Uniquely realizable graphs

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Kyoto, August 6, 2019

A very general question

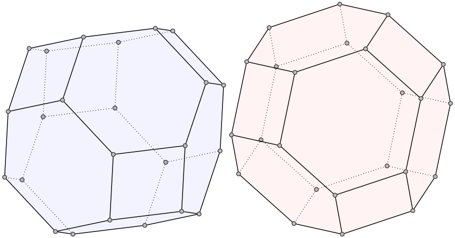
Consider a configuration of objects (bodies, lines, points, etc.) in $\mathbb{R}^d$ and a set of geometric properties satisfied by certain pairs (or subsets) of objects – for example, their distance, angle, direction, etc.

Is there another (non-congruent) configuration of the same objects which satisfies the same set of geometric properties?

Cauchy's theorem

Theorem (Cauchy, 1813)

Let P_1 and P_2 be convex polyhedra in $\mathbb{R}^3$ whose graphs are isomorphic and for which corresponding faces are pairwise congruent. Then P_1 and P_2 are congruent.

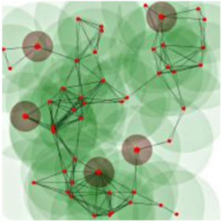


Points and distances

Consider a configuration of n points $p_1, p_2, \ldots, p_n$ in $\mathbb{R}^d$ and a fixed subset E of the $\binom{n}{2}$ pairwise distances.

Is there a non-congruent configuration $q_1, q_2, \ldots, q_n$ in $\mathbb{R}^d$ in which $||p_i - p_j|| = ||q_i - q_j||$ holds for all $ij \in E$?

Localizable sensor networks



A subset of the pairwise distances may uniquely determine the configuration and hence the location of each sensor (provided we have some anchor nodes whose location is known).

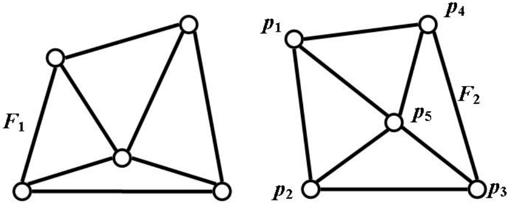
Bar-and-joint frameworks

A d-dimensional (bar-and-joint) *framework* is a pair (G, p), where $G = (V, E)$ is a graph and p is a map from V to $\mathbb{R}^d$. We consider the framework to be a straight line *realization* of G in $\mathbb{R}^d$.

Two realizations (G, p) and (G, q) of G are *equivalent* if $||p(u) - p(v)|| = ||q(u) - q(v)||$ holds for all pairs u, v with $uv \in E$, where $||.||$ denotes the Euclidean norm in $\mathbb{R}^d$. Frameworks (G, p), (G, q) are *congruent* if $||p(u) - p(v)|| = ||q(u) - q(v)||$ holds for all pairs u, v with $u, v \in V$.

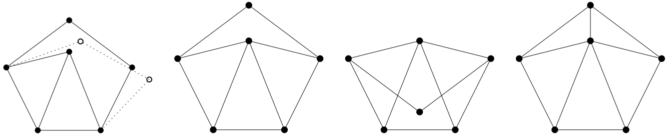
Globally rigid frameworks

We say that (G, p) is *globally rigid* in $\mathbb{R}^d$ if every d-dimensional framework which is equivalent to (G, p) is congruent to (G, p).



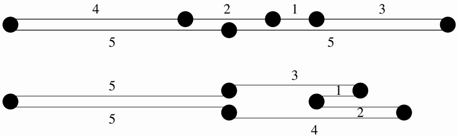
Rigid frameworks

The framework (G, p) is *rigid* if there exists an $\epsilon > 0$ such that, if (G, q) is equivalent to (G, p) and $||p(u) - q(u)|| < \epsilon$ for all $v \in V$, then (G, q) is congruent to (G, p). Equivalently, the framework is rigid if every continuous deformation that preserves the edge lengths results in a congruent framework.



Generic frameworks

Testing rigidity is NP-hard for $d \geq 2$ (T.G. Abbot, 2008). Testing global rigidity is NP-hard for $d \geq 1$ (J.B. Saxe, 1979).



A d-dimensional framework (G, p) is said to be *generic* if the set of the $d|V(G)|$ coordinates of the points is algebraically independent over the rationals.

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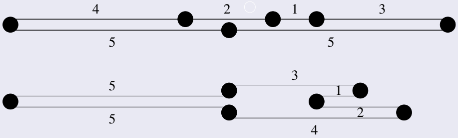

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Rigid and globally rigid graphs

The rigidity (resp. global rigidity) of frameworks in $\mathbb{R}^d$ is a generic property: there exists a rigid (resp. globally rigid) generic framework (G, p) if and only if every generic framework (G, p) is rigid (resp. globally rigid).

L. Asimow and B. Roth (1979),
R. Connelly (2005), S. Gortler, A. Healy and D. Thurston (2010).

We say that the graph G is *rigid* (*globally rigid*) in $\mathbb{R}^d$ if every (or equivalently, if some) generic realization of G in $\mathbb{R}^d$ is rigid.

Combinatorial (global) rigidity

- Characterize the rigid graphs in $\mathbb{R}^d$,
- Characterize the globally rigid graphs in $\mathbb{R}^d$,
- Find an efficient deterministic algorithm for testing these properties,
- Obtain further structural results (maximal rigid subgraphs, maximal globally rigid clusters, globally linked pairs of vertices, etc.)
- Solve the related optimization problems (e.g. make the graph rigid or globally rigid by pinning a smallest vertex set or adding a smallest edge set)

The rigidity matrix

The *rigidity matrix* $R(G,p)$ of framework (G,p) is a matrix of size $|E| \times d|V|$ in which the row corresponding to edge uv contains $p(u)-p(v)$ in the d-tuple of columns of u, $p(v)-p(u)$ in the d-tuple of columns of v, and the remaining entries are zeros. For example, the graph G with $V(G)=\{u,v,x,y\}$ and $E(G)=\{uv,vx,ux,xy\}$ has the following rigidity matrix:

$$\begin{array}{c} \\ uv \\ vx \\ ux \\ xy \end{array} \begin{array}{c} \begin{array}{cccc} \quad u \quad & \quad v \quad & \quad x \quad & \quad y \quad \end{array} \\ \begin{pmatrix} p(u)-p(v) & p(v)-p(u) & 0 & 0 \\ 0 & p(v)-p(x) & p(x)-p(v) & 0 \\ p(u)-p(x) & 0 & p(x)-p(u) & 0 \\ 0 & 0 & p(x)-p(y) & p(y)-p(x) \end{pmatrix} \end{array}.$$

The rigidity matroid $\mathcal{R}_d(G)$

The rank of $R(G, p)$ is the same for all generic realizations. This gives rise to a matroid $\mathcal{R}_d(G)$ defined on the edge set of G in which a set of edges is independent if the corresponding rows of $R(G, p)$ are linearly independent for a generic p.

A graph G on $n \geq d$ vertices is rigid in $\mathbb{R}^d$ if and only if the rank of its rigidity matroid is equal to $d|V| - \binom{d+1}{2}$.

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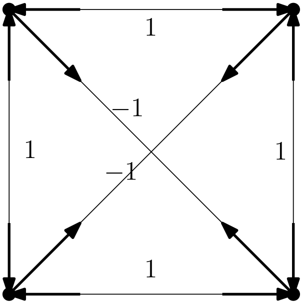
Equilibrium stresses

The function $\omega : e \in E \mapsto \omega_e \in \mathbb{R}$ is an *equilibrium stress* on framework (G, p) if for each vertex u we have

$$\sum_{v \in N(u)} \omega_{uv}(p(v) - p(u)) = 0. \tag{1}$$



The stress matrix and global rigidity

The *stress matrix* Ω of ω is a symmetric matrix of size $|V| \times |V|$ in which all row (and column) sums are zero and

$$\Omega[u, v] = -\omega(uv). \tag{2}$$

Theorem (B. Connelly, 2005, S. Gortler, A. Healy, D. Thurston, 2010)

Let (G, p) be a generic framework in $\mathbb{R}^d$ on $n \geq d + 2$ vertices. Then (G, p) is globally rigid in $\mathbb{R}^d$ if and only if (G, p) has an equilibrium stress ω for which the rank of the associated stress matrix Ω is $n - d - 1$.

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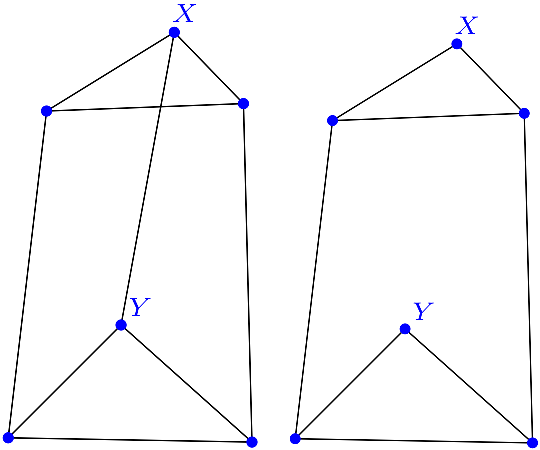
Globally rigid graphs - necessary conditions

We say that G is *redundantly rigid in* $\mathbb{R}^d$ if $G - e$ is rigid for all $e \in E(G)$.

Theorem (B. Hendrickson, 1992)

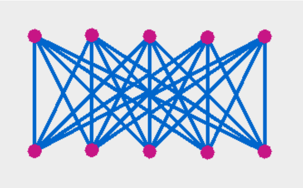
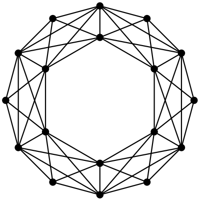
Let G be a globally rigid graph in $\mathbb{R}^d$. Then either G is a complete graph on at most $d + 1$ vertices, or G is
(i) $(d + 1)$-connected, and
(ii) redundantly rigid in $\mathbb{R}^d$.

A minimally rigid graph in $\mathbb{R}^2$



The prism is rigid but not globally rigid in the plane.

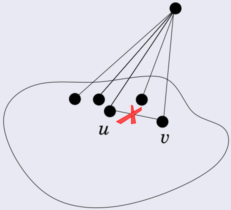
Higher dimensions

B. Connelly (1991)

T.J., C. Király, and S. Tanigawa (2016)

Global rigidity - sufficient conditions



The d-dimensional *extension* operation: delete uv and add a new vertex of degree $d+1$ connected to u, v and some more vertices.

Theorem (B. Connelly, 1989, 2005), (B. Jackson, T.J., Z. Szabadka, 2006)

Suppose that G can be obtained from K_{d+2} by extensions and edge additions. Then G is globally rigid in $\mathbb{R}^d$.

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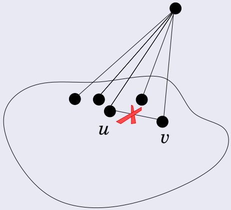

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Suppose that G can be obtained from K_{d+2} by extensions and edge additions. Then G is globally rigid in $\mathbb{R}^d$.

Global rigidity in the plane

Theorem (B. Jackson, T. J., 2005)

Let G be a 3-connected and redundantly rigid graph in $\mathbb{R}^2$ on at least four vertices. Then G can be obtained from K_4 by extensions and edge-additions.

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A graph G on $n \geq 4$ vertices is globally rigid in $\mathbb{R}^2$ if and only if G is 3-connected and redundantly rigid.

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Sufficient conditions - connectivity

Theorem (L. Lovász, Y. Yemini, 1982)

If a graph G is 6-vertex-connected, then G is redundantly rigid in $\mathbb{R}^2$.

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If a graph G is 6-vertex-connected, then G is globally rigid in $\mathbb{R}^2$.

Conjecture (L. Lovász and Y. Yemini, 1982, B. Connelly, T.J., W. Whiteley, 2013)

For every $d \geq 1$ there exists an integer f_d (resp. g_d) such that every f_d-connected graph is rigid (resp. every g_d-connected graph is globally rigid) in $\mathbb{R}^d$.

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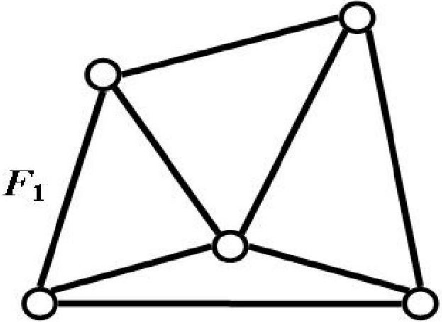
Sufficient conditions - vertex-redundant rigidity

We say that G is *vertex-redundantly rigid* in $\mathbb{R}^d$ if $G - v$ is rigid in $\mathbb{R}^d$ for all $v \in V(G)$.

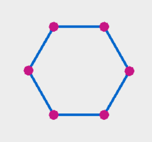
Theorem (S. Tanigawa, 2015)

If G is vertex-redundantly rigid in $\mathbb{R}^d$ then it is globally rigid in $\mathbb{R}^d$.



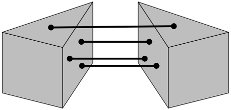
Tree-connectivity

Let $M = (V, E)$ be a multigraph. We say that M is *k-tree-connected* if M contains k edge-disjoint spanning trees. If $M - e$ contains k edge-disjoint spanning trees for all $e \in E$ then M is called *highly k-tree-connected*.
For a multigraph H and integer k we use kH to denote the multigraph obtained from H by replacing each edge e of H by k parallel copies of e.

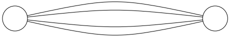


$5G$ is 6-tree-connected.

Body-bar frameworks



A d-dimensional *body-bar framework* is a structure consisting of rigid bodies in d-space in which some pairs of bodies are connected by bars. The bars are pairwise disjoint. Two bodies may be connected by several bars. In the underlying multigraph of the framework the vertices correspond to the bodies and the edges correspond to the bars.



Rigid and globally rigid generic body-bar frameworks

Theorem (T-S. Tay, 1989, W. Whiteley, 1988)

A generic body-bar framework with underlying multigraph $H = (V, E)$ is rigid in $\mathbb{R}^d$ if and only if H is $\binom{d+1}{2}$-tree-connected.

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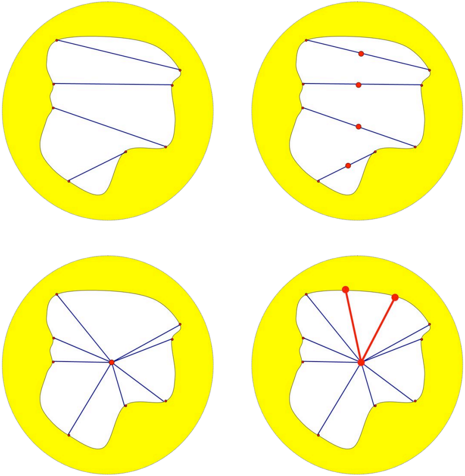
Pinching edges



The pinching operation ($m = 6, k = 4$).

Another constructive characterization

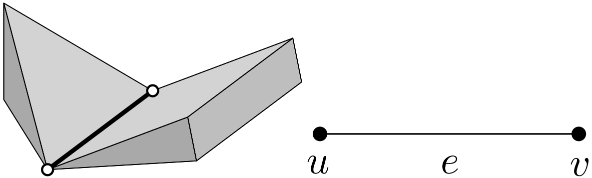
Theorem (A. Frank, L. Szegő 2003)

A multigraph H is highly m-tree-connected if and only if H can be obtained from a vertex by repeated applications of the following operations:

- (i) adding an edge (possibly a loop),
- (ii) pinching k edges ($1 \leq k \leq m-1$) with a new vertex z and adding $m-k$ new edges connecting z with existing vertices.

Body-hinge frameworks

A d-dimensional *body-hinge framework* is a structure consisting of rigid bodies in d-space in which some pairs of bodies are connected by a hinge, restricting the relative position of the corresponding bodies. Each hinge corresponds to a $(d-2)$-dimensional affine subspace. In the underlying multigraph of the framework the vertices correspond to the bodies and the edges correspond to the hinges.



Bodies connected by a hinge in 3-space and the corresponding edge of the underlying multigraph.

Rigid and globally rigid generic body-hinge frameworks

Theorem (T-S. Tay, 1989, W. Whiteley, 1988)

A generic body-hinge framework with underlying multigraph $H = (V, E)$ is rigid in $\mathbb{R}^d$ if and only if $(\binom{d+1}{2} - 1)H$ is $\binom{d+1}{2}$-tree-connected.

Theorem (T.J., C. Király, S. Tanigawa, 2016)

Let $H = (V, E)$ be a multigraph and $d \geq 3$. Then the body-hinge graph G_H is globally rigid in $\mathbb{R}^d$ if and only if $(\binom{d+1}{2} - 1)H$ is highly $\binom{d+1}{2}$-tree-connected.

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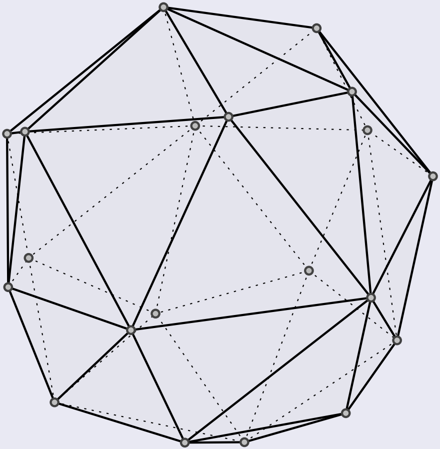
Convex polyhedra with triangular faces



A convex polyhedron with triangular faces

Triangulations are rigid

Corollary

Let P be a convex polyhedron with triangular faces and let $(G(P), p)$ be the corresponding bar-and-joint realization of its graph in three-space. Then $(G(P), p)$ is rigid.

Proof sketch

Consider a continuous motion of the vertices of $(G(P), p)$ which preserves the edge lengths. Then it must also preserve the faces as well as the convexity in a small enough neighbourhood. Thus it results in a congruent realization by Cauchy's theorem.

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The graphs of convex polyhedra

Theorem (Steinitz)

A graph G is the graph of some convex polyhedron P in $\mathbb{R}^3$ if and only if G is 3-connected and planar.

Theorem (Steinitz, 1906)

A graph G is the graph of some convex polyhedron P in $\mathbb{R}^3$ with triangular faces if and only if G is a maximal planar graph.

We shall simply call a maximal planar graph (or planar triangulation) a *triangulation*.

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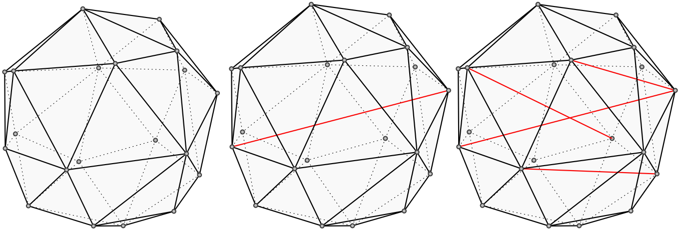
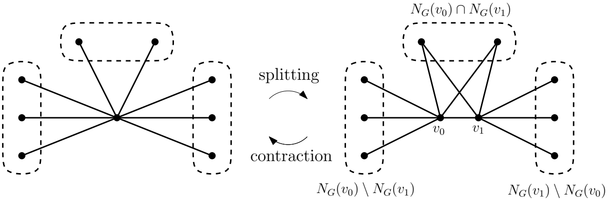
Braced triangulations

We call a graph $H = (V, E + B)$ a *braced triangulation* if it is obtained from a triangulation $G = (V, E)$ by adding a set B of new edges (called *bracing edges*). In the special case when $|B| = 1$ we say that H is a *uni-braced* triangulation.



Braced triangulations

Vertex splitting



Theorem (T.J. and S. Tanigawa, 2017)

Suppose that G can be obtained from K_5 by a sequence of non-trivial vertex splitting operations. Then G is globally rigid in $\mathbb{R}^3$.

The inductive construction

Theorem (T.J. and S. Tanigawa, 2017)

Let H be a 4-connected uni-braced triangulation. Then H can be obtained from K_5 by a sequence of non-trivial 2-vertex splitting operations.

Theorem (T.J. and S. Tanigawa, 2017)

Every 4-connected uni-braced triangulation is globally rigid in $\mathbb{R}^3$.

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Further results

Theorem (T.J. and S. Tanigawa, 2017)

Let G be a triangulation and let $\{u, v\}$ be a pair of non-adjacent vertices of G. Then $\{u, v\}$ is globally loose.

Theorem (Whiteley, 1988)

Let H be a graph with a quadrilateral hole and a quadrilateral block, obtained from a triangulation by removing an edge and adding a new edge between adjacent triangles. Then H is rigid in $\mathbb{R}^3$ if and only if there exist 4 vertex-disjoint paths between the hole and the block.

Conjecture (T.J. and S. Tanigawa, 2017)

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Further results

Theorem (T.J. and S. Tanigawa, 2017)

Let G be a triangulation and let $\{u, v\}$ be a pair of non-adjacent vertices of G. Then $\{u, v\}$ is globally loose.

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Globally rigid triangulations of other surfaces

Theorem (Barnette, 1990)

(a) Every 4-connected triangulation of the projective plane can be obtained from K_6 or K_7 minus a triangle by nontrivial vertex-splitting operations.
(b) Every 4-connected triangulation of the torus can be obtained from one of 21 graphs by nontrivial vertex-splitting operations.

Theorem (T.J. and S. Tanigawa, 2017)

Suppose that G is a 4-connected triangulation of the torus or the projective plane. Then G is globally rigid in $\mathbb{R}^3$.

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Scalar product rigidity vs. global completability

We say that two d-dimensional frameworks (G, q) and (G, p) are *equivalent* if

$$\langle p_i, p_j \rangle = \langle q_i, q_j \rangle \qquad \text{(for all } ij \in E(G)) \tag{3}$$

and that they are *congruent* if (3) holds for every pair i, j in $V(G)$ (including i, j with $i = j$). This is equivalent to saying that $q_i = Ap_i$ for all $i \in V$ for some fixed $d \times d$ orthogonal matrix A.

We say that a d-dimensional framework (G, p) is *globally completable* in $\mathbb{R}^d$ if for every d-dimensional framework (G, q) which is equivalent to (G, p) we have that (G, q) and (G, p) are congruent.

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The PSD matrix completion problem

Let M be an $n \times n$ positive semidefinite matrix of rank d. Then $M = P^\top P$ for some $d \times n$ matrix P and hence it can be defined by specifying n points in $\mathbb{R}^d$, corresponding to the columns of P. Consider a partially filled $n \times n$ positive semidefinite matrix M of rank d. The given entries define a graph $G = (V, E)$ on $V = \{1, 2, ...n\}$ in which two vertices i, j are adjacent if and only if $M[i, j]$ is given. The completion is unique if and only if (G, p) is globally completable in $\mathbb{R}^d$.

Generic global completability

Theorem (B. Jackson, T.J., S. Tanigawa, 2016)

Global completability is not a generic property in $\mathbb{R}^2$.

We say that a graph G is *globally completable* in $\mathbb{R}^d$ if *every* generic realization of G in $\mathbb{R}^d$ is globally completable.

Theorem (B. Jackson, T.J., S. Tanigawa, 2016)

Let $G = (V, E)$ be a simple graph on n vertices. Suppose that $\delta(G) \geq \lceil n/2 \rceil + 3$. Then G is globally completable in $\mathbb{R}^2$.

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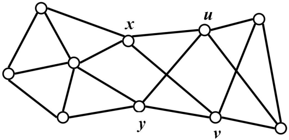
Globally linked pairs

We say that a pair $\{u, v\}$ of vertices of G is *globally linked* in G in $\mathbb{R}^d$ if for all generic d-dimensional realizations (G, p) we have that the distance between $q(u)$ and $q(v)$ is the same in all realizations (G, q) equivalent with (G, p).



Globally linked pairs in minimally rigid graphs

Theorem (B. Jackson, T.J., Z. Szabadka (2014))

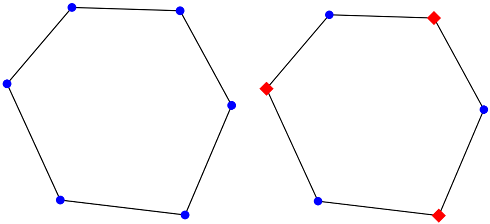
Let $G = (V, E)$ be a minimally rigid graph in $\mathbb{R}^2$ and $u, v \in V$. Then $\{u, v\}$ is globally linked in G in $\mathbb{R}^2$ if and only if $uv \in E$.

Pinning sets

Let (G, p) be a d-dimensional framework and $T \subseteq V(G)$. We say that P is a *pinning set (with respect to rigidity)* if there is no continuous motion of (G, p) which leaves all vertices $v \in T$ fixed.

Let (G, p) be a d-dimensional framework and $T \subseteq V(G)$. We say that P is a *pinning set (with respect to global rigidity)* if every equivalent realization (G, q) with $q(v) = p(v)$ for all $v \in P$ satisfies $q(v) = p(v)$ for all $v \in V(G)$.

Smallest pinning sets



Pinning a six-cycle in the plane with respect to rigidity.

Theorem (L. Lovász (1980))

The smallest pinning set of a two-dimensional framework (G, p), with respect to rigidity, can be found in polynomial time.

Unlabeled global rigidity

Theorem (S.J. Gortler, L. Theran, D.P. Thurston (2018)

Let G be globally rigid in $\mathbb{R}^d$, $d \geq 2$, and let (G, p) be a generic realization in $\mathbb{R}^d$. Consider a graph H with the same number of vertices and edges and suppose that some realization (H, q) of H has the same set of edge lengths. Then G and H are isomorphic and p and q are congruent (after relabeling).

Length equivalent frameworks

We say that two frameworks (G, p) and (H, q) are *length-equivalent* (*under the bijection* ψ) if there is a bijection ψ between the edge sets of G and H such that for every edge e of G, the length of e in (G, p) is equal to the length of $\psi(e)$ in (H, q).

If G and H have the same number of vertices then we say that they have the same *order*.

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Generic unlabeled global rigidity

Theorem (S. Gortler, L. Theran, D. Thurston, 2018)

Let $G = (V, E)$ be globally rigid in $\mathbb{R}^d$ on at least $d + 2$ vertices, where $d \geq 2$, and let (G, p) be a d-dimensional generic realization of G. Suppose that (H, q) is another d-dimensional framework such that G and H have the same order and (G, p) is length-equivalent to (H, q). Then there is a graph isomorphism $\varphi : V(G) \to V(H)$ which induces ψ, that is, for which $\psi(uv) = \varphi(u)\varphi(v)$ for all $uv \in E$. In particular, G and H are isomorphic and the frameworks (G, p) and (H, q) are congruent after relabeling, i.e. (G, p) is congruent to $(G, q \circ \varphi)$.

Complete graphs

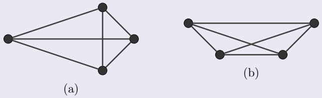


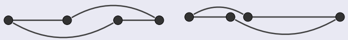
Figure 1: Two non-equivalent frameworks with the same edge measurement sets.

Weak and strong reconstructibility

Let (G, p) be a d-dimensional generic (complex) realization of the graph G. We say that (G, p) is *weakly reconstructible* if whenever (H, q) is a d-dimensional generic complex framework such that G and H have the same order and (G, p) is length-equivalent to (H, q), then H is isomorphic to G.

Let (G, p) be a d-dimensional generic (complex) realization of the graph G. We say that (G, p) is *strongly reconstructible* if for every d-dimensional generic complex framework (H, q) which is length-equivalent to (G, p) under some bijection ψ and has the same order, there is an isomorphism $\varphi : G \to H$ for which $\psi(uv) = \varphi(u)\varphi(v)$ for all $uv \in E$.

Four-cycle



(a) (b)

Figure 1: Realizations of C_4 with the same edge measurements where the mapping between corresponding edges does not arise from a graph isomorphism. This shows that C_4 is not strongly reconstructible in one dimension.

Weakly and strongly reconstructible graphs

A graph G is said to be *(generically) weakly reconstructible* (respectively *strongly reconstructible*) in $\mathbb{C}^d$ if every d-dimensional generic realization (G, p) of G is weakly (respectively strongly) reconstructible.

Theorem (S. Gortler, L. Theran, D. Thurston, 2018)

Let G be a graph on $n \geq d + 2$ vertices, where $d \geq 2$. Suppose that G is globally rigid in $\mathbb{R}^d$. Then G is strongly reconstructible in $\mathbb{C}^d$.

Families of non-globally rigid (non-rigid) WR graphs

We call a graph G on n vertices and m edges *maximally non-globally rigid* (resp. *maximally non-rigid*) if it is not globally rigid (resp. not rigid) but every graph on n vertices and with more than m edges is globally rigid (resp. rigid).

Theorem

Let G be a graph on n vertices and $d \geq 2$ a fixed dimension. Suppose that G is the only maximally non-globally rigid graph on n vertices (up to isomorphism). Then G is weakly reconstructible in $\mathbb{C}^d$.

Theorem

Let G be a graph on n vertices and $d \geq 1$ a fixed dimension. Suppose that G is the only maximally non-rigid graph on n vertices (up to isomorphism). Then G is weakly reconstructible in $\mathbb{C}^d$.

Maximally non-globally rigid graphs

Theorem

Let H be the extension of K_{n-1} by a vertex of degree d, where $d \geq 1$ and $n \geq d+2$. Then H is the unique maximal non-globally rigid graph in d dimensions on n vertices.

Theorem

Let H be the extension of K_{n-1} by a single vertex of degree $d-1$ for some $d \geq 1$ and $n \geq d+3$. Then H is the unique maximally non-rigid graph in d dimensions on n vertices.

Examples of WR graphs

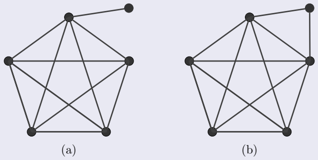


Figure 1: The maximum size non-rigid and non-globally rigid graphs in the plane on 6 vertices.

The measurement variety

Let G be a graph with n vertices and m edges. Fix an ordering of the edges and consider the (rigidity) map $m_{d,G}$ from $\mathbb{C}^{nd}$ to $\mathbb{C}^m$, where the i-th coordinate of $m_{d,G}(p)$ (which corresponds to some edge uv of G) is $m_{uv}(p)$, that is, the squared complex edge length of uv in the d-dimensional complex realization (G, p).

Definition

The *d-dimensional measurement variety* of a graph G (on n vertices), denoted by $M_{d,G}$, is the Zariski closure of $m_{d,G}(\mathbb{C}^{nd})$.

Weak reconstructibility and the measurement variety

We say that $M_{d,G}$ *uniquely determines* the graph G if whenever $M_{d,G} = M_{d,H}$ for some graph H with the same order as G, we have that H is isomorphic to G.

Theorem

Let G be a graph and $d \geq 1$ be fixed. The following are equivalent.
1. G is (generically) weakly reconstructible in $\mathbb{C}^d$.
2. There exists some generic d-dimensional framework (G, p) which is weakly reconstructible.
3. $M_{d,G}$ uniquely determines G.

Strong reconstructibility and the measurement variety

Theorem

Let G be a graph and $d \geq 1$ be fixed. The following are equivalent.
1. G is (generically) strongly reconstructible in $\mathbb{C}^d$.
2. There exists some generic d-dimensional framework (G, p) which is strongly reconstructible.
3. $M_{d,G}$ uniquely determines G and whenever $M_{d,G}$ is invariant under a permutation ψ of the edges of G, ψ is induced by a graph automorphism.
4. Whenever $M_{d,G} = M_{d,H}$ under an edge bijection ψ for some graph H with the same order as G, ψ is induced by a graph isomorphism.

The measurement variety and the rigidity matroid

Lemma

Let G be a graph on n vertices. Then

$$\dim(M_{d,G}) = r_d(G). \tag{4}$$

In particular, when $n \geq d+1$ we have $\dim(M_{d,G}) \leq nd - \binom{d+1}{2}$ and equality holds if and only if G is rigid in d dimensions.

Theorem

Let G be a graph with m edges. Then G is independent in d dimensions if and only if $M_{d,G} = \mathbb{C}^m$.

The measurement variety and the rigidity matroid

Theorem

Let $G = (V, E)$ be a graph, and suppose that $M_{d,G} = M_{d,G_1} \oplus M_{d,G_2}$ for some subgraphs $G_i = (V, E_i)$ for $i = 1, 2$. Then $\mathcal{R}_d(G) = \mathcal{R}_d(G_1) \oplus \mathcal{R}_d(G_2)$.

Theorem

Let G be a graph on n vertices with edges $e_1, \ldots, e_m$ and let $G' = G - e_m$. Then e_m is a bridge of $\mathcal{R}_d(G)$ if and only if $M_{d,G} = M_{d,G'} \oplus \mathbb{C}$.

The measurement variety and the rigidity matroid

Theorem

Let G and H be graphs with the same number of edges and suppose that $M_{d,G} = M_{d,H}$ under some edge bijection ψ. Then this edge bijection defines an isomorphism between the d-dimensional rigidity matroids of G and H.

Theorem

Let G be a graph that is uniquely (up to isomorphism) determined by its d-dimensional rigidity matroid among graphs on the same number of vertices. Then G is weakly reconstructible in d dimensions.

WR-1

Theorem

Let G be a graph that is not 2-connected and has at least two edges. Then G is weakly reconstructible in one dimension if and only if one of the following holds:
1. G is isomorphic to a 2-connected, weakly reconstructible graph H plus some isolated vertices.
2. G is isomorphic to the 1-sum of two connected vertex-transitive graphs.

Theorem

The Graph Isomorphism problem is polynomially reducible to the problem of testing weak reconstructibility in $\mathbb{C}^1$.

WR-2

Theorem

Let G be a non-redundant rigid graph in $\mathbb{R}^2$. Then G is weakly reconstructible in $\mathbb{C}^2$ if and only if G can be obtained by taking the 1-sum of two complete graphs K_r and K_s and then adding an edge, where $r, s \geq 2$, and if $s = 3$ (resp. $r = 3$) then $r = 2$ (resp. $s = 2$) holds.

Theorem

The Graph Isomorphism problem is polynomially reducible to the problem of testing weak reconstructibility in $\mathbb{C}^2$.

Bridge invariant graphs

Definition

Suppose that G has at least $d+2$ vertices and let e be a bridge in $\mathcal{R}_d(G)$. Then there is another edge e' that we can add to the flexible graph $G-e$ to obtain a graph $H = G - e + e'$ which is again rigid. In this case we say that H is obtained from G by a *bridge replacement* operation.

Definition

A graph G is called *bridge invariant* if every sequence of bridge replacement operations starting from G leads to a graph isomorphic to G.

Bridge invariant graphs in the pane

Theorem

Let G be a non-redundant rigid graph in $\mathbb{R}^2$ on $n \geq 3$ vertices. Then G is bridge invariant if and only if it satisfies one of the following properties:
1. G is isomorphic to a degree-2-extension of K_{n-1},
2. G is the cone graph of a connected graph obtained from two disjoint vertex-transitive graphs on at least three vertices by adding an edge e.

SR-1, SR-2

Theorem

Let G be a graph on at least four vertices and without isolated vertices. Then G is strongly reconstructible in $\mathbb{C}^1$ if and only if it is 3-connected.

Theorem

Let G be a graph on at least four vertices and without isolated vertices. Then G is strongly reconstructible in $\mathbb{C}^2$ if and only if it is globally rigid in $\mathbb{R}^2$.

Real reconstructibility

Theorem

Let (G, p) be a generic framework in $\mathbb{R}^1$ that is weakly (respectively strongly) reconstructible in $\mathbb{R}^1$. Then (G, p) is weakly (resp. strongly) reconstructible in $\mathbb{C}^1$.

Thank you.