大談話会 091118

# Random point fields revisited: Poissonian Fock spaces, Gibbs measures, fermion (determinantal) processes etc.

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空間上に可算個の粒子をランダムにばらまいたものを確率点場（random point field）または点過程（point process）という。これを巡る数学は、1970 年代までは確率論と数理統計学、統計力学に限定されていた（下記の例１－２など）が、近年、無限対称群の表現論やグラフ理論、また共形場理論その他から新たな‘素材’（例３－５など）を得て急速な展開を見せ始めている。他方、最近、その記述方法に関して見通しよいと確信する枠組を見出した。歴史（回顧）も踏まえつつ、その一端を紹介したい。

例１　Poisson 測度
　１ａ　複合 Poisson 過程 etc.
　１ｂ　Sinai-Volkoviski の理想気体モデル（一般に、平衡過程）
例２　Gibbs 測度
　２ａ　Dobrushin-Lanford-Ruelle 測度
　２ｂ　Sinai-Bowen 測度 etc.
例３　fermion（行列式）過程
　３ａ　Random matrix（乱雑行列）：GUE と Dyson dynamics
　３ｂ　GAF（Gauss 分布に従う係数をもつ解析関数）の零点分布
　３ｃ　Young 図形、Maya 図形等の上の確率とその極限
　３ｄ　spanning trees 上の一様分布 etc.
例４ boson（パーマネント）過程、$\alpha$ 行列式過程　(例３の一般化)
例５　Schramm-Loewner 方程式（SLE) 関連
　Markov loop（vs. loop-erased Markov chain）, etc.

Notations. For a polish space $R$,

$$C_c(R) = \{f \in C(R) | \text{ compact support}\},$$

$$C_c'(R) = \{\text{Radon measures on } R\} \supset C_c'(R)_+,$$

$$Q(R) = \{\xi \in C_c'(R)_+ | \text{integer-valued}\} \quad \text{(locally finite configuration space)}$$

$$\hat{R} = \bigcup_{\Lambda \subset R, \text{compact}} Q(\Lambda) = \{X = \sum_{i=1}^{n} \delta_{x_i} | x_i \in R,\ n \geq 0\} \quad \text{(finite configuration space)},$$

$$\langle \xi, f \rangle = \sum_i f(x_i) \qquad \text{if} \quad \xi = \sum_i \delta_{x_i} \in Q(R), f \in C_c(R),$$

$$\hat{\lambda} = \sum_{n=0}^{\infty} \frac{1}{n!} \lambda^{\otimes n} \quad (\text{on } \hat{R}\ !)$$

$$X \le \xi \Longleftrightarrow X(x) \le \xi(x)(\forall x \in R),$$

$$\xi_* = \sum_{X \in \hat{R}, X \le \xi} \delta_X \in Q(\hat{R}) \quad (\text{lift of } \xi \in Q(R) \text{ onto } \hat{R}).$$

# 1 はじめに

# 2 Poisson measures: 無限系の " 一様分布 " とは何か？

Exc. Let $f$ be a continuous function with compact support on the real line. Show that

$$\Big(\frac{1}{2L}\int_{[-L,L]} e^{-f(x)}dx\Big)^N \longrightarrow \exp \rho \int_{-\infty}^{\infty} (e^{-f(x)} - 1)dx$$

as $L \to \infty$ and $N/L \to \rho > 0$.

**Definition 1.** *A probability Borel measure $\pi_\lambda$ on $Q(R)$ is called Poisson measure with intensity $\lambda \in C_c'(R)_+$ if its Laplace transform is given by*

$$\int_{Q(R)} \pi_\lambda(d\xi) e^{-\langle \xi, f \rangle} = \exp \int_R \lambda(dx)(e^{-f(x)} - 1) \qquad f \in C_c(R).$$

# 3 Gibbs measures: 無限剛球系の " 一様分布 " とは何か？

**Definition 2.** *(Dobrushin-Lanford-Ruelle) A probability Borel measure $\mu$ on $Q(R)$ is called Gibbs measure with energy function $U(X|\xi)$ if its conditional measure is given by*

$$\mu(\cdot|\mathcal{F}_{\Lambda^c})(\xi) = q^U_{\Lambda,\xi}(\cdot)$$

*if $\Lambda \subset R$ is compact where $\mathcal{F}_{\Lambda^c}$ is the sigma-algebra genrated by the configurations outside $\Lambda^c$ and*

$$q^U_{\Lambda,\xi}(dX) = \frac{1}{Z^U_{\Lambda,\xi}} e^{-U(X|\xi)} \hat{\lambda}(dX)$$

*with normalizing constant $Z^U_{\Lambda,\xi}$.*

The energy function $U$ is necessarily a cocycle: $U(X+Y|\xi) = U(Y|\xi+X) + U(X|\xi)$. Moreover, it must satisfy certain growth condition and the condition:

$$U(X|\xi) < \infty \textit{ iff } X + \xi \in Q_U \quad (\exists Q_U \subset Q(R)).$$

Exc. Find $Q_U$ for the hard sphere.

## 4 Correlation functions and Palm mesures

**Definition 3.** *A probability measure $\mu$ on $Q(R)$ is said to admit $\rho$ as its correlation functions with respect to $\lambda$ if*

$$\Big(\int_{Q(R)} \mu(d\xi)\xi_*\Big)(dX) = \rho(X)\hat{\lambda}(dX)$$

*($\rho|_{R^n}$ is the usual $n$-point correlation function.)*

Exc. Show that the Poisson measure $\pi_\lambda$ admits constant function 1 as its correlation function.

**Definition 4.** *A probability measure $\mu$ on $Q(R)$ is said to admit $\mu^X,\ X \in \hat{R}$ as its Palm measures if they satify the disintegration formula*

$$\int_{Q(R)} \mu(d\xi) \int_{\hat{R}} \xi_*(dX)F(X,\xi) = \int_{\hat{R}} \rho(X)\hat{\lambda}(dX) \int_{Q(R)} \mu(d\xi)^X F(X,\xi+X).$$

Exc. Show that the Palm measures of a Poisson measure are the original Poisson measure.
Facts: (i) $(\mu^X)^Y = \mu^{X+Y}$.
(ii) Let $\rho^X$ be correlation functions of $\mu^X$. Then,

$$\rho^X(Y) = \rho(X+Y)/\rho(X)$$

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## 5 Poissonian Fock space

$$H = L^2(R,\lambda), \quad \mathbf{H} = L^2(\hat{R},\hat{\lambda}), \quad \mathcal{H} = L^2(Q(R),\pi_\lambda).$$

Fact: The Hilbert space $\mathbf{H}$ is the Fock space over $H$ and is isomorphic to the Hilbert space $\mathcal{H}$.

Let $\mathbf{H}_0$ be the set of those $\mathbf{f} \in \mathbf{H}$ with "compact support" and of exponential growth.

**Theorem 1.** *For $\mathbf{f} \in \mathbf{H}_0$, define*

$$S\mathbf{f}(X) = \sum_{Y\le X} \binom{X}{Y}\mathbf{f}(Y), \quad T\mathbf{f}(X) = \int_{\hat{R}} \mathbf{f}(X+Y)\hat{\lambda}(dY), \quad J\mathbf{f}(X) = (-1)^{|X|}\mathbf{f}(X).$$

*Then, $S^{-1} = JSJ$, $\quad T^{-1} = JTJ$, $T : \mathbf{H}_0 \to \mathbf{H}_0$ and, formally, $T = S^*$ in $\mathbf{H}$.*

**Theorem 2.** *For $\mathbf{f} \in \mathbf{H}_0$, define*

$$\mathbf{I}(\mathbf{f})(\xi) = \langle \xi_*, T^{-1}\mathbf{f}\rangle \qquad \xi \in Q(R).$$

*Then, $\mathbf{I}$ can be uniquely extended to the unitary isomorphism from $\mathbf{H}$ onto $\mathcal{H}$.*

Key to the proof. Given a $\phi$ on $R$, define a function $\hat{\phi}$ on $\hat{R}$ by

$$\hat{\phi}(X) = \prod_{x \in R} \phi(x)^{X(x)}.$$

If $\phi - 1 \in C_c(R)$, then $\hat{\phi}$ is an eigenfunction of $T$:

$$T\hat{\phi}(X) = \alpha\hat{\phi}(X) \text{ with } \alpha = T\hat{\phi}(0) = \exp \int_R (\phi(x) - 1)\lambda(dx).$$

# 6 More about $T$

## 6.1 相関関数と密度関数の関係式は簡潔に書くことができる。

**Theorem 3.** *Let $\mu$ a probability measure on $Q(R)$ and assume that it admits correlation function $\rho$ w.r.t. $\lambda$ and that its restriction $\mu_\Lambda$ to $Q(\Lambda)$ is abosolutelly continuous to $\hat{\lambda}$. Then the density is given by*

$$\sigma_\Lambda = T^{-1}(\rho\chi_\Lambda)$$

*where $\chi_\Lambda(X) = 1(\sup X \subset \Lambda)$.*

## 6.2 作用素 $T$ で保存される関数族は他にもある。

以下が、積分作用素に付随する点過程（fermion, permanent 過程など）の存在証明（白井一高橋 1999, 2002）の本質部分。

**Theorem 4.** *For an integral operator $K$ define*

$$\mathbf{d}_K(X) = \det_\alpha(K(x_i, x_j)) \qquad \text{if } \ X = \sum \delta_{x_i}$$

*where*

$$\det_\alpha(A) = \sum_{\tau \in S_n} (-1)^{n-\nu(\tau)} \prod_{i=1}^{n} a_{i\tau(i)}$$

*($\nu(\tau)$ is the number of cycles in permutation $\tau$.)*

*which is the $\alpha$-determinant:* $\det_{-1} = \det, \det_1 = \mathrm{perm}$. *Assume that $K$ is of trace class and let* $\mathrm{Det}(I - \alpha K)$ *be the Fredholm determinant. Then,*

$$T\mathbf{d}_K = Det(I - \alpha K)^{-1/\alpha}\mathbf{d}_J$$

*where $J = -\alpha(I - \alpha K)^{-1}K$.*

参考文献

Y.Takahashi, Random point fields revisited: Fock space associated with Poisson measures, fermion(determinatal) processes, and Gibbs measures, RIMS-1681 Oct.,2009