Third International Conference on
Geometry, Integrability and Quantization
June 14–23, 2001, Varna, Bulgaria
Ivaïlo M. Mladenov and Gregory L. Naber, Editors
Coral Press, Sofia 2001, pp 209–223

DEGENERATE CURVES IN PSEUDO-EUCLIDEAN SPACES OF INDEX TWO

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Abstract. We study degenerate curves in pseudo-Euclidean spaces of index two by introducing the Cartan reference along a degenerate curve. We obtain several different types of degenerate curves and present existence, uniqueness and congruence theorems. We also give some examples of such a curves in low dimensions.

1. Introduction

The aim of this paper is to find a good Frenet frame for degenerate curves in pseudo-Euclidean spaces of index two. The study of this type of curves is motivated because of the growing importance that degenerate geometry (null curves, null hypersurfaces, etc) plays in mathematical physics (see for instance [2, 7–10]). Null curves in Lorentzian (index one) space forms has been studied by several authors ([1, 3, 5]) due to its importance in General Relativity. It is well known the important role played by the anti de Sitter space, so we focus on ambient spaces of index two. A first approach to this question has been made by Duggal and Jin, [4], from a different point of view.

Here, we are going to study degenerate curves in pseudo-Euclidean spaces of index two from a mathematical viewpoint.

2. Preliminaries

Let V be an n-dimensional real vector space endowed with a symmetric bilinear mapping $g\colon V \times V \to \mathbb{R}$. We will say that g is degenerate on V if there exists a vector $\xi \neq 0$ of V such that

$$g(\xi, v) = 0\,, \quad \text{for all } v \in V$$

otherwise, g is said to be non-degenerate.

The **radical** of (V, g) is the subspace of V defined by

$$\operatorname{Rad} V = \{\xi \in V\,;\, g(\xi, v) = 0 \ \text{ for all } v \in V\}\,.$$

It is clear that V is non-degenerate if and only if $\operatorname{Rad} V = \{\mathbf{0}\}$.

A **pseudo-Euclidean space** (V, g) will be an n-dimensional real vector space V equipped with a symmetric non-degenerate bilinear map g. The dimension q of the largest subspace $W \subset V$ on which $g|_W$ is definite negative is called the index of g on V. (V, g) will be denoted by $\mathbb{R}^n_q$.

Let $B = \{V_1, \ldots, V_n\}$ be an ordered basis of a pseudo-Euclidean space and let r_i and q_i be the dimension of the radical and the index of $\operatorname{span}\{V_1, \ldots, V_i\}$ for all i, respectively. The sequences $\{r_i\,;\, 0 \leq i \leq n\}$ and $\{q_i\,;\, 0 \leq i \leq n\}$, where $r_0 = q_0 = 0$, will be called the *nullity degree sequence* and the *index sequence* of the basis B.

It is easy to see that $|r_i - r_{i-1}|$ and $q_i - q_{i-1}$ are either 0 or 1, for all $i = 1, \ldots, n$, as well as $r_n = 0$ and $q_n = q$.

Definition 2.1. *Let $B = \{V_1, \ldots, V_n\}$ be an ordered basis of a pseudo-Euclidean space and let $\{r_i\,;\, 1 \leq i \leq n\}$ be the nullity degree sequence. The positive number*

$$r = \frac{1}{2} \sum_{i=1}^{n} |r_i - r_{i-1}|$$

*is said to be the **degeneration degree** of the basis B.*

The following result, that extends the Gram–Smidt's orthonormalization method, will be used in next sections.

Lemma 2.1. *Let $(E, \langle\,,\rangle)$ be a bilinear space and let F be a hyperplane. Suppose that $F = F_1 \perp F_2$, where $F_1 = \operatorname{span}\{L_1, \ldots, L_r\}$ is totally lightlike and F_2 is non-degenerate. Then we have:*

i) *If $\dim \operatorname{Rad}(E) = r + 1$ ($F_1 \subsetneq \operatorname{Rad}(E)$), there exists a null vector L (not unique) such that*

$$E = F_1 \perp F_2 \perp \operatorname{span}\{L\}\,.$$

ii) *If $\dim \operatorname{Rad}(E) = r$ ($F_1 = \operatorname{Rad}(E)$), there exists a non-null unit vector V such that*

$$E = F_1 \perp F_2 \perp \operatorname{span}\{V\}\,.$$

Moreover, if $\operatorname{Rad}(E) = \{\mathbf{0}\}$, then V is unique (up to the sign).

iii) *If* $\dim \mathrm{Rad}(E) = r - 1$ $(\mathrm{Rad}(E) \subsetneq F_1)$, *there exists a null vector* N_j *such that* $\langle L_j, N_j\rangle = \eta$, $\eta = \pm 1$, *and*

$$E = (\mathrm{span}\{L_j\} \oplus \mathrm{span}\{N_j\}) \perp \mathrm{span}\{L_1, \ldots, \widehat{L}_j, \ldots, L_r\} \perp F_2 \,.$$

Furthermore, if $\mathrm{Rad}(E) = \{\mathbf{0}\}$, *then* N_j *is unique.*

Definition 2.2. *A basis* $B = \{L_1, N_1, \ldots, L_r, N_r, W_1, \ldots, W_m\}$ *of* $\mathbb{R}^n_q$, *with* $2r \leq 2q \leq n$ *and* $m = n - 2r$, *is said to be* pseudo-orthonormal *if it satisfies the following conditions:*

$$\begin{aligned} \langle L_i, L_j\rangle = \langle N_i, N_j\rangle = 0\,, \qquad & \langle L_i, N_j\rangle = \eta_i \delta_{ij}\,, \\ \langle L_i, W_\alpha\rangle = \langle N_i, W_\alpha\rangle = 0\,, \qquad & \langle W_\alpha, W_\beta\rangle = \varepsilon_\alpha \delta_{\alpha\beta}\,, \end{aligned}$$

where $i, j \in \{1, \ldots, r\}$, $\eta_i = \langle L_i, N_i\rangle = \pm 1$, $\alpha, \beta \in \{1, \ldots, m\}$, $\varepsilon_\alpha = -1$ *if* $1 \leq \alpha \leq q - r$ *and* $\varepsilon_\alpha = 1$ *if* $q - r + 1 \leq \alpha \leq m$.

Corollary 2.1. *Let* $B = \{V_1, \ldots, V_n\}$ *be an ordered basis of a pseudo-Euclidean space and let* r *be the degeneration degree of* B. *Then:*

i) r *is well-defined, that is, it is an integer.*

ii) $r \leq q$, *where* q *is the index of* V.

Proof: We know that $r_0 = r_n = 0$ and sequence $\{r_i\}$ satisfies that either $r_i = r_{i-1} + 1$, or $r_i = r_{i-1} - 1$ or $r_i = r_{i-1}$. Then, from Lemma 2.1, we get a pseudo-orthonormal basis $\mathcal{C} = \{C_1, \ldots, C_n\}$ satisfying that $\mathrm{span}\{V_1, \ldots, V_i\} = \mathrm{span}\{C_1, \ldots, C_i\}$, for all $i = 1, \ldots, n$, and

$$C_i = \begin{cases} W_i & \text{if } r_i - r_{i-1} = 0\,, \\ L_i & \text{if } r_i - r_{i-1} = 1\,, \\ N_i & \text{if } r_i - r_{i-1} = -1\,, \end{cases}$$

where $\langle W_i, W_i\rangle = \pm 1$ and $\langle L_i, L_i\rangle = \langle N_i, N_i\rangle = 0$. Then (i) is clear. To show (ii), first observe that $r = \mathrm{card}\,\{i_k\,;\, C_{i_k} = L_{i_k}\}$. Now, for all $L_{i_k} \in \mathcal{C}$, there exists $N_{j_k} \in \mathcal{C}$, with k in $\{1, \ldots, r\}$, verifying that $\mathrm{span}\{L_{i_k}, N_{j_k}\}$ is a hyperbolic plane. Then $r \leq q$.

3. Frenet References Along Degenerate Curves

Let $\mathbb{R}^n_2$ be a pseudo-Euclidean space of index two and let $\gamma\colon I \to \mathbb{R}^n_2$ be a differentiable curve in $\mathbb{R}^n_2$. Assume that $\mathcal{A} = \{\gamma'(t), \ldots, \gamma^{(n)}(t)\}$ is a linearly independent system for all $t \in I$, and, for all i, $r_i(t)$ and $q_i(t)$ are constant for all $t \in I$, where $\{r_i(t)\,;\, 0 \leq i \leq n\}$ and $\{q_i(t)\,;\, 0 \leq i \leq n\}$ stand for the nullity degree and index sequences of the basis $\mathcal{A}$. In this case, these sequences will be called nullity degree and index sequences of the curve γ, respectively,

and the degeneration degree r ($= \text{const}$) of $\mathcal{A}$ will be called the degeneration degree of the curve γ.

Definition 3.1. *With the above notations, a curve $\gamma\colon I \to \mathbb{R}^n_2$ is said to be a* ***degenerate curve*** *if $r > 0$. We will say that two degenerate curves C and $\bar{C}$ are of the same type if $r_i = \bar{r}_i$ and $q_i = \bar{q}_i$, for all i.*

The relation "to be of the same type" defines an equivalence relation and each equivalence class defines a type of degenerate curves.

From definition and Corollary 2.1, the degeneration degree of a degenerate curve in a pseudo-Euclidean space of index two satisfies $0 < r \leq 2$. Observe that the index sequence is very conditioned by the nullity degree sequence. Indeed, two curves C and $\bar{C}$ with degeneration degree two are of the same type if and only if they have the same nullity degree sequence.

Remark 3.1. *The nullity degree and index sequences, as well as the degeneration degree, of a degenerate curve do not depend on the chosen parameter and they are invariant under pseudo-Euclidean transformations.*

Observe that we are dealing not only with null curves, but also spacelike and timelike ones. Now we aim to classify degenerate curves depending on the nullity degree and index sequences, said otherwise, to classify the types. To do that, we need to pseudo-orthonormalize the basis $\{\gamma'(t), \ldots, \gamma^{(i)}(t)\}$, for all $i = 1, \ldots, n$, such as in Corollary 2.1. The pseudo-orthonormal bases obtained are just the Frenet references.

We will consider two cases according to whether the degeneration degree r is one or two.

3.1. Degenerate Curves in $\mathbb{R}^n_2$ with Degeneration Degree One

In this case we will get a family-type of degenerate curves. The method to construct a Frenet frame is quite similar to that used in [6]. It can be proved that the only nullity degree sequences are of the form $\{0, \ldots, 0, 1, 1, 0, \ldots, 0\}$, where $1, 1, 0 \ldots$ can be moved along the sequence. The possible Frenet equations are as follows:

Family I

$$
\begin{aligned}
\gamma' &= \bar{\varepsilon}_1 \bar{k}_1 \bar{W}_1 \,, \\
\bar{W}_1' &= \bar{\varepsilon}_2 \bar{k}_2 \bar{W}_2 \,, \\
\bar{W}_i' &= -\bar{\varepsilon}_{i-1} \bar{k}_i \bar{W}_{i-1} + \bar{\varepsilon}_{i+1} \bar{k}_{i+1} \bar{W}_{i+1} \,, \quad 2 \leq i \leq s-2 \,, \\
\bar{W}_{s-1}' &= -\bar{\varepsilon}_{s-2} \bar{k}_{s-1} \bar{W}_{s-2} + \bar{\eta}_s \bar{k}_s \bar{L}_s \,,
\end{aligned}
$$

$$
\begin{aligned}
\bar{L}'_s &= \bar{\eta}_s \bar{k}_{s+1} \bar{L}_s + \bar{\varepsilon}_{s+1} \bar{k}_{s+2} \bar{W}_{s+1}\,, \\
\bar{W}'_{s+1} &= \bar{\eta}_s \bar{k}_{s+3} \bar{L}_s - \bar{\eta}_s \bar{k}_{s+2} \bar{N}_s\,, \\
\bar{N}'_s &= -\bar{\varepsilon}_{s-1} \bar{k}_s \bar{W}_{s-1} - \bar{\eta}_s \bar{k}_{s+1} \bar{N}_s - \bar{\eta}_{s+1} \bar{k}_{s+3} \bar{W}_{s+1} + \bar{\varepsilon}_{s+2} \bar{k}_{s+4} \bar{W}_{s+2}\,, \\
\bar{W}'_{s+2} &= -\bar{\eta}_s \bar{k}_{s+4} \bar{L}_s + \bar{\varepsilon}_{s+3} \bar{k}_{s+5} \bar{W}_{s+3}\,, \\
\bar{W}'_i &= -\bar{\varepsilon}_{i-1} \bar{k}_{i+2} \bar{W}_{i-1} + \bar{\varepsilon}_{i+1} \bar{k}_{i+3} \bar{W}_{i+1}\,, \quad s+3 \leq i \leq n-2\,, \\
\bar{W}'_{n-1} &= -\bar{\varepsilon}_{n-2} \bar{k}_{n+1} \bar{W}_{n-2}
\end{aligned}
$$

where $\bar{\eta}_j = \langle \bar{L}_j, \bar{N}_j \rangle = \pm 1$ and $\bar{\varepsilon}_j = \langle \bar{W}_j, \bar{W}_j \rangle = \pm 1$, existing only one j_0 such that $\bar{\varepsilon}_{j_0} = -1$.

3.2. Degenerate Curves in $\mathbb{R}^n_2$ with Degeneration Degree Two

We will find two family-types of curves depending on the nullity degree sequence is given by $\{0, \ldots, 0, 1, 1, 0, \ldots, 0, 1, 1, 0, \ldots, 0\}$ or $\{0, \ldots, 0, 1, 2, 2, 1, 0, \ldots, 0\}$. To do that we proceed as follows.

Assume that $r_1 = r_2 = \cdots = r_{s-1} = 0$. By an iterative process, using Lemma 1, we obtain a set $\{\bar{W}_1, \ldots, \bar{W}_{s-1}\}$ of orthonormal spacelike vector fields along γ. Now suppose that $r_s = 1$. From Lemma 2.1 and Corollary 2.1 the possible cases are collected in Figure 1. We will rule out those ones which are not admissible.

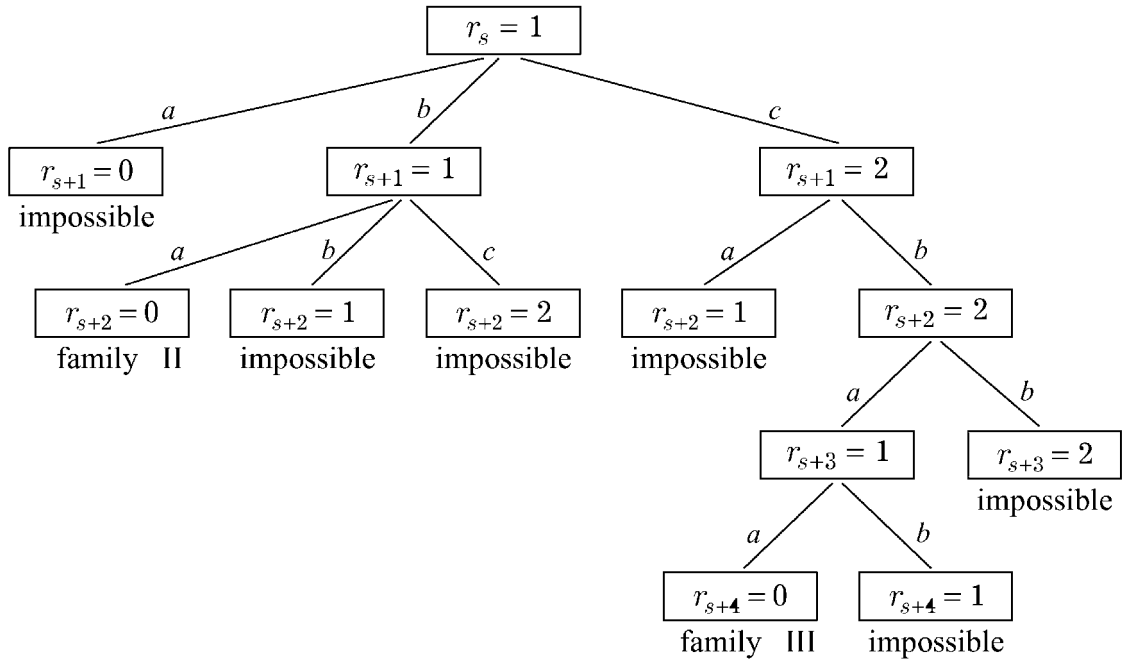

Figure 1. Tree of possibilities

Way *a*: We have the following equations:

$r_1 = 0$	$\gamma' = \bar{k}_1 \bar{W}_1$
$r_2 = 0$	$4\bar{W}_1' = \bar{k}_2 \bar{W}_2$
$r_{i+1} = 0$	$\bar{W}_i' = -\bar{k}_i \bar{W}_{i-1} + \bar{k}_{i+1} \bar{W}_{i+1}$
$r_s = 1$	$\bar{W}_{s-1}' = -\bar{k}_{s-1} \bar{W}_{s-2} + \bar{\eta}_s \bar{k}_s \bar{L}_s$
$r_{s+1} = 0$	$\bar{L}_s' = \bar{\eta}_s \bar{k}_{s+1} \bar{L}_s$

It is clear that $\bar{L}_s \in \mathrm{span}\{\gamma', \ldots, \gamma^{(s)}\}$, so we write $\bar{L}_s = \lambda_1 \gamma' + \cdots + \lambda_s \gamma^{(s)}$, with $\lambda_s \neq 0$. Then $\bar{L}_s' = \cdots + \lambda_s \gamma^{(s+1)} = \bar{\eta}_s \bar{k}_{s+1} \bar{L}_s$ and $\gamma^{(s+1)} \in \mathrm{span}\{\gamma', \ldots, \gamma^{(s)}\}$, which is a contradiction.

Way *bb*: Now we obtain:

$r_s = 1$	$\bar{W}_{s-1}' = -\bar{k}_{s-1} \bar{W}_{s-2} + \bar{\eta}_s \bar{k}_s \bar{L}_s$
$r_{s+1} = 1$	$\bar{L}_s' = \bar{\eta}_s \bar{k}_{s+1} \bar{L}_s + \bar{k}_{s+2} \bar{W}_{s+1}$

Since $r_{s+2} = 1$, then $\mathrm{Rad}(E_{s+2}) = \mathrm{span}\{\bar{L}_s\}$ and $\langle \bar{L}_s, \gamma^{(s+1)} \rangle = \langle \bar{L}_s, \gamma^{(s+2)} \rangle = 0$. We deduce that $\langle \bar{L}_s', \gamma^{(s+1)} \rangle = 0$ and, using the above equations, we get $\langle \bar{W}_{s+1}, \gamma^{(s+1)} \rangle = 0$. Therefore $\bar{W}_{s+1} \in \mathrm{Rad}(E_{s+1})$, which can not hold.

Way *bc*: We find that:

$r_s = 1$	$\bar{W}_{s-1}' = -\bar{k}_{s-1} \bar{W}_{s-2} + \bar{\eta}_s \bar{k}_s \bar{L}_s$
$r_{s+1} = 1$	$\bar{L}_s' = \bar{\eta}_s \bar{k}_{s+1} \bar{L}_s + \bar{k}_{s+2} \bar{W}_{s+1}$
$r_{s+2} = 2$	$\bar{W}_{s+1}' = \bar{\eta}_s \bar{k}_{s+3} \bar{L}_s + \bar{\eta}_{s+1} \bar{k}_{s+4} \bar{L}_{s+1}$

Then we write

$$0 \neq \bar{k}_{s+2} = \langle \bar{L}_s', \bar{W}_{s+1} \rangle = -\langle \bar{W}_{s+1}', \bar{L}_s \rangle = 0\,,$$

getting again a contradiction.

Way *ca*: We have:

$r_s = 1$	$\bar{W}_{s-1}' = -\bar{k}_{s-1} \bar{W}_{s-2} + \bar{\eta}_s \bar{k}_s \bar{L}_s$
$r_{s+1} = 2$	$\bar{L}_s' = \bar{\eta}_s \bar{k}_{s+1} \bar{L}_s + \bar{\eta}_{s+1} \bar{k}_{s+2} \bar{L}_{s+1}$
$r_{s+2} = 1$	$\bar{L}_{s+1}' = \bar{\eta}_s \bar{k}_{s+3} \bar{L}_s + \bar{\eta}_{s+1} \bar{k}_{s+4} \bar{L}_{s+1} + \bar{\eta} \bar{k} \bar{N}$

We obtain that either $\bar{N} = \bar{N}_{s+1}$ or $\bar{N} = \bar{N}_s$. In the first case we find $\bar{k} = \langle \bar{L}_{s+1}', \bar{L}_{s+1} \rangle = 0$ and in the second one we have $\bar{k} = \langle \bar{L}_{s+1}', \bar{L}_s \rangle = -\langle \bar{L}_{s+1}, \bar{L}_s' \rangle = 0$. In any case $\bar{k} = 0$, which can not be hold.

Way *cbb*: Now the equations are:

$r_s = 1$	$\bar{W}'_{s-1} = -\bar{k}_{s-1}\bar{W}_{s-2} + \bar{\eta}_s\bar{k}_s\bar{L}_s$
$r_{s+1} = 2$	$\bar{L}'_s = \bar{\eta}_s\bar{k}_{s+1}\bar{L}_s + \bar{\eta}_{s+1}\bar{k}_{s+2}\bar{L}_{s+1}$
$r_{s+2} = 2$	$\bar{L}'_{s+1} = \bar{\eta}_s\bar{k}_{s+3}\bar{L}_s + \bar{\eta}_{s+1}\bar{k}_{s+4}\bar{L}_{s+1} + \bar{k}_{s+5}\bar{W}_{s+2}$

Working as above, we get again a contradiction.

Way *cbab*: The Frenet equation write down as follows:

$r_s = 1$	$\bar{W}'_{s-1} = -\bar{k}_{s-1}\bar{W}_{s-2} + \bar{\eta}_s\bar{k}_s\bar{L}_s$
$r_{s+1} = 2$	$\bar{L}'_s = \bar{\eta}_s\bar{k}_{s+1}\bar{L}_s + \bar{\eta}_{s+1}\bar{k}_{s+2}\bar{L}_{s+1}$
$r_{s+2} = 2$	$\bar{L}'_{s+1} = \bar{\eta}_s\bar{k}_{s+3}\bar{L}_s + \bar{\eta}_{s+1}\bar{k}_{s+4}\bar{L}_{s+1} + \bar{k}_{s+5}\bar{W}_{s+2}$
$r_{s+3} = 1$	$\bar{W}'_{s+2} = \bar{\eta}_s\bar{k}_{s+6}\bar{L}_s + \bar{\eta}_{s+1}\bar{k}_{s+7}\bar{L}_{s+1} - \bar{\eta}\bar{k}_{s+5}\bar{N}$
$r_{s+4} = 1$	$\bar{N}' = \bar{\eta}\bar{k}_{s+8}\bar{L} - \bar{k}_{s+7}\bar{W}_{s+2} + \bar{\eta}\bar{k}\bar{N} + \bar{k}_{s+9}\bar{W}_{s+3}$

Two possibilities can be given: (i) $\bar{N} = \bar{N}_s$, and therefore $\bar{L} = \bar{L}_{s+1}$, $\bar{\eta} = \bar{\eta}_{s+1}$ and $\bar{k} = \bar{k}_{s+1}$; and (ii) $\bar{N} = \bar{N}_{s+1}$, and therefore $\bar{L} = \bar{L}_s$, $\bar{\eta} = \bar{\eta}_s$ and $\bar{k} = \bar{k}_{s+4}$.

In any case, we find a contradiction.

Hence, we have only to consider two admissible families. As for Family III, it is clear that $r_i = 0$, for $i \geq s+4$. So the Frenet reference is given by

$$\{\bar{W}_1, \ldots, \bar{W}_{s-1}, \bar{L}_s, \bar{L}_{s+1}, \bar{W}_{s+2}, \bar{N}_{s+1}, \bar{N}_s, \bar{W}_{s+3}, \ldots, \bar{W}_{n-2}\}\,.$$

As for Family II, calling $s_1 = s$, there is only one $s_2 \geq s_1 + 3$ satisfying $r_{s_2} = 1$ and $r_i = 0$, for $i = s_1 + 2, \ldots, s_2 - 1$. We also have that $r_{s_2+1} = 1$ and $r_i = 0$, for all $i \geq s_2 + 2$. Therefore, the Frenet reference for this Family is of the form:

$$\{\bar{W}_1, \ldots, \bar{W}_{s_1-1}, \bar{L}_{s_1}, \bar{W}_{s_1+1}, \bar{N}_{s_1}, \bar{W}_{s_1+2}, \ldots \\ \ldots, \bar{W}_{s_2-1}, \bar{L}_{s_2}, \bar{W}_{s_2+1}, \bar{N}_{s_2}, \bar{W}_{s_2+2}, \ldots, \bar{W}_{n-2}\}\,.$$

Summing up, the general Frenet equations for degenerate curves in $\mathbb{R}^n_2$ with degeneration degree two state as follows:

Family II

$$\begin{aligned}
\gamma' &= \bar{k}_1\bar{W}_1 \\
\bar{W}'_1 &= \bar{k}_2\bar{W}_2 \\
\bar{W}'_i &= -\bar{k}_i\bar{W}_{i-1} + \bar{k}_{i+1}\bar{W}_{i+1}\,, \quad 2 \leq i \leq s-2 \\
\bar{W}'_{s_1-1} &= -\bar{k}_{s_1-1}\bar{W}_{s_1-2} + \bar{\eta}_{s_1}\bar{k}_{s_1}\bar{L}_{s_1} \\
\bar{L}'_{s_1} &= \bar{\eta}_{s_1}\bar{k}_{s_1+1}\bar{L}_{s_1} + \bar{k}_{s_1+2}\bar{W}_{s_1+1}
\end{aligned}$$

$$
\begin{aligned}
\bar{W}'_{s_1+1} &= \bar{\eta}_{s_1}\bar{k}_{s_1+3}\bar{L}_{s_1} - \bar{\eta}_{s_1}\bar{k}_{s_1+2}\bar{N}_{s_1}\\
\bar{N}'_{s_1} &= -\bar{k}_{s_1}\bar{W}_{s_1-1} - \bar{\eta}_{s_1}\bar{k}_{s_1+1}\bar{N}_{s_1} - \bar{k}_{s_1+3}\bar{W}_{s_1+1} + \bar{k}_{s_1+4}\bar{W}_{s_1+2}\\
\bar{W}'_{s_1+2} &= -\bar{\eta}_{s_1}\bar{k}_{s_1+4}\bar{L}_{s_1} + \bar{k}_{s_1+5}\bar{W}_{s_1+3}\\
\bar{W}'_{i} &= -\bar{k}_{i+2}\bar{W}_{i-1} + \bar{k}_{i+3}\bar{W}_{i+1}\,, \quad s_1+3 \le i \le s_2-2\\
\bar{W}'_{s_2-1} &= -\bar{k}_{s_2+1}\bar{W}_{s_2-2} + \bar{\eta}_{s_2}\bar{k}_{s_2+2}\bar{L}_{s_2}\\
\bar{L}'_{s_2} &= \bar{\eta}_{s_2}\bar{k}_{s_2+3}\bar{L}_{s_2} + \bar{k}_{s_2+4}\bar{W}_{s_2+1}\\
\bar{W}'_{s_2+1} &= \bar{\eta}_{s_2}\bar{k}_{s_2+5}\bar{L}_{s_2} - \bar{\eta}_{s_2}\bar{k}_{s_2+4}\bar{N}_{s_2}\\
\bar{N}'_{s_2} &= -\bar{k}_{s_2+2}\bar{W}_{s_2-1} - \bar{\eta}_{s_2}\bar{k}_{s_2+3}\bar{N}_{s_2} - \bar{k}_{s_2+5}\bar{W}_{s_2+1} + \bar{k}_{s_2+6}\bar{W}_{s_2+2}\\
\bar{W}'_{s_2+2} &= -\bar{\eta}_{s_2}\bar{k}_{s_2+6}\bar{L}_{s_2} + \bar{k}_{s_2+7}\bar{W}_{s_2+3}\\
\bar{W}'_{i} &= -\bar{k}_{i+4}\bar{W}_{i-1} + \bar{k}_{i+5}\bar{W}_{i+1}\,, \quad s_2+3 \le i \le n-3\\
\bar{W}'_{n-2} &= -\bar{k}_{n+2}\bar{W}_{n-3}
\end{aligned}
$$

Family III

$$
\begin{aligned}
\gamma' &= \bar{k}_1\bar{W}_1\\
\bar{W}'_1 &= \bar{k}_2\bar{W}_2\\
\bar{W}'_i &= -\bar{k}_i\bar{W}_{i-1} + \bar{k}_{i+1}\bar{W}_{i+1}\,, \quad 2 \le i \le s-2\\
\bar{W}'_{s-1} &= -\bar{k}_{s-1}\bar{W}_{s-2} + \bar{\eta}_s\bar{k}_s\bar{L}_s\\
\bar{L}'_s &= \bar{\eta}_s\bar{k}_{s+1}\bar{L}_s + \bar{\eta}_{s+1}\bar{k}_{s+2}\bar{L}_{s+1}\\
\bar{L}'_{s+1} &= \bar{\eta}_s\bar{k}_{s+3}\bar{L}_s + \bar{\eta}_{s+1}\bar{k}_{s+4}\bar{L}_{s+1} + \bar{k}_{s+5}\bar{W}_{s+2}\\
\bar{W}'_{s+2} &= \bar{\eta}_s\bar{k}_{s+6}\bar{L}_s + \bar{\eta}_{s+1}\bar{k}_{s+7}\bar{L}_{s+1} - \bar{\eta}_{s+1}\bar{k}_{s+5}\bar{N}_{s+1}\\
\bar{N}'_{s+1} &= \bar{\eta}_s\bar{k}_{s+8}\bar{L}_s - \bar{k}_{s+7}\bar{W}_{s+2} - \bar{\eta}_{s+1}\bar{k}_{s+4}\bar{N}_{s+1} - \bar{\eta}_s\bar{k}_{s+2}\bar{N}_s\\
\bar{N}'_s &= -\bar{k}_s\bar{W}_{s-1} - \bar{\eta}_{s+1}\bar{k}_{s+8}\bar{L}_{s+1} - \bar{\eta}_{s+1}\bar{k}_{s+3}\bar{N}_{s+1}\\
&\quad - \bar{\eta}_s\bar{k}_{s+1}\bar{N}_s - \bar{k}_{s+6}\bar{W}_{s+2} + \bar{k}_{s+9}\bar{W}_{s+3}\\
\bar{W}'_{s+3} &= \bar{\eta}_s\bar{k}_{s+9}\bar{L}_s + \bar{k}_{s+10}\bar{W}_{s+4}\\
\bar{W}'_i &= -\bar{k}_{i+6}\bar{W}_{i-1} + \bar{k}_{i+7}\bar{W}_{i+1}\,, \quad s+4 \le i \le n-3\\
\bar{W}'_{n-2} &= -\bar{k}_{n+4}\bar{W}_{n-3}
\end{aligned}
\tag{1}
$$

where $\bar{\eta}_j = \langle \bar{L}_j, \bar{N}_j \rangle$.

4. The Cartan Reference of a Degenerate Curve

As we have seen, the Frenet equations for degenerate curves are quite complicated and involve too many curvature functions. In the non-degenerate case,

it is well-known that choosing an arbitrary parameter t, there exists only one Frenet reference satisfying the Frenet equations. In particular, if one chooses the arclength parameter, one obtains the usual curvature functions. However, this is not true here. Actually, for null curves it does not exit the arclength parameter, so we have to define a new one as follows.

Definition 4.1. *Let* $\gamma\colon I \to \mathbb{R}^n_2$ *be a differentiable curve, parametrized by* t*, satisfying that* $\langle \gamma^{(i)}(t), \gamma^{(i)}(t)\rangle = 0$ *for* $i = 1, \ldots, m-1$*, and* $\langle \gamma^{(m)}(t), \gamma^{(m)}(t)\rangle = \pm 1$*. Then* t *is said to be* the pseudo-arclength parameter.

Even though we have chosen the pseudo-arclength parameter, we can not assure the uniqueness of this Frenet reference. Then, we wondered whether there exist any "canonical" Frenet reference, in the following sense:

1) It is unique, that is, if we have references B and $\bar{B}$ satisfying the same equations, then $B = \bar{B}$.
2) The number of the corresponding curvature functions is minimal.
3) The corresponding curvature functions are invariant under pseudo-Euclidean transformations.

Theorem 4.1. *Let* $\gamma\colon I \to \mathbb{R}^n_2$ *be a degenerate curve and suppose that* $T_{\gamma(t)}\mathbb{R}^n_2$ *is spanned by* $\{\gamma'(t), \gamma''(t), \ldots, \gamma^{(n)}(t)\}$ *for all* t*. Then there exists only one (up the orientation) Frenet reference verifying the above conditions. Furthermore, the corresponding curvature functions are given by one of the following set of equations*

Family I

Null curves	*Non-null curves*
$\gamma' = L_1$	$\gamma' = W_1$
$L_1' = \mu_2\varepsilon_2 W_2$	$W_1' = \varepsilon_2 k_1 W_2$
$W_2' = \eta_1 k_1 L_1 - \mu_2\eta_1 N_1$	$W_i' = \varepsilon_{i-1}k_{i-1}W_{i-1} + \varepsilon_{i+1}k_i W_{i+1}$
$N_1' = -\varepsilon_2 k_1 W_2 + \varepsilon_3 k_2 W_3$	$W_{s-1}' = -\varepsilon_{s-2}k_{s-2}W_{s-2} + \mu_s\eta_s L_s$
$W_3' = -\eta_1 k_2 L_1 + \varepsilon_4 k_3 W_4$	$L_s' = \varepsilon_{s+1}k_{s-1}W_{s+1}$
$W_i' = -\varepsilon_{i-1}k_{i-1}W_{i-1} + \varepsilon_i k_i W_{i+1}$	$W_{s+1}' = \eta_s k_s L_s - \eta_s k_{s-1} N_s$
$W_{n-1}' = -\varepsilon_{n-2}k_{n-2}W_{n-2}$	$N_s' = -\mu_s\varepsilon_{s-1}W_{s-1} - \varepsilon_{s+1}k_s W_{s+1} + \varepsilon_{s+2}k_{s+1}W_{s+2}$
	$W_{s+2}' = -\eta_s k_{s+1} L_s + \varepsilon_{s+3}k_{s+2}W_{s+3}$
	$W_i' = \varepsilon_{i-1}k_{i-1}W_{i-1} + \varepsilon_{i+1}k_i W_{i+1}$
	$W_{n-1}' = -\varepsilon_{n-2}k_{n-2}W_{n-2}$

Family II

Null curves	*Spacelike curves*
$\gamma' = L_1$	$\gamma' = W_1$
$L_1' = W_2$	$W_1' = k_1 W_2$
$W_2' = \eta_1 k_1 L_1 - \eta_1 N_1$	$W_i' = k_{i-1} W_{i-1} + k_i W_{i+1}$
$N_1' = -k_1 W_2 + k_2 W_3$	$W_{s_1-1}' = -k_{s_1-2} W_{s_1-2} + \mu_{s_1} \eta_{s_1} L_{s_1}$
$W_3' = -\eta_1 k_2 L_1 + k_3 W_4,$	$L_{s_1}' = k_{s_1-1} W_{s_1+1}$
$W_i' = -k_{i-1} W_{i-1} + k_i W_{i+1}$	$W_{s_1+1}' = \eta_{s_1} k_{s_1} L_{s_1} - \eta_{s_1} k_{s_1-1} N_{s_1}$
$W_{s-1}' = -k_{s-2} W_{s-2} + \mu_s \eta_s L_s$	$N_{s_1}' = -\eta_{s_1} W_{s_1-1} - k_{s_1} W_{s_1+1} + k_{s_1+1} W_{s_1+2}$
$L_s' = k_{s-1} W_{s+1}$	$W_{s_1+2}' = -\eta_{s_1} k_{s_1+1} L_{s_1} + k_{s_1+2} W_{s_1+3}$
$W_{s+1}' = \eta_s k_s L_s - \eta_s k_{s-1} N_s$	$W_i' = k_{i-1} W_{i-1} + k_i W_{i+1}$
$N_s' = -\mu_s W_{s-1} - k_s W_{s+1} + k_{s+1} W_{s+2}$	$W_{s_2-1}' = -k_{s_2-2} W_{s_2-2} + \mu_{s_2} \eta_{s_2} L_{s_2}$
$W_{s+2}' = -\eta_s k_{s+1} L_s + k_{s+2} W_{s+3}$	$L_{s_2}' = k_{s_2-1} W_{s_2+1}$
$W_j' = -k_{j-1} W_{j-1} + k_j W_{j+1}$	$W_{s_2+1}' = \eta_{s_2} k_{s_2} L_{s_2} - \eta_{s_2} k_{s_2-1} N_{s_2}$
$W_{n-2}' = -k_{n-3} W_{n-3}$	$N_{s_2}' = -\eta_{s_2} W_{s_2-1} - k_{s_2} W_{s_2+1} + k_{s_2+1} W_{s_2+2}$
	$W_{s_2+2}' = -\eta_{s_2} k_{s_2+1} L_{s_2} + k_{s_2+2} W_{s_2+3}$
	$W_i' = k_{i-1} W_{i-1} + k_i W_{i+1}$
	$W_{n-2}' = -k_{n-3} W_{n-3}$

Family III

Null curves	*Spacelike curves*
$\gamma' = L_1$	$\gamma' = W_1$
$L_1' = \mu_2 \eta_2 L_2$	$W_1' = k_1 W_2$
$L_2' = W_3$	$W_i' = -k_{i-1} W_{i-1} + k_i W_{i+1}$
$W_3' = \eta_2 k_1 L_2 - \eta_2 N_2$	$W_{s-1}' = -k_{s-2} W_{s-2} + \mu_s \eta_s L_s$
$N_2' = \eta_1 k_2 L_1 - \mu_2 \eta_1 N_1 - k_1 W_3$	$L_s' = \mu_{s+1} \eta_{s+1} L_{s+1}$
$N_1' = -\eta_2 k_2 L_2 + k_3 W_4$	$L_{s+1}' = k_{s-1} W_{s+2}$
$W_4' = -\eta_1 k_3 L_1 + k_4 W_5$	$W_{s+2}' = \eta_{s+1} k_s L_{s+1} - \eta_{s+1} k_{s-1} N_{s+1}$
$W_i' = -k_{i-1} W_{i-1} + k_i W_{i+1}$	$N_{s+1}' = \eta_s k_{s+1} L_s - k_s W_{s+2} - \mu_{s+1} \eta_s N_s$
$W_{n-2}' = -k_{n-3} W_{n-3}$	$N_s' = -\eta_{s+1} k_{s+1} L_{s+1} - \mu_s W_{s-1} + k_{s+2} W_{s+3}$
	$W_{s+3}' = -\eta_s k_{s+2} L_s + k_{s+3} W_{s+4}$
	$W_i' = -k_{i-1} W_{i-1} + k_i W_{i+1}$
	$W_{n-2}' = -k_{n-3} W_{n-3}$

where $\mathrm{e}_j = \langle W_j, W_j \rangle$, $\eta_j = \langle L_j, N_j \rangle$ *and* $\mu_j = \pm 1$. *Moreover, we can choose* η_j *and* μ_j *so that* $\{\gamma', \ldots, \gamma^{(i)}\}$ *and* $\{C_1, \ldots, C_i\}$ *have the same orientation, for all* $i = 1, \ldots, n-1$, *and* $\{C_1, \ldots, C_n\}$ *is positively oriented, where* $\{C_1, \ldots, C_n\}$ *represents a Frenet reference as above.*

Proof: For families I and II we follow the ideas contained in [6]. As for Family III, let $\bar{B}$ and B^* be two Frenet references where we have chosen the pseudo-arclength parameter and let $\bar{k}_s = \mu_s$ and $k_s^* = \mu_s$, where $\mu_s = \pm 1$. Then we have the following bases

$$\bar{B} = \{\bar{W}_1, \ldots, \bar{W}_{s-1}, \bar{L}_s, \bar{L}_{s+1}, \bar{W}_{s+2}, \bar{N}_{s+1}, \bar{N}_s, \bar{W}_{s+3}, \ldots, \bar{W}_{n-2}\}$$

and

$$B^* = \{\bar{W}_1, \ldots, \bar{W}_{s-1}, \bar{L}_s, L_{s+1}^*, W_{s+2}^*, N_{s+1}^*, N_s^*, W_{s+3}^*, \ldots, W_{n-2}^*\}$$

with curvatures $\{\bar{k}_1 = 1, \bar{k}_2, \ldots, \bar{k}_s = \mu_s, \bar{k}_{s+1}, \ldots, \bar{k}_m\}$ and $\{\bar{k}_1 = 1, \bar{k}_2, \ldots, \bar{k}_s = \mu_s, k_{s+1}^*, \ldots, k_m^*\}$, respectively. □

As $\{\bar{L}_s, \bar{L}_{s+1}, \bar{W}_{s+2}, \bar{N}_{s+1}, \bar{N}_s\}$ and $\{\bar{L}_s, L_{s+1}^*, W_{s+2}^*, N_{s+1}^*, N_s^*\}$ are pseudo-orthonormal and they have the same orientation, there exist a matrix $P = (p_{ij})$ such that

$$\begin{pmatrix} \bar{L}_s \\ L_{s+1}^* \\ W_{s+2}^* \\ N_{s+1}^* \\ N_s^* \end{pmatrix} =$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ p_{21} & p_{22} & 0 & 0 & 0 \\ p_{31} & p_{32} & 1 & 0 & 0 \\ p_{41} & -\frac{1}{2}\frac{p_{32}^2}{p_{22}} & -\frac{p_{32}}{p_{22}} & \frac{1}{p_{22}} & 0 \\ -p_{21}p_{41} - \frac{1}{2}p_{31}^2 & \frac{1}{2}\frac{p_{21}p_{32}^2}{p_{22}} - p_{31}p_{32} - p_{22}p_{41} & \frac{p_{21}p_{32}}{p_{22}} - p_{31} & -\frac{p_{21}}{p_{22}} & 1 \end{pmatrix} \begin{pmatrix} \bar{L}_s \\ \bar{L}_{s+1} \\ \bar{W}_{s+2} \\ \bar{N}_{s+1} \\ \bar{N}_s \end{pmatrix}$$

By choosing $p_{22} = \dfrac{\bar{k}_{s+2}}{\mu_{s+1}}$ and $p_{21} = \dfrac{\bar{k}_{s+1}}{\mu_{s+1}}$, and using the Frenet equations (1), a straightforward computation leads to $k_{s+1}^* = 0$ and $k_{s+2}^* = \mu_{s+1}$. Therefore the problem can be reduced to the bases

$$\bar{B} = \{\bar{W}_1, \ldots, \bar{W}_{s-1}, \bar{L}_s, \bar{L}_{s+1}, \bar{W}_{s+2}, \bar{N}_{s+1}, \bar{N}_s, \bar{W}_{s+3}, \ldots, \bar{W}_{n-2}\}$$

and

$$B^* = \{\bar{W}_1, \ldots, \bar{W}_{s-1}, \bar{L}_s, \bar{L}_{s+1}, W_{s+2}^*, N_{s+1}^*, N_s^*, W_{s+3}^*, \ldots, W_{n-2}^*\}$$

where the curvatures are given by $\{\bar{k}_1 = 1, \bar{k}_2, \ldots, \bar{k}_s = 1, \bar{k}_{s+1} = 0, \bar{k}_{s+2} = 1, \bar{k}_{s+3}, \ldots, \bar{k}_m\}$ and $\{\bar{k}_1 = 1, \bar{k}_2, \ldots, \bar{k}_s = 1, \bar{k}_{s+1} = 0, \bar{k}_{s+2} = 1, k_{s+3}^*, \ldots, k_m^*\}$, respectively.

Now, the pseudo-ortonormal bases are related by

$$\begin{pmatrix} \bar{L}_s \\ \bar{L}_{s+1} \\ W^*_{s+2} \\ N^*_{s+1} \\ N^*_s \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ p_{31} & p_{32} & 1 & 0 & 0 \\ p_{41} & -\frac{1}{2}p_{32}^2 & -p_{32} & 1 & 0 \\ -\frac{1}{2}p_{31}^2 & -p_{31}p_{32} - p_{41} & -p_{31} & 0 & 1 \end{pmatrix} \begin{pmatrix} \bar{L}_s \\ \bar{L}_{s+1} \\ \bar{W}_{s+2} \\ \bar{N}_{s+1} \\ \bar{N}_s \end{pmatrix}$$

Choose $p_{31} = \frac{\bar{k}_{s+3}}{\bar{k}_{s+5}}$ and $p_{32} = \frac{\bar{k}_{s+4}}{\bar{k}_{s+5}}$ and use the Frenet equations to get $k^*_{s+3} = 0$ and $k^*_{s+4} = 0$. Therefore, we can now suppose that $\bar{W}_{s+2} = W^*_{s+2}$. We have again reduced the problem to a simpler one. Working as above, taking $p_{41} = \frac{\bar{k}_{s+6}}{\bar{k}_{s+5}}$, we show that $k^*_{s+6} = 0$. We only have to rename curvatures and use a suitable notation. Concerning to the orientation we stand out three cases corresponding to the three family-types.

Family I: There exist only one j_0 such that $\mathrm{e}_{j_0} = -1$, so we have several possibilities.

If $j_0 \leq s-1 < n-3$, we take $\mu_s = \eta_s = -1$. If $j_0 \leq s-1 = n-3$, we choose $\mu_s = \eta_s = \pm 1$ depending on $\{\gamma^{(i)}\}_{1\leq i\leq n}$ is negatively or positively oriented, respectively. In these cases we have $k_{j_0-1} < 0$ and $k_j > 0$ for all $j \neq \{j_0 - 1, s\}$.

If $j_0 = s+1 < n-1$, we choose $\mu_s = \eta_s = 1$. If $j_0 = s+1 = n-1$, then we take $\mu_s = \eta_s = \pm 1$ depending on $\{\gamma^{(i)}\}_{1\leq i\leq n}$ is positively or negatively oriented, respectively. Now we obtain $k_{s-1} < 0$ and $k_j > 0$ for all $j \neq \{s-1, s\}$.

Finally, if $j_0 > s+1$ take $\mu_s = \eta_s = -1$ to get $k_{j_0-1} < 0$ and $k_j > 0$ for all $j \neq \{j_0 - 1, s\}$.

Family II: If $s_2 < n-3$ choose $\mu_{s_1} = \mu_{s_2} = \eta_{s_1} = \eta_{s_2} = -1$. If $s_2 = n-3$ take $\mu_{s-1} = \eta_{s_1} = -1$, and $\mu_{s_2} = \eta_{s_2} = \pm 1$ depending on $\{\gamma^{(i)}\}_{1\leq i\leq n}$ is negatively or positively oriented, respectively. Then $k_j > 0$ for all $j \neq \{s_1, s_2\}$.

Family III: If $s < n-4$ take $\mu_{s+1} = \eta_{s+1} = -1$ and $\mu_s = \eta_s = 1$. If $s = n-4$ choose $\mu_{s+1} = \eta_{s+1} = -1$, and $\mu_s = \eta_s = \pm 1$ depending on $\{\gamma^{(i)}\}_{1\leq i\leq n}$ is positively or negatively oriented, respectively. Therefore $k_j > 0$ for all $j \neq \{s, s+1\}$.

The uniqueness follows now from Lemma 2.1.

Definition 4.2. *A degenerate curve γ satisfying the above conditions is said to be a* degenerate Cartan curve. *The reference and curvature functions given by those equations will be called the* Cartan reference *and* Cartan curvatures *of γ, respectively.*

Corollary 4.1. *The number of Cartan curvatures of a degenerate curve* $\gamma\colon I \to \mathbb{R}^n_2$ *is* $n-r-1$, *where* r *is the degeneration degree of* γ.

Hence, degenerate curves with degeneration degree one (resp. two) have $n-2$ (resp. $n-3$) Cartan curvatures.

5. Congruence Theorems for a Degenerate Cartan Curve in a Pseudo-Euclidean Space of Index Two

The following question naturally arises: Let $\mathcal{C}$ be a reference satisfying the Cartan equations for certain functions k_j. Is there a degenerate Cartan curve γ whose Cartan reference is $\mathcal{C}$ and his Cartan curvatures are k_j? If it is affirmative, is that curve unique?

The answer is affirmative and the result sets out as follows.

Theorem 5.1. *Let* $k_1, \ldots, k_m\colon [-\delta, \delta] \to \mathbb{R}$ *be differentiable functions. Let* p *be a point of* $\mathbb{R}^n_2$ *and let* $\mathcal{C}_0$ *be an admissible pseudo-orthonormal basis of* $T_p\mathbb{R}^n_2$ *with degeneration degree* 1 *or* 2, *according to* $m = n-2$ *or* $m = n-3$, *respectively. Then there exists a unique degenerate Cartan curve* γ *in* $\mathbb{R}^n_2$, *with* $\gamma(0) = p$ *and the same nullity degree and index sequences that* $\mathcal{C}_0$, *whose Cartan reference at* p *is just* $\mathcal{C}_0$.

Proof: See [5] and [6]. □

Theorem 5.2. (Congruence Theorem) *Let* C *and* $\bar{C}$ *be two degenerate Cartan curves which are of the same type and have the same Cartan curvatures* $\{k_1, \ldots, k_m\}$, *where* $k_i\colon [-\delta, \delta] \to \mathbb{R}$ *are differentiable functions. Then there exists a pseudo-Euclidean transformation of* $\mathbb{R}^n_2$ *which maps bijectively* C *into* $\bar{C}$.

Remark 5.1. *The same results can be easily obtained in the de Sitter space* $\mathbb{S}^n_2$ *and in the anti de Sitter space* $\mathbb{H}^n_2$. *With some extra effort they can be extended to higher dimensions.*

6. Examples

Example 6.1.
Spacelike degenerate curves in $\mathbb{R}^5_2$ *with degeneration degree 1,* $k_1 = \sigma > 0$, $k_2 = 0$, $k_3 = -1$, $\varepsilon_4 = -1$ *and nullity degree sequence* $\{0,1,1,0,0\}$:

$$\gamma(t) = \left(\frac{\sigma^2 t^5}{120}, \frac{\sigma t^2(t^2+1)}{4\sqrt{6}}, \frac{\sigma t^3}{6}, \frac{\sigma t^2(t^2-1)}{4\sqrt{6}}, t\left(\frac{\sigma^2 t^4}{120} + 1\right)\right).$$

Example 6.2.
Timelike degenerate curves in $\mathbb{R}^5_2$ *with degeneration degree 1,* $k_1 = \sigma > 0$, $k_2 = 0$, $k_3 = 1$ *and nullity degree sequence* $\{0,1,1,0,0\}$*:*

$$\gamma(t) = \left(t\left(\frac{\sigma^2 t^4}{120} - 1\right), \frac{\sigma t^2(t^2+1)}{4\sqrt{6}}, \frac{\sigma t^3}{6}, \frac{\sigma t^2(t^2-1)}{4\sqrt{6}}, \frac{\sigma^2 t^5}{120} \right).$$

Example 6.3.
Null curves in $\mathbb{R}^5_2$ *with degeneration degree 1,* $k_1 = \sigma^2$, $k_2 = -2\sigma^2$ *and* $k_3 = \sqrt{2}\sigma > 0$, $\varepsilon_3 = -1$ *and nullity degree sequence* $\{1,1,0,0,0\}$*:*

$$\gamma(t) = \left(\left(\frac{1}{\sqrt{2}\sigma} - \frac{\sigma^3 t^4}{30\sqrt{2}} \right) t, \frac{\sigma^2 t^4}{12}, \frac{t^2}{2}\left(1 - \frac{\sigma^2 t^2}{6}\right), -\frac{t}{\sqrt{2}\sigma}, \frac{\sigma^3 t^5}{30\sqrt{2}} \right)$$

Example 6.4.
A null curve in $\mathbb{R}^5_2$ *with degeneration degree 2 and* $k_1 = k_2 = 0$*:*

$$\gamma(t) = \left(\frac{t(1-t^4)}{4\sqrt{15}}, \frac{t^2(1+t^2)}{4\sqrt{6}}, \frac{t^3}{6}, \frac{t^2(1-t^2)}{4\sqrt{6}}, \frac{t(1+t^4)}{4\sqrt{15}} \right)$$

Since it is similar to the null cubic of $\mathbb{R}^3_1$*, we will call it the* null quintic *of* $\mathbb{R}^5_2$.

Acknowledgments

This research has been partially supported by DGICYT grant PB97-0784. The second author is supported by a FPPI Grant, Program PG, Ministerio de Educación y Ciencia

References

[1] Bejancu A., *Lightlike Curves in Lorentz Manifolds*, Publ. Math. Debrecen, **44** (1994) 145–155.

[2] Bejancu A., Ferrández A. and Lucas P., *A New Viewpoint on Geometry of a Lightlike Hypersurface in Semi-Euclidean Space*, Saitama Mathematical Journal **16** (1998) 31–38.

[3] Bonnor W., *Null Curves in a Minkowski Spacetime*, Tensor, N. S. **20** (1969) 229–242.

[4] Duggal K. and Jin D., *Geometry of Null Curves*, Math. J. Toyama Univ. **22** (1999) 95–120.

[5] Ferrández A., Giménez A. and Lucas P., *Null Helices in Lorentzian Space Forms*, to appear in Int. J. Mod. Phys. A, 2001.

[6] Ferrández A., Giménez A. and Lucas P., *s-Degenerate Curves in Lorentzian Space Forms*, Preprint, 2001.

[7] Galloway G., *Maximum Principles For Null Hypersurfaces and Null Splitting Theorems*, Ann. Henri Poincaré **1** (2000) 543–567.

[8] Nersessian A. and Ramos E., *Massive Spinning Particles and the Geometry of Null Curves*, Phys. Lett. B **445** (1998) 123–128.

[9] Nurowski P. and Robinson D., *Intrinsic Geometry of a Null Hypersurface*, Class. Quantum Grav. **17** (2000) 4065–4084.

[10] Samuel J. and Nityananda R., *Transport Along Null Curves*, J. Phys. A: Math. Gen. **33** (2000) 2895–2905.