離散凸解析とマッチングモデル その5：総括

Matching with partially ordered contracts

joint work with R. Farooq and T. Fleiner

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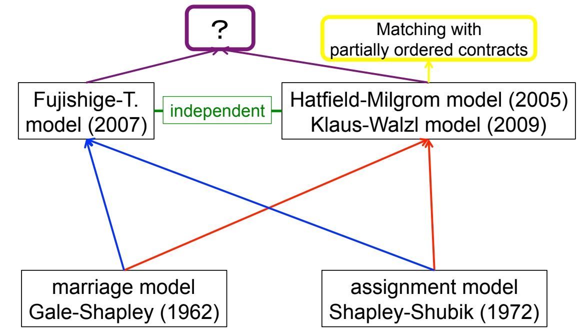
Two-sided matching markets



Contracts (Hatfield-Milgrom model)

- D is a finite set of doctors
- H is a finite set of hospitals
- X is a finite set of contracts:
  - each contract $x \in X$ is associated with a doctor $D(x)$ and a hospital $H(x)$, and includes additional info. e.g., working days and salary between $D(x)$ and $H(x)$, etc.
  - for $k \in D \cup H$ and for $Y \subseteq X$,

    $Y_k = \{x \in Y \mid D(x) = k$ or $H(x) = k\}$

Example A

- $D = \{ d, \underline{d} \}$, $H = \{ h \}$
- $X = \{x_4, \underline{x}_4, x_3, \underline{x}_3, x_2, \underline{x}_2\}$

  x_i represents i days job for d per week

  $\underline{x}_j$ represents j days job for $\underline{d}$ per week
- h's preference

  $\{x_3, \underline{x}_3\} > \{x_4, \underline{x}_2\} > \{x_2, \underline{x}_4\} > \{x_3, \underline{x}_2\} > \{x_2, \underline{x}_3\}$
  $> \{x_2, \underline{x}_2\} > \{x_4\} > \{\underline{x}_4\} > \{x_3\} > \{\underline{x}_3\} > \{x_2\} > \{\underline{x}_2\}$
- C_H does not satisfy substitutability

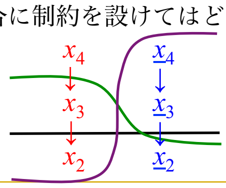
  $R_H(x_4, \underline{x}_4, x_3, x_2, \underline{x}_2) = \{\underline{x}_4, x_3, x_2\} \not\subseteq \{x_4, \underline{x}_4, x_2, \underline{x}_2\} = R_H(X)$

Example A からの教訓

- C_H does not satisfy substitutability

  $R_H(x_4, \underline{x}_4, x_3, x_2, \underline{x}_2) = \{\underline{x}_4, x_3, x_2\} \not\subseteq \{x_4, \underline{x}_4, x_2, \underline{x}_2\} = R_H(X)$
- $\underline{d}$ は4日働けるならば，3日働くこともできるだろう

  すなわち$\{x_4, \underline{x}_4, x_3, x_2, \underline{x}_2\}$という選択肢が妥当でない
- Xの部分集合に制約を設けてはどうか？



Partially ordered contracts

- $(X, \prec)$ is a finite partially ordered set of contracts
- $D(x) \neq D(y)$ or $H(x) \neq H(y) \Rightarrow x, y$: incomparable
- $x \prec y$ means
  - if doctor $D(y)$ can select y, she can also select x but not both
  - if hospital $H(y)$ can select y, it can also select x but not both
- Hence, the domain and range of a choice function should be the lower ideals and the antichains of $(X, \prec)$, respectively.

Lower ideals & Antichains

- $Y \subseteq X$ is a lower ideal of $(X,\prec) \Leftrightarrow$

  $x \prec y \in Y \;\Rightarrow\; x \in Y$

  $\mathcal{L}(X)$: the set of all lower ideals of $(X,\prec)$
- $Y \subseteq X$ is an antichain of $(X,\prec) \Leftrightarrow$

  $x,\, y \in Y \;\Rightarrow\; x,\, y$: incomparable

  $\mathcal{A}(X)$: the set of all antichains of $(X,\prec)$

Lower ideals & Antichains

- $\mathcal{L}(X)$: the set of all lower ideals of $(X,\prec)$
- $\mathcal{A}(X)$: the set of all antichains of $(X,\prec)$
- $\mathrm{Li}(Y) = \{x \in X \mid x \prec y$ for some $y \in Y\}$ $\quad (Y \subseteq X)$

  $\mathrm{Max}(Y) = \{y \in Y \mid y \prec x \in L \Rightarrow y = x\}$ $\quad (Y \subseteq X)$
- $\mathrm{Li} : \mathcal{A}(X) \to \mathcal{L}(X)$

  $\mathrm{Max} : \mathcal{L}(X) \to \mathcal{A}(X)$

  $\mathrm{Li}(\mathrm{Max}(L)) = L \quad (L \in \mathcal{L}(X))$

  $\mathrm{Max}(\mathrm{Li}(A)) = A \quad (A \in \mathcal{A}(X))$

Choice function

- $k \in D \cup H$ has a choice fn $C_k : \mathcal{L}(X) \to \mathcal{A}(X)$ with

  $C_k(Y) \subseteq Y_k \qquad (Y \subseteq \mathcal{L}(X))$
- $C_D(Y) = \bigcup_{i \in D} C_i(Y) \qquad (Y \subseteq \mathcal{L}(X))$

  $C_H(Y) = \bigcup_{j \in H} C_j(Y) \qquad (Y \subseteq \mathcal{L}(X))$

Remark: $C_D(Y),\, C_H(Y) \in \mathcal{A}(X)$

Choice function & Rejection fn

- $C_D,\, C_H : \mathcal{L}(X) \to \mathcal{A}(X)$
- $C^*_D(Y) = \mathrm{Li}(C_D(Y)), \quad C^*_H(Y) = \mathrm{Li}(C_H(Y))$

  $C^*_D,\, C^*_H : \mathcal{L}(X) \to \mathcal{L}(X)$

  $C_D(Y) = \mathrm{Max}(C^*_D(Y)), \quad C_H(Y) = \mathrm{Max}(C^*_H(Y))$
- $R_D(Y) = Y - C^*_D(Y) \qquad (Y \subseteq \mathcal{L}(X))$

  $R_H(Y) = Y - C^*_H(Y) \qquad (Y \subseteq \mathcal{L}(X))$

Consistency & Substitutability

- C^*_D and C^*_H satisfy consistency:

  $Z, Y \in \mathcal{L}(X),\; C^*(Y) \subseteq Z \subseteq Y \Rightarrow C^*(Z) = C^*(Y)$
- C^*_D and C^*_H satisfy substitutability:

  $Z, Y \in \mathcal{L}(X),\; Z \subseteq Y \;\Rightarrow\; C^*(Y) \cap Z \subseteq C^*(Z)$

  $\Updownarrow$

  $Z, Y \in \mathcal{L}(X), \quad Z \subseteq Y \;\Rightarrow\; R(Z) \subseteq R(Y)$

Pairwise Stability (1)

- $Y \subseteq X$ is a pairwise stable allocation if
  - $C_D(\mathrm{Li}(Y)) = Y$ and $C_H(\mathrm{Li}(Y)) = Y$
  - for $x \in X - Y$,

    $x \notin C_D(\mathrm{Li}(Y \cup \{x\}))$ or

    $x \notin C_H(\mathrm{Li}(Y \cup \{x\}))$

Pairwise Stability (2)

Lemma A:

If $Y_D = X - \mathrm{Ui}(R_H(Y_H))$ and $Y_H = X - \mathrm{Ui}(R_D(Y_D))$

then $\mathrm{Max}(Y_D \cap Y_H)$ is pairwise stable.

$\mathrm{Ui}(Y) = \{y \in X \mid x \prec y \text{ for some } x \in Y\} \qquad (Y \subseteq X)$

Pairwise Stability (3)

Lemma A:

If $Y_D = X - \mathrm{Ui}(R_H(Y_H))$ and $Y_H = X - \mathrm{Ui}(R_D(Y_D))$

then $\mathrm{Max}(Y_D \cap Y_H)$ is pairwise stable.

Lemma B:

If Y is pairwise stable, then there exist Y_D and Y_H s.t.

$Y_D = X - \mathrm{Ui}(R_H(Y_H))$, $Y_H = X - \mathrm{Ui}(R_D(Y_D))$,

$Y = \mathrm{Max}(Y_D \cap Y_H)$.

Existence of Stable Outcomes

- partial order $\geq$ on $\mathcal{L}(X) \times \mathcal{L}(X)$ s.t.

  $(Y, Z) \geq (Y', Z') \Leftrightarrow Y' \subseteq Y$ and $Z \subseteq Z'$
- $F(Y,Z) = (F_1(Z), F_2(F_1(Z))) \qquad (Y, Z \subseteq X)$

  where $F_1(Z) = X - \mathrm{Ui}(R_H(Z))$, $F_2(Z) = X - \mathrm{Ui}(R_D(Z))$
- F is monotone on a complete lattice $(\mathcal{L}(X) \times \mathcal{L}(X), \geq)$:

  $(Y, Z) \geq (Y', Z') \Rightarrow F(Y, Z) \geq F(Y', Z')$
- by Tarski's fixed point theorem, there exists (Y, Z) with

  $(Y, Z) = F(Y, Z)$

  $= (X - \mathrm{Ui}(R_H(Z)), X - \mathrm{Ui}(R_D(Y)))$

Example A再び

- $D = \{ d, \underline{d} \}, \quad H = \{ h \}$
- $X = \{x_2 \prec x_3 \prec x_4,\ \underline{x}_2 \prec \underline{x}_3 \prec \underline{x}_4\}$
- h's preference:

  $\{x_3, \underline{x}_3\} > \{x_4, \underline{x}_2\} > \{x_2, \underline{x}_4\} > \{x_3, \underline{x}_2\} > \{x_2, \underline{x}_3\}$
  $> \{x_2, \underline{x}_2\} > \{x_4\} > \{\underline{x}_4\} > \{x_3\} > \{\underline{x}_3\} > \{x_2\} > \{\underline{x}_2\}$
- $C_H : \mathcal{L}(X) \to \mathcal{A}(X)$ is Substitutable

C_H	$\varnothing$	$\underline{2}$	$\underline{23}$	$\underline{234}$
$\varnothing$	$\varnothing$	$\underline{2}$	$\underline{3}$	$\underline{4}$
2	2	$2\underline{2}$	$2\underline{3}$	$2\underline{4}$
23	3	$3\underline{2}$	$3\underline{3}$	$3\underline{3}$
234	4	$4\underline{2}$	$3\underline{3}$	$3\underline{3}$

R_H	$\varnothing$	$\underline{2}$	$\underline{23}$	$\underline{234}$
$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$
2	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$
23	$\varnothing$	$\varnothing$	$\varnothing$	$\underline{4}$
234	$\varnothing$	$\varnothing$	4	$4\underline{4}$

Example A再び

- $D = \{ d, \underline{d} \}, \quad H = \{ h \}$
- $X = \{x_2 \prec x_3 \prec x_4,\ \underline{x}_2 \prec \underline{x}_3 \prec \underline{x}_4\}$
- h's preference: $C_H : \mathcal{L}(X) \to \mathcal{A}(X)$

C_H	$\varnothing$	$\underline{2}$	$\underline{23}$	$\underline{234}$
$\varnothing$	$\varnothing$	$\underline{2}$	$\underline{3}$	$\underline{4}$
2	2	$2\underline{2}$	$2\underline{3}$	$2\underline{4}$
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234	4	$4\underline{2}$	$3\underline{3}$	$3\underline{3}$

R_H	$\varnothing$	$\underline{2}$	$\underline{23}$	$\underline{234}$
$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$
2	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$
23	$\varnothing$	$\varnothing$	$\varnothing$	$\underline{4}$
234	$\varnothing$	$\varnothing$	4	$4\underline{4}$

- doctors' preferences: $x_4 > x_3 > x_2, \qquad \underline{x}_4 > \underline{x}_3 > \underline{x}_2$
- $\{x_3, \underline{x}_3\}$ is stable

今後の課題

- F-TのモデルとH-Mのモデルを包含するモデルの構築
- 安定割当が存在するための必要十分条件は？
- 最後のモデルは操作不可能性を持つか？
- 最後のモデルの応用は？