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MEMBRANE APPROACH TO BALLOONS AND SOME RELATED SURFACES

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Abstract. The well known Laplace–Young equation asserts that the pressure difference across the film or a membrane in a equilibrium is proportional to the mean curvature with a proportionality constant the surface tension of the interface. Here we present two variants of this equation leading to the surfaces of Delaunay and the mylar balloon and in this way provide their nonvariational characterization.

1. Introduction

The equations describing axisymmetric membranes are used not only in Biology but also in many other areas including the design and development of scientific balloons. The balloons are used by many space agencies to carry out researches in the upper stratosphere. The equilibrium equations of such surfaces are expressed as some boundary value problems. In order to study all external and internal forces acting on to the surface we are lead to many different cases of equilibrium conditions. Some of them can be solved exactly to determine the shape. The physical meaning of these parameters plays a very important role. The most crucial quantities as the membrane weight density, circumferential and meridional stresses, the differential pressure could variate. Guided by mechanical ideas we will derive two classes of shapes having quite interesting geometrical properties.

2. Axisymmetric Membranes

As usual we will think of the axisymmetric surface $\mathcal{S}$ by specifying its meridional section, i.e., a curve $u \longrightarrow (r(u), z(u))$ in the XOZ plane, assuming that u is the so called natural parameter provided by the corresponding arc length. We will denote the total arc length by L. The surface $\mathcal{S}$ can be presented in the ordinary

Euclidean space $\mathbb{R}^3$, with a fixed orthonormal basis $(\mathbf{i}, \mathbf{j}, \mathbf{k})$, by making use of the parameter u and the angle v specifying the rotation of the XOY plane via the vector-valued function

$$\mathbf{x}(u,v) = r(u)\mathbf{e}_1(v) + z(u)\mathbf{e}_3(v), \qquad 0 < u \leq L, \quad 0 \leq v < 2\pi. \tag{1}$$

Here the vector $\mathbf{e}_1(v)$ is the new position of $\mathbf{i}$ after the rotation at angle v

$$\mathbf{e}_1(v) = \cos v\mathbf{i} + \sin v\mathbf{j}. \tag{2}$$

Since the rotation is around the third axis $\mathbf{k}$ the vector representing it in (1) is a constant, i.e., $\mathbf{e}_3(v) = \text{const} = \mathbf{k}$. The pair $\{\mathbf{e}_1, \mathbf{e}_3\}$ can be completed to the orthonormal basis set $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ in $\mathbb{R}^3$. The third vector $\mathbf{e}_2(v)$ is introduced as a cross product of the vectors $\mathbf{e}_3(v)$ and $\mathbf{e}_1(v)$, i.e.,

$$\mathbf{e}_2(v) = \mathbf{e}_3(v) \times \mathbf{e}_1(v) = \mathbf{k} \times \mathbf{e}_1(v) = -\sin v\mathbf{i} + \cos v\mathbf{j}\,.$$

More detailed specification of the surface require to find some other important characteristics of the generating curve. This relies mostly on the derivatives of $\mathbf{x}(u,v)$. E.g., the tangent vector at each point of the generating curve is given of the first derivative with respect to u

$$\mathbf{t}(u,v) = \mathbf{x}_u(u,v) = r'(u)\mathbf{e}_1(v) + z'(u)\mathbf{k}. \tag{3}$$

In equation (3), and elsewhere in this paper, the prime denotes a derivative with respect to the meridional arc length u. Let us introduce also $\theta(u)$, which measures the angle between the tangent vector $\mathbf{t}$ and $\mathbf{k}$. Then, the coordinates $r(u)$ and $z(u)$ depend on $\theta(u)$ through the equations

$$r'(u) = \cos\theta(u) \tag{4}$$

$$z'(u) = -\sin\theta(u). \tag{5}$$

Using these equations we can express the tangent vector as

$$\mathbf{t}(u,v) = \cos\theta(u)\mathbf{e}_1(v) - \sin\theta(u)\mathbf{k}. \tag{6}$$

By differentiating the last relation with respect to the parameter u we get

$$\mathbf{x}_{uu} = -\theta'(u)(\sin\theta(u)\mathbf{e}_1(v) + \cos\theta(u)\mathbf{k}). \tag{7}$$

Next, we compute the first and second order derivatives of $\mathbf{x}(u,v)$ upon v

$$\mathbf{x}_v = r(u)(\mathbf{e}_1(v))_v = r(u)\mathbf{e}_2(v) \tag{8}$$

$$\mathbf{x}_{vv} = r(u)(\mathbf{e}_2(v))_v = -r(u)\mathbf{e}_1(v) \tag{9}$$

and finally, the mixed derivative

$$\mathbf{x}_{uv} = \mathbf{x}_{vu} = \cos\theta(u)\mathbf{e}_2(v). \tag{10}$$

Another significant object that we will need to know is the outward normal vector $\mathbf{n}(u, v)$. We can present it as a cross product of the vectors $\mathbf{t}(u, v)$ and $\mathbf{e}_2(v)$, i.e.,

$$\mathbf{n}(u, v) = \mathbf{t}(u, v) \times \mathbf{e}_2(v) = \sin\theta(u)\mathbf{e}_1(v) + \cos\theta(u)\mathbf{k}. \tag{11}$$

The last couple of relations are sufficient to obtain the coefficients of the first,

$$E = \mathbf{x}_u^2 = 1, \qquad F = \mathbf{x}_u.\mathbf{x}_v = 0, \qquad G = \mathbf{x}_v^2 = r^2(u). \tag{12}$$

and that of the second fundamental form of $\mathcal{S}$, namely

$$L = \mathbf{n}.\mathbf{x}_{uu} = -\theta'(s), \quad M = \mathbf{n}.\mathbf{x}_{uv} = 0, \quad N = \mathbf{n}.\mathbf{x}_{vv} = -r(u)\sin\theta(u). \tag{13}$$

Having them one can find easily the **mean curvature** H (meaning average) of the membrane surface under consideration. Using the standard formula for H which appears in the textbooks on classical differential geometry (see, e.g., [7, 13, 17])

$$H = \frac{LG - 2MF + NE}{2(EG - F^2)} = \frac{1}{2}(k_\mu + k_\pi)$$

where

$$k_\mu = L/E = -\theta'(u) \qquad \text{and} \qquad k_\pi = N/G = -\frac{\sin\theta(u)}{r(u)} \tag{14}$$

are the so called principal curvatures along meridional, respectively parallel directions, so that the mean curvature can be finally expressed in the form

$$H = -\frac{1}{2}\left(\theta'(u) + \frac{\sin\theta(u)}{r(u)}\right). \tag{15}$$

3. Equilibrium Equations

The results obtained in the previous section will help us to find the shape's equilibrium conditions. For that purpose we will consider the forces acting on the surface. The internal forces are

$$\mathbf{f}_1(u, v) = \sigma_m(u)\mathbf{t}(u, v) \qquad \text{and} \qquad \mathbf{f}_2(u, v) = \sigma_c(u)\mathbf{e}_2(v). \tag{16}$$

In the left hand side of equation (16) $\sigma_m(u)$ means the **meridional stress resultant** and in the right one $\sigma_c(u)$ is the **circumferential stress resultant** (see [1] for more details). Let us mention that the situation when $\sigma_c(u) \equiv 0$ is referred in ballooning literature as the **natural shape model**.

The external forces depend on the pressure and the density of the membrane's material, namely,

$$\mathbf{f}(u, v) = p(u)\mathbf{n}(u, v) - w(u)\mathbf{k}. \tag{17}$$

Here $p(u)$ is the hydrostatic differential pressure and $w(u)$ is the weight density of the film.

The balancing of the internal and external forces led us to the following equilibrium equations

$$(\sigma_m(u)r(u)\mathbf{t})_u - \sigma_c(u)\mathbf{e}_1(v) + r(u)\mathbf{f}(u,v) = 0. \tag{18}$$

The above vectorial equation projected onto $\mathbf{n}$ and $\mathbf{t}$ gives us respectively

$$(\sigma_m(u)r(u))\frac{\mathrm{d}\theta}{\mathrm{d}u} = -w(u)r(u)\cos\theta(u) - \sigma_c(u)\sin\theta(u) + p(u)r(u) \tag{19}$$

$$\frac{\mathrm{d}(\sigma_m(u)r(u))}{\mathrm{d}u} = -w(u)r(u)\sin\theta(u) + \sigma_c(u)\cos\theta(u). \tag{20}$$

4. Shapes and Related Surfaces

The first profound analysis on the axisymmetric balloon shapes was done in the period between 1960 and 1970 by J. Smalley. For this purpose he implemented numeric models on a digital computer (all relevant references on the subject can be found in [2]).

As Smalley's considerations were of numerical origin it seems worth to look for those models possessing analytical solutions. Despite that the system governing these shapes is highly nonlinear we have been successful in finding a few exact solutions which will be presented below.

These solutions are retrieved by neglecting some of the parameters from the equilibrium equations. Starting with the case where the film weight contribution is assumed to be zero, i.e., supposing that $w(u) \equiv 0$ we will have instead the equations (19) and (20) the system

$$-(\sigma_m(u)r(u))\frac{\mathrm{d}\theta}{\mathrm{d}u} = \sigma_c(u)\sin\theta(u) - p(u)r(u) \tag{21}$$

$$\frac{\mathrm{d}(\sigma_m(u)r(u))}{\mathrm{d}u} = \sigma_c(u)\cos\theta(u). \tag{22}$$

Following the geometrical relation (4), the second equation in this system implies that the meridional and circumferential stresses are constant and of the same magnitude, i.e., $\sigma_m(u) = \sigma_c(u) = \sigma = \mathrm{const}$, while the first equation (21) can be recognized as the mean curvature of $\mathcal{S}$, namely

$$H = -\frac{p(u)}{2\sigma}. \tag{23}$$

If we continue with examination of the case where the hydrostatic pressure is also a constant, i.e., $p(u) = p_o = \mathrm{const}$, we end up with a surface of constant mean curvature

$$H = -\frac{p_o}{2\sigma} = \mathrm{const}\,. \tag{24}$$

This class of surfaces was isolated many years ago by Delaunay [3], following a genuine geometrical argument – all they are just the traces of the foci of the non-degenerate conics when they roll along a straight line in a plane (**roulettes** in French). On the other side, these surfaces of revolution have a minimal lateral area at a fixed volume as proved by Sturm in an Appendix to the same paper. That in turn revealed why these surfaces make their appearance as soap bubbles, liquid drops [4, 6, 12] or cells under compression [5, 18] and now as balloons shape. The list of all Delaunay's surfaces includes cylinders of radius R and mean curvature $H = 1/2R$, spheres of radius R and mean curvature $H = 1/R$, catenoids of mean curvature $H = 0$, and nodoids and unduloids of constant non-zero mean curvatures with all their profile curves shown in Fig. 1.

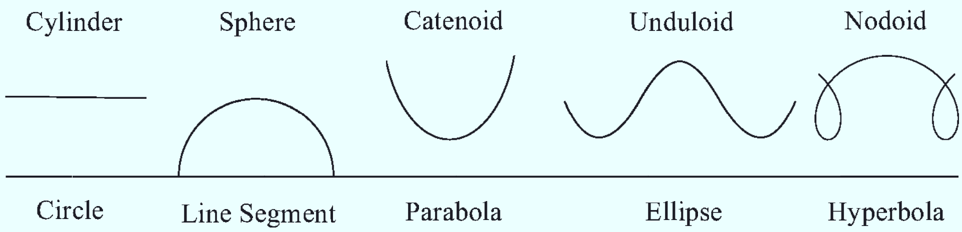

Figure 1. The profile curves of Delaunay's surfaces obtained by rolling the conics listed below the horizontal line on it.

5. Nodoids and Unduloids

In this section we will derive the analytical description of the last two and most interesting cases from the Delaunay's list. We start with the system formed by the equations (21) and (22) which ensures the geometrical relation

$$\sin\theta(u) = \frac{\mathring{p}r}{2} + \frac{C}{r} \tag{25}$$

where C is some integration constant. Combined with (4) this leads to the equation

$$\frac{\mathrm{d}r(u)}{\mathrm{d}u} = \cos\theta(u) = \frac{1}{2r}\sqrt{-\mathring{p}^2r^4 + 4(1-\mathring{p}C)r^2 - 4C^2} \tag{26}$$

in which the variables can be separated, i.e.,

$$\frac{2r\mathrm{d}r}{\sqrt{-\mathring{p}^2r^4 + 4(1-\mathring{p}C)r^2 - 4C^2}} = \mathrm{d}u. \tag{27}$$

Introducing $\xi = r^2$ we end up with the task for the evaluation of the elementary integral (on the left) written below

$$\int \frac{\mathrm{d}\xi}{\sqrt{(c^2-\xi)(\xi-a^2)}} = \int \mathrm{d}u = u + \phi \tag{28}$$

where

$$a = \frac{1 - \sqrt{1 - 2\mathring{p}C}}{\mathring{p}}, \qquad c = \frac{1 + \sqrt{1 - 2\mathring{p}C}}{\mathring{p}} \qquad \text{and} \qquad \phi \in \mathbb{R} \tag{29}$$

is some integration constant.

After some calculations the result can be written in the form

$$\xi(u) = r^2(u) = [(c^2 - a^2)\sin u + (c^2 + a^2)]/2 \tag{30}$$

in which the integration constant is omitted as it is inessential for our further considerations.

In order to find the generating curve we have to solve also the equation (5) which in view of the above notation reads

$$\frac{\mathrm{d}z}{\mathrm{d}r} = -\tan\theta = -\frac{\mathring{p}r^2 + 2C}{\sqrt{(c^2 - r^2)(r^2 - a^2)}} \tag{31}$$

and, therefore,

$$z(r) = -\int \frac{(\mathring{p}r^2 + 2C)\mathrm{d}r}{\sqrt{(c^2 - r^2)(r^2 - a^2)}}. \tag{32}$$

The integral on the right hand side can be uniformized by performing the change

$$r(t) = c\,\mathrm{dn}(t, k) \tag{33}$$

where $\mathrm{dn}(t, k)$ is one of the Jacobian elliptic function of the argument t and the **elliptic module** k (details about elliptic functions, their integrals, and properties can be found in [8]). Choosing k to be $\sqrt{c^2 - a^2}/c$ we get

$$z(t) = c\mathring{p}\int \mathrm{dn}^2(t, k)\mathrm{d}t + \frac{2C}{c}\int \mathrm{d}t \tag{34}$$

and consequently

$$z(t) = c\mathring{p}E\left(\mathrm{am}(t, k), k\right) + \frac{2C}{c}F\left(\mathrm{am}(t, k), k\right). \tag{35}$$

Finally, equations (30) and (33) will be compatible if the natural parameter u and the uniformizing parameter t are related by the equation

$$\sin u = 1 - 2\,\mathrm{sn}^2(t, k). \tag{36}$$

Let us mention also that another pair of formulas in place of (33) and (34) used for drawing Fig. 2 and Fig. 3 has been derived following the variational approach in [10].

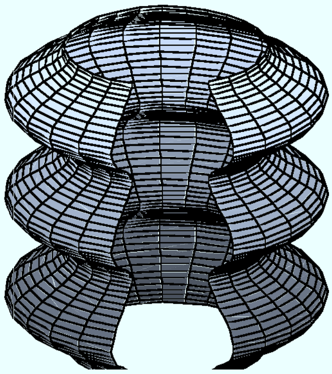

Figure 2. The unduloid surface.

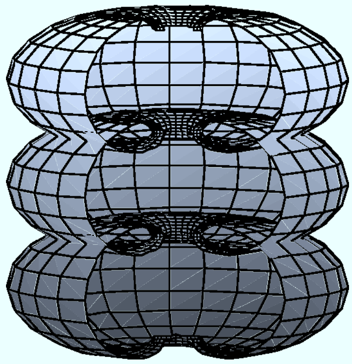

Figure 3. The shape of the nodoid.

6. The Mylar Balloon

This is another case in which the system of equations (19) and (20) can be solved up to the very end. Assuming that $w(u) = \sigma_c(u) = 0$ and that the hydrostatic pressure $p(u) = p_o$ is a non-zero constant, the initial system (19) and (20) reduces to the equations

$$(\sigma_m(u)r(u))\frac{\mathrm{d}\theta(u)}{\mathrm{d}u} = p_o r(u) \tag{37}$$

$$\frac{\mathrm{d}(\sigma_m(u)r(u))}{\mathrm{d}u} = 0. \tag{38}$$

The second equation above tells us that $\sigma_m(u)r(u)$ is a constant quantity. Following Gibbons [4] we introduce the meridional stress resultant $\mathring{\sigma}$ on the equator of the balloon, i.e., the points for which $r(u) = a$, $z(u) = 0$ and rewrite the above integral in the form

$$\sigma_m(u) = \frac{a\mathring{\sigma}}{r(u)}. \tag{39}$$

This allows as to rewrite the first equation (37) as

$$\frac{\mathrm{d}\theta(u)}{\mathrm{d}u} = \mathring{p}r(u), \qquad \mathring{p} = \frac{p_o}{a\mathring{\sigma}}. \tag{40}$$

If we combine this equation with (4) we get the following geometrical relation

$$r^2(u) = \frac{2}{\mathring{p}}\sin\theta(u). \tag{41}$$

This last relation, as we shall see, characterizes uniquely the surface in question. Let us start with solving (41) for $r(u)$. After that we replace the result in (40) and in this way obtain a differential equation with separated variables

$$\frac{\mathrm{d}\theta}{\sqrt{\sin\theta}} = \sqrt{2\overset{\circ}{p}}\,\mathrm{d}u. \tag{42}$$

Next we introduce $\eta = \sin\theta$ which transforms the left hand side into

$$\frac{\mathrm{d}\eta}{\sqrt{\eta(1-\eta^2)}}$$

and this suggest a new change $\eta = \zeta^2$ of the independent variable η which gives

$$\frac{\mathrm{d}\eta}{\sqrt{\eta(1-\eta^2)}} = \frac{2\mathrm{d}\zeta}{\sqrt{1-\zeta^4}} = \frac{2\mathrm{d}\zeta}{\sqrt{(1+\zeta^2)(1-\zeta^2)}}.$$

By all these changes equation (42) reduces to the form

$$\frac{2\mathrm{d}\zeta}{\sqrt{(1+\zeta^2)(1-\zeta^2)}} = \sqrt{2\overset{\circ}{p}}\,\mathrm{d}u. \tag{43}$$

However, this is a standard elliptic integral (cf. [8]) which can be inverted directly

$$\zeta(u) = -\operatorname{cn}\left(\sqrt{\overset{\circ}{p}}u, \frac{1}{\sqrt{2}}\right).$$

As a consequence we have also

$$\sin\theta(u) = \frac{\overset{\circ}{p}}{2}r^2(u) = \zeta^2(u) = \operatorname{cn}^2\left(\sqrt{\overset{\circ}{p}}u, \frac{1}{\sqrt{2}}\right) \tag{44}$$

and therefore

$$r(u) = \sqrt{\frac{2}{\overset{\circ}{p}}}\operatorname{cn}\left(\sqrt{\overset{\circ}{p}}u, \frac{1}{\sqrt{2}}\right). \tag{45}$$

Here $\operatorname{cn}(t,k)$ is the so called **Jacobian cosine elliptic function**.
In order to find $z(u)$ we make use of (5) and (44) which lead to

$$\frac{\mathrm{d}z(u)}{\mathrm{d}u} = -\operatorname{cn}^2\left(\sqrt{\overset{\circ}{p}}u, \frac{1}{\sqrt{2}}\right). \tag{46}$$

Details about the integration of the above equation can be found in [11] and the result is

$$z(u) = -\frac{2}{\sqrt{\overset{\circ}{p}}}\left[E\left(\operatorname{am}\left(\sqrt{\overset{\circ}{p}}u, \frac{1}{\sqrt{2}}\right), \frac{1}{\sqrt{2}}\right) - \frac{1}{2}F\left(\operatorname{am}\left(\sqrt{\overset{\circ}{p}}u, \frac{1}{\sqrt{2}}\right), \frac{1}{\sqrt{2}}\right)\right]. \tag{47}$$

If we compare the obtained parametrization of the profile curve $(r(u), z(u))$ provided by (45) and (47) with that one in [11] we can easily conclude that we are dealing here with the mylar balloon. The profile curve and the surface generated by them are shown in Fig. 4 and Fig. 5. For commercial purposes the just mentioned mylar balloon is fabricated from two circular disks of mylar, sewing them along their boundaries and then inflating. Surprisingly enough, these balloons are

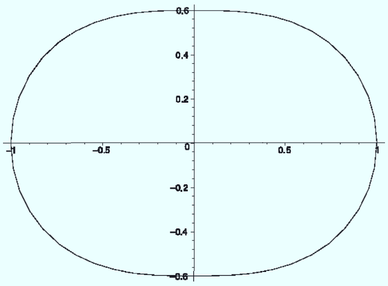

Figure 4. The profile of the mylar balloon in XOZ plane.

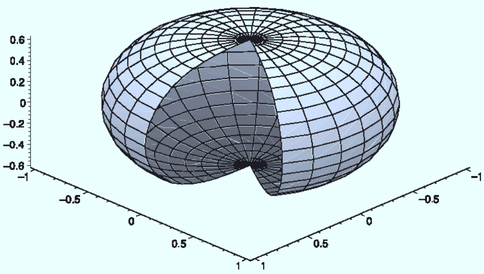

Figure 5. An open part of the mylar balloon.

not spherical as one naïvely might expect from the well-known fact that the sphere possesses the maximal volume for a given surface area. An experimental fact like this suggests a mathematical problem regarding the exact shape of the balloon when it is fully inflated.

This problem was first spelled out by Paulsen in a variational setting [14] while here we have provided in fact its non-variational characterization. One should mention also the remarkable scale invariance (i.e., independence of the actual size) of the thickness to diameter ratio of the inflated balloon which turns out to be with a good approximation equal to 0.599. Another important fact about this surface is the very simple expression for its area given by the formula $\mathcal{A} = \pi^2 a^2$ where a is the radius of the inflated balloon. In some sense all these nice properties are due to the remarkable property which specifies uniquely the mylar balloon as the only surface of revolution for which the principal curvatures k_μ and k_π obey to the equation

$$k_\mu = 2k_\pi. \tag{48}$$

As has been noticed by Gibbons [4] this (Weingarten) property can be derived within membrane approach as well by rewriting (41) in the form

$$-\mathring{p}r(u) = -2\frac{\sin\theta(u)}{r(u)}. \tag{49}$$

Taking into account (40) along with the definitions of the principal curvatures given in (14) amounts directly to the equality (48).

Detailed differential-geometric proofs of the other facts mentioned above and many additional comments can be found in the already cited papers [10] and [11].

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