Overconvergence Phenomena and Grouping in Exponential Representation of Solutions of Linear Differential Equations of Infinite Order

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This paper is dedicated to the memory
of the late Professor Gian-Carlo Rota.

Introduction

This paper tries to shed some light on the very classical topic of overconvergence for Dirichlet series, by employing results in the theory of infinite order differential operators with constant coefficients ([3], [13]). The possibility of linking infinite order differential operators with gap theorems and related subjects such as overconvergence phenomena was first suggested by Ehrenpreis in [9], but in a form which could not be fully exploited. On the other hand, the proof of Theorem 1 in [13] indicates that if some lacunary Dirichlet series happens to be analytically extended beyond a point on the line of absolute convergence, then it is automatically extended analytically to a wider region and, in particular, its abscissa of holomorphy may be different from its abscissa of absolute convergence. We note that the circle of convergence coincides with that of holomorphy for a Taylor series. This fact indicates that the situation we discuss here is tied up with some subtle features of the behavior of the frequencies λ_n of the Dirichlet series $\sum a_n e^{-\lambda_n z}$ in question (see Remark 2 of Section 1 for the concrete description of the subleties). One natural question, then, might be whether we can describe the analytically continued function"concretely" in terms of the starting convergent Dirichlet series. Such a description, if it exists, may be highly transcendental; for example, let us consider the Dirichlet series

$$\eta(z) = \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n^z}$$

which is actually equal to $(1 - 2^{1-z})\zeta(z)$, with $\zeta(z)$ denoting the zeta function of Riemann (cf.e.g. [10, page 30]). It is known that $\eta(z)$ determines an entire function and that its value outside the domain of convergence can be found through the functional equation it satisfies. In the situation we consider in this article, however, we can describe the analytically continued object in a straightforward manner; the description is given by an appropriate grouping of the summation of the Dirichlet series in question. Such a clear result is obtained just because we consider a lacunary series. On the other hand, enlarging the domain of holomorphy of a series by using a suitably grouped sum is exactly the classical subject of overconvergence studied by Ostrowski ([17]), Bernstein ([7]), and others. In a word, our main result (Theorem 2.1) asserts that, if the frequencies λ_n of the Dirichlet series in question are positive and sufficiently lacunary, then the abscissa of holomorphy of the series coincides with the abscissa of overconvergence (in the terminology of [7]). See [7] for commonly used notions and notations concerning Dirichlet series.

Our strategy of the proof is as follows: we first construct an infinite order differential operator $P(\frac{d}{dz})$ that annihilates the Dirichlet series $f(z) = \sum a_n e^{-\lambda_n z}$; here the operator P is really "differential" in the sense that it acts on the sheaf

of holomorphic functions as a sheaf homomorphism, and this important property of P follows from the lacunary character of λ_n. If the Dirichlet series can be extended analytically beyond some point on the line of absolute convergence $\{z \in \mathbb{C};\ \mathrm{Re}\ z = c\}$, the equation $P(\frac{d}{dz})f(z) = 0$ enables us to extend $f(z)$ analytically to $\{z \in \mathbb{C};\ \mathrm{Re}\ z > h\}$ with $h < c$, and the resulting holomorphic function on $\{z \in \mathbb{C};\ \mathrm{Re}\ z > h\}$ can be represented by a grouped sum

$$\sum_{j=0}^{+\infty} \left(\sum_{k_j \leq n < k_j+1} a_n e^{-\lambda_n z} \right).$$

In extending the domain of holomorphy we use the reality of λ_n and, to find an appropriate grouping of exponentials, we use the fundamental principle for infinite order differential equations with constant coefficients. This approach neatly explains the nature of the problem; the gap structure of λ_n alone does not play a decisive role, though important, in the study of the overconvergence phenomena of this sort, as Bernstein himself observes ([7, page 45]). In our approach, the overconvergence phenomena we encounter ("ultraconvergence serrée" in the terminology of [7]) are linked with the (non-)interpolating property of the characteristic variety of the equation $Pf = 0$. (See [6] and [3] for the non-interpolating property of a characteristic variety.)

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1 Dirichlet Series and Infinite Order Differential Equations

In this paper we will restrict our attention to those Dirichlet series which arise in natural fashion as solutions of infinite order differential equations. The link between these two notions is made explicit in the following Fundamental Principle, essentially due to Berenstein and Taylor ([6]) (but see also Kawai and Struppa ([14]), Meril and Struppa ([16]), and finally Berenstein, Kawai and Struppa ([3])). We will state the result for the case of one variable, though it extends (with suitable modifications) to systems in several variables. We note that no slowly decreasing assumption is needed in the one-dimensional case as shown in [2, Theorem 6.4.8], though the known result for the higher dimensional case requires it. ([3])

Theorem 1.1 *Let P be a linear differential operator of infinite order and with constant coefficients, and let V denote its characteristic variety i.e., $\{\zeta \in \mathbb{C} : P(\zeta) = 0\} = \{\alpha_k \in \mathbb{C} : |\alpha_1| \leq |\alpha_2| \leq \ldots\}$; assume that all roots of P are simple. Then there exists a sequence of indices $0 = k_1 < k_2 < \ldots$ such that every entire solution f of*

$$P\left(\frac{d}{dz}\right) f = 0$$

can be written as

$$f(z) = \sum_{n \geq 1} \left(\sum_{k_n \leq k < k_{n+1}} a_k e^{\alpha_k z} \right) \tag{1}$$

for a suitable sequence of complex coefficients $\{a_k\}$ which satisfies the following growth condition: define by V_n the variety of points α_k such that $k_n \leq k < k_{n+1}$, by a_n the set $a_n := \{a_{k_n}, \ldots a_{k_{n+1}-1}\}$, and (for $t_n = k_{n+1} - k_n$) construct the $t_n \times t_n$ matrix

$$J^n = \begin{bmatrix} 1 & 1 & 1 & \ldots \\ 0 & \alpha_{k_n+1} - \alpha_{k_n} & \alpha_{k_n+2} - \alpha_{k_n} & \vdots \\ 0 & 0 & (\alpha_{k_n+1} - \alpha_{k_n})(\alpha_{k_n+2} - \alpha_{k_n}) & \vdots \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix};$$

then for any $D > 0$ and any point $\beta_n \in V_n$, it is

$$\sum_{n \geq 1} \left(\sum_{j=1}^{t_n} |c_j^{(n)}| \right) e^{D|\beta_n|} < +\infty \tag{2}$$

where $\underline{c}^{(n)} = \left(c_1^{(n)}, \ldots, c_{t_n}^{(n)}\right)$ *is defined by* $\underline{c}^{(n)} := J^{(n)} \cdot a_n$. *The convergence of the series (1) is uniform on every compact subset of* $\mathbb{C}$. *The grouping in (1) is determined by* $P(\zeta)$ *(through the minimum modulus theorem), and is independent of the specific solution. Conversely, every function* $f(z)$ *represented as in (1) is an entire solution of* $P\left(\frac{d}{dz}\right) f = 0$, *provided that the sequence* $\{a_k\}$ *satisfies (2).*

Remark 1 This theorem shows how overconvergence phenomena for Dirichlet series arise naturally in connection with infinite order differential operators. Indeed, Theorem 1.1 can also be stated and proved for solutions which are only holomorphic in an open convex set Ω. The statement of the corresponding version of the result can be given verbatim, with the only exception that condition (2) will now be replaced by

$$\sum_{n \geq 1} \left(\sum_{j=1}^{t_n} |c_j^{(n)}| \right) e^{H_K(\beta_n)} < +\infty \tag{3}$$

where K is any compact convex set contained in Ω and $H_K(\zeta) = \sup_{z \in K} Re\, \langle z, \zeta \rangle$ is its support function.

Remark 2 We wish to point out that the groupings which appear in the statement of Theorem 1.1 cannot be avoided in general. The example we provide here is taken from [18], to which we refer for details. Consider a sequence $\{\alpha_k\}$, $|\alpha_k| \nearrow +\infty$ such that $\sum \frac{1}{|\alpha_n|} < \infty$ and that $Re\ \alpha_n > 0$. Then we know, that there exists an infinite order differential operator P_1 for whose symbol $\{\alpha_k\}$ is the zero-variety. Construct now $\beta_k = \alpha_k + e^{-|\alpha_k|^2}$; this gives rise to another differential operator P_2, whose symbol has the zero-variety $\{\beta_k\}$. If we consider the differential operator

$$P\left(\frac{d}{dz}\right) := P_1\left(\frac{d}{dz}\right) P_2\left(\frac{d}{dz}\right),$$

it is not difficult [18] to show that every entire solution of

$$P\left(\frac{d}{dz}\right) f = 0$$

can be represented by the grouped series

$$f(z) = \sum_{k=1}^{+\infty} \left(A_k e^{\alpha_k z} + B_k e^{\beta_k z} \right).$$

In particular, by choosing $A_k = -B_k = 1$, we obtain the entire solution

$$f(z) = \sum_{k=1}^{+\infty} \left(e^{\alpha_k z} - e^{\beta_k z} \right);$$

this last series, however, is not entire any longer if the groupings are eliminated.

Remark 3 The way in which the groupings appear in Theorem 1.1 is only evident if one examines carefully the proof of the theorem itself. It is, however, possible to get some intuitive idea about their nature by the above example.

In particular, one can show that if for two sequences $\{\alpha_k\}$ and $\{\beta_k\}$ we have that , for every ε, $C > 0$, it is

$$|\alpha_k - \beta_k| < \varepsilon e^{-C|\alpha_k|}, \tag{4}$$

then we can conclude that, for all k sufficiently large, α_k and β_k will indeed belong to the same grouping. The proof of this fact is a consequence of the geometry of the regions of the form

$$S(P;\varepsilon,C) := \left\{\zeta \in \mathbb{C} : |P(\zeta)| < \varepsilon e^{-C|\zeta|}\right\}$$

which play a crucial role in the proof of Theorem 1.1 (see [2], [3], [6]). The reader will note that our choice in Remark 2 was, in fact, directly violating (4). On the other hand, given P, it is possible to find $C > 0$ such that the diameters of the components of $S(P;\varepsilon,C)$ containing α_k is bounded by $Ce^{C|\alpha_k|}$ (with C independent of k). Then if we have two sequences $\{\alpha_k\}$ and $\{\beta_k\}$ such that

$$|\alpha_k - \beta_k| > Ce^{C|\alpha_k|},$$

we know that the points α_k and β_k will be in different groupings.

2 Overconvergence Phenomena in One Variable

The overconvergence theorems which we will prove depend on two key ingredients. The first is the representation theorem described in Section 1. The second is a result on the invertibility of an infinite order differential operator, as a holomorphic microlocal operator. (We refer the reader to [11] for definitions and terminology from microlocal analysis.)

Let $P\left(\frac{d}{dz}\right) = \sum_{j=0}^{+\infty} a_j \frac{d^j}{dz^j}$ be an infinite order differential operator. We can consider $P\left(\frac{d}{dz}\right)$ as a holomorphic microlocal operator, and as such we may study its invertibility at a point (z_0, ζ_0) of the cotangent bundle $T^*\mathbb{C} \cong \mathbb{C} \times (\mathbb{C} \setminus \{\zeta = 0\})$. A well known ([1]) result claims that $P\left(\frac{d}{dz}\right)$ is in fact invertible on a neighborhood of (z_0, ζ_0) if $P(\zeta)$ never vanishes on the set

$$\left\{\zeta \in \mathbb{C} : \left|\frac{\zeta}{|\zeta|} - \frac{\zeta_0}{|\zeta_0|}\right| < \delta, |\zeta| \gg 1\right\},$$

for some $\delta > 0$. Hence a holomorphic solution of the equation $Pu = 0$ defined on $\{z : \mathrm{Re}\,\langle z_0, \zeta_0\rangle > 0\}$ can be extended across the hypersurface $\{z : \mathrm{Re}\,\langle z_0, \zeta_0\rangle = 0\}$ ([12]; Theorem 4.5.1.). This extension result is an important ingredient in the proof of [13].

Theorem 2.1 *Let $\{\lambda_n\}$ be a sequence of mutually distinct real numbers such that $\sum |\frac{1}{\lambda_n}| < +\infty$. Suppose that the Dirichlet series*

$$\sum_{n=0}^{+\infty} a_n e^{-\lambda_n z}$$

has abscissa of absolute convergence equal to zero. Suppose, moreover, that the series is holomorphic in a neighborhood of the origin. Then the series extends holomorphically to $\{z \in \mathcal{C} : Re\ z > -\delta\}$ for some $\delta > 0$ and it can be expressed there as a grouped sum, i.e.

$$\sum_{j=0}^{+\infty} \left(\sum_{k_j \leq n < k_{j+1}} a_n e^{-\lambda_n z} \right).$$

Proof. By the hypothesis on $\{\lambda_n\}$, $P(\zeta) = \prod_{n=0}^{+\infty} \left(1 + \frac{\zeta}{\lambda_n}\right)$ determines an infraexponential entire function, and hence $P\left(\frac{d}{dx}\right)$ is a linear differential operator of infinite order. It is clear that $\{\lambda_n\}$ is the zero-variety of $P\left(\frac{d}{dz}\right)$. Since $P\left(\frac{d}{dz}\right)$ annihilates $a_n e^{-\lambda_n z}$ for every n, the series $f(z) = \sum_{n=0}^{+\infty} a_n e^{-\lambda_n z}$, absolutely convergent in the half-plane $\{z \in \mathcal{C} : \text{Re } z > 0\}$ is a solution of the infinite order differential equation

$$P\left(\frac{d}{dz}\right) f = 0$$

in that same half-plane. On the other hand, since f extends analytically near the origin, and since all frequencies $\{\lambda_n\}$ are real, one can show [13] that f can actually be continued analytically to a function $\tilde{f}$ holomorphic in $\{z \in \mathcal{C} : \text{Re } z > -\delta\}$, for some $\delta > 0$. But $\tilde{f}$ coincides with f in $\{z \in \mathcal{C} : \text{Re } z > 0\}$ where $P\left(\frac{d}{dz}\right) f = 0$; by analytic continuation one deduces that $P\left(\frac{d}{dz}\right) \tilde{f} = 0$ in $\{z \in \mathcal{C} : \text{Re } z > -\delta\}$. Since, this last set is convex, the representation theorem given in Section 1 shows that $\tilde{f}$ has an exponential representation

$$\sum a_n e^{-\lambda_n z}.$$

However, since the original series has abscissa of absolute convergence equal to zero, the convergence of the representation of $\tilde{f}$ requires groupings and the theorem is proved. ■

Remark 4 We note that the restriction on the reality of the frequencies $\{\lambda_n\}$ can be readily weakened (see [13] and [15]).

Our approach, though very simple, also allows us to quickly obtain some other results due to Cramer and Pólya, and described in Chapter III of Bernstein's book [7].

Theorem 2.2 *Let $f(z) = \sum_{n=1}^{+\infty} a_n e^{-\lambda_n z}$ be a Dirichlet series with mutually distinct frequencies λ_n such that $\sum \frac{1}{|\lambda_n|} < \infty$, and assume that it is absolutely convergent in $\{Re\ z > 0\}$. If φ is an entire function of exponential type D, i.e.*

$$|\varphi(\zeta)| \leq A_\varepsilon e^{(D+\varepsilon)|z|},$$

then the Dirichlet series

$$\sum_{n=1}^{+\infty} \varphi(-\lambda_n) a_n e^{-\lambda_n z}$$

converges absolutely in $\{Re\ z > D\}$. In addition, if f can be analytically continued in a neighborhood of the origin, then for some $\delta > 0$, a suitably grouped series

$$\sum_{j=0}^{+\infty} \left(\sum_{k_j \leq n < k_{j+1}} \varphi(-\lambda_n) a_n e^{-\lambda_n z} \right)$$

converges in $\{z \in \mathbb{C} : Re\ z > D - \delta\}$.

Proof. Given the growth condition of φ, we see that there exists an analytic functional μ, carried by the disk centered at the origin and with radius D, such that its Borel transform $\hat{\mu}$ coincides with φ. Since, the convolution operator $\mu*$ acts continuously from the space of holomorphic functions in $\{\mathrm{Re}\ z > 0\}$ to the space of holomorphic functions in $\{\mathrm{Re}\ z > D\}$, we have that

$$\mu * f = \sum a_n \mu * (e^{-\lambda_n z}) = \sum \hat{\mu}(-\lambda_n) a_n e^{-\lambda_n z}.$$

The result now follows from the fact that $\mu * f$ satisfies the same infinite order differential equation that f satisfies. Since the groupings in Theorem 1.1 do not depend on the chosen solution, this concludes the proof. ■

Remark 5 The above result indicates that groupings are not destroyed by convolution operators; an intuitive explanation for this fact, a priori surprising, is that frequencies λ_n, λ_m in the same grouping are so close that, thanks to the growth order condition on φ, $\varphi(-\lambda_n)$ and $\varphi(-\lambda_m)$ are sufficiently close in order not to change the convergence of the series.

3 The Higher Dimensional Case

We wish to conclude this paper by showing how, at least in some cases, it is possible to obtain a higher dimensional analogue of Theorem 2.1.

As a way of introduction, one may consider a simple two-dimensional case. Let $\{\alpha_k\}$ and $\{\beta_k\}$ be real sequences such that $\frac{\log k}{\alpha_k} \to 0$, $\frac{\log k}{\beta_k} \to 0$ and consider a Dirichlet series

$$\sum_{k=1}^{+\infty} a_k e^{-\alpha_k z - \beta_k \omega}$$

in the two complex variables (z,ω). If we assume that both sequences $\frac{\log|a_k|}{\alpha_k}$ and $\frac{\log|a_k|}{\beta_k}$ converge to zero as $k \to +\infty$, it is easy to verify that $(0,0)$ is an abscissa of convergence in the sense that the series converges in

$$\{(z,\omega) \in \mathbb{C}^2 : \mathrm{Re}\ z > 0, , \mathrm{Re}\ \omega > 0\}$$

but, for any $\varepsilon > 0$, it does not converge neither in

$$\{(z,\omega) \in \mathbb{C}^2 : \mathrm{Re}\ z > -\varepsilon, \mathrm{Re}\ \omega > 0\}$$

nor in

$$\{(z,\omega) \in \mathbb{C}^2 : \mathrm{Re}\ z > 0, \mathrm{Re}\ \omega > -\varepsilon\}.$$

If both $\sum |\alpha_k|^{-\rho}$ and $\sum |\beta_k|^{-\rho}$ are convergent for some $\rho < 1$, the Dirichlet series f in question satisfies simultaneous equations

$$\mathrm{II}\left(1 + \frac{(\frac{\partial}{\partial z})}{\alpha_k}\right) f = \mathrm{II}\left(1 + \frac{(\frac{\partial}{\partial \omega})}{\beta_k}\right) f = 0.$$

Furthermore, these two operators $\mathrm{II}\left(1 + \frac{(\frac{\partial}{\partial z})}{\alpha_k}\right)$ and $\mathrm{II}\left(1 + \frac{(\frac{\partial}{\partial \omega})}{\beta_k}\right)$ determine a locally slowly decreasing ideal ([14], Example 3.1). The reality of $\{\alpha_k\}$ and $\{\beta_k\}$ then allows us to extend f to the region

$$\{(z,\omega) \in \mathbb{C}^2 : \mathrm{Re}\ z > -\delta, \mathrm{Re}\ \omega > -\delta\}$$

for some $\delta > 0$ if f can be analytically continued onto a neighborhood of the origin. (See [[13], Theorem 2].) Since a representation theorem for holomorphic solutions of such systems holds as well (see the recent work of Berenstein and the authors [3], particularly Theorem 3.6), at least under the locally slowly decreasing condition, one can give the following result (see [3] for the terminology).

Theorem 3.1 *Suppose that a sequence of mutually distinct real numbers $\{\alpha_k\}$ (resp., $\{\beta_k\}$) satisfies $\sum |\alpha_k|^{-\rho} < +\infty$ (resp. $\sum |\beta_k|^{-\rho} < +\infty$) for some $\rho < 1$. Assume that the series*

$$f(z,\omega) = \sum a_k e^{-\alpha_k z - \beta_k \omega}$$

has $(0,0)$ as abscissa of convergence in the sense described above. Then if the function f admits a holomorphic extension near the origin, it is possible to find $\delta > 0$ such that f extends to a function $\tilde{f}$, holomorphic in $\{(z,\omega) \in \mathbb{C}^2 : Re\ z > -\delta,\ Re\ \omega > -\delta\}$ and such that $\tilde{f}$ is represented, there, by a grouped series

$$\tilde{f}(z,\omega) = \sum_{k=1}^{+\infty} \left(\sum_{j \in J_k} a_j e^{-\alpha_j z - \beta_j \omega} \right).$$

A more general formulation can be given by defining, for a Dirichlet series

$$\sum_{\alpha \in I\!N^n} a_\alpha \exp\left(-\langle \lambda_\alpha, z \rangle\right)$$

in n variables, $\lambda_\alpha \in I\!R^n$, a notion of "associated convergence abscissa" as follows: we say that $\vec{c} = (c_1, \ldots, c_n)$ is an associated convergence abscissa for $\sum_{\alpha \in I\!N^n} a_\alpha \exp\left(-\langle \lambda_\alpha, z \rangle\right)$ if the series is absolutely convergent if Re $z_j > c_j$ for $j = 1, \ldots, n$ and if the series is not convergent if Re $z_j < c_j$ for $j = 1, \ldots, n$ (note the corresponding notion of "associated convergence radii" which is used for Taylor series in several complex variables). One can then state, in this case, a theorem in all similar to Theorem 3.1.

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