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GEOMETRIC SYMMETRY GROUPS, CONSERVATION LAWS AND GROUP-INVARIANT SOLUTIONS OF THE WILLMORE EQUATION

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Abstract. The present paper is concerned with the geometric (point) Lie symmetry groups of the Willmore equation – the Euler-Lagrange equation associated with the Willmore functional. The ten-parameter group of special conformal transformations in three-dimensional Euclidean space, which is known to be the symmetry group of the Willmore functional, is recognized as the largest group of geometric transformations admitted by the Willmore equation in Mongé representation. The conserved currents of ten linearly independent conservation laws, which correspond to the variational symmetries of the Willmore equation and hold on its smooth solutions, are derived. All types of non-equivalent group-invariant solutions of the Willmore equation are identified, an optimal system of one-dimensional subalgebras being given together with the invariants of the corresponding one-parameter groups, up to one exception. Special attention is paid to the rotationally-invariant (axially-symmetric) solutions.

1. Introduction

The so-called Willmore functional

$$\mathcal{W} = \int_{\mathcal{S}} H^2 \, \mathrm{d}A \tag{1}$$

which assigns to each surface $\mathcal{S}$ its total squared mean curvature H (here $\mathrm{d}A$ is the area element on the surface) has drawn much attention after the appearance of Willmore's paper [18] in 1965. In this work, Willmore proposed to study the surfaces providing extremum to the functional (1), which are now referred to as Willmore surfaces. These surfaces obey the corresponding Euler-Lagrange equation

$$\Delta H + 2(H^2 - K)H = 0 \tag{2}$$

which will be further referred to as Willmore equation. Here Δ is the Laplace-Beltrami operator on the surface $\mathcal{S}$ and K is the Gaussian curvature of $\mathcal{S}$. Apparently, according to Thomsen [16], Schadow was the first who had derived this equation in 1922 as the Euler-Lagrange equation for the variational problem

$$\int_{\mathcal{S}} \left(\frac{1}{R_1} - \frac{1}{R_2}\right)^2 \mathrm{d}A \tag{3}$$

where $1/R_1$ and $1/R_2$ are the two principal curvatures of the surface $\mathcal{S}$. This variational problem is studied in Thomsen's paper [16] devoted to the conformal geometry. There, a reference to the aforementioned (and probably unpublished) result by Schadow was given. Actually, the Lagrangian densities of the functionals (1) and (3) are proportional up to the divergence term $2K$ and that is why they lead to the same Euler-Lagrange equation.

The study of the extremals of the functional (1), i.e. the Willmore surfaces, turned out to be of great importance not only for differential geometry (in connection with the Willmore problem and conformal geometry) but also for the 2D string theory and 2D gravity based on the Polyakov integral over surfaces (see [14]). In these theories, the functional (1) is known as the Polyakov's extrinsic action. The properties of the Polyakov's extrinsic action as well as various generalizations, such as the Polyakov-Kleinert rigid string action [15], [9]

$$\mathcal{A} = \int_{\mathcal{S}} \left(\alpha H^2 + \gamma\right) \mathrm{d}A \quad \alpha,\ \gamma - \text{constant} \tag{4}$$

for instance, have been studied in a number of papers (see the review paper [4]).

The functional (1) has found application in biophysics too. In Helfrich theory [5], the bending energy of a homogeneous vesicle membrane is assumed to be given by the functional

$$\mathcal{F}_c = k \int_{\mathcal{S}} H^2 \,\mathrm{d}A + \bar{k} \int_{\mathcal{S}} K \,\mathrm{d}A$$

where k and $\bar{k}$ are real constants representing the bending and Gaussian rigidity of the membrane. The equilibrium shape of the vesicle is supposed to be determined by the extremals of the Helfrich curvature free energy (shape energy)

$$\mathcal{F} = k \int_{\mathcal{S}} (H - H_s)^2 \,\mathrm{d}A + \bar{k} \int_{\mathcal{S}} K \,\mathrm{d}A + \lambda \int_{\mathcal{S}} \mathrm{d}A + \Delta p \int \mathrm{d}V \tag{5}$$

where $\mathrm{d}V$ is the volume element, H_s, λ and Δp are real constants and denote the so-called spontaneous curvature, tensile stress and osmotic pressure difference between the outer and inner media. The corresponding Euler-Lagrange equation

$$k\Delta H + 2k\,(H - H_s)\,(H^2 - K) - 2k\,(H - H_s)^2\, H - 2\lambda H + \Delta p = 0 \tag{6}$$

(derived in [20], [21]) is referred to as the Helfrich's membrane shape equation. There is a vast amount of papers in which the extremals of functional (5), i.e. the solutions of equation (6) are studied (see e.g. [10], [22] and the references therein).

2. Willmore Equation in Mongé Representation

Let (x^1, x^2, x^3) be a fixed right-handed rectangular Cartesian coordinate system in the three-dimensional Euclidean space $\mathbb{R}^3$ in which a surface $\mathcal{S}$ is immersed, and let this surface be given by the equation

$$\mathcal{S} : x^3 = w(x^1, x^2), \quad (x^1, x^2) \in \Omega \subset \mathbb{R}^2 \tag{7}$$

where $w : \mathbb{R}^2 \to \mathbb{R}$ is a single-valued and smooth function possessing as many derivatives as may be required on the domain Ω. Let us take x^1, x^2 to serve as Gaussian coordinates on the surface $\mathcal{S}$. Then, relative to this coordinate system, the components of the first fundamental tensor $g_{\alpha\beta}$, the second fundamental tensor $b_{\alpha\beta}$, and the alternating tensor $\varepsilon^{\alpha\beta}$ of $\mathcal{S}$ are given by the expressions

$$g_{\alpha\beta} = \delta_{\alpha\beta} + w_\alpha w_\beta, \quad b_{\alpha\beta} = g^{-1/2} w_{\alpha\beta}, \quad \varepsilon^{\alpha\beta} = g^{-1/2} \mathrm{e}^{\alpha\beta} \tag{8}$$

where

$$g = \det(g_{\alpha\beta}) = 1 + (w_1)^2 + (w_2)^2 \tag{9}$$

while $\delta_{\alpha\beta}$ will denote the Kronecker delta symbol and $\mathrm{e}^{\alpha\beta}$ is the surface alternating symbol. The contravariant components $g^{\alpha\beta}$ of the first fundamental tensor read

$$g^{\alpha\beta} = g^{-1}\delta^{\alpha\beta} + \varepsilon^{\alpha\mu}\varepsilon^{\beta\nu} w_\mu w_\nu = g^{-1}\left(\delta^{\alpha\beta} + \mathrm{e}^{\alpha\mu}\mathrm{e}^{\beta\nu} w_\mu w_\nu\right). \tag{10}$$

Here and in what follows: Greek indices have the range 1, 2, and the usual summation convention over a repeated index is employed, $w_{\alpha_1 \ldots \alpha_k}$ $(k = 1, 2, \ldots)$ denote the k-th order partial derivatives of the function w with respect to the variables x^1 and x^2, i.e.,

$$w_{\alpha_1 \alpha_2 \ldots \alpha_k} = \frac{\partial^k w}{\partial x^{\alpha_1} \ldots \partial x^{\alpha_k}}, \quad k = 1, 2, \ldots .$$

The mean curvature H of the surface $\mathcal{S}$ and its Gaussian curvature K are given as follows

$$H = \frac{1}{2} g^{\alpha\beta} b_{\alpha\beta}, \qquad K = \frac{1}{2} \varepsilon^{\alpha\mu}\varepsilon^{\beta\nu} b_{\alpha\beta} b_{\mu\nu} \tag{11}$$

that is,

$$H = \frac{1}{2} g^{-3/2}\left(\delta^{\alpha\beta} w_{\alpha\beta} + \mathrm{e}^{\alpha\mu}\mathrm{e}^{\beta\nu} w_{\alpha\beta} w_\mu w_\nu\right), \; K = \frac{1}{2} g^{-2} \mathrm{e}^{\alpha\mu}\mathrm{e}^{\beta\nu} w_{\alpha\beta} w_{\mu\nu}. \tag{12}$$

In the above Mongé representation the Willmore functional (1) reads

$$\begin{aligned} \mathcal{W} &= \iint_\Omega H^2 g^{1/2}\, \mathrm{d}x^1\, \mathrm{d}x^2 \\ &= \iint_\Omega \frac{1}{4} g^{-5/2} \left(\delta^{\alpha\beta} w_{\alpha\beta} + \mathrm{e}^{\alpha\mu}\mathrm{e}^{\beta\nu} w_{\alpha\beta} w_\mu w_\nu\right)^2 \mathrm{d}x^1\, \mathrm{d}x^2. \end{aligned} \tag{13}$$

The application of the Euler operator

$$E = \frac{\partial}{\partial w} - D_\mu \frac{\partial}{\partial w_\mu} + D_\mu D_\nu \frac{\partial}{\partial w_{\mu\nu}} - \cdots$$

where

$$D_\alpha = \frac{\partial}{\partial x^\alpha} + w_\alpha \frac{\partial}{\partial w} + w_{\alpha\mu} \frac{\partial}{\partial w_\mu} + w_{\alpha\mu\nu} \frac{\partial}{\partial w_{\mu\nu}} + w_{\alpha\mu\nu\sigma} \frac{\partial}{\partial w_{\mu\nu\sigma}} + \cdots$$

denote the total derivative operators, on the Lagrangian density

$$L = H^2 g^{1/2} = \frac{1}{4} g^{-5/2} \left(\delta^{\alpha\beta} w_{\alpha\beta} + \mathrm{e}^{\alpha\mu}\mathrm{e}^{\beta\nu} w_{\alpha\beta} w_\mu w_\nu\right)^2 \tag{14}$$

of the Willmore functional (13) leads, after taking into account expressions (8–10) and (12), to the expression

$$E(L) = \Delta H + 2(H^2 - K)H.$$

Actually, in Mongé representation, the Willmore equation $E(L) = 0$ is to be regarded as a fourth-order partial differential equation in two independent variables x^1, x^2 and one dependent variable w – the displacement field. This equation belongs to the class of equations of the form

$$\mathcal{E} \equiv \frac{1}{2} g^{-1/2} g^{\alpha\beta} g^{\mu\nu} w_{\alpha\beta\mu\nu} + \Phi\left(x_1, x_2, w, w_1, \ldots, w_{222}\right) = 0 \tag{15}$$

where $\Phi\left(x_1, x_2, w, w_1, \ldots, w_{222}\right)$ is a differential function depending on the independent and dependent variables and the derivatives of the dependent variable up to third order. Indeed, using expressions (8–10), (12) and the well-known formula

$$\Delta = g^{-1/2} \frac{\partial}{\partial x^\alpha}\left(g^{1/2} g^{\alpha\beta} \frac{\partial}{\partial x^\beta}\right) = g^{\alpha\beta} \frac{\partial^2}{\partial x^\alpha \partial x^\beta} + g^{-1/2} \frac{\partial}{\partial x^\alpha}\left(g^{1/2} g^{\alpha\beta}\right) \frac{\partial}{\partial x^\beta}$$

one can represent the Willmore equation (2) in the form (15).

It should be remarked that there are other equations of form (15) which have attracted much attention in differential geometry, theoretical physics and biology. Among them one can find, for instance, the equations for the Willmore surfaces in the three-dimensional sphere S^3 and in three dimensional manifolds of constant negative curvature (see [8]), the Euler-Lagrange equation for the Polyakov-Kleinert rigid string action (4) and the Helfrich's membrane shape equation (6).

3. Symmetry Groups

The main objective of the present Section is to establish, following [12], [13] and [7], the invariance properties of the Willmore equation (2) relative to local one-parameter Lie groups of local point transformations acting on open subsets of the three-dimensional Euclidean space $\mathbb{R}^3$, with coordinates (x^1, x^2, w), representing the involved independent and dependent variables x^1, x^2 and w, respectively. For that purpose Lie infinitesimal technique is used and the results obtained are expressed in terms of the infinitesimal generators (operators) of the respective groups. In the present case, the latter are vector fields on $\mathbb{R}^3$ of the form

$$\mathbf{v} = \xi^\mu \frac{\partial}{\partial x^\mu} + \eta \frac{\partial}{\partial w} \tag{16}$$

where ξ^μ and η are functions of the variables x^1, x^2 and w.

The infinitesimal criterion of invariance

$$\mathrm{pr}^{(4)}\,\mathbf{v}\,(\mathcal{E}) = 0 \quad \text{whenever} \quad \mathcal{E} = 0 \tag{17}$$

where $\mathrm{pr}^{(n)}\,\mathbf{v}$ denotes the n-th prolongation of the vector field $\mathbf{v}$ (see [12]), leads, through the standard computational procedure (see, e.g. [13] or [12]), to the following result.

Proposition 1. *The ten-parameter Lie group G_{SCT} of special conformal transformations in $\mathbb{R}^3$ (whose basic generators $\mathbf{v}_j$, $j = 1, \ldots, 10$, their characteristics, commutators and corresponding finite transformations and invariants are given in Table 1, Table 2 and Table 3 listed below) is the largest group of geometric transformations of the involved independent and dependent variables that a generic equation of form (15) could admit.*

Remark 1. *Let us denote by L_{SCT} the Lie algebra corresponding to the group G_{SCT}, i.e. L_{SCT} is the ten-dimensional Lie algebra spanned by the vector fields $\mathbf{v}_j$, $j = 1, \ldots, 10$. Actually, the group G_{SCT} is a representation of the Lie group $O(4,1)$ in the vector space $\mathbb{R}^3$, which corresponds to the action of $O(4,1)$ on $\mathbb{R}^3$ determined by the representation L_{SCT} of its Lie algebra $o(4,1)$ in $\mathbb{R}^3$.*

Proposition 2. *In Mongé representation, the Willmore equation (2) admits all the transformations of the group G_{SCT}.*

It should be noticed that the geometric symmetries of a system of partial differential equations equivalent to the Helfrich's membrane shape equation (6), including the Willmore equation (2) as a special case, are studied in [11]. It seems that the symmetry groups obtained in this paper can be interpreted as generalized symmetries of the Willmore equation (2) in Mongé representation, but this matter remains to be clarified.

Table 1. Generators and characteristics of the group of special conformal transformations in $\mathbb{R}^3$. Here and below the following notations are used: $\chi^1 = (x^1)^2 - (x^2)^2 - w^2$, $\chi^2 = (x^2)^2 - (x^1)^2 - w^2$ and $\chi^3 = w^2 - (x^2)^2 - (x^1)^2$.

Generators	**Characteristics**
Translations	
$\mathbf{v}_1 = \frac{\partial}{\partial x^1}$	$Q_1 = -w_1$
$\mathbf{v}_2 = \frac{\partial}{\partial x^2}$	$Q_2 = -w_2$
$\mathbf{v}_3 = \frac{\partial}{\partial w}$	$Q_3 = 1$
Rotations	
$\mathbf{v}_4 = -x^2\frac{\partial}{\partial x^1} + x^1\frac{\partial}{\partial x^2}$	$Q_4 = x^2 w_1 - x^1 w_2$
$\mathbf{v}_5 = -w\frac{\partial}{\partial x^1} + x^1\frac{\partial}{\partial w}$	$Q_5 = x^1 + w w_1$
$\mathbf{v}_6 = -w\frac{\partial}{\partial x^2} + x^2\frac{\partial}{\partial w}$	$Q_6 = x^2 + w w_2$
Dilatation	
$\mathbf{v}_7 = x^1\frac{\partial}{\partial x^1} + x^2\frac{\partial}{\partial x^2} + w\frac{\partial}{\partial w}$	$Q_7 = w - x^1 w_1 - x^2 w_2$
Inversions	
$\mathbf{v}_8 = \chi^1\frac{\partial}{\partial x^1} + 2x^1x^2\frac{\partial}{\partial x^2} + 2x^1 w\frac{\partial}{\partial w}$	$Q_8 = 2x^1 w - \chi^1 w_1 - 2x^1x^2 w_2$
$\mathbf{v}_9 = 2x^2x^1\frac{\partial}{\partial x^1} + \chi^2\frac{\partial}{\partial x^2} + 2x^2 w\frac{\partial}{\partial w}$	$Q_9 = 2x^2 w - 2x^1x^2 w_1 - \chi^2 w_2$
$\mathbf{v}_{10} = 2x^1 w\frac{\partial}{\partial x^1} + 2x^2 w\frac{\partial}{\partial x^2} + \chi^3\frac{\partial}{\partial w}$	$Q_{10} = \chi^3 - 2x^1 w w_1 - 2x^2 w w_2$

Table 2. Commutator table

	$\mathbf{v}_1$	$\mathbf{v}_2$	$\mathbf{v}_3$	$\mathbf{v}_4$	$\mathbf{v}_5$	$\mathbf{v}_6$	$\mathbf{v}_7$	$\mathbf{v}_8$	$\mathbf{v}_9$	$\mathbf{v}_{10}$
$\mathbf{v}_1$	0	0	0	$\mathbf{v}_2$	$\mathbf{v}_3$	0	$\mathbf{v}_1$	$2\mathbf{v}_7$	$-2\mathbf{v}_4$	$-2\mathbf{v}_5$
$\mathbf{v}_2$	0	0	0	$-\mathbf{v}_1$	0	$\mathbf{v}_3$	$\mathbf{v}_2$	$2\mathbf{v}_4$	$2\mathbf{v}_7$	$-2\mathbf{v}_6$
$\mathbf{v}_3$	0	0	0	0	$-\mathbf{v}_1$	$-\mathbf{v}_2$	$\mathbf{v}_3$	$2\mathbf{v}_5$	$2\mathbf{v}_6$	$2\mathbf{v}_7$
$\mathbf{v}_4$	$-\mathbf{v}_2$	$\mathbf{v}_1$	0	0	$-\mathbf{v}_6$	$\mathbf{v}_5$	0	$-\mathbf{v}_9$	$\mathbf{v}_8$	0
$\mathbf{v}_5$	$-\mathbf{v}_3$	0	$\mathbf{v}_1$	$\mathbf{v}_6$	0	$-\mathbf{v}_4$	0	$-\mathbf{v}_{10}$	0	$\mathbf{v}_8$
$\mathbf{v}_6$	0	$-\mathbf{v}_3$	$\mathbf{v}_2$	$-\mathbf{v}_5$	$\mathbf{v}_4$	0	0	0	$-\mathbf{v}_{10}$	$\mathbf{v}_9$
$\mathbf{v}_7$	$-\mathbf{v}_1$	$-\mathbf{v}_2$	$-\mathbf{v}_3$	0	0	0	0	$\mathbf{v}_8$	$\mathbf{v}_9$	$\mathbf{v}_{10}$
$\mathbf{v}_8$	$-2\mathbf{v}_7$	$-2\mathbf{v}_4$	$-2\mathbf{v}_5$	$\mathbf{v}_9$	$\mathbf{v}_{10}$	0	$-\mathbf{v}_8$	0	0	0
$\mathbf{v}_9$	$2\mathbf{v}_4$	$-2\mathbf{v}_7$	$-2\mathbf{v}_6$	$-\mathbf{v}_8$	0	$\mathbf{v}_{10}$	$-\mathbf{v}_9$	0	0	0
$\mathbf{v}_{10}$	$2\mathbf{v}_5$	$2\mathbf{v}_6$	$-2\mathbf{v}_7$	0	$-\mathbf{v}_8$	$-\mathbf{v}_9$	$-\mathbf{v}_{10}$	0	0	0

Here, the entry in row i and column j represents the commutator $[\mathbf{v}_i, \mathbf{v}_j]$.

Table 3. Finite transformations and invariants. Here, the entries in the "Finite Transformations" column give the transformed points $\exp(\varepsilon \mathbf{v}_j)(x^1, x^2, w)$ and $\chi = (r^2 + w^2)$.

Groups	Finite Transformations	Invariants
$G(\mathbf{v}_1)$	$(x^1 + \varepsilon, x^2, w)$	$I_1 = w$, $I_2 = x^2$
$G(\mathbf{v}_2)$	$(x^1, x^2 + \varepsilon, w)$	$I_1 = w$, $I_2 = x^1$
$G(\mathbf{v}_3)$	$(x^1, x^2, w + \varepsilon)$	$I_1 = x^1$, $I_2 = x^2$
$G(\mathbf{v}_4)$	$(x^1 \cos\varepsilon - x^2 \sin\varepsilon, x^2 \cos\varepsilon + x^1 \sin\varepsilon, w)$	$I_1 = w$, $I_2 = r$
$G(\mathbf{v}_5)$	$(x^1 \cos\varepsilon - w \sin\varepsilon, x^2, w \cos\varepsilon + x^1 \sin\varepsilon)$	$I_1 = (x^1)^2 + w^2$, $I_2 = x^2$
$G(\mathbf{v}_6)$	$(x^1, x^2 \cos\varepsilon - w \sin\varepsilon, w \cos\varepsilon + x^2 \sin\varepsilon)$	$I_1 = (x^2)^2 + w^2$ $I_2 = x^1$
$G(\mathbf{v}_7)$	$(\mathrm{e}^{\varepsilon} x^1, \mathrm{e}^{\varepsilon} x^2, \mathrm{e}^{\varepsilon} w)$	$I_1 = \frac{w}{x^1}$, $I_2 = \frac{x^2}{x^1}$
$G(\mathbf{v}_8)$	$\left(\frac{x^1 - \varepsilon\chi}{1 - 2\varepsilon x^1 + \varepsilon^2\chi}, \frac{x^2}{1 - 2\varepsilon x^1 + \varepsilon^2\chi}, \frac{w}{1 - 2\varepsilon x^1 + \varepsilon^2\chi}\right)$	$I_1 = \frac{x^2}{\chi}$, $I_2 = \frac{w}{\chi}$
$G(\mathbf{v}_9)$	$\left(\frac{x^1}{1 - 2\varepsilon x^2 + \varepsilon^2\chi}, \frac{x^2 - \varepsilon\chi}{1 - 2\varepsilon x^2 + \varepsilon^2\chi}, \frac{w}{1 - 2\varepsilon x^2 + \varepsilon^2\chi}\right)$	$I_1 = \frac{x^1}{\chi}$, $I_2 = \frac{w}{\chi}$
$G(\mathbf{v}_{10})$	$\left(\frac{x^1}{1 - 2\varepsilon w + \varepsilon^2\chi}, \frac{x^2}{1 - 2\varepsilon w + \varepsilon^2\chi}, \frac{w - \varepsilon\chi}{1 - 2\varepsilon w + \varepsilon^2\chi}\right)$	$I_1 = \frac{x^1}{\chi}$, $I_2 = \frac{x^2}{\chi}$

4. Conservation Laws

A particular interest exists for the variational symmetries of equation (2) – the Lie groups generated by the so-called infinitesimal divergence symmetries (see Definition 4.33 in [12]) of any variational functional with (2) as the associated Euler-Lagrange equation. Note that if two functionals lead to the same Euler-Lagrange equation, then they have the same collection of infinitesimal divergence symmetries. This interest is motivated by the fact that, in virtue of Bessel-Hagen's extension of Noether's theorem, each variational symmetry of a given self-adjoint equation corresponds to a conservation law admitted by the smooth solutions of the equation. Thus, if a vector field $\mathbf{v}$ of form (16) is found to generate a variational symmetry of equation (2), then Bessel-Hagen's extension of Noether's theorem implies the existence of a conserved current, which, in the present case, is a couple of differential functions P^α (i.e. functions depending on the independent and dependent variables and the derivatives of the dependent variable) such that

$$D_\alpha P^\alpha = QE(L) \tag{18}$$

where Q is the characteristic of the vector field $\mathbf{v}$. By definition

$$Q = \eta - w_\mu \xi^\mu. \tag{19}$$

The total divergence of the conserved current P^α vanishes on the smooth solutions of equation (2) and so we have the conservation law

$$D_\alpha P^\alpha = 0 \tag{20}$$

where (18) is its expression in characteristic form, and Q – its characteristic.

To derive the conservation laws of the foregoing type, one can proceed by first determining the variational symmetries of the equation considered on the ground of the invariance criterion

$$\mathrm{pr}^{(2)}\,\mathbf{v}(L) + (D_\mu \xi^\mu)L = D_\mu B^\mu$$

where B^α are certain differential functions. Then using their characteristics (19) we find, from equality (18), explicit expressions for the corresponding conserved currents P^α.

It is well-known (see [19]), that the Willmore functional (1) is invariant under the conformal transformations of a closed surface S. This follows from the invariance of the functional

$$\int_S (H^2 - K)\,\mathrm{d}A \tag{21}$$

under the group of conformal transformations (see [17], [3]) and the Gauss-Bonnet theorem which states that the area-integral over the Gaussian curvature is a topological invariant.

All vector fields $\mathbf{v}_j$, $j = 1, \ldots, 10$ are variational symmetries of the Willmore equation (2) and hence, ten linearly independent conservation laws of the form (20)

$$D_\alpha P_j^\alpha = 0, \quad j = 1, \ldots, 10$$

exist that hold on its smooth solutions. The corresponding conserved currents are

$$\begin{aligned} P_j^\alpha &= N_j^\alpha L, \quad j = 1, \ldots, 7 \\ P_8^\alpha &= N_8^\alpha L - Q^{\alpha 1} \\ P_9^\alpha &= N_9^\alpha L - Q^{\alpha 2} \\ P_{10}^\alpha &= N_{10}^\alpha L + \frac{2}{\sqrt{g}}\delta^{\alpha\mu} w_\mu \end{aligned}$$

where

$$N_j^\alpha = \xi_j^\alpha - \frac{1}{2}Q_j D_\mu \frac{\partial}{\partial w_{\alpha\mu}} - \frac{1}{2}Q_j D_\mu \frac{\partial}{\partial w_{\mu\alpha}} + \frac{1}{2}(D_\mu Q_j)\frac{\partial}{\partial w_{\alpha\mu}} + \frac{1}{2}(D_\mu Q_j)\frac{\partial}{\partial w_{\mu\alpha}}$$

are the so-called Noether operators (cf. [7]), corresponding to the vector fields $\mathbf{v}_j$ with characteristics Q_j, $j = 1, \ldots, 10$, and

$$Q^{\alpha\beta} = -\frac{2}{\sqrt{g}} \mathrm{e}^{\alpha\mu} \mathrm{e}^{\beta\nu} g_{\mu\nu}.$$

Note that in the above notations

$$H = D_\mu \left(\frac{1}{2\sqrt{g}} \delta^{\alpha\mu} w_\mu \right), \qquad 4H\delta^{\alpha\mu} w_\mu = D_\mu Q^{\alpha\mu}.$$

5. Group-invariant Solutions

Once a group G is found to be a symmetry group of a given differential equation, it is possible to look for the so-called group-invariant (G-invariant) solutions of the equation – the solutions, which are invariant under the transformations of the symmetry group G (see [12], [13]). The main advantage that one can gain when looking for this kind of particular solutions of the given differential equation consists in the fact that each group-invariant solution is determined by a reduced equation obtained by a symmetry reduction of the original one and involves less independent variables than the latter.

Let $G(\mathbf{v})$ be a one parameter group generated by a vector field $\mathbf{v}$ belonging to the Lie algebra L_{SCT}, that is $\mathbf{v}$ is a linear combination of the vector fields $\mathbf{v}_j$, $j = 1, \ldots, 10$,

$$\mathbf{v} = \sum_{j=1}^{10} c_j \mathbf{v}_j \tag{22}$$

where c_j, $j = 1, \ldots, 10$, are real numbers – the components of the vector field $\mathbf{v}$ with respect to the basic vector fields $\mathbf{v}_j$. Then, $G(\mathbf{v})$ is a symmetry group of the Willmore equation (2) and so one can look for the $G(\mathbf{v})$-invariant solutions of this equation. For that purpose, first one should find a complete set of functionally independent invariants of the group $G(\mathbf{v})$. In the present case this is a set of two functionally independent functions $I_\alpha(x^1, x^2, w)$ such that

$$\mathbf{v} I_\alpha = 0$$

where the vector field $\mathbf{v}$ is regarded as an operator acting on the functions $\zeta : \mathbb{R}^3 \to \mathbb{R}$. Then, if the necessary condition for the existence of group invariant solutions is satisfied, that in the present case means

$$\operatorname{rank} \left(\frac{\partial I_\alpha}{\partial w} \right) = 1 \tag{23}$$

one can find $G(\mathbf{v})$-invariant solutions in the form

$$U = U(s), \quad U = I_1, \quad s = I_2 \tag{24}$$

and where it is assumed that $\partial I_1/\partial w \neq 0$.

The complete sets of functionally independent invariants of the one-parameter groups $G(\mathbf{v}_j)$, generated by the basic vector fields $\mathbf{v}_j$, $j = 1, \ldots, 10$, are given in Table 3. Evidently, only the invariants of the group $G(\mathbf{v}_3)$ do not satisfy the necessary condition (23) for the existence of group-invariant solutions. The invariants of the rest of the groups $G(\mathbf{v}_j)$ can be used to construct the corresponding group-invariant solutions of the form (24).

On the other hand, each vector field $\mathbf{v}$ of the form (22), i.e. $\mathbf{v} \in L_{SCT}$, can be mapped by a suitable inner automorphism (adjoint map) of the algebra L_{SCT} (whose adjoint representation is given in Table 4) to one of the following representatives of conjugacy classes of the one-dimensional subalgebras of the algebra L_{SCT}:

$$\begin{aligned} &\langle \mathbf{v}_1 \rangle,\ \langle \mathbf{v}_4 \rangle,\ \langle \mathbf{v}_4 \pm \mathbf{v}_3 \rangle,\ \langle \mathbf{v}_7 \rangle,\ \langle \mathbf{v}_7 + a_1\mathbf{v}_4 \rangle,\ \langle \mathbf{v}_7 + a_2\mathbf{v}_4 \pm \mathbf{v}_1 \rangle, \\ &\langle \mathbf{v}_{10} \rangle,\ \langle \mathbf{v}_{10} + a_3\mathbf{v}_4 \rangle,\ \langle \mathbf{v}_{10} + a_4\mathbf{v}_4 \pm \mathbf{v}_3 \rangle,\ \langle \mathbf{v}_{10} + a_5\mathbf{v}_4 + a_6\mathbf{v}_3 \pm \mathbf{v}_1 \rangle, \end{aligned} \tag{25}$$

where $a_1, \ldots, a_6$ are real numbers.

In other words, the vector fields (25) constitute an optimal system of one-dimensional subalgebras and therefore the essentially different group-invariant solutions correspond to the groups are generated by the vector fields (25). The invariants of the groups generated by the vector fields $\mathbf{v}_1$, $\mathbf{v}_4$, $\mathbf{v}_7$ and $\mathbf{v}_{10}$ are given in Table 3. In Table 5 below one can find the invariants of the groups generated by the rest of the vector fields of the optimal system (25) except for those corresponding to the vector fields $\mathbf{v}_{10} + a_5\mathbf{v}_4 + a_6\mathbf{v}_3 \pm \mathbf{v}_1$. These remain to be found.

6. Willmore Surfaces in Revolution

In this Section we are looking for the rotationally-invariant solutions to the Willmore equation (2), i.e. for its solutions of the form

$$w = w(r), \qquad r = \sqrt{(x^1)^2 + (x^2)^2}.$$

Note that r and w are two functionally independent invariants of the operator $\mathbf{v}_4$ generating the one-parameter group of rotations admitted by the equation considered. After such a symmetry reduction, the Willmore equation (2) takes the form

$$\begin{aligned} \mathcal{R} \equiv\ & (2r^3 + 4r^3w_r^2 + 2r^3w_r^4)w_{rrrr} + (4r^2 + 8r^2w_r^2 + 4r^2w_r^4 - 20r^3w_rw_{rr} \\ & -20r^3w_r^3w_{rr})w_{rrr} - 5r^2(3w_r + 3w_r^3 + rw_{rr} - 6rw_r^2w_{rr})w_{rr}^2 \\ & +(rw_r^6 - 2r - 3rw_r^2)w_{rr} + 2w_r + 7w_r^3 + 9w_r^5 + 5w_r^7 + w_r^9 = 0 \end{aligned} \tag{26}$$

where

$$w_r = \frac{\mathrm{d}w}{\mathrm{d}r}, \quad w_{rr} = \frac{\mathrm{d}^2w}{\mathrm{d}r^2}, \quad w_{rrr} = \frac{\mathrm{d}^3w}{\mathrm{d}r^3}, \quad w_{rrrr} = \frac{\mathrm{d}^4w}{\mathrm{d}r^4}.$$

Table 4. Adjoint representation. Here, the (i, j)-th entry gives the adjoint action $\mathrm{Ad}(\exp(\varepsilon \mathbf{v}_i))\mathbf{v}_j$.

Ad↗	$\mathbf{v}_1$	$\mathbf{v}_2$	$\mathbf{v}_3$
$\mathbf{v}_1$	$\mathbf{v}_1$	$\mathbf{v}_2$	$\mathbf{v}_3$
$\mathbf{v}_2$	$\mathbf{v}_1$	$\mathbf{v}_2$	$\mathbf{v}_3$
$\mathbf{v}_3$	$\mathbf{v}_1$	$\mathbf{v}_2$	$\mathbf{v}_3$
$\mathbf{v}_4$	$\mathbf{v}_1 \cos\varepsilon + \mathbf{v}_2 \sin\varepsilon$	$\mathbf{v}_2 \cos\varepsilon - \mathbf{v}_1 \sin\varepsilon$	$\mathbf{v}_3$
$\mathbf{v}_5$	$\mathbf{v}_1 \cos\varepsilon + \mathbf{v}_3 \sin\varepsilon$	$\mathbf{v}_2$	$\mathbf{v}_3 \cos\varepsilon - \mathbf{v}_1 \sin\varepsilon$
$\mathbf{v}_6$	$\mathbf{v}_1$	$\mathbf{v}_2 \cos\varepsilon + \mathbf{v}_3 \sin\varepsilon$	$\mathbf{v}_3 \cos\varepsilon - \mathbf{v}_2 \sin\varepsilon$
$\mathbf{v}_7$	$\mathbf{v}_1 \mathrm{e}^{\varepsilon}$	$\mathbf{v}_2 \mathrm{e}^{\varepsilon}$	$\mathbf{v}_3 \mathrm{e}^{\varepsilon}$
$\mathbf{v}_8$	$\mathbf{v}_1 + 2\varepsilon\mathbf{v}_7 + \varepsilon^2\mathbf{v}_8$	$\mathbf{v}_2 + 2\varepsilon\mathbf{v}_4 - \varepsilon^2\mathbf{v}_9$	$\mathbf{v}_3 + 2\varepsilon\mathbf{v}_5 - \varepsilon^2\mathbf{v}_{10}$
$\mathbf{v}_9$	$\mathbf{v}_1 - 2\varepsilon\mathbf{v}_4 - \varepsilon^2\mathbf{v}_8$	$\mathbf{v}_2 + 2\varepsilon\mathbf{v}_7 + \varepsilon^2\mathbf{v}_9$	$\mathbf{v}_3 + 2\varepsilon\mathbf{v}_6 - \varepsilon^2\mathbf{v}_{10}$
$\mathbf{v}_{10}$	$\mathbf{v}_1 - 2\varepsilon\mathbf{v}_5 - \varepsilon^2\mathbf{v}_8$	$\mathbf{v}_2 - 2\varepsilon\mathbf{v}_6 - \varepsilon^2\mathbf{v}_9$	$\mathbf{v}_3 + \varepsilon 2\mathbf{v}_7 + \varepsilon^2\mathbf{v}_{10}$

Ad↗	$\mathbf{v}_4$	$\mathbf{v}_5$	$\mathbf{v}_6$
$\mathbf{v}_1$	$\mathbf{v}_4 - \varepsilon\mathbf{v}_2$	$\mathbf{v}_5 - \varepsilon\mathbf{v}_3$	$\mathbf{v}_6$
$\mathbf{v}_2$	$\mathbf{v}_4 + \varepsilon\mathbf{v}_1$	$\mathbf{v}_5$	$\mathbf{v}_6 - \varepsilon\mathbf{v}_3$
$\mathbf{v}_3$	$\mathbf{v}_4$	$\mathbf{v}_5 + \varepsilon\mathbf{v}_1$	$\mathbf{v}_6 + \varepsilon\mathbf{v}_2$
$\mathbf{v}_4$	$\mathbf{v}_4$	$\mathbf{v}_5 \cos\varepsilon + \mathbf{v}_6 \sin\varepsilon$	$\mathbf{v}_6 \cos\varepsilon - \mathbf{v}_5 \sin\varepsilon$
$\mathbf{v}_5$	$\mathbf{v}_4 \cos\varepsilon - \mathbf{v}_6 \sin\varepsilon$	$\mathbf{v}_5$	$\mathbf{v}_6 \cos\varepsilon + \mathbf{v}_4 \sin\varepsilon$
$\mathbf{v}_6$	$\mathbf{v}_4 \cos\varepsilon + \mathbf{v}_5 \sin\varepsilon$	$\mathbf{v}_5 \cos\varepsilon - \mathbf{v}_4 \sin\varepsilon$	$\mathbf{v}_6$
$\mathbf{v}_7$	$\mathbf{v}_4$	$\mathbf{v}_5$	$\mathbf{v}_6$
$\mathbf{v}_8$	$\mathbf{v}_4 - \varepsilon\mathbf{v}_9$	$\mathbf{v}_5 - \varepsilon\mathbf{v}_{10}$	$\mathbf{v}_6$
$\mathbf{v}_9$	$\mathbf{v}_4 + \varepsilon\mathbf{v}_8$	$\mathbf{v}_5$	$\mathbf{v}_6 - \varepsilon\mathbf{v}_{10}$
$\mathbf{v}_{10}$	$\mathbf{v}_4$	$\mathbf{v}_5 + \varepsilon\mathbf{v}_8$	$\mathbf{v}_6 + \varepsilon\mathbf{v}_9$

Ad↗	$\mathbf{v}_7$	$\mathbf{v}_8$	$\mathbf{v}_9$	$\mathbf{v}_{10}$
$\mathbf{v}_1$	$\mathbf{v}_7 - \varepsilon\mathbf{v}_1$	$\mathbf{v}_8 - 2\varepsilon\mathbf{v}_7 + \varepsilon^2\mathbf{v}_1$	$\mathbf{v}_9 + 2\varepsilon\mathbf{v}_4 - \varepsilon^2\mathbf{v}_2$	$\mathbf{v}_{10} + 2\varepsilon\mathbf{v}_5 - \varepsilon^2\mathbf{v}_3$
$\mathbf{v}_2$	$\mathbf{v}_7 - \varepsilon\mathbf{v}_2$	$\mathbf{v}_8 - 2\varepsilon\mathbf{v}_4 - \varepsilon^2\mathbf{v}_1$	$\mathbf{v}_9 - 2\varepsilon\mathbf{v}_7 + \varepsilon^2\mathbf{v}_2$	$\mathbf{v}_{10} + 2\varepsilon\mathbf{v}_6 - \varepsilon^2\mathbf{v}_3$
$\mathbf{v}_3$	$\mathbf{v}_7 - \varepsilon\mathbf{v}_3$	$\mathbf{v}_8 - 2\varepsilon\mathbf{v}_5 - \varepsilon^2\mathbf{v}_1$	$\mathbf{v}_9 - 2\varepsilon\mathbf{v}_6 - 2\varepsilon^2\mathbf{v}_2$	$\mathbf{v}_{10} - 2\varepsilon\mathbf{v}_7 + \varepsilon^2\mathbf{v}_3$
$\mathbf{v}_4$	$\mathbf{v}_7$	$\mathbf{v}_8 \cos\varepsilon + \mathbf{v}_9 \sin\varepsilon$	$\mathbf{v}_9 \cos\varepsilon - \mathbf{v}_8 \sin\varepsilon$	$\mathbf{v}_{10}$
$\mathbf{v}_5$	$\mathbf{v}_7$	$\mathbf{v}_8 \cos\varepsilon + \mathbf{v}_{10} \sin\varepsilon$	$\mathbf{v}_9$	$\mathbf{v}_{10} \cos\varepsilon - \mathbf{v}_8 \sin\varepsilon$
$\mathbf{v}_6$	$\mathbf{v}_7$	$\mathbf{v}_8$	$\mathbf{v}_9 \cos\varepsilon + \mathbf{v}_{10} \sin\varepsilon$	$\mathbf{v}_{10} \cos\varepsilon - \mathbf{v}_9 \sin\varepsilon$
$\mathbf{v}_7$	$\mathbf{v}_7$	$\mathbf{v}_8 \mathrm{e}^{-\varepsilon}$	$\mathbf{v}_9 \mathrm{e}^{-\varepsilon}$	$\mathbf{v}_{10} \mathrm{e}^{-\varepsilon}$
$\mathbf{v}_8$	$\mathbf{v}_7 + \varepsilon\mathbf{v}_8$	$\mathbf{v}_8$	$\mathbf{v}_9$	$\mathbf{v}_{10}$
$\mathbf{v}_9$	$\mathbf{v}_7 + \varepsilon\mathbf{v}_9$	$\mathbf{v}_8$	$\mathbf{v}_9$	$\mathbf{v}_{10}$
$\mathbf{v}_{10}$	$\mathbf{v}_7 + \varepsilon\mathbf{v}_{10}$	$\mathbf{v}_8$	$\mathbf{v}_9$	$\mathbf{v}_{10}$

Table 5. Functionally independent invariants.

Generator	Sets of Functionally Independent Invariants $\{I_1, I_2\}$
$\mathbf{v}_4 \pm \mathbf{v}_3$	$I_1 = w \mp \arctan\left(\frac{x^2}{x^1}\right)$ $I_2 = r$
$\mathbf{v}_7 + a_1\mathbf{v}_4$	$I_1 = w\mathrm{e}^{-J_2}$ $I_2 = \arctan\left(\dfrac{a_1x^1 + x^2}{x^1 - a_1x^2}\right) - a_1 \ln r$ $J_2 = \dfrac{1}{1+a_1^2}\left(a_1 \arctan\left(\dfrac{a_1x_1 + x_2}{x_1 - a_1x_2}\right) + \ln r\right)$
$\mathbf{v}_7 + a_2\mathbf{v}_4 \pm \mathbf{v}_1$	$I_1 = w\mathrm{e}^{-J_2}$ $I_2 = \arctan\left(\dfrac{a_2x^1 + x^2}{x^1 - a_2x^2 \pm 1}\right)$ $-a_2 \ln \sqrt{(1+a_2^2)r^2 \pm 2(x^1 - a_2x^2) + 1}$ $J_2 = \dfrac{1}{1+a_2^2}\left(a_2 \arctan\left(\dfrac{a_2x^1 + x^2}{x^1 - a_2x^2 \pm 1}\right)\right.$ $\left. + \ln \sqrt{(1+a_2^2)r^2 \pm 2(x^1 - a_2x^2) + 1}\right)$
$\mathbf{v}_{10} + a_3\mathbf{v}_4$	$I_1 = \dfrac{w}{r^2 + w^2} - \dfrac{1}{a_3} \arctan\left(\dfrac{x^2}{x^1}\right)$ $I_2 = \dfrac{r}{r^2 + w^2}$
$\mathbf{v}_{10} + a_4\mathbf{v}_4 + \mathbf{v}_3$	$I_1 = \dfrac{1}{2} \arctan\left(\dfrac{2w}{r^2 + w^2 - 1}\right) - \dfrac{1}{a_4} \arctan\left(\dfrac{x^2}{x^1}\right)$ $I_2 = \dfrac{r}{r^2 + w^2 + 1}$
$\mathbf{v}_{10} + a_4\mathbf{v}_4 - \mathbf{v}_3$	$I_1 = \dfrac{1}{2} \operatorname{arctanh}\left(\dfrac{2w}{r^2 + w^2 + 1}\right) - \dfrac{1}{a_4} \arctan\left(\dfrac{x^2}{x^1}\right)$ $I_2 = \dfrac{r}{r^2 + w^2 - 1}$

At the same time, expressions (12) for the mean and Gaussian curvatures take the form

$$H = \frac{1}{2r}\frac{rw_{rr} + w_r^3 + w_r}{(1+w_r^2)^{3/2}}, \qquad K = \frac{1}{r}\frac{w_{rr}w_r}{(1+w_r^2)^2}. \tag{27}$$

The reduced Willmore equation (26) is the Euler-Lagrange equation for the functional with Lagrangian density

$$L_{\mathcal{R}} = \frac{1}{4r}\frac{(rw_{rr} + w_r^3 + w_r)^2}{(1 + w_r^2)^{5/2}}.$$

Indeed, applying the Euler operator, which in this case reads

$$E = \frac{\partial}{\partial w} - D_r\frac{\partial}{\partial w_r} + D_r D_r\frac{\partial}{\partial w_{rr}} - \cdots$$

where

$$D_r = \frac{\partial}{\partial r} + w_r\frac{\partial}{\partial w} + w_{rr}\frac{\partial}{\partial w_r} + w_{rrr}\frac{\partial}{\partial w_{rr}} + w_{rrrr}\frac{\partial}{\partial w_{rrr}} + \cdots$$

to the differential function $L_{\mathcal{R}}$ one can easily check that $E[L_{\mathcal{R}}] = \mathcal{R}$. The above Lagrangian $L_{\mathcal{R}}$ is independent of the variable w and so one can reduce its order by one introducing the new dependent variable $v = w_r$ along $v_r = \mathrm{d}v/\mathrm{d}r$ and therefore

$$\tilde{L}_{\mathcal{R}} = \frac{1}{4r}\frac{(rv_r + v^3 + v)^2}{(1 + v^2)^{5/2}}. \tag{28}$$

The Euler-Lagrange equation for the functional with Lagrangian density (28) is

$$E[\tilde{L}_{\mathcal{R}}] = \frac{\mathcal{G}}{r}(1 + v^2)^{-7/2} = 0 \tag{29}$$

where

$$\mathcal{G} = -2r^2(v^2 + 1)v_{rr} + 5r^2 v v_r^2 - 2r(v^2 + 1)v_r + v^7 + 4v^5 + 5v^3 + 2v$$

and $v_{rr} = \mathrm{d}^2 v/\mathrm{d}r^2$. Then, every solution $w(r)$ to the reduced Willmore equation (26) corresponds to a solution $v(r)$ of the second-order equation

$$E[\tilde{L}_{\mathcal{R}}] = \lambda, \quad \lambda = \text{constant} \tag{30}$$

and can be recovered by the quadrature

$$w(r) = \int v(r, \lambda)\,\mathrm{d}r + C, \quad C = \text{constant}.$$

In this sense, the fourth-order equation (26) is relegated to a second-order one.

In the special case $\lambda = 0$, equation (30) coincides with (29) and may be written in the form

$$\mathcal{G} = -2r^2(v^2+1)v_{rr} + 5r^2 v v_r^2 - 2r(v^2+1)v_r + v^7 + 4v^5 + 5v^3 + 2v = 0 \tag{31}$$

since $(1/r)(1 + v^2)^{-7/2} \neq 0$. Equation (31) is scaling-invariant and hence under the change of the variables

$$v = y, \quad \rho = \ln r, \quad Y = \frac{\mathrm{d}\rho}{\mathrm{d}y}$$

transforms into the first order equation

$$\frac{\mathrm{d}Y}{\mathrm{d}y} + \frac{1}{2}\frac{y^7 + 4y^5 + 5y^3 + 2y}{y^2+1}Y^3 + \frac{5}{2}\frac{y}{y^2+1}Y = 0.$$

This is a Bernoulli-type equation which can be integrated by quadratures and its general solution is

$$Y(y) = \pm\frac{1}{(y^2+1)\sqrt{y^2 + A\sqrt{y^2+1}}}, \qquad A = \text{constant}. \tag{32}$$

Now, going back to the variables r and v we can express the general solution of the equation $E[\tilde{L}_{\mathcal{R}}] = 0$ in terms of the relation

$$r = R\exp\left(\pm\int\frac{\mathrm{d}v}{(v^2+1)\sqrt{v^2 + A\sqrt{v^2+1}}}\right), \qquad R = \text{constant} > 0. \tag{33}$$

In the case $A = 0$, we get spheres

$$v = \pm\frac{r}{\sqrt{R^2 - r^2}}, \quad w = \mp\sqrt{R^2 - r^2} + C, \quad H = \mp\frac{1}{R}, \quad K = \frac{1}{R^2}$$

and catenoids

$$v = \pm\frac{R}{\sqrt{r^2 - R^2}}, \quad w = R\ln\left(r \pm \sqrt{r^2 - R^2}\right) + C, \quad H = 0, \quad K = -\frac{R^2}{r^4}.$$

The integral in (33) can be written in terms of the Jacobian elliptic functions and the elliptic integral of the third kind $\Pi(\varphi, n, k)$ as follows (for more details cf. [6] and the Appendix)

$$\int\frac{\mathrm{d}v}{(v^2+1)\sqrt{v^2 + A\sqrt{v^2+1}}} = \begin{cases} I_+(v) & \text{if } A = +4/a^2 > 0,\ |v| < 2\sqrt{2 + \sqrt{a^2+4}}/a^2 \\ I_-(v) & \text{if } A = -4/a^2 < 0,\ |v| > 2\sqrt{2 + \sqrt{a^2+4}}/a^2 \end{cases}$$

where

$$I_+(v) = \frac{a}{(a^2+4)^{1/4}} \times \left[u_+ - \frac{4}{\sqrt{a^4+4}+a^2+2} \Pi \left(\mathrm{am}(u_+,k), \frac{\sqrt{a^4+4}+a^2}{\sqrt{a^4+4}+a^2+2}, k \right) \right]$$

$$I_-(v) = \frac{a}{(a^2+4)^{1/4}} \times \left[u_- - \frac{2(\sqrt{4+a^4}-a^2)}{\sqrt{a^4+4}-a^2+2} \Pi \left(\mathrm{am}(u_-,k), \frac{2}{\sqrt{a^4+4}-a^2+2}, k \right) \right]$$

$$u_+ = \mathrm{cn}^{-1}\left(\frac{(\sqrt{1+v^2}-1)\sqrt{2}}{v\sqrt{\sqrt{a^4+4}+a^2}}, k \right), \quad u_- = \mathrm{cn}^{-1}\left(\frac{v\sqrt{\sqrt{a^4+4}-a^2}}{(\sqrt{1+v^2}-1)\sqrt{2}}, k \right)$$

and

$$k = \sqrt{\frac{a^2+\sqrt{a^4+4}}{2\sqrt{a^4+4}}}.$$

Then, relation (33) becomes

$$r(v) = \begin{cases} R\exp\{\pm I_+(v)\} & \text{if } A = +4/a^2 > 0,\ |v| < 2\sqrt{2+\sqrt{a^2+4}}/a^2 \\ R\exp\{\pm I_-(v)\} & \text{if } A = -4/a^2 > 0,\ |v| < 2\sqrt{2+\sqrt{a^2+4}}/a^2. \end{cases} \tag{34}$$

The above results can be formulated in another form. Consider the following normal system of two ordinary differential equations

$$\begin{aligned} \frac{\mathrm{d}w}{\mathrm{d}r} &= v \\ \frac{\mathrm{d}v}{\mathrm{d}r} &= \pm\frac{1}{r}(v^2+1)\sqrt{v^2+A\sqrt{v^2+1}} \end{aligned} \tag{35}$$

which is equivalent to the single second-order equation

$$\frac{\mathrm{d}^2w}{\mathrm{d}r^2} \pm \frac{1}{r}\left[\left(\frac{\mathrm{d}w}{\mathrm{d}r}\right)^2 + 1 \right] \sqrt{\left(\frac{\mathrm{d}w}{\mathrm{d}r}\right)^2 + A\sqrt{\left(\frac{\mathrm{d}w}{\mathrm{d}r}\right)^2+1}} = 0. \tag{36}$$

The substitution (35) into the expression $\mathcal{R}$ leads to $\mathcal{R} = 0$ and thus shows that each solution of system (35) or equation (36) is a solution of the reduced Willmore equation (26). In this way, we have obtained a special class of solutions to equation (26), i.e. a special class of Willmore surfaces. Substituting (35) into expressions (27) one can see that the mean and Gaussian curvatures of a surface belonging to this special class are given as follows

$$H = \frac{1}{2r}\frac{v \pm \sqrt{v^2 + A\sqrt{v^2+1}}}{\sqrt{v^2+1}}, \qquad K = \pm\frac{1}{r^2}\frac{v\sqrt{v^2 + A\sqrt{v^2+1}}}{v^2+1} \tag{37}$$

where v is any solution of system (35). These functions are depicted in Fig. 1 using the explicit expressions (34). Various branches shown there correspond to the different choices of the signs of A (sub-index) and that ones in (37) (upper-index). The above curves are obtained with $R = a = 1$. Integrating (numerically) the system (35) one can find the profile curves of the Willmore surfaces shown in Fig. 2. Again, various branches correspond to the different choices of the signs of A (sub-index) and the exponent (upper-index) in (33) which are coherent with the respective signs in (35) and (37). The concrete curves are obtained with the same values of the parameters R and a chosen to produce Fig. 1.

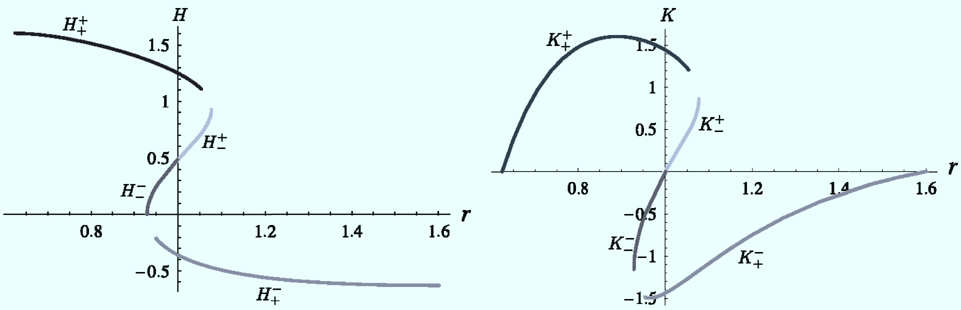

Figure 1. The mean H and Gaussian K curvatures as functions of r. Various branches correspond to the different choices of signs of A (sub-index) and that ones in (37) (upper-index). The above curves are obtained with $R = a = 1$

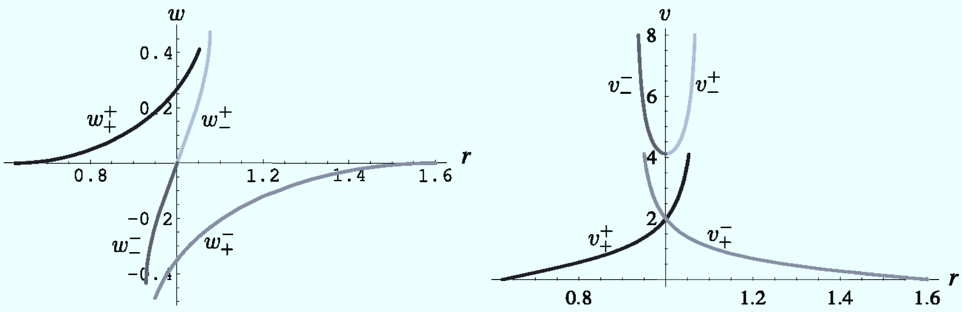

Figure 2. Profile curves (on the left) and their slopes (on the right). Various branches correspond to the different choices of signs of A (sub-index) and the exponent (upper-index) in (33) which are in agreement with the signs in (35) and (37). Curves are drawn with $R = a = 1$

7. Concluding Remarks

In this paper, Lie transformation group methods have been applied to the class of nonlinear fourth-order partial differential equations (15). This class of equations is of interest in differential geometry, mathematical biophysics and string theory since it comprises, for instance, the Willmore equation, the Helfrich's membrane shape equation, the field equations associated with the Polyakov-Kleinert rigid string action and their generalizations. The standard computational procedure shows that the ten-parameter Lie group G_{SCT} of special conformal transformations in $\mathbb{R}^3$ is the largest group of geometric transformations of the involved independent and dependent variables that a generic equation of form (15) could admit. In this way, the group G_{SCT}, which is known to be the symmetry group of the Willmore functional, is recognized to be the largest group of geometric transformations admitted by the Willmore equation (2) in Mongé representation. These results are presented in Section 2. In Section 3, the conserved currents of ten linearly independent conservation laws, which correspond (by virtue of Bessel-Hagen's extension of Noether's theorem) to the symmetries of the Willmore equation, are derived in explicite form. In Section 4, a classification of the group-invariant solutions of the Willmore equation provided by an optimal system of one-dimension subalgebras of the symmetry algebra L_{SCT} is presented. The invariants of the corresponding one-parameter groups are found (up to one exception) and listed so as to be readily applicable for constructing the respective group-invariant solutions of the Willmore equation.

Appendix A. Elliptic Functions and Integrals

Standard integration techniques allow us to find closed form expressions (in terms of trigonometric functions, exponentials and logarithms) for any integral of the form

$$\int \mathcal{R}(z, w)\,\mathrm{d}z, \qquad w^2 = P(z) \tag{38}$$

where $\mathcal{R}(z, w)$ is a rational function and $P(z)$ is a linear or quadratic polynomial. However, if we wish to handle polynomials of higher degree and in particular, when $P(z)$ is cubic or quadratic, then the required functions are called *elliptic functions*. It is easy to prove that every integral of the form (38), where $P(z)$ is a third or a fourth degree polynomial, can be reduced to a linear combination of integrals leading to elementary functions and the following three integrals which are called respectively **elliptic integrals** of the **first**, **second**, and **third kind**

$$\int \frac{\mathrm{d}z}{\sqrt{(1-z^2)(1-k^2z^2)}}, \quad \int \frac{\sqrt{1-k^2z^2}}{\sqrt{1-z^2}}\,\mathrm{d}z, \quad \int \frac{\mathrm{d}z}{(1-nz^2)\sqrt{(1-z^2)(1-k^2z^2)}}.$$

Here the number k is called the **modulus** of these integrals and the number n is called the **parameter** of the integral of the third kind. By means of the substitution $z = \sin\varphi$, the above elliptic integrals can be reduced to their normal trigonometric form

$$\int \mathrm{d}\varphi\sqrt{1-k^2\sin^2\varphi}, \quad \int \sqrt{1-k^2\sin^2\varphi}\,\mathrm{d}\varphi, \quad \int \frac{\mathrm{d}\varphi}{(1-n\sin^2\varphi)\sqrt{1-k^2\sin^2\varphi}}.$$

The corresponding definite elliptic integrals (when the lower limit of integration is taken to be zero) are denoted respectively as

$$\int_0^{\sin\varphi} \frac{\mathrm{d}z}{\sqrt{(1-z^2)(1-k^2z^2)}} = \int_0^{\varphi} \frac{\mathrm{d}\alpha}{\sqrt{1-k^2\sin^2\alpha}} = F(\varphi,k) \tag{39}$$

$$\int_0^{\sin\varphi} \frac{\sqrt{1-k^2z^2}}{\sqrt{1-z^2}}\,\mathrm{d}z = \int_0^{\varphi} \sqrt{1-k^2\sin^2\alpha}\,\mathrm{d}\alpha = E(\varphi,k) \tag{40}$$

and

$$\int_0^{\sin\varphi} \frac{\mathrm{d}z}{(1-nz^2)\sqrt{(1-z^2)(1-k^2z^2)}} = \int_0^{\varphi} \frac{\mathrm{d}\alpha}{(1-n\sin^2\alpha)\sqrt{1-k^2\sin^2\alpha}} = \Pi(\varphi,n,k).$$

These integrals are called also **incomplete elliptic integrals** of the **first**, **second** and **third kind**, respectively. When the upper limit of integration φ for the integrals in (39) and (40) is chosen to be $\pi/2$ they are called **complete elliptic integrals** of the **first**, **second** and **third kind** and are denoted as

$$K(k) = F(\pi/2,k), \quad E(k) = E(\pi/2,k) \ \text{ and } \ \Pi(n,k) = \Pi(\pi/2,n,k). \tag{41}$$

The inverse functions of the elliptic integrals are called **elliptic functions**. E.g., if

$$u = F(\varphi,k) = \int_0^{\varphi} \frac{\mathrm{d}\alpha}{\sqrt{1-k^2\sin^2\alpha}} \tag{42}$$

is the incomplete elliptic integral of the first kind, then φ is called the **amplitude** of u and denoted (following Jacobi) as

$$\varphi = am(u,k). \tag{43}$$

The Jacobian elliptic functions are introduced via (43) and the following formulas

$$\begin{aligned} \mathrm{sn}(u,k) &= \sin\varphi = \sin am(u,k) \\ \mathrm{cn}(u,k) &= \cos\varphi = \cos am(u,k) \\ \mathrm{dn}(u,k) &= \sqrt{1-k^2\sin^2\varphi} = \sqrt{1-k^2\,\mathrm{sn}^2(u,k)}. \end{aligned} \tag{44}$$

It should be noted that while $am(u,k)$ is the inverse function of $u = F(\varphi, k)$, the inversion of

$$u = F(z,k) = \int_0^z \frac{\mathrm{d}\zeta}{\sqrt{(1-\zeta^2)(1-k^2\zeta^2)}} \tag{45}$$

is furnished by $\mathrm{sn}(u,k)$, i.e., one has $F(\mathrm{sn}(u,k),k) = u$.

More details about elliptic integrals and functions can be found in [6] and [2] and references therein.

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