfaces satisfy the above conditions.

The importance of such surfaces arises when we are interested in even dimensional Ricci flat geometries. By the utility the metric g of these surfaces we shall give a construction (without solving any further differential equations) of the metric of a $2N$ dimensional Ricci flat (pseudo-)Riemannian geometries. Ricci flat geometries are important not only in differential geometry and general relativity but also in gravitational instantons and in brane solutions of string theory [6].

2. Locally Conformal Minimal Surfaces

Let ϕ be a differentiable function of $x^1 = x$ and $x^2 = y$ and S_0 be the surface in a three dimensional manifold M_3 with a pseudo-Euclidean metric g_3 defined through $\mathrm{d}s^2 = g_{0\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu + \epsilon(\,\mathrm{d}x^3)^2$, where $\mu, \nu = 1, 2$, $\epsilon = \pm 1$ and g_0 is a constant everywhere in M_3, invertible, symmetric 2×2 matrix. In this work we assume Einstein summation convention, i. e., the repeated indices are summed up. Let S_0 be given as the graph of the function ϕ, i. e., $S_0 = \{(x^1, x^2, x^3) \in M_3\,;\, x^3 = \phi(x^1, x^2)\}$. Then the metric on S_0 is given by

$$h_{\mu\nu} = g_{0\mu\nu} + \epsilon\phi_\mu\phi_\nu\,. \tag{2.1}$$

Since $\det h = (\det g_0)\rho$ where

$$\rho = 1 + \epsilon g_0^{\mu\nu}\phi_\mu\phi_\nu\,, \tag{2.2}$$

then h is everywhere (except at those points where $\rho = 0$) invertible. Its inverse is given by

$$h^{\mu\nu} = g_0^{\mu\nu} - \frac{\epsilon}{\rho}\phi^\mu\phi^\nu \tag{2.3}$$

where $g_0^{\mu\nu}$ are the components of the inverse matrix g_0^{-1} of g_0. Here the indices are lowered and raised by the metric g_0 and its inverse g_0^{-1} respectively. For instance, $\phi^\mu{}_\nu = g_0^{\mu\alpha}\phi_{\alpha\nu}$. The Ricci tensor corresponding to the metric in (2.1) is given by

$$r_{\mu\nu} = \frac{\epsilon}{\rho}(\nabla^2\phi)\phi_{\mu\nu} - \frac{\epsilon}{\rho}\phi^\alpha_\mu\phi_{\nu\alpha} + \frac{1}{4\rho^2}\rho_\mu\rho_\nu \tag{2.4}$$

where

$$\nabla^2\phi = h^{\mu\nu}\phi_{\mu\nu} = g_0^{\mu\nu}\phi_{\mu\nu} - \frac{1}{2\rho}\phi^\alpha\rho_\alpha\,. \tag{2.5}$$

The Ricci scalar or the Gaussian curvature K and the mean curvature H are obtained as

$$K = \frac{\epsilon}{\rho^2}[(\phi^\alpha_\alpha)^2 - \phi^{\alpha\beta}\phi_{\alpha\beta}] \tag{2.6}$$

$$H = \frac{1}{\sqrt{\rho}} h^{\mu\nu}\phi_{\mu\nu} \,. \tag{2.7}$$

The following equation is valid only for the case of two dimensional geometries.

$$\phi_{\alpha\mu}\phi_{\beta\gamma} - \phi_{\alpha\beta}\phi_{\mu\gamma} = -\lambda_0(g_{0\alpha\mu}g_{0\beta\gamma} - g_{0\alpha\beta}g_{0\gamma\mu}) \tag{2.8}$$

where

$$\lambda_0 = \frac{1}{2}[\phi^{\alpha\beta}\phi_{\alpha\beta} - (\phi^\alpha_\alpha)^2] \,. \tag{2.9}$$

Contracting this equation with $g^{\alpha\beta}$ leads to

$$\phi^\alpha_\mu\phi_{\alpha\nu} - \phi^\alpha_\alpha\phi_{\mu\nu} = \lambda_0 g_{0\mu\nu} \,.$$

We also have

$$r_{\alpha\beta} = \frac{K}{2}h_{\alpha\beta} \,, \qquad \lambda_0 = -\frac{\epsilon}{2}\rho^2 K \,.$$

For the minimal surfaces we have $H = 0$ and the following important properties of the metric $h_{\alpha\beta}$ on S_0 [7]

$$\partial_\alpha[\sqrt{\rho}h^{\alpha\beta}\partial_\beta\phi] = 0 \,, \tag{2.10}$$

$$\partial_\alpha(\sqrt{\rho}h^{\alpha\beta}) = 0 \,. \tag{2.11}$$

We now define surfaces which are locally conformal to minimal surfaces. Let S be such a surface, i. e., locally conformal to S_0. Then the metric on S is given by

$$g_{\alpha\beta} = \frac{1}{\sqrt{\rho}}h_{\alpha\beta} \,. \tag{2.12}$$

It is clear that $\det g = \det g_0 \neq 0$. In the sequel we shall assume that the surface S_0 is minimal and hence the metric defined on it satisfies all the equivalent conditions in (2.10) and (2.11). The corresponding Ricci tensor of g is given as

$$R_{\alpha\beta} = r_{\alpha\beta} - (\nabla^2_g\psi_0)g_{\alpha\beta} \tag{2.13}$$

where $\psi_0 = -\frac{1}{4}\log(\rho)$ and ∇^2_g is the Laplace–Beltrami operator with respect to the metric g. Then we have

Proposition 1. *The following equation is an identity related to the conformal surface S.*

$$R = -\frac{1}{4} g^{\alpha\beta}\, \mathrm{Tr}[\partial_\alpha g^{-1} \partial_\beta g]\,. \tag{2.14}$$

Here g is the 2×2 matrix of $g_{\alpha\beta}$ and g^{-1} is its inverse. The operation Tr *is the standard trace operation for matrices.*

In the following parts of the work we need some harmonic functions with respect to the metric g. For this purpose we introduce some vectors on S. Let $v_\alpha = (1,0)$, $v'_\alpha = (0,1)$ and $u^\alpha = (1,0)$, $u'^\alpha = (0,1)$. We now define some functions over S.

$$\xi_1 = g^{\alpha\beta} v_\alpha v_\beta\,, \qquad \xi_2 = g^{\alpha\beta} v'_\alpha v'_\beta\,, \tag{2.15}$$

$$w_1 = \sqrt{\rho}\, g_{\alpha\beta} u^\alpha u^\beta\,, \qquad w_2 = \sqrt{\rho}\, g_{\alpha\beta} u'^\alpha u'^\beta\,. \tag{2.16}$$

It is now easy to prove

Proposition 2.

$$\nabla_g^2 \zeta - a_0 R = -a_0 \sqrt{\rho} K\,, \tag{2.17}$$

$$\nabla_g^2 \psi_1 - (a_1 + a_2) R = 0\,, \tag{2.18}$$

$$\nabla_g^2 \psi_2 - 2(b_1 + b_2) R = -(b_1 + b_2)\sqrt{\rho} K \tag{2.19}$$

where

$$\zeta = \frac{a_0}{2} \log(\rho)\,, \tag{2.20}$$

$$\psi_1 = a_1 \log(\xi_1) + a_2 \log(\xi_2)\,, \tag{2.21}$$

$$\psi_2 = b_1 \log(w_1) + b_2 \log(w_2)\,. \tag{2.22}$$

Here a_0, a_1, a_2, b_1, and b_2 are arbitrary constants.

The function μ defined by $\mu = (b_1 + b_2)\zeta - a_0 \psi_2$ satisfies similar equation as ψ_1

$$\nabla_g^2 \mu = -a_0 (b_1 + b_2) R\,. \tag{2.23}$$

Hence we have two different solutions of the equation

$$\nabla_g^2 \sigma = -\frac{c}{4} g^{\alpha\beta}\, \mathrm{Tr}[(\partial_\alpha g^{-1}) \partial_\beta g]\,, \tag{2.24}$$

for some function σ. If $\sigma = \psi_1$ then $c = a_1 + a_2$, if $\sigma = \mu$ then $c = -a_0(b_1 + b_2)$. It is straightforward to show that

$$\xi_1 = \frac{w_2}{\det g_0 \sqrt{\rho}}, \qquad \xi_2 = \frac{w_1}{\det g_0 \sqrt{\rho}}. \tag{2.25}$$

Hence ψ_1 will not be considered as an independent function. It is interesting and important to note that under the minimality condition the matrix g satisfies the following condition as well.

Proposition 3. *Minimality of S_0, $H = 0$, also implies a sigma model [7], [8] like equation for g, i. e.,*

$$\partial_\alpha [g^{\alpha\beta} g^{-1} \partial_\beta g] = 0\,. \tag{2.26}$$

Proof: The metric $g_{\alpha\beta}$ and its inverse $g^{\alpha\beta}$ are written in a nice form

$$g_{\alpha\beta} = \frac{1}{\sqrt{\rho}}(g_{0\alpha\beta} + \epsilon\phi_\alpha\phi_\beta)\,, \tag{2.27}$$

$$g^{\alpha\beta} = \sqrt{\rho}(g_0^{\alpha\beta} - \frac{\epsilon}{\rho}\phi^\alpha\phi^\beta) \tag{2.28}$$

where $g_{0\alpha\beta}$ are the components of the matrix g_0. The minimality condition $H = 0$ reduces to $g^{\alpha\beta}\phi_{\alpha\beta} = 0$ or

$$\phi^\alpha{}_\alpha = \frac{\phi^\alpha \rho_\alpha}{2\rho}\,. \tag{2.29}$$

This condition also implies

$$\partial_\mu g^{\mu\nu} = 0\,. \tag{2.30}$$

Hence the sigma model equation (2.26) to be proved takes the form

$$h^{\mu\nu}\partial_\nu[g^{\alpha\gamma}\partial_\mu g_{\gamma\beta}] = 0\,, \tag{2.31}$$

where $h_{\alpha\beta} = \sqrt{\rho} g_{\alpha\beta}$. It is straightforward to show that

$$\begin{aligned}(g^{-1}\partial_\mu g)^\alpha_\beta &= g^{\alpha\gamma}\partial_\mu g_{\beta\gamma} \\ &= -\frac{1}{2}\frac{\rho_\mu}{\rho}\delta^\alpha_\beta - \frac{\epsilon}{2}\frac{\rho_\mu}{\rho}\phi^\alpha\phi_\beta + \epsilon\phi_\beta\phi^\alpha{}_\mu + \frac{\epsilon}{\rho}\phi^\alpha\phi_{\mu\beta}\,,\end{aligned} \tag{2.32}$$

Using the identity (2.8) and the minimality condition (2.29) we obtain the following

$$\rho_\mu\phi_{\beta\gamma} - \rho_\beta\phi_{\mu\gamma} = 2\epsilon\lambda_0(\phi_\mu g_{0\beta\gamma} - \phi_\beta g_{0\gamma\mu})\,, \tag{2.33}$$

$$\rho^\mu\phi_{\mu\beta} = \phi^\alpha{}_\alpha\rho_\beta - 2\epsilon\lambda_0\phi_\beta\,, \tag{2.34}$$

$$h^{\alpha\beta}\partial_\alpha\left(\frac{1}{\rho}\partial_\beta\rho\right)+\frac{2\lambda_0}{\rho^2}(1+\rho)=0\,. \tag{2.35}$$

Utilizing these identities we get

$$h^{\alpha\beta}\phi^\mu{}_{\alpha\beta}=-\frac{2\epsilon\lambda_0}{\rho}\phi^\mu\,, \tag{2.36}$$

$$h^{\alpha\beta}\phi_{\mu\alpha}\phi^\nu{}_\beta=-\frac{\lambda_0}{\rho}\delta^\nu_\mu-\frac{\epsilon\lambda_0}{\rho}\phi^\nu\phi_\mu\,, \tag{2.37}$$

$$h^{\alpha\beta}\partial_\alpha(\phi^\nu{}_\beta\phi_\mu)=-\frac{\lambda_0}{\rho}\delta^\nu_\mu-\frac{3\epsilon\lambda_0}{\rho}\phi^\nu\phi_\mu\,, \tag{2.38}$$

$$\rho_\alpha h^{\alpha\beta}\partial_\beta(\phi^\nu\phi_\mu)=-4\epsilon\lambda_0\phi^\nu\phi_\mu\,. \tag{2.39}$$

Now applying ∂_ν to (2.32) then multiplying by $h^{\mu\nu}$ and using the above identities (by virtue of the minimality condition (2.29)) it is easy to show (2.31). □

Hence for every minimal surface S_0 and its metric h we have a conformally related surface S with metric $g=\frac{h}{\sqrt{\rho}}$ ($\det h=\rho\det g_0$) satisfying the conditions

$$R+\frac{1}{4}g^{\alpha\beta}\,\mathrm{Tr}[\partial_\alpha g^{-1}\partial_\beta g]=0\,, \tag{2.40}$$

$$\partial_\alpha[g^{\alpha\beta}g^{-1}\partial_\beta g]=0\,. \tag{2.41}$$

Here g has determinant equals to $\det g_0$ which is a nonzero constant. This does not violate the covariance of our formulation because we could formulate everything in terms of the metric h of the minimal surfaces S_0 but the above identities become lengthy and complicated. We loose no generality by using surfaces S and the metric g on them.

3. Ricci Flat Pseudo-Riemannian Geometries

We start first with four dimensions. Let the metric of the four dimensional manifold M_4 be given by

$$\mathrm{d}s^2=\mathrm{e}^{2\psi}g_{\alpha\beta}\,\mathrm{d}x^\alpha\,\mathrm{d}x^\beta+\epsilon_1 g_{\alpha\beta}\,\mathrm{d}y^\alpha\,\mathrm{d}y^\beta \tag{3.1}$$

where ψ is a function of x^α and $\epsilon_1=\pm1$. The local coordinates of M_4 are denoted as $x^a=(x^\alpha,y^\alpha)$, $a=1,\ldots,4$.

Proposition 4. *The Ricci flat equations $R_{ab}=0$ for the metric* (3.1) *are given in two sets. One set is satisfied identically due to the Proposition 3 above and the second one is given by*

$$\nabla_g^2\psi=0\,. \tag{3.2}$$

There are two independent functions satisfying the above Laplace equation, ϕ and μ. Using (2.23) we find that $\psi = e_0\phi + e_1\mu$ where e_0 and e_2 are arbitrary constants and $b_2 = -b_1$. Combining all these constants we find that

$$\mathrm{e}^{2\psi} = \mathrm{e}^{2e_0\phi} w_1^{-2m_1} w_2^{-2m_2} \tag{3.3}$$

where m_1 and m_2 are constants satisfying $m_1 + m_2 = 0$. Then the line element (3.1) becomes

$$\mathrm{d}s^2 = \frac{\mathrm{e}^{2e_0\phi}}{w_1^{2m_1} w_2^{2m_2}} \frac{h_{\alpha\beta}\,\mathrm{d}x^\alpha\,\mathrm{d}x^\beta}{\sqrt{\rho}} + \frac{h_{\alpha\beta}\,\mathrm{d}y^\alpha\,\mathrm{d}y^\beta}{\sqrt{\rho}} \tag{3.4}$$

where ϕ satisfies the minimality condition ($H = 0$) (2.7) which is explicitly given by

$$[k_2 + \epsilon(\phi_y)^2]\phi_{xx} - 2[k_0 + \epsilon\phi_x\phi_y]\phi_{xy} + [k_1 + \epsilon(\phi_x)^2]\phi_{yy} = 0 \tag{3.5}$$

where we take $(g_0)_{11} = k_1$, $(g_0)_{01} = k_0$, $(g_0)_{22} = k_2$ and assume that $\det(g_0) = k_1k_2 - k_0^2 \neq 0$. Hence the functions w_1 and w_2 are given explicitly as

$$w_1 = k_1 + \epsilon(\phi_x)^2, \qquad w_2 = k_2 + \epsilon(\phi_y)^2\,. \tag{3.6}$$

The metric in (3.4) with $e_0 = 0, m_1 = m_2 = 0$ reduces to an instanton metric [10].

We shall now generalize Proposition 4 for an arbitrary even dimensional pseudo-Riemannian geometry. Let M_{2+2n} be a $2 + 2n$ dimensional manifold with a metric

$$\mathrm{d}s^2 = \mathrm{e}^{2\Phi} g_{\alpha\beta}\,\mathrm{d}x^\alpha\,\mathrm{d}x^\beta + G_{AB}\,\mathrm{d}y^A\,\mathrm{d}y^B \tag{3.7}$$

where the local coordinates of M_{2+2n} are given by $x^{\alpha+A} = (x^\alpha, y^A)$, $A = 1, 2, \ldots, 2n$, Φ and G_{AB} are functions of x^α alone. The Einstein equations are given in the following proposition

Proposition 5. *The Ricci flat equations for the metric in (3.7) are given by*

$$\partial_\alpha[g^{\alpha\beta}G^{-1}\partial_\beta G] = 0\,, \tag{3.8}$$

$$\nabla_g^2\Phi = \frac{1}{8}g^{\alpha\beta}\,\mathrm{Tr}[(\partial_\alpha G^{-1})\partial_\beta G] + \frac{R}{2} \tag{3.9}$$

where G is $2n\times 2n$ matrix of G_{AB} and G^{-1} is its inverse.

Let us choose G as a block diagonal matrix and each block is the 2×2 matrix g. This means that the metric in (3.7) reduces to a special form

$$\mathrm{d}s^2 = \mathrm{e}^{2\Phi} g_{\alpha\beta}\,\mathrm{d}x^\alpha\,\mathrm{d}x^\beta + \epsilon_1 g_{\alpha\beta}\,\mathrm{d}y_1^\alpha\,\mathrm{d}y_1^\beta + \cdots + \epsilon_n g_{\alpha\beta}\,\mathrm{d}y_n^\alpha\,\mathrm{d}y_n^\beta \tag{3.10}$$

where the local coordinates of M_{2+2n} are given by $x^{\alpha+A} = (x^\alpha, y_1^\alpha, \ldots, y_n^\alpha)$, $\epsilon_i = \pm 1$, $i = 1, 2, \ldots, n$. Then we have the following theorem

Theorem 1. *For every two dimensional minimal surface* S_0 *immersed in a three dimensional manifold* M_3 *there corresponds a* $2N = 2 + 2n$*-dimensional Ricci flat (pseudo-)Riemannian geometry with the metric given in (* 3.10*) with*

$$e^{2\Phi} = e^{2\psi} w_1^{-2n_1} w_2^{-2n_2} \rho^{n_1+n_2} \tag{3.11}$$

where ψ *is given in (*3.3*),* w_1 *and* w_2 *are given in (*3.6*),* n_1 *and* n_2 *satisfy*

$$n_1 + n_2 = \frac{n-1}{2}\,. \tag{3.12}$$

Proof: Using Proposition 5 for the metric (3.10) the Ricci flat equations reduce to the following equation

$$\nabla_g^2 \Phi = \frac{n-1}{8} g^{\alpha\beta}\, \mathrm{Tr}[(\partial_\alpha g^{-1})\partial_\beta g] \tag{3.13}$$

By using (2.24) and letting $a_0 b_1 = n_1$, $a_0 b_2 = n_2$ and $\Phi = \mu + \psi$ we find (3.11) with the condition (3.12). Here ψ is a harmonic function (3.2) with respect to the metric $g_{\alpha\beta}$. A solution of this function is given in the previous section in (3.3). Metric functions ψ, w_1, w_2 and $g_{\alpha\beta}$ are expressed explicitly in terms the function ϕ and its derivatives ϕ_x and ϕ_y. This means that for each solution ϕ of (3.5) there exists a $2N$-dimensional metric (3.10). □

The dimension of the manifold is $4(1 + n_1 + n_2)$. Here $n = 1$ or $n_1 + n_2 = 0$ corresponds to the four dimensional case. The signature of the geometry depends on the signature of S. If S has zero signature then M_{2N} has also zero signature, but if the signature of S is 2 then the signature of M_{2N} is $2(1 + \epsilon_1 + \cdots + \epsilon_n)$.

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