

## ONE DIMENSIONAL QUASI-EXACTLY SOLVABLE DIFFERENTIAL EQUATIONS

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**Abstract.** In this paper by means of similarity transformation we find some one-dimensional quasi-exactly solvable differential equations and their related Hamiltonians which appear in physical problems. We have provided also two examples with application of these differential equations.

### 1. Introduction

During the last decade a remarkable new class of quasi-exactly solvable spectral problems was introduced in [5]. These occupy an intermediate position between exactly solvable and unsolvable models in the sense that exact solution in an algebraic form exists only for a part of the spectrum.

In this paper we suggest a generalization of Bender-Dunne [1] approach to possible one-dimensional elliptic quasi-exactly solvable second order differential equations.

For this purpose, and with an attention to applications of elliptic potential we are motivated to obtain generalized master functions  $A(x)$  that lead to elliptic quasi-exactly solvable potentials. By appropriate choice of the generalized master function  $A(x)$  we obtain some one dimensional quasi-exactly solvable potentials that in all cases are functions of **Jacobi elliptic function**. These functions are periodic functions.

The paper is organized as follows: In Section 2 we show that we can generalize the usual quadratic master function to a master function of at most four order polynomials, then the most general elliptic quasi-exactly solvable differential operators related to generalized master function of degree  $k = 3$  and  $k = 4$  are given. Also by expanding their solutions in powers of  $x$ , we get three-term and four-term recursion relations among their coefficients, where Bender–Dunne factorization follows

through imposing the quasi-exactly solvability conditions and in Section 3 we derive all one-dimensional elliptic quasi-exactly solvable differential equations for  $k = 3$  and  $k = 4$  and respectively the relative quantum Hamiltonian via prescription of references [3, 4]. Finally, in Section 4, as an example, we derive Lamé potential from the special case of the potential which is given by the generalized master function  $A(x) = 4x(1 - x)(1 - k^2x)$ .

## 2. Quasi-Exactly Solvable Differential Equations Associated with Generalized Master Function

In the following, by generalizing master function of order up to two to polynomial of order up to  $k$  together with the non-negative weight function  $W(x)$ , defined on the interval  $(a, b)$  such that  $\frac{1}{W(x)} \frac{d}{dx}(A(x)W(x))$  is a polynomial of degree at most  $k - 1$ , we can define the operator

$$L = -\frac{1}{W(x)} \frac{d}{dx} \left( A(x)W(x) \frac{d}{dx} \right) + B(x) \tag{1}$$

where  $B(x)$  is a polynomial of order up to  $k - 2$ . The interval  $(a, b)$  is chosen so that, we have  $A(a)W(a) = A(b)W(b) = 0$ . It is straightforward to show that the above defined operator  $L$  is a self-adjoint linear operator which maps a given polynomial of order  $m$  to another polynomial of order  $m + k - 2$ . Now, by an appropriate choice of  $B(x)$  and weight function  $W(x)$ , the operator  $L$  can have an invariant subspace of polynomials of order up to  $n$ . Then by choosing the set of orthogonal polynomials  $\{\phi_0, \phi_1, \dots, \phi_n\}$  defined in the interval  $(a, b)$  with respect to the weight function  $W(x)$

$$\int_a^b \phi_m(x)\phi_n(x)W(x) dx = 0 \quad \text{for } m \neq n \tag{2}$$

as a basis, the matrix elements of the operator  $L$  on this base will have the following block diagonal form

$$L_{ij} = 0 \quad \text{if } \{i \leq n \text{ and } j \geq n + 1\} \quad \text{or} \quad \{i \geq n + 1 \text{ and } j \leq n\}. \tag{3}$$

Since, according to the well known theorem of orthogonal polynomials,  $\phi_n(x)$  is orthogonal to any polynomial of order up to  $n - 1$  and, therefore, for the matrix  $L$  we get

$$L = \begin{bmatrix} M & 0 \\ 0 & N \end{bmatrix} \tag{4}$$

where  $M$  is an  $(n + 1) \times (n + 1)$  matrix with matrix elements

$$M_{ij} = \int_a^b W(x)\phi_1(x)L(x)\phi_j(x) dx, \quad i, j = 0, 1, 2, \dots, n \tag{5}$$

and  $N$  is an infinite matrix element defined as above with  $i, j \geq n + 1$ .

The block-diagonal form of the operator  $L$  indicates that by diagonalizing the  $(n + 1) \times (n + 1)$  matrix  $M$ , we can find  $n + 1$  eigenvalues of the operator  $L$  together with the related eigenfunctions as linear functions of orthogonal polynomials  $\{\phi_0, \phi_1, \dots, \phi_n\}$ .

In order to determine the appropriate  $B(x)$  and  $W(x)$  for given generalized master function  $A(x)$ , we use the Taylor expansion of these functions

$$A(x) = \sum_{i=0}^k \frac{A^{(i)}(0)}{i!} x^i, \quad \text{with} \quad A^{(i)}(0) = \left. \frac{d^i A(x)}{dx^i} \right|_{x=0} \quad (6)$$

$$\frac{(A(x)W(x))'}{W(x)} = \sum_{i=0}^{k-1} \frac{\left(\frac{(AW)'}{W}\right)^{(i)}(0)}{i!} x^i \quad (7)$$

and, therefore,

$$\begin{aligned} \left(\frac{(AW)'}{W}\right)^{(i)}(0) &= \left. \frac{d^i \left(\frac{(A(x)W(x))'}{W(x)}\right)}{dx^i} \right|_{x=0} \\ B(x) &= \sum_{i=0}^{k-2} \frac{B^{(i)}(0)}{i!} x^i, \quad \text{with} \quad B^{(i)}(0) = \left. \frac{d^i B(x)}{dx^i} \right|_{x=0}. \end{aligned} \quad (8)$$

Then, the existence of invariant subspace built by the polynomials of order  $n$  of the operator  $L$  leads to the following linear equations between the coefficients of the above Taylor expansions

$$-\frac{A^{(i+2)}}{(i+2)!} l(l-1) - \frac{\left(\frac{(AW)'}{W}\right)^{(i+1)}}{(i+1)!} l + \frac{B^{(i)}}{i!} = 0 \quad (9)$$

where

$$\begin{cases} l = n & \text{and} \quad i = 1, 2, \dots, k-2 \\ l = n-1 & \text{and} \quad i = 2, 3, \dots, k-2 \\ \dots\dots\dots \\ l = n-k+4 & \text{and} \quad i = k-3, k-2 \\ l = n-k+3 & \text{and} \quad i = k-2. \end{cases} \quad (10)$$

The number of above equations for a given value of  $k$  is  $\frac{(k-1)(k-2)}{2}$ . If we are to determine only the unknown function  $B(x)$  without having any further constraint on the weight function  $W(x)$ , then the above  $\frac{(k-1)(k-2)}{2}$  equations should be satisfied with  $(k-2)$  coefficients of Taylor expansion of  $B$  as the only unknowns, since  $B^{(0)}$  can be absorbed in the eigenspectrum operator  $L$ . Therefore, we are left with  $k-2$  unknowns to be determined, where the compatibility of equations (9) require that  $k=3$  at most. On the other hand, if we add the coefficients of Taylor

expansions of  $A(x)$  and  $\frac{(A(x)W(x))'}{W(x)}$  to our list of unknowns, (to be determined by solving equations (9)), then their compatibility conditions require that

$$3(k - 1) \geq \frac{(k - 1)(k - 2)}{2} \tag{11}$$

or  $k \leq 8$ , where further investigations show that we can have at most  $k = 4$ , since for  $k \geq 5$  the coefficients  $A^{(k)}(0)$  and  $\left(\frac{(A(x)W(x))'}{W(x)}\right)^{(k-1)}(0)$  will vanish. Below we summarize the above-mentioned discussion for  $k = 3$  and  $k = 4$ , separately.

**2.1. The Case  $k = 3$**

In this case,  $B(x)$  is a second order polynomial where  $B^{(1)}$  can be determined by solving equations (9)

$$B^{(1)} = \frac{n}{2} \left( \frac{A^{(3)}(0)}{3}(n - 1) + \left(\frac{(AW)'}{W}\right)^{(2)} \right) \tag{12}$$

which is the only unknown in this case.

**2.2. The Case  $k = 4$**

Again, solving the equation (9) leads to

$$B^{(1)} = \frac{n}{2} \left( \frac{A^{(3)}(0)}{3}(n - 1) + \left(\frac{(AW)'}{W}\right)^{(2)} \right) \tag{13}$$

$$B^{(2)} = -\frac{A^{(4)}}{12}n(n - 1) \tag{14}$$

and

$$\left(\frac{(AW)'}{W}\right)^{(3)} = -\frac{A^{(4)}}{2}(n - 1). \tag{15}$$

Here, besides having a constraint over the second order polynomial  $B(x)$ , we have to put further constraints on the weight function  $W(x)$  given in (15).

Definitely, we can determine  $n + 1$  eigenvalues of the operator  $L$ , simply by diagonalizing the  $(n + 1) \times (n + 1)$  matrix  $M$ , since it is a self-adjoint operator in Hilbert space of polynomials and it has a block diagonal form given in (4).

As we are going to see at the end of this section, we can determine its eigenspectrum analytically, using some recursion relations.

### 2.3. Recursion Relations

Now we show that the eigenfunction of the operator  $L$  is a generating function for a new set of polynomials  $P_m(E)$  where the eigenfunction equation of the operator  $L$  leads to the recursion relations between these polynomials. Quasi-exact solvable constraints (9) will lead to their factorization, that is,  $P_{n+N+1}(E) = P_{n+1}(E)Q_N$  for  $N \geq 0$ , where the roots of polynomials  $P_{n+1}(E)$  turn out to be the eigenvalues of the operator  $L$ . To achieve these results, first we expand  $\psi(x)$ , the eigenfunction of  $L$ , as

$$\psi(x) = \sum_{m=0}^{\infty} P_m(E)x^m \quad (16)$$

where the eigenfunction equation

$$L\psi(x) = E\psi(x) \quad (17)$$

can be expressed as

$$\begin{aligned} -A(x) \sum_{m=2}^{\infty} m(m-1)P_m(E)x^{m-2} - \frac{(AW)'}{W} \int_{m=1}^{\infty} mP_m(E)x^{m-1} dx \\ + B(x) \sum_{m=0}^{\infty} P_m(E)x^m = E \sum_{m=0}^{\infty} P_m(E)x^m \end{aligned} \quad (18)$$

and this leads to the following recursion relations for the coefficients  $P_m(E)$

$$\begin{aligned} & \left( A^{(1)}(m+1)(m+2) + \left( \frac{(AW)'}{W} \right)^{(0)} (m+2) \right) P_{m+2}(E) \\ & + \left( \frac{A^{(2)}}{2!} m(m+1) + \left( \frac{(AW)'}{W} \right)^{(1)} (m+1) + E \right) P_{m+1}(E) \\ & + \left( \frac{A^{(3)}}{3!} m(m-1) + \frac{\left( \frac{(AW)'}{W} \right)^{(2)}}{2!} m - B^{(1)} \right) P_m(E) \\ & + \left( \frac{A^{(4)}}{4!} (m-1)(m-2) + \frac{\left( \frac{(AW)'}{W} \right)^{(3)}}{3!} m - \frac{B^{(2)}}{2!} \right) P_{m-1}(E) = 0. \end{aligned} \quad (19)$$

Below we investigate recursion relations which are obtained in the cases when  $k = 3$  (cubic  $A(x)$ ) and  $k = 4$  (quartic  $A(x)$ ), separately.

**Cubic A:**

In this case the four-term general recursion relation reduces to the following three-term recursion relation

$$\begin{aligned} & \left( A^{(1)}(m+1)(m+2) + \left( \frac{(AW)'}{W} \right)^{(0)} (m+2) \right) P_{m+2}(E) \\ & + \left( \frac{A^{(2)}}{2!} m(m+1) + \left( \frac{(AW)'}{W} \right)^{(1)} (m+1) + E \right) P_{m+1}(E) \\ & + \left( \frac{A^{(3)}}{3!} m(m-1) + \frac{\left( \frac{(AW)'}{W} \right)^{(2)}}{2!} m - B^{(1)} \right) P_m(E) = 0. \end{aligned} \quad (20)$$

In order to have finite eigenspectrum, that is, quasi-integrable differential equation, the above recursion relation should be truncated at some value of  $m = n$ , which is obviously possible by an appropriate choice of

$$B^{(1)} = \frac{n}{2} \left( \frac{A^{(3)}(0)}{3} (n-1) + \left( \frac{(AW)'}{W} \right)^{(2)} \right) \quad (21)$$

and this is in agreement with the results of previous subsection.

Using the recursion relations (20) with  $B^{(1)}$  given in (21), we get a factorization of the polynomial  $P_{n+N+1}(E)$  for  $N \geq 0$  in terms of  $P_{n+1}(E)$  as follows

$$P_{n+N+1}(E) = P_{n+1}(E)Q_N(E), \quad N \geq 0 \quad (22)$$

where, by choosing the eigenvalue  $E$  as a root of the polynomials  $P_{n+1}(E)$ , all polynomials of order higher than  $n$  will vanish.

By using equations (16) we obtain the eigenfunctions  $\psi_i(x)$

$$\psi_i(x) = \sum_{m=0}^n P_m(E_i)x^m, \quad i = 0, 1, \dots, n \quad (23)$$

where  $E_i$  are roots of the polynomial  $P_{n+1}(E)$ .

The above eigenfunctions are polynomials of order  $n$ , hence they have at most  $n$  roots in the interval  $(a, b)$ , where, according to the well-known oscillation and comparison theorem for the second-order linear differential equation [2] these numbers order the eigenvalues according to the number of roots of corresponding eigenfunctions. Therefore, we can say that the eigenvalues thus obtained are the first  $n + 1$  eigenvalues of the operator  $L$ . Using the recursion relations (20), we can evaluate the polynomials  $P_m(E)$  in term of  $P_0(E)$ , where we have chosen  $P_0(E) = 1$ . Following the above scheme we have evaluated the first five polynomials shown in the Appendix.

**Quartic A:**

Again in order to truncate the recursion relations (19) and to factorize the polynomials  $P_{n+N+1}(E)$  in terms of  $P_{n+1}(E)$ , we should have

$$B^{(1)} = \frac{n}{2} \left( \frac{A^{(3)(0)}}{3} (n-1) + \left( \frac{(AW)'}{W} \right)^{(2)} \right) \quad (24)$$

$$\frac{B^{(2)}}{2!} = \frac{A^{(4)}}{4!} (n-1)(n-2) + \frac{\left( \frac{(AW)'}{W} \right)^{(3)}}{3!} n \quad (25)$$

and

$$\frac{B^{(2)}}{2!} = \frac{A^{(4)}}{4!} n(n-1) + \frac{\left( \frac{(AW)'}{W} \right)^{(3)}}{3!} (n+1). \quad (26)$$

Solving the above equations we get

$$B^{(2)} = -\frac{A^{(4)}}{12} n(n-1) \quad (27)$$

and

$$\left( \frac{(AW)'}{W} \right)^{(3)} = -\frac{A^{(4)}}{2} (n-1). \quad (28)$$

The equations (24), (27) and (28) are the same equations which are required for the reduction of the operator  $L$  to its block diagonal form.

Again the roots of the polynomial  $P_{n+1}$  will correspond to  $n+1$  eigenvalues of the differential operator  $L$  with eigenfunctions which can be expressed in term of  $P_m(E_i)$  for  $m \leq n$ , where polynomials  $P_m(E)$  can be obtained from recursion relation by choosing  $P_0 = 1$  and  $P_{-1} = 0$ .

### 3. Quasi-Exactly Potential Associated with Generalized Master Function

As in [3, 4], writing

$$\psi(t) = A^{1/4}(x)W^{1/2}(x)\phi(x) \quad (29)$$

by a change of the variable  $\frac{dx}{dt} = \sqrt{A(x)}$ , the eigenvalue equation for the operator  $L$  reduces to the Schrödinger equation

$$H(t)\psi(t) = E\psi(t) \quad (30)$$

with the same eigenvalue  $E$  and  $\psi(t)$  given in (30), in terms of eigenfunction of  $L$ , where  $H(t) = -\frac{d^2}{dt^2} + V(t)$  is the similarity transformation of  $L(x)$  defined as

$$H(t) = A^{1/4}(x)W^{1/2}(x)L(x)A^{-1/4}(x)W^{-1/2}(x) \quad (31)$$

with

$$V(t) = -\frac{3}{16} \frac{\dot{A}^2(t)}{A^2(t)} - \frac{1}{4} \frac{\dot{W}^2(t)}{W^2(t)} + \frac{1}{4} \frac{\dot{A}(t)\dot{W}(t)}{A(t)W(t)} + \frac{1}{4} \frac{\ddot{A}(t)}{A(t)} + \frac{1}{2} \frac{\ddot{W}(t)}{W(t)} + B(t) \quad (32)$$

and

$$V(x) = \frac{\ddot{A}^2(x)}{4} - \frac{\dot{A}^2(x)}{16A(x)} - \frac{A(x)\dot{W}(x)^2}{4W^2(x)} + \frac{A(x)\ddot{W}(x)}{2W(x)} + \frac{\dot{A}(x)\dot{W}(x)}{2W(x)} + B(x).$$

It is also straightforward to show that

$$\int \phi(t)H(t)\psi(t) dt = \int_a^b W(x)\psi(x)L(x)\psi(x) dx. \tag{33}$$

Hence block diagonalization of  $L$  leads to block-diagonalization of  $H$ .

### 3.1. Elliptic Quasi-Exactly Solvable Potential

The starting point to find elliptic quasi-exactly solvable potential is generalized master function  $A(x)$ , as mentioned before. Therefore, the selection of master function  $A$  which leads to elliptic potential, is very important. Considering the relation  $\frac{dx}{dt} = \sqrt{A(x)}$ , we select the master function so that  $x$  comes into the form of elliptic Jacobi functions. The weight function  $W(x)$  related to the given master function  $A(x)$  of order three and four can be obtained so that the polynomial  $\frac{1}{W} \frac{d}{dx}(AW)$  to be of order two or three, respectively.

After determining  $B_1$  and  $B_2$  from equations (13) and (14), the function  $B(x)$  can be obtained easily

$$B(x) = B_1x + \frac{1}{2!}B_2x^2.$$

Now, we can determine operator  $L$  and potential  $V(t)$  by knowing  $A$ ,  $W$  and  $B$ .

The interval  $(a, b)$  for  $x$  is chosen so that to have  $A(a)W(a) = A(b)W(b) = 0$ , and the interval of the parameters  $\alpha, \beta, \gamma$  and  $\delta$  such that  $A(x)W(x)$  has not any singularity and also  $A(a)W(a) = A(b)W(b) = 0$  and equation (28) are conserved.

We introduce the possible 24 generalized master functions  $A(x)$  of order three and four in Table 1 below.

### 4. Example

As an example we are going to obtain the Lamé potential. For this purpose we consider the generalized master function  $A(x) = 4x(1-x)(1-k^2x)$ ,  $x = \text{sn}^2(t, k)$  where its corresponding differential equation  $L$ , weight function  $W(x)$ , polynomial  $B$ , potential  $V$  and the interval of  $x$  are given bellow.

$$W = x^\alpha(1-x)^\beta(1-k^2x)^\gamma, \quad 0 \leq x \leq 1, \quad 0 < k < 1, \\ \alpha > -1, \quad \beta > -1, \quad -\infty < \gamma < \infty$$

$$B = 4nk^2(n+2+\alpha+\beta+\gamma)x$$



Cubic $A$	$x$	Quartic $A$	$x$
$4x(1-x)(1-k^2x)$	$\operatorname{sn}^2(t, k)$	$(x^2 - k^2)(x^2 - 1)$	$\frac{\operatorname{dn}(t, k)}{\operatorname{cn}(t, k)}$
$4x(1+x)(1+(1-k^2)x)$	$\frac{\operatorname{sn}^2(t, k)}{\operatorname{cn}^2(t, k)}$	$(1+x^2)(1-k^2+x^2)$	$\frac{\operatorname{cn}(t, k)}{\operatorname{sn}(t, k)}$
$4x(1+k^2x)(1+(k^2-1)x)$	$\frac{\operatorname{sn}^2(t, k)}{\operatorname{dn}^2(t, k)}$	$(x^2-1)(1-k^2-x^2)$	$\operatorname{dn}(t, k)$
$4x(x-1)(x-k^2)$	$\frac{1}{\operatorname{sn}^2(t, k)}$	$(x^2-1)(x^2-k^2)$	$\frac{1}{\operatorname{sn}(t, k)}$
$4x(x-1)((1-k^2)x+k^2)$	$\frac{1}{\operatorname{cn}^2(t, k)}$	$(1-x^2)((1-k^2)x^2-1)$	$\frac{1}{\operatorname{dn}(t, k)}$
$4x(1-x)(1-k^2+k^2x)$	$\operatorname{cn}^2(t, k)$	$(x^2-1)((1-k^2)x^2+k^2)$	$\frac{1}{\operatorname{cn}(t, k)}$
$4x(1-x)(k^2-1+x)$	$\operatorname{dn}^2(t, k)$	$(1-x^2)(1-k^2x^2)$	$\operatorname{sn}(t, k)$
$4x(x-1)((k^2-1)x+1)$	$\frac{1}{\operatorname{dn}^2(t, k)}$	$(1-x^2)(1-k^2+k^2x^2)$	$\operatorname{cn}(t, k)$
$4x(1+x)(1-k^2+x)$	$\frac{\operatorname{cn}^2(t, k)}{\operatorname{sn}^2(t, k)}$	$(k^2+x^2)(k^2-1+x^2)$	$\frac{\operatorname{dn}(t, k)}{\operatorname{sn}(t, k)}$
$4x(k^2x-1)(x-1)$	$\frac{\operatorname{cn}^2(t, k)}{\operatorname{dn}^2(t, k)}$	$(k^2+x^2)(k^2-1+x^2)$	$\frac{\operatorname{dn}(t, k)}{\operatorname{sn}(t, k)}$
$4x(x-k^2)(x-1)$	$\frac{\operatorname{dn}^2(t, k)}{\operatorname{cn}^2(t, k)}$	$(1+k^2x^2)(1-(1-k^2)x^2)$	$\frac{\operatorname{sn}(t, k)}{\operatorname{dn}(t, k)}$
$4x(k^2+x)(x+k^2-1)$	$\frac{\operatorname{dn}^2(t, k)}{\operatorname{sn}^2(t, k)}$	$(1+x^2)(1+(1-k^2)x^2)$	$\frac{\operatorname{sn}(t, k)}{\operatorname{cn}(t, k)}$

**Table 1.** Cubic and Quartic Master Functions

$$L = -4x(1-x)(1-k^2x) \frac{d^2}{dx^2} - [4k^2(3 + \alpha + \beta + \gamma)x^2 + (-8k^2 - 8 - 4\alpha k^2 - 4\alpha - 4\beta - 4\gamma k^2)x + 4 + 4\alpha] \frac{d}{dx} + 4nk^2(n + 2 + \alpha + \beta + \gamma)x$$

$$V = \frac{1}{4(1-k^2)} \left( (3C_4 + C_3)\operatorname{cn}^2(t, k) - C_4\operatorname{cn}^4(t, k) + \frac{C_4\operatorname{dn}^4(t, k)}{k^4} - \left( \frac{3C_4}{k^4} + \frac{C_3}{k^2} \right) \operatorname{dn}^2(t, k) - \left( \frac{C_4}{k^4} + \frac{C_3}{k^2} + C_2 + k^2C_1 + k^4C_0 \right) \frac{1}{\operatorname{dn}^2(t, k)} + \frac{C_0}{4\operatorname{sn}^2(t, k)} + \frac{3(1+k^2)C_4}{4k^4} + \frac{C_3}{2k^2} \right)$$

$$C_1 = -8\beta - 8\alpha k^2 - 8\gamma k^2 - 8\alpha - 4k^2 - 8\alpha\beta - 8\alpha^2 k^2 - 8\alpha\gamma k^2 - 8\alpha^2 - 4$$

$$C_2 = 32\alpha k^2 + 24\beta k^2 + 24\gamma k^2 + 26k^2 + 4\beta^2 + 4\alpha^2 + 8\alpha\beta + 4k^4 + 8\alpha \\ + 16nk^2\gamma + 8\beta\gamma k^2 + 16\alpha\beta k^2 + 8\gamma k^4 + 32nk^2 + 4\alpha^2 k^4 + 16\alpha^2 k^2 \\ + 8\alpha k^4 + 4\gamma^2 k^4 + 16n^2 k^2 + 8\beta + 4 + 16\alpha\gamma k^2 + 16nk^2\beta + 16nk^2\alpha \\ + 8\alpha\gamma k^4$$

$$C_3 = -4k^2(6\alpha k^2 + 4\beta k^2 + 4\alpha n + 4\beta n + 6\gamma k^2 + 4\gamma + 5k^2 + 2k^2\gamma^2 + 4\gamma n \\ + 2\beta^2 + 2\alpha^2 + 4\alpha\beta + 6\alpha + 4nk^2\gamma + 4n^2 + 2\beta\gamma k^2 + 2\alpha\beta k^2 + 2\beta\gamma \\ + 8n + 8nk^2 + 2\alpha^2 k^2 + 4n^2 k^2 + 6\beta + 2\alpha\gamma + 5 + 4\alpha\gamma k^2 + 4nk^2\beta \\ + 4nk^2\alpha)$$

$$C_4 = k^4(2\gamma + 5 + 4n + 2\beta + 2\alpha)(2\gamma + 3 + 4n + 2\beta + 2\alpha)$$

$$C_0 = 4\alpha^2 - 1.$$

Let us restrict ourselves to the case in which the parameters  $\alpha, \beta, \gamma$  are

$$\alpha = \beta = \gamma = -\frac{1}{2}. \quad (34)$$

The relative potential of the generalized master function  $A(x)$  reduces to

$$V(x) = 2n(2n + 1)k^2 x^2 \quad (35)$$

which is exactly the **Lame potential**.

Below we obtain the low lying eigenvalues and eigenstates for this potential. In order to find the eigenvalues and eigenstates for  $n = 1$ , first we obtain from  $P_2 = 0$  the eigenvalues  $E_1$  and  $E_2$

$$P_2 = \frac{E^2}{24} - \frac{(k^2 + 1)E}{6} + \frac{k^2}{2} \\ E_1 = 2k^2 + 2 - 2\sqrt{k^4 - k^2 + 1} \\ E_2 = 2k^2 + 2 + 2\sqrt{k^4 - k^2 + 1}.$$

Now from  $\psi_i(x) = \sum_{m=0}^n P_m(E_i)x^m$ , we can obtain the eigenstates  $\psi_1$  and  $\psi_2$  as given below

$$\psi_1(x) = 1 + 2 \left( k^2 + 1 + \sqrt{k^4 - k^2 + 1} \right) x^2 \\ \psi_2(x) = 1 + 2 \left( k^2 + 1 - \sqrt{k^4 - k^2 + 1} \right) x^2.$$

Similarly for  $n = 2$  with  $P_3 = 0$  we obtain  $E_1, E_2, E_3$  and relative eigenstates as

$$P_3 = -\frac{1}{720}E^3 + \frac{(1 + k^2)E^2}{36} - \frac{k^2(4k^2 + 21)E}{45} + \frac{8k^2(k^2 + 1)}{9} \\ E_1 = -\frac{20}{3} - \frac{20}{3}k^2$$

$$\begin{aligned}
E_2 &= \frac{10}{3} + \frac{10}{3}k^2 + 2\sqrt{9k^4 - 4k^2 + 9} \\
E_3 &= \frac{10}{3} + \frac{10}{3}k^2 - 2\sqrt{9k^4 - 4k^2 + 9} \\
\psi_1(x) &= 1 + \frac{10}{3}(1 + k^2)x^2 + \frac{1}{27}(80k^4 + 205k^2 + 80)x^4 \\
\psi_2(x) &= -\frac{2}{3} - \frac{5}{3}k^2 - \sqrt{9k^4 - 4k^2 + 9}x^2 \\
&\quad + \frac{1}{27} \left( 6\sqrt{9k^4 - 4k^2 + 9}(1 + k^2) + 38k^4 + 22k^2 + 38 \right) x^4 \\
\psi_3(x) &= -\frac{2}{3} - \frac{5}{3}k^2 - \sqrt{9k^4 - 4k^2 + 9}x^2 \\
&\quad + \frac{1}{27} \left( -6\sqrt{9k^4 - 4k^2 + 9}(1 + k^2) + 38k^4 + 22k^2 + 38 \right) x^4.
\end{aligned}$$

### Appendix: The First Four Polynomials $P_n(E)$ for $k = 3$

To abbreviate, we set  $F^{(i)} = \left( \frac{AW'}{W} \right)^{(i)}$ .

$$P_0 = 1$$

$$P_1 = -\frac{E}{F^0}$$

$$P_2 = \frac{1}{2} \frac{B^1 F^0 + E F^1 + E^2}{F^0(A^1 + F^0)}$$

$$\begin{aligned}
P_3 &= -(2EB^{(1)}A^{(1)} + A^{(2)}E^2 + 2F^{(1)}B^{(0)}F^{(0)} \\
&\quad + A^{(2)}B^{(1)}F^{(0)} + A^{(2)}EF^{(1)} + 3EB^{(1)}F^{(0)} + E^3 \\
&\quad + 2EF^{(1)^2} + 3F^{(1)}E^2 - EF^{(2)}A^{(1)} - EF^{(2)}F^{(0)}) \\
&\quad / (6F^{(0)}2A^{(1)^2} + 3A^{(1)}F^{(0)} + F^{(0)^2})
\end{aligned}$$

$$\begin{aligned}
P_4 &= (-A^{(3)}EF^{(1)}F^{(0)} + 4A^{(2)}E^3 + 6F^{(1)}E^3 + 6EF^{(1)^3} + 11F^{(1)^2}E^2 \\
&\quad + 3B^{(1)^2}F^{(0)^2} - 2A^{(3)}E^2A^{(1)} + 3A^{(2)^2}B^{(1)}F^{(0)} + 3A^{(2)^2}EF^{(1)} \\
&\quad + 6F^{(1)^2}B^{(1)}F^{(0)} + 8E^2B^{(1)}A^{(1)} + 6E^2B^{(1)}F^{(0)} - 7E^2F^{(2)}A^{(1)} \\
&\quad - 4E^2F^{(2)}F^{(0)} + 9A^{(2)}EF^{(1)^2} + 13A^{(2)}F^{(1)}E^2 - 3F^{(2)}B^{(1)}F^{(0)^2} \\
&\quad + 6A^{(1)}B^{(1)^2}F^{(0)} - 2A^{(3)}EF^{(1)}A^{(1)} + 6A^{(2)}EB^{(1)}A^{(1)} \\
&\quad + 9A^{(2)}F^{(1)}B^{(1)}F^{(0)} + 10A^{(2)}EB^{(1)}F^{(0)} - 3A^{(2)}EF^{(2)}A^{(1)})
\end{aligned}$$

$$\begin{aligned}
& - 3A^{(2)}EF^{(2)}F^{(0)} + 12F^{(1)}EB^{(1)}A^{(1)} + 14F^{(1)}EB^{(1)}F^{(0)} \\
& - 9F^{(1)}EF^{(2)}A^{(1)} - 6F^{(1)}EF^{(2)}F^{(0)} - 6A^{(1)}F^{(2)}B^{(1)}F^{(0)} \\
& - 2A^{(1)}A^{(3)}B^{(1)}F^{(0)} - A^3B^1F^{(0)^2} - A^{(3)}E^2F^{(0)} + 3A^{(2)^2}E^2 + E^4 \\
& / (24F^{(0)}(6A^{(1)^3} + 11A^{1^2}F^{(0)} + 6A^{(1)}F^{(0)^2} + F^{(0)^3})).
\end{aligned}$$

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