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Quasi M-convex and L-convex Functions — Quasiconvexity in Discrete Optimization —*

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Abstract

We introduce two classes of discrete quasiconvex functions, called quasi M-convex and L-convex functions, by generalizing the concepts of M-convexity and L-convexity due to Murota (1996, 1998). We investigate the structure of quasi M-convex and L-convex functions with respect to level sets, and show that various greedy algorithms work for the minimization of quasi M-convex and L-convex functions.

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Contents

1 Introduction

The concept of convexity for sets and functions plays a central role in continuous optimization (or nonlinear programming with continuous variable), and has various applications in the areas of mathematical economics, engineering, operations research, etc. [2, 19, 22]. The importance of convexity relies on the fact that a local minimum of a convex function is also a global minimum. Due to this property, we can find a global minimum of a convex function by iteratively moving in descent directions, i.e., so-called descent algorithms work for the convex function minimization. Therefore, convexity for a function is a sufficient condition for the success of descent methods. Most of descent methods, however, work for a fairly larger class of functions called quasiconvex functions.

A function $f : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ is said to be quasiconvex if it satisfies

$$f(\alpha x + (1-\alpha)y) \leq \max\{f(x), f(y)\} \qquad (\forall x, y \in \mathrm{dom}\, f,\ 0 < \forall\alpha < 1),$$

and semistrictly quasiconvex if it satisfies

$$f(\alpha x + (1-\alpha)y) < \max\{f(x), f(y)\} \qquad (\forall x, y \in \mathrm{dom}\, f \text{ with } f(x) \neq f(y),\ 0 < \forall\alpha < 1),$$

where $\mathrm{dom}\, f = \{x \in \mathbf{R}^n \mid f(x) < +\infty\}$.

It is easy to see that convexity implies semistrict quasiconvexity, and semistrict quasiconvexity implies quasiconvexity under the assumption of lower semicontinuity. Although (semistrict) quasiconvexity is a weaker property than convexity, it still has nice properties as follows:

- A strict local minimum of a quasiconvex function is also a strict global minimum.
- A local minimum of a semistrictly quasiconvex function is also a global minimum.
- Level sets of quasiconvex functions are convex sets.

Due to these properties, quasiconvexity also plays an important role in continuous optimization. See [1] for more accounts on quasiconvexity.

Remark 1.1. In the literature, semistrictly quasiconvex functions above are sometimes called "strictly quasiconvex functions," "explicitly quasiconvex functions," etc. In this paper, we follow the terminology in Avriel et al. [1]. □

Remark 1.2. A function $f : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ is said to be strictly quasiconvex if it satisfies

$$f(\alpha x + (1-\alpha)y) < \max\{f(x), f(y)\} \qquad (\forall x, y \in \mathrm{dom}\, f \text{ with } x \neq y,\ 0 < \forall\alpha < 1).$$

The concept of strictly quasiconvex functions is a generalization of that of strictly convex functions. A strictly quasiconvex function attains its minimum at only one point if the minimum exists. It is clear that any strictly quasiconvex function is semistrictly quasiconvex. □

In the area of discrete optimization, on the other hand, discrete analogues of convexity, or "discrete convexity" for short, have been considered, with a view to identifying the discrete structure that guarantees the success of descent methods, i.e., the so-called "greedy algorithms." Examples of discrete convexity are "discretely-convex functions" by Miller [11], "integrally-convex functions" by Favati–Tardella [5], and "M-convex and L-convex functions" by Murota [12, 13, 14, 15, 16] as well as their variants called "M$^\natural$-convex functions" by Murota–Shioura [17] and "L$^\natural$-convex functions" by Fujishige–Murota [7].

A function $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ is called M-convex if $\mathrm{dom}\, f = \{x \in \mathbf{Z}^V \mid f(x) < +\infty\} \neq \emptyset$ and f satisfies the following property:

(M-EXC) $\forall x, y \in \mathrm{dom}\, f$, $\forall u \in \mathrm{supp}^+(x - y)$, $\exists v \in \mathrm{supp}^-(x - y)$ such that

$$f(x) + f(y) \geq f(x - \chi_u + \chi_v) + f(y + \chi_u - \chi_v), \tag{1.1}$$

where $\mathrm{supp}^+(x - y) = \{w \in V \mid x(w) > y(w)\}$, $\mathrm{supp}^-(x - y) = \{w \in V \mid x(w) < y(w)\}$, and $\chi_w \in \{0, 1\}^V$ is the characteristic vector of $w \in V$. A function $g : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ is called L-convex if $\mathrm{dom}\, g \neq \emptyset$ and g satisfies the following properties:

(SBM) g is submodular, i.e., for all $p, q \in \mathbf{Z}^V$ we have

$$g(p) + g(q) \geq g(p \wedge q) + g(p \vee q),$$

(TRF) $\exists r \in \mathbf{R}$ such that $g(p + \lambda \mathbf{1}) = g(p) + \lambda r \quad (\forall p \in \mathbf{Z}^V,\ \forall \lambda \in \mathbf{Z})$,

where $p \wedge q, p \vee q \in \mathbf{Z}^V$ are defined by

$$(p \wedge q)(w) = \min\{p(w), q(w)\}, \qquad (p \vee q)(w) = \max\{p(w), q(w)\} \qquad (w \in V).$$

M-convex and L-convex functions have various properties as discrete convex functions:

(i) A local minimum of an M-convex/L-convex function is also a global minimum.
(ii) M-convex/L-convex functions can be extended to ordinary convex functions.
(iii) Various duality theorems hold.
(iv) M-convex and L-convex functions are conjugate to each other.

In particular, the property (i) shows that greedy algorithms work for the M-convex/L-convex function minimization. However, we see from results in continuous optimization that strong properties such as M-convexity/L-convexity are not required for the success of greedy algorithms, and that some properties like "quasi M-convexity/L-convexity" will suffice.

The main aim of this paper is to introduce the concepts of quasi M-convex and L-convex functions by generalizing those of M-convexity and L-convexity.

Table 1: Possible sign patterns of $\Delta f(x; v, u)$ and $\Delta f(y; u, v)$ in (M-EXC)

$\Delta f(x; v, u) \setminus \Delta f(y; u, v)$	$-$	0	$+$
$-$	○	○	○
0	○	○	$\times$
$+$	○	$\times$	$\times$

○ $\cdots$ possible, $\times \cdots$ impossible

To extend the concept of M-convexity to quasi M-convexity, we relax the condition (1.1) while keeping the possible sign patterns of values $\Delta f(x; v, u) = f(x - \chi_u + \chi_v) - f(x)$ and $\Delta f(y; u, v) = f(y + \chi_u - \chi_v) - f(y)$ in mind. Table 1 shows the possible sign patterns of those values. Let $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ be a function. Then, we call f *quasi M-convex* if $\mathrm{dom}\, f \neq \emptyset$ and it satisfies (QM):

(QM) $\forall x, y \in \mathrm{dom}\, f$, $\forall u \in \mathrm{supp}^+(x - y)$, $\exists v \in \mathrm{supp}^-(x - y)$ such that
$$\Delta f(x; v, u) \leq 0 \qquad \text{or} \qquad \Delta f(y; u, v) \leq 0.$$

Similarly, we call f *semistrictly quasi M-convex* if $\mathrm{dom}\, f \neq \emptyset$ and it satisfies (SQM):

(SSQM) $\forall x, y \in \mathrm{dom}\, f$, $\forall u \in \mathrm{supp}^+(x - y)$, $\exists v \in \mathrm{supp}^-(x - y)$ such that
$$\text{(i) } \Delta f(x; v, u) \geq 0 \Longrightarrow \Delta f(y; u, v) \leq 0, \qquad \text{and}$$
$$\text{(ii) } \Delta f(y; u, v) \geq 0 \Longrightarrow \Delta f(x; v, u) \leq 0.$$

We introduce the concept of quasi L-convex functions by generalizing the submodularity of functions to quasi-submodularity. We consider two different generalizations of the submodularity (SBM):

(QSB) For all $p, q \in \mathbf{Z}^V$ we have $g(p \wedge q) \leq g(p)$ or $g(p \vee q) \leq g(q)$.
(SSQSB) For all $p, q \in \mathbf{Z}^V$ we have both (i) and (ii):
(i) $g(p \vee q) \geq g(q) \Longrightarrow g(p \wedge q) \leq g(p)$, and (ii) $g(p \wedge q) \geq g(p) \Longrightarrow g(p \vee q) \leq g(q)$.

We call a function $g : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ quasi-submodular (resp. semistrictly quasi-submodular) if it satisfies (QSB) (resp. (SSQSB)). We define a quasi L-convex (resp. semistrictly quasi L-convex) function as a function $g : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ with $\mathrm{dom}\, g \neq \emptyset$ satisfying (QSB) (resp. (SSQSB)) and (TRF).

Remark 1.3. The condition (SSQSB) was introduced by Milgrom–Shannon [10], in which a function $g : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ is called quasi-supermodular if the function $-g$ satisfies (SSQSB) above. We adopt the terminology "semistrict quasi-submodularity" for the property (SSQSB) in view of our results shown in Section 5. $\square$

The organization of this paper is as follows. We first review some fundamental results on M-convex and L-convex functions in Section 2. In Sections 3 and 5, we show some properties of level sets of quasi M-convex/L-convex functions and prove that the classes of quasi M-convex/L-convex functions are closed under various fundamental operations. These results justify the definitions of quasi M-convex/L-convex functions. Finally, we show that various greedy algorithms work for the minimization of (semistrictly) quasi M-convex/L-convex functions in Sections 4 and 6. We also show proximity theorems on (semistrictly) quasi M-convex/L-convex functions, which guarantee the applicability of the so-called "scaling technique" to the quasi M-convex/L-convex function minimization.

The concepts of M$^\natural$-convex functions by Murota–Shioura [17] and L$^\natural$-convex functions by Fujishige–Murota [7] can be also extended to quasi M$^\natural$-convex and L$^\natural$-convex functions, and the results in this paper can be restated in obvious ways in terms of quasi M$^\natural$-convex and L$^\natural$-convex functions.

2 Review of Fundamental Results on M-convexity/L-convexity

2.1 Definitions and Notation

We denote by $\mathbf{R}$ the set of reals, and by $\mathbf{Z}$ the set of integers. Also, we denote by $\mathbf{R}_{++}$ the set of positive reals. For any finite set X, its cardinality is denoted by $|X|$. Throughout this paper, we assume that V is a nonempty finite set with $|V| = n(> 0)$. The characteristic vector of a subset $X \subseteq V$ is denoted by χ_X $(\in \{0,1\}^V)$, i.e.,

$$\chi_X(w) = \begin{cases} 1 & (w \in X), \\ 0 & (w \in V \setminus X). \end{cases}$$

In particular, we use the notation $\mathbf{0} = \chi_\emptyset$, $\mathbf{1} = \chi_V$.

Let $x = (x(w) \mid w \in V) \in \mathbf{R}^V$. We define

$$\begin{aligned} \mathrm{supp}^+(x) &= \{v \in V \mid x(v) > 0\}, & \mathrm{supp}^-(x) &= \{v \in V \mid x(v) < 0\}, \\ \|x\|_1 &= \sum_{v \in V} |x(v)|, & \|x\|_\infty &= \max_{v \in V} |x(v)|, \\ \langle p, x\rangle &= \sum_{v \in V} p(v)x(v) \qquad (p \in \mathbf{R}^V), & x(X) &= \sum_{v \in X} x(v) \qquad (X \subseteq V). \end{aligned}$$

For any $p, q \in \mathbf{R}^V$, $p \wedge q$ and $p \vee q$ denote the vectors in $\mathbf{R}^V$ such that

$$(p \wedge q)(w) = \min\{p(w), q(w)\}, \quad (p \vee q)(w) = \max\{p(w), q(w)\} \qquad (w \in V).$$

For $a : V \to \mathbf{Z} \cup \{-\infty\}$ and $b : V \to \mathbf{Z} \cup \{+\infty\}$ with $a(v) \leq b(v)$ $(v \in V)$, we define the interval $[a, b]$ $(\subseteq \mathbf{Z}^V)$ by

$$[a, b] = \{x \in \mathbf{Z}^V \mid a(v) \leq x(v) \leq b(v) \ (v \in V)\}.$$

Let $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$. The *effective domain* $\mathrm{dom}\, f$ of f is defined by

$$\mathrm{dom}\, f = \{x \in \mathbf{Z}^V \mid f(x) < +\infty\}.$$

We denote by $\arg\min f$ the set of the minimizers of f, i.e.,

$$\arg\min f = \{x \in \mathbf{Z}^V \mid f(x) \leq f(y) \ (\forall y \in \mathbf{Z}^V)\}.$$

For any $\alpha \in \mathbf{R} \cup \{+\infty\}$, the *level set* $\mathrm{L}(f, \alpha)$ is defined by

$$\mathrm{L}(f, \alpha) = \{x \in \mathbf{Z}^V \mid f(x) \leq \alpha\}.$$

Note that $\arg\min f = \mathrm{L}(f, \inf f)$ and $\mathrm{dom}\, f = \mathrm{L}(f, +\infty)$ are special cases of level sets. For any vector $p \in \mathbf{R}^V$, the function $f[p] : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ is given by

$$f[p](x) = f(x) + \sum_{v \in V} p(v)x(v) \quad (x \in \mathbf{Z}^V). \tag{2.1}$$

For a set $S \subseteq \mathbf{Z}^V$, the *indicator function* $\delta_S : \mathbf{Z}^V \to \{0, +\infty\}$ of S is given by

$$\delta_S(x) = \begin{cases} 0 & (x \in S), \\ +\infty & (x \notin S). \end{cases}$$

We define (semistrict) quasiconvexity for functions $\varphi : \mathbf{Z} \to \mathbf{R} \cup \{+\infty\}$ in the following way: we call a function φ *quasiconvex* if it satisfies

$$\varphi(\beta) \leq \max\{\varphi(\alpha_1), \varphi(\alpha_2)\} \qquad (\forall \alpha_1, \alpha_2, \beta \in \mathbf{Z} \text{ with } \alpha_1 < \beta < \alpha_2), \tag{2.2}$$

and *semistrictly quasiconvex* if it is a quasiconvex function and satisfies

$$\varphi(\beta) < \max\{\varphi(\alpha_1), \varphi(\alpha_2)\} \qquad (\forall \alpha_1, \alpha_2, \beta \in \mathbf{Z} \text{ with } \alpha_1 < \beta < \alpha_2,\ \varphi(\alpha_1) \neq \varphi(\alpha_2)). \tag{2.3}$$

Remark 2.1. For a function $f : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$, semistrict quasiconvexity implies quasi convexity under the assumption of lower semicontinuity [1, 2]. For a function $\varphi : \mathbf{Z} \to \mathbf{R} \cup \{+\infty\}$, on the other hand, the property (2.3) alone does not imply the quasiconvexity in general. It is convenient for our subsequent development to assume quasiconvexity in the definition of semistrict quasiconvexity for φ. $\square$

Theorem 2.2. *Let $\varphi : \mathbf{Z} \to \mathbf{R} \cup \{+\infty\}$.*
(i) *φ is quasiconvex $\iff$ for any $\alpha_1, \alpha_2 \in \mathrm{dom}\,\varphi$ with $\alpha_1 < \alpha_2$ we have*

$$\min\{\varphi(\alpha_1+1), \varphi(\alpha_2-1)\} \le \max\{\varphi(\alpha_1), \varphi(\alpha_2)\}.$$

(ii) *Under quasiconvexity (2.2), the condition (2.3) for φ is equivalent to the following condition:*

$$\min\{\varphi(\alpha_1+1), \varphi(\alpha_2-1)\} < \max\{\varphi(\alpha_1), \varphi(\alpha_2)\} \quad (\forall \alpha_1, \alpha_2 \in \mathrm{dom}\,\varphi \text{ with } \alpha_1 < \alpha_2,\ \varphi(\alpha_1) \ne \varphi(\alpha_2)).$$

(iii) *φ is semistrictly quasiconvex $\iff$ for any $\alpha_1, \alpha_2 \in \mathrm{dom}\,\varphi$ with $\alpha_1 < \alpha_2$ we have both*

$$\varphi(\alpha_1+1) \ge \varphi(\alpha_1) \Longrightarrow \varphi(\alpha_2-1) \le \varphi(\alpha_2), \quad \text{and} \quad \varphi(\alpha_2-1) \ge \varphi(\alpha_2) \Longrightarrow \varphi(\alpha_1+1) \le \varphi(\alpha_1).$$

Proof. See Appendix. □

A function $\varphi : \mathbf{R} \to \mathbf{R} \cup \{+\infty\}$ is said to be *nondecreasing* if $\varphi(\alpha) \le \varphi(\beta)$ holds for all $\alpha, \beta \in \mathbf{R}$ with $\alpha < \beta$, and *strictly increasing* if for all $\alpha, \beta \in \mathbf{R}$ with $\alpha < \beta$ we have either $\varphi(\alpha) < \varphi(\beta)$ or $\varphi(\alpha) = \varphi(\beta) = +\infty$.

For a set T, a *total order* on T, denoted by $\preceq$, is a binary relation satisfying the conditions (i) $\forall a \in T$, $a \preceq a$, (ii)$a \preceq b$, $b \preceq a \Longrightarrow a = b$, (iii) $a \preceq b$, $b \preceq c \Longrightarrow a \preceq c$, and (iv) $\forall a, b \in T$, $a \preceq b$ or $b \preceq a$. We call such a pair $(T, \preceq)$ a *totally ordered set.* For $a, b \in T$, we denote $a \succeq b$ if $b \preceq a$, and $a \prec b$ if $a \preceq b$ and $a \ne b$.

For the set of real vectors $\mathbf{R}^n$, the *lexicographic order* is the total order $\preceq_{\mathrm{lex}}$ defined as follows: for $a, b \in \mathbf{R}^n$, $a \preceq_{\mathrm{lex}} b$ if either $a = b$ or there exists some $k \in \{1, 2, \cdots, n\}$ such that $a_i = b_i$ for $i = 1, \cdots, k-1$ and $a_k < b_k$.

2.2 M-convex Functions

A function $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ is called *M-convex* if $\mathrm{dom}\, f \ne \emptyset$ and f satisfies the following property:

(M-EXC) $\forall x, y \in \mathrm{dom}\, f$, $\forall u \in \mathrm{supp}^+(x-y)$, $\exists v \in \mathrm{supp}^-(x-y)$ such that

$$f(x) + f(y) \ge f(x - \chi_u + \chi_v) + f(y + \chi_u - \chi_v). \tag{2.4}$$

For any $x \in \mathrm{dom}\, f$ and $u, v \in V$, the *directional difference* of f at x w.r.t. u and v by

$$\Delta f(x; u, v) = f(x + \chi_u - \chi_v) - f(x).$$

Then, the inequality (2.4) can be rewritten as follows in terms of directional differences:

$$\Delta f(x; v, u) + \Delta f(y; u, v) \le 0.$$

M-convex functions can be characterized by the following (seemingly) weaker property:

(M-EXC$_\mathbf{w}$) $\forall x, y \in \mathrm{dom}\, f$ with $x \neq y$, $\exists u \in \mathrm{supp}^+(x-y)$, $\exists v \in \mathrm{supp}^-(x-y)$ satisfying (2.4).

Theorem 2.3 ([13, Th. 3.1]). *For* $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$, *we have* (M-EXC) $\iff$ (M-EXC$_\mathrm{w}$).

We also define the set version of M-convexity as follows. A set $B \subseteq \mathbf{Z}^V$ is called *M-convex* if $B \neq \emptyset$ and it satisfies

(B-EXC) $\forall x, y \in B$, $\forall u \in \mathrm{supp}^+(x-y)$, $\exists v \in \mathrm{supp}^-(x-y)$ such that $x - \chi_u + \chi_v \in B$ and $y + \chi_u - \chi_v \in B$.

An M-convex set is nothing but (the set of integral vectors in) an integral base polyhedron [6]. It is easy to see that

- $B \subseteq \mathbf{Z}^V$ satisfies (B-EXC) $\iff$ δ_B satisfies (M-EXC),
- $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ satisfies (M-EXC) $\Longrightarrow$ $\mathrm{dom}\, f$ satisfies (B-EXC).

For $x \in B$ and $u, v \in V$, the *exchange capacity* associated with x, v and u is defined as

$$\tilde{c}_B(x, v, u) = \max\{\alpha \in \mathbf{R} \mid x + \alpha(\chi_v - \chi_u) \in B\}.$$

M-convex sets can be characterized also by the following (seemingly) weaker property:

(B-EXC$_\mathbf{w}$) $\forall x, y \in B$ with $x \neq y$, $\exists u \in \mathrm{supp}^+(x-y)$, $\exists v \in \mathrm{supp}^-(x-y)$ such that $x - \chi_u + \chi_v \in B$ and $y + \chi_u - \chi_v \in B$.

Theorem 2.4 ([23]). *For* $B \subseteq \mathbf{Z}^V$, *we have* (B-EXC) $\iff$ (B-EXC$_\mathrm{w}$).

A variant of M-convex functions, called M$^\natural$-convex function, was introduced in [17]. A function $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ is called M$^\natural$*-convex* if the function $\tilde{f} : \mathbf{Z}^{\tilde{V}} \to \mathbf{R} \cup \{+\infty\}$ defined by

$$\tilde{f}(x_0, x) = \begin{cases} f(x) & (x_0 = -x(V)), \\ +\infty & (x_0 \neq -x(V)), \end{cases} \qquad ((x_0, x) \in \mathbf{Z}^{\tilde{V}})$$

is an M-convex function, where $\tilde{V} = \{v_0\} \cup V$. M$^\natural$-convex functions are essentially equivalent to M-convex functions, whereas the class of M$^\natural$-convex functions properly contains that of M-convex functions; i.e.,

$$\mathcal{M}_n^\natural \simeq \mathcal{M}_{n+1}, \qquad \mathcal{M}_n \subseteq \mathcal{M}_n^\natural,$$

where $\mathcal{M}_n$ (resp. $\mathcal{M}_n^\natural$) denotes the class of M-convex (resp. M$^\natural$-convex) functions defined over $\mathbf{Z}^n$.

2.3 L-convex Functions

Let $g : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$. We call g *submodular* if it satisfies the property (SBM):

(SBM) $\quad g(p) + g(q) \geq g(p \wedge q) + g(p \vee q) \quad (p, q \in \mathbf{Z}^V).$

A function g is called *L-convex* if $\mathrm{dom}\, g \neq \emptyset$ and it satisfies (SBM) and (TRF):

(TRF) $\quad \exists r \in \mathbf{R}$ such that $g(p + \lambda \mathbf{1}) = g(p) + \lambda r \quad (\forall p \in \mathrm{dom}\, g,\ \forall \lambda \in \mathbf{Z}).$

We also define the set version of L-convexity as follows. A set $D \subseteq \mathbf{Z}^V$ is called *L-convex* if $D \neq \emptyset$ and it satisfies (DL) and (TRS):

(DL) $\quad p, q \in D \Longrightarrow p \wedge q, p \vee q \in D,$
(TRS) $\quad p \in D, \lambda \in \mathbf{Z} \Longrightarrow p + \lambda \mathbf{1} \in D.$

It is easy to see that

- $D \subseteq \mathbf{Z}^V$ satisfies (DL) (resp. (TRS)) $\iff$ δ_D satisfies (SBM) (resp. (TRF)),
- $g : \mathbf{Z}_V \to \mathbf{R} \cup \{+\infty\}$ satisfies (SBM) (resp. (TRF)) $\Longrightarrow$ $\mathrm{dom}\, g$ satisfies (DL) (resp. (TRS)).

A variant of L-convex functions, called L$^\natural$-convex function, was introduced in [7]. A function $g : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ is called *L$^\natural$-convex* if the function $\tilde{g} : \mathbf{Z}^{\tilde{V}} \to \mathbf{R} \cup \{+\infty\}$ defined by

$$\tilde{g}(p_0, p) = g(p - p_0 \mathbf{1}) \quad ((p_0, p) \in \mathbf{Z}^{\tilde{V}})$$

is L-convex, where $\tilde{V} = \{v_0\} \cup V$. We see that L$^\natural$-convex functions are essentially the same as L-convex functions, while the class of L$^\natural$-convex functions properly contains that of L-convex functions; i.e.,

$$\mathcal{L}_n^\natural \simeq \mathcal{L}_{n+1}, \qquad \mathcal{L}_n \subseteq \mathcal{L}_n^\natural,$$

where $\mathcal{L}_n$ (resp. $\mathcal{L}_n^\natural$) denotes the class of L-convex (resp. L$^\natural$-convex) functions defined over $\mathbf{Z}^n$.

3 Quasi M-convex Functions

3.1 Definitions of Quasi M-convex Functions

Let $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$. Then, we call f *quasi M-convex* if $\mathrm{dom}\, f \neq \emptyset$ and it satisfies (QM):

(QM) $\forall x, y \in \mathrm{dom}\, f$, $\forall u \in \mathrm{supp}^+(x - y)$, $\exists v \in \mathrm{supp}^-(x - y)$ such that

$$\Delta f(x; v, u) \leq 0 \qquad \text{or} \qquad \Delta f(y; u, v) \leq 0.$$

Similarly, we call f *semistrictly quasi M-convex* if $\mathrm{dom}\, f \neq \emptyset$ and it satisfies (SSQM):

(SSQM) $\forall x, y \in \mathrm{dom}\, f$, $\forall u \in \mathrm{supp}^+(x-y)$, $\exists v \in \mathrm{supp}^-(x-y)$ such that
$$\text{(i) } \Delta f(x; v, u) \geq 0 \Longrightarrow \Delta f(y; u, v) \leq 0, \qquad \text{and}$$
$$\text{(ii) } \Delta f(y; u, v) \geq 0 \Longrightarrow \Delta f(x; v, u) \leq 0.$$

Note that (SSQM) can be rewritten as follows:

$\forall x, y \in \mathrm{dom}\, f$, $\forall u \in \mathrm{supp}^+(x-y)$, $\exists v \in \mathrm{supp}^-(x-y)$ satisfying at least one of the following:
(i) $\Delta f(x; v, u) < 0$, (ii) $\Delta f(y; u, v) < 0$, (iii) $\Delta f(x; v, u) = \Delta f(y; u, v) = 0$.

We also consider weaker properties than (QM) and (SSQM):

($\mathbf{QM_w}$) $\forall x, y \in \mathrm{dom}\, f$ with $x \neq y$, $\exists u \in \mathrm{supp}^+(x-y)$, $\exists v \in \mathrm{supp}^-(x-y)$ such that
$$\Delta f(x; v, u) \leq 0 \qquad \text{or} \qquad \Delta f(y; u, v) \leq 0.$$

($\mathbf{SSQM_w}$) $\forall x, y \in \mathrm{dom}\, f$ with $x \neq y$, $\exists u \in \mathrm{supp}^+(x-y)$, $\exists v \in \mathrm{supp}^-(x-y)$ such that
$$\text{(i) } \Delta f(x; v, u) \geq 0 \Longrightarrow \Delta f(y; u, v) \leq 0, \qquad \text{and}$$
$$\text{(ii) } \Delta f(y; u, v) \geq 0 \Longrightarrow \Delta f(x; v, u) \leq 0.$$

The set version of quasi M-convexity can be obtained by translating the properties (QM) and ($\mathrm{QM_w}$) for the indicator function $\delta_B : \mathbf{Z}^V \to \{0, +\infty\}$ of a set $B \subseteq \mathbf{Z}^V$ in terms of B.

(Q-EXC) $\forall x, y \in B$, $\forall u \in \mathrm{supp}^+(x-y)$, $\exists v \in \mathrm{supp}^-(x-y)$ such that
$$x - \chi_u + \chi_v \in B \qquad \text{or} \qquad y + \chi_u - \chi_v \in B.$$

($\mathbf{Q\text{-}EXC_w}$) $\forall x, y \in B$ with $x \neq y$, $\exists u \in \mathrm{supp}^+(x-y)$, $\exists v \in \mathrm{supp}^-(x-y)$ such that
$$x - \chi_u + \chi_v \in B \qquad \text{or} \qquad y + \chi_u - \chi_v \in B.$$

It may be noted that the properties (Q-EXC) and ($\mathrm{Q\text{-}EXC_\mathbf{w}}$) are labeled (EXC) and ($\mathrm{EXC_\mathbf{w}}$) in [21], respectively.

The following properties for $B \subseteq \mathbf{Z}^V$ can be shown easily from the fact that $\Delta\delta_B(x; v, u) \in \{0, +\infty\}$ for $x \in B$ and $u, v \in V$.

- ($\mathrm{Q\text{-}EXC_w}$) for B $\Longleftrightarrow$ ($\mathrm{QM_w}$) for δ_B,
- (Q-EXC) for B $\Longleftrightarrow$ (QM) for δ_B,
- (B-EXC) for B $\Longleftrightarrow$ (SSQM) for δ_B $\Longleftrightarrow$ ($\mathrm{SSQM_w}$) for δ_B.

We show some examples of quasi M-convex functions below.

Example 3.1. Let $\varphi : \mathbf{Z} \to \mathbf{R} \cup \{+\infty\}$. We define $f : \mathbf{Z}^2 \to \mathbf{R} \cup \{+\infty\}$ by

$$f(x_1, x_2) = \begin{cases} \varphi(x_1) & (x_1 + x_2 = 0), \\ +\infty & (x_1 + x_2 \neq 0). \end{cases} \tag{3.1}$$

By Theorem 2.2, f satisfies (QM) (or (QM$_{\mathbf{w}}$)) if and only if φ is quasiconvex, and f satisfies (SSQM) (or (SSQM$_{\mathbf{w}}$)) if and only if φ is semistrictly quasiconvex. $\square$

Example 3.2. Let $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ be an M-convex function, and $\varphi : \mathbf{R} \to \mathbf{R} \cup \{+\infty\}$ be a nondecreasing function. We define the function $\tilde{f} : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ by

$$\tilde{f}(x) = \begin{cases} \varphi(f(x)) & (x \in \mathrm{dom}\, f), \\ +\infty & (x \notin \mathrm{dom}\, f). \end{cases} \tag{3.2}$$

Then, $\tilde{f}$ satisfies (QM). Furthermore, if φ is strictly increasing, then $\tilde{f}$ satisfies (SSQM). $\square$

Example 3.3. Let $B \subseteq \mathbf{Z}^V$ be an M-convex set, $p \in \mathbf{R}^V$, and $\alpha \in \mathbf{R}$. Then, the set $S = \{x \in B \mid \langle p, x\rangle \leq \alpha\}$ satisfies (Q-EXC). Moreover, the function $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ with $\mathrm{dom}\, f = S$ defined by $f(x) = \langle p, x\rangle$ $(x \in S)$ satisfies (SSQM). $\square$

Remark 3.4. The concept of (semistrict) quasi M-convexity can be naturally extended to functions $f : S \to T$ with $S \subseteq \mathbf{Z}^V$ and a totally ordered set T with total order $\preceq$. For example, the property (SSQM) is rewritten for such functions as follows:

> $\forall x, y \in S$, $\forall u \in \mathrm{supp}^+(x - y)$, $\exists v \in \mathrm{supp}^-(x - y)$ such that
> (i) if either $x - \chi_u + \chi_v \notin S$, or $x - \chi_u + \chi_v \in S$ and $f(x - \chi_u + \chi_v) \succeq f(x)$, then $y + \chi_u - \chi_v \in S$ and $f(y + \chi_u - \chi_v) \preceq f(y)$, and
> (ii) if either $y + \chi_u - \chi_v \notin S$, or $y + \chi_u - \chi_v \in S$ and $f(y + \chi_u - \chi_v) \succeq f(y)$, then $x - \chi_u + \chi_v \in S$ and $f(x - \chi_u + \chi_v) \preceq f(x)$.

It is easy to see that the properties of (semistrictly) quasi M-convex functions shown in this paper still hold true. For simplicity and convenience, however, we assume in this paper that the codomain of a function is $\mathbf{R} \cup \{+\infty\}$. $\square$

Example 3.5. Suppose that $V = \{1, 2, \cdots, n\}$ $(n \geq 1)$. Let $a : V \to \mathbf{Z} \cup \{-\infty\}$, $b : V \to \mathbf{Z} \cup \{+\infty\}$, and $\alpha \in \mathbf{Z}$ satisfy $a(i) \leq b(i)$ $(i \in V)$ and $\sum_{i \in V} a(i) \leq \alpha \leq \sum_{i \in V} b(i)$. For $i \in V$, let $f_i : [a(i), b(i)] \to \mathbf{R}$ be a semistrictly quasiconvex function. We define $B \subseteq \mathbf{Z}^V$ and $f : B \to \mathbf{R}^V$ by

$$B = \{x \in [a, b] \mid x(V) = \alpha\}, \qquad f(x) = (f_i(x(i)) \mid i \in V) \ \ (x \in B),$$

where the total order $\preceq$ on the codomain $\mathbf{R}^V$ of f is defined by the lexicographic order. Then, f satisfies (SSQM) in the extended sense (see Remark 3.4).

Proof. Let $x, y \in B$ be distinct vectors. Also, let $u \in \mathrm{supp}^+(x-y)$, $v \in \mathrm{supp}^-(x-y)$ be any elements, and w.l.o.g. assume that $u < v$. Then, we have $x-\chi_u+\chi_v \in B$ and $y+\chi_u-\chi_v \in B$. If $f_u(x(u)-1) < f_u(x(u))$ or $f_u(y(u)+1) < f_u(y(u))$ holds, then we have $f(x-\chi_u+\chi_v) \prec f(x)$ or $f(y+\chi_u-\chi_v) \prec f(y)$. Otherwise, we have $f_u(x(u)-1) = f_u(x(u))$ and $f_u(y(u)+1) = f_u(y(u))$ by Theorem 2.2. If $f_v(x(v)+1) < f_v(x(v))$ or $f_v(y(v)-1) < f_v(y(v))$ holds, then we have $f(x-\chi_u+\chi_v) \prec f(x)$ or $f(y+\chi_u-\chi_v) \prec f(y)$. Otherwise, we have $f_v(x(v)+1) = f_v(x(v))$ and $f_v(y(v)-1) = f_v(y(v))$, from which follows $f(x-\chi_u+\chi_v) = f(x)$ and $f(y+\chi_u-\chi_v) = f(y)$. □

The relationship among various quasi M-convexity for sets and functions is summarized as follows. Note that the claim (i) of Theorem 3.6 is already shown in [21, Remark 11].

Theorem 3.6. *Let $S \subseteq \mathbf{Z}^V$ and $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$. Then, we have*

$$\begin{array}{cccccc}
\text{(i)} & \text{(B-EXC)} & \Longrightarrow & \text{(Q-EXC)} & & \\
& \Updownarrow & & \Downarrow & & \\
& (\text{B-EXC}_\text{w}) & \Longrightarrow & (\text{Q-EXC}_\text{w}). & & \\
\text{(ii)} & \text{(M-EXC)} & \Longrightarrow & \text{(SSQM)} & \Longrightarrow & \text{(QM)} \\
& \Updownarrow & & \Downarrow & & \Downarrow \\
& (\text{M-EXC}_\text{w}) & \Longrightarrow & (\text{SSQM}_\text{w}) & \Longrightarrow & (\text{QM}_\text{w}).
\end{array}$$

Remark 3.7. The converses of the statements "(B-EXC$_\text{w}$) $\Longrightarrow$ (Q-EXC)" and "(Q-EXC) $\Longrightarrow$ (Q-EXC$_\text{w}$)" do not hold in general (see [21, Remark 11]). This fact shows that neither of the implications "(QM) $\Longrightarrow$ (SSQM$_\text{w}$)" and "(QM$_\text{w}$) $\Longrightarrow$ (QM)" hold.

In the following, we present several examples to show that implications not mentioned in Theorem 3.6 (ii) do not hold in general.

[(SSQM$_\text{w}$) $\not\Rightarrow$ (QM)] The function $f_1 : \mathbf{Z}^6 \to \mathbf{R} \cup \{+\infty\}$ given by

$$\begin{aligned}
&\mathrm{dom}\, f_1 = \{(1,1,1,0,0,0), (0,1,1,1,0,0), (0,0,1,1,1,0), (0,0,0,1,1,1)\}, \\
&f_1(x) = x_1 + x_2 + x_3 \ (x \in \mathrm{dom}\, f_1)
\end{aligned}$$

satisfies (SSQM$_\text{w}$) and not (QM) since $\mathrm{dom}\, f_1$ does not satisfy (Q-EXC) (see Theorem 3.11 below).

[(SSQM) $\not\Rightarrow$ (M-EXC)] The function $f_2 : \mathbf{Z}^2 \to \mathbf{R} \cup \{+\infty\}$ given by

$$\mathrm{dom}\, f_2 = \{(0,0), (1,-1), (2,-2)\}, \quad f_2(0,0) = 0, \quad f_2(1,-1) = 2, \quad f_2(2,-2) = 3$$

satisfies (SSQM) and not (M-EXC). □

The property (QM$_\text{w}$) is equivalent to each of the following (seemingly) weaker conditions.

$$\max\{f(x), f(y)\} \geq \min_{\substack{u \in \mathrm{supp}^+(x-y) \\ v \in \mathrm{supp}^-(x-y)}} \{f(x-\chi_u+\chi_v), f(y+\chi_u-\chi_v)\} \qquad (\forall x, y \in \mathrm{dom}\, f), \tag{3.3}$$

$$f(x) \geq \min_{\substack{u \in \mathrm{supp}^+(x-y) \\ v \in \mathrm{supp}^-(x-y)}} f(x-\chi_u+\chi_v) \qquad (\forall x, y \in \mathrm{dom}\, f \text{ with } f(x) \geq f(y)). \tag{3.4}$$

Theorem 3.8. *For $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$, we have* (QM$_\mathrm{w}$) $\Longleftrightarrow$ *(3.3)* $\Longleftrightarrow$ *(3.4).*

Proof. It is easy to see that (3.4) implies both (QM$_\mathrm{w}$) and (3.3). Hence, we prove "(QM$_\mathrm{w}$) $\Longrightarrow$ (3.4)" and "(3.3) $\Longrightarrow$ (3.4)" below.

Suppose that $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ satisfies (QM$_\mathrm{w}$) or (3.3). Let $x, y \in \mathrm{dom}\, f$ be vectors such that $f(x) \geq f(y)$. We show by induction on the value $||x - y||_1$ that there exist some $u \in \mathrm{supp}^+(x-y)$ and $v \in \mathrm{supp}^-(x-y)$ such that $\Delta f(x; v, u) \leq 0$. We may assume $||x-y||_1 > 2$, since otherwise the claim holds obviously.

Suppose first that f satisfies (QM$_\mathrm{w}$). Then, there exist some $u \in \mathrm{supp}^+(x - y)$ and $v \in \mathrm{supp}^-(x - y)$ such that $\Delta f(x; v, u) \leq 0$ or $\Delta f(y; u, v) \leq 0$. If the latter holds, then we have $f(x) \geq f(y')$ for $y' = y + \chi_u - \chi_v$ and $||x - y'||_1 < ||x - y||_1$. Hence, the inductive hypothesis yields $\Delta f(x; v', u') \leq 0$ for some $u' \in \mathrm{supp}^+(x - y') \subseteq \mathrm{supp}^+(x - y)$ and $v' \in \mathrm{supp}^-(x - y') \subseteq \mathrm{supp}^-(x - y)$.

We next suppose that f satisfies (3.3). Then, there exist some $u \in \mathrm{supp}^+(x - y)$ and $v \in \mathrm{supp}^-(x - y)$ such that $\Delta f(x; v, u) \leq 0$ or $f(y + \chi_u - \chi_v) \leq f(x)$. If the latter holds, then we have $f(x) \geq f(y')$ for $y' = y + \chi_u - \chi_v$ and $||x - y'||_1 < ||x - y||_1$. Hence, the inductive hypothesis yields $\Delta f(x; v', u') \leq 0$ for some $u' \in \mathrm{supp}^+(x - y') \subseteq \mathrm{supp}^+(x - y)$ and $v' \in \mathrm{supp}^-(x - y') \subseteq \mathrm{supp}^-(x - y)$. □

3.2 Level Sets of Quasi M-convex Functions

We show various properties of level sets of quasi M-convex functions.

The following two theorems claim that level sets of quasi M-convex functions have quasi M-convexity. Furthermore, the weaker version of quasi M-convexity (QM$_\mathrm{w}$) for functions can be characterized by quasi M-convexity (Q-EXC$_\mathrm{w}$) of level sets.

Lemma 3.9 ([21]). *Let $B \subseteq \mathbf{Z}^V$.*
(i) *If B satisfies* (Q-EXC$_\mathrm{w}$)*, then $x(V) = y(V)$ for all $x, y \in \mathrm{dom}\, f$.*
(ii) (Q-EXC$_\mathrm{w}$) *is equivalent to the following property:*

> **(Q-EXC$_{\mathbf{w+}}$)** $\forall x, y \in B$, $x \neq y$, $\exists u \in \mathrm{supp}^+(x - y)$, $\exists v \in \mathrm{supp}^-(x - y)$ such that $x - \chi_u + \chi_v \in B$.

Theorem 3.10. *A function $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ satisfies* (QM$_\mathrm{w}$) *if and only if the level set* $\mathrm{L}(f, \alpha)$ *satisfies* (Q-EXC$_\mathrm{w}$) *for all $\alpha \in \mathbf{R} \cup \{+\infty\}$. In particular, if f satisfies* (QM$_\mathrm{w}$)*, then* $\mathrm{dom}\, f$ *and* $\arg\min f$ *satisfy* (Q-EXC$_\mathrm{w}$).

Proof. ["only if" part] Let $\alpha \in \mathbf{R} \cup \{+\infty\}$, and $x, y \in \mathrm{L}(f, \alpha)$ be vectors with $x \neq y$. Applying (QM$_\mathrm{w}$) to x and y, we have $\Delta f(x; v, u) \leq 0$ or $\Delta f(y; u, v) \leq 0$ for some $u \in \mathrm{supp}^+(x - y)$ and $v \in \mathrm{supp}^-(x - y)$. Therefore, we have $x - \chi_u + \chi_v \in \mathrm{L}(f, \alpha)$ or $y + \chi_u - \chi_v \in \mathrm{L}(f, \alpha)$.

["if" part] Let $x, y \in \mathrm{dom}\, f$ be distinct vectors, and assume $f(x) \geq f(y)$. By Lemma 3.9 (ii), the level set $\mathrm{L}(f, f(x))$ satisfies (Q-EXC$_{\mathrm{w+}}$), from which follows $x - \chi_u + \chi_v \in \mathrm{L}(f, f(x))$ for some $u \in \mathrm{supp}^+(x-y)$ and $v \in \mathrm{supp}^-(x-y)$, i.e., $f(x - \chi_u + \chi_v) \leq f(x)$ holds. □

Theorem 3.11. *Let* $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ *satisfy* (QM). *Then, the level set* $\mathrm{L}(f, \alpha)$ *satisfies* (Q-EXC) *for all* $\alpha \in \mathbf{R} \cup \{+\infty\}$. *In particular,* $\mathrm{dom}\, f$ *and* $\arg\min f$ *satisfy* (Q-EXC).

Proof. The proof is similar to that for the "only if" part of Theorem 3.10. □

Theorem 3.12. *Let* $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$. *Suppose that the level set* $\mathrm{L}(f, \alpha)$ *satisfies* (B-EXC) *for all* $\alpha \in \mathbf{R} \cup \{+\infty\}$. *Then,* f *satisfies* (QM).

Remark 3.13. The converse of the statement of Theorem 3.11 does not hold in general, as shown in the following example.

Let $V = \{a, b, c, d\}$, and $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ be a function given by

$$\begin{aligned} &\mathrm{dom}\, f = \{x \in \{0,1\}^V \mid x_a + x_b + x_c + x_d = 2\}, \\ &f(1,1,0,0) = 1, \quad f(1,0,1,0) = f(1,0,0,1) = 2, \\ &f(0,0,1,1) = 3, \quad f(0,1,0,1) = f(0,1,1,0) = 4. \end{aligned}$$

We can easily check that for any $\alpha \in \mathbf{R} \cup \{+\infty\}$ the level set $\mathrm{L}(f, \alpha)$ satisfies (Q-EXC). Let $x = (1,1,0,0)$, $y = (0,0,1,1)$, and $u = b \in \mathrm{supp}^+(x-y)$. Then, for any $v \in \mathrm{supp}^-(x-y) = \{c, d\}$ we have

$$2 = f(x - \chi_u + \chi_v) > f(x) = 1, \quad 4 = f(y + \chi_u - \chi_v) > f(y) = 3.$$

Hence, (QM) does not hold for f. Note that the level set $\mathrm{L}(f, 3)$ does not satisfy (B-EXC). □

Remark 3.14. A function does not necessarily satisfy (SSQM$_{\mathrm{w}}$) even if every level set satisfies (B-EXC), as shown in the following example. Let $f : \mathbf{Z}^2 \to \mathbf{R} \cup \{+\infty\}$ be a function given by

$$\mathrm{dom}\, f = \{(0,0), (1,-1), (2,-2)\}, \qquad f(0,0) = 0, \quad f(1,-1) = f(2,-2) = 1.$$

Every level set of f satisfies (B-EXC), but (SSQM$_{\mathrm{w}}$) does not holds for f. □

Theorem 3.15. *If* $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ *satisfies* (SSQM$_{\mathrm{w}}$), *then* $\arg\min f$ *satisfies* (B-EXC).

An M-convex function can be characterized by quasi M-convexity of level sets of functions perturbed by linear functions. Recall the definition of a function $f[p]$ in (2.1).

Theorem 3.16 ([21, Th. 1]). *Let* $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$. *Then,*

$$\begin{aligned} f \text{ satisfies (M-EXC)} &\iff \forall p \in \mathbf{R}^V, \forall \alpha \in \mathbf{R} \cup \{+\infty\}, \mathrm{L}(f[p], \alpha) \text{ satisfies (Q-EXC)} \\ &\iff \forall p \in \mathbf{R}^V, \forall \alpha \in \mathbf{R} \cup \{+\infty\}, \mathrm{L}(f[p], \alpha) \text{ satisfies (Q-EXC}_{\mathrm{w}}). \end{aligned}$$

Combining Theorems 3.10 and 3.16, we see the following:

Corollary 3.17. *Let* $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$. *Then,*

$$f \text{ satisfies (M-EXC)} \iff \forall p \in \mathbf{R}^V,\ f[p] \text{ satisfies (QM)} \iff \forall p \in \mathbf{R}^V,\ f[p] \text{ satisfies (QM}_{\mathrm{w}}).$$

3.3 Operations for Quasi M-convex Functions

The class of (semistrictly) quasi M-convex functions is closed under several fundamental operations.

Let $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$. For any subset $U \subseteq V$, define $f_U : \mathbf{Z}^U \to \mathbf{R} \cup \{+\infty\}$ by

$$f_U(y) = f(y, \mathbf{0}_{V \setminus U}) \qquad (y \in \mathbf{Z}^U),$$

where $\mathbf{0}_{V\setminus U} \in \mathbf{Z}^{V \setminus U}$ denotes the zero vector. For any functions $a : V \to \mathbf{Z} \cup \{-\infty\}$ and $b : V \to \mathbf{Z} \cup \{+\infty\}$, define $f_a^b : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ by

$$f_a^b(x) = \begin{cases} f(x) & (x \in [a, b]), \\ +\infty & (\text{otherwise}). \end{cases} \tag{3.5}$$

Theorem 3.18. *Let* $(*\mathrm{QM}_*)$ *denote one of* (QM), $(\mathrm{QM_w})$, (SSQM), *and* $(\mathrm{SSQM_w})$, *and* $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ *be a function with the property* $(*\mathrm{QM}_*)$.
(i) *For any* $a \in \mathbf{Z}^V$ *and* $\nu > 0$, *the functions* $\nu \cdot f(a - x)$ *and* $\nu \cdot f(a + x)$ *satisfy* $(*\mathrm{QM}_*)$ *as a function in* x.
(ii) *For any* $U \subseteq V$, *the function* $f_U : \mathbf{Z}^U \to \mathbf{R} \cup \{+\infty\}$ *satisfies* $(*\mathrm{QM}_*)$.
(iii) *For any* $a : V \to \mathbf{Z} \cup \{-\infty\}$ *and* $b : V \to \mathbf{Z} \cup \{+\infty\}$ *with* $a \le b$, *the function* $f_a^b : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ *satisfies* $(*\mathrm{QM}_*)$.
(iv) *Let* $f_i : \mathbf{Z}^{V_i} \to \mathbf{R}_{++} \cup \{+\infty\}$ $(i = 1, 2)$ *be functions with* $(*\mathrm{QM}_*)$. *Then, the function* $f : \mathbf{Z}^{V_1} \times \mathbf{Z}^{V_2} \to \mathbf{R}_{++} \cup \{+\infty\}$ *defined by*

$$f(x_1, x_2) = f_1(x_1) f_2(x_2) \quad ((x_1, x_2) \in \mathbf{Z}^{V_1} \times \mathbf{Z}^{V_2})$$

satisfies $(*\mathrm{QM}_*)$.

Proof. We prove (iv) only. We consider the case when $(*\mathrm{QM}_*) = (\mathrm{SSQM})$. Let $x = (x_1, x_2), y = (y_1, y_2) \in \operatorname{dom} f_1 \times \operatorname{dom} f_2$, and let $u \in \operatorname{supp}^+(x - y)$, where $u \in \operatorname{supp}^+(x_1 - y_1)$ w.l.o.g. Then, there exists $v \in \operatorname{supp}^-(x_1 - y_1)$ such that

$$\Delta f_1(x_1; v, u) \ge 0 \Longrightarrow \Delta f_1(y_1; u, v) \le 0, \quad \text{and} \quad \Delta f_1(y_1; u, v) \ge 0 \Longrightarrow \Delta f_1(x_1; v, u) \le 0.$$

This implies that

$$\Delta f(x; v, u) \ge 0 \Longrightarrow \Delta f(y; u, v) \le 0, \quad \text{and} \quad \Delta f(y; u, v) \ge 0 \Longrightarrow \Delta f(x; v, u) \le 0.$$

Hence, (SSQM) holds for f. □

Remark 3.19. The class of (semistrictly) quasi M-convex functions is not closed under addition; in particular, it is not closed under the addition of a linear function, as shown in the following example.

Let $\varphi : \mathbf{Z} \to \mathbf{Z}$ be a function such that

$$\varphi(\alpha) = \begin{cases} 3\alpha & (\alpha < 0), \\ \alpha & (\alpha \geq 0), \end{cases}$$

and define $f : \mathbf{Z}^2 \to \mathbf{R} \cup \{+\infty\}$ by (3.1). It is easy to see that f satisfies (SSQM) (and not (M-EXC)). We also define a linear function $g : \mathbf{Z}^2 \to \mathbf{R} \cup \{+\infty\}$ by

$$g(x_1, x_2) = \begin{cases} -2x_1 & (x_1 + x_2 = 0), \\ +\infty & (x_1 + x_2 \neq 0). \end{cases}$$

Then, we have

$$f(x_1, x_2) + g(x_1, x_2) = \begin{cases} +x_1 & (x_1 + x_2 = 0,\ x_1 < 0), \\ -x_1 & (x_1 + x_2 = 0,\ x_1 \geq 0), \\ +\infty & (x_1 + x_2 \neq 0), \end{cases}$$

which satisfies neither (SSQM) nor (QM$_{\mathrm{w}}$). □

Theorem 3.20. *For $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ and $\varphi : \mathbf{R} \to \mathbf{R} \cup \{+\infty\}$, define $\tilde{f} : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ by (3.2).*
(i) *If f satisfies* (QM) *(resp.* (QM$_{\mathrm{w}}$)*) and φ is nondecreasing, then $\tilde{f}$ satisfies* (QM) *(resp.* (QM$_{\mathrm{w}}$)*).*
(ii) *If f satisfies* (SSQM) *(resp.* (SSQM$_{\mathrm{w}}$)*) and φ is strictly increasing, then $\tilde{f}$ satisfies* (SSQM) *(resp.* (SSQM$_{\mathrm{w}}$)*).*

Remark 3.21. A quasi M-convex function $\tilde{f} : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ is not necessarily represented in the form (3.2) with an M-convex function $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ and a nondecreasing function $\varphi : \mathbf{R} \to \mathbf{R} \cup \{+\infty\}$. As an example, let us consider a function $\tilde{f} : \mathbf{Z}^3 \to \mathbf{R} \cup \{+\infty\}$ given by

$$\begin{aligned} &\operatorname{dom} \tilde{f} = \{(0,0,0), (1,0,-1), (2,0,-2), (2,1,-3)\}, \\ &\tilde{f}(x_1, x_2, x_3) = -2x_1 + x_2 \quad (x \in \operatorname{dom} \tilde{f}), \end{aligned}$$

which satisfies (SSQM). Suppose that $\tilde{f}(x) = \varphi(f(x))$ $(x \in \mathbf{Z}^V)$. Since φ is nondecreasing, we must have

$$f(2,0,-2) < f(2,1,-3) < f(1,0,-1) < f(0,0,0) < f(1,1,-2),$$

which implies

$$f(0,0,0) + f(2,1,-3) < f(1,0,-1) + f(1,1,-2).$$

Hence, (M-EXC) for f does not hold for $x = (0,0,0)$ and $y = (2,1,-3)$. □

Theorem 3.22. *Let $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ and $g : \mathbf{Z}^V \to \mathbf{R} \cup \{-\infty\}$ be functions such that $g(x) > 0$ for all $x \in \mathrm{dom}\, f$. If the function $f(\cdot) - \alpha g(\cdot)$ satisfies* (QM$_\mathrm{w}$) *for all $\alpha \in \mathbf{R} \cup \{+\infty\}$, then the function $r : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ given by*

$$r(x) = \begin{cases} f(x)/g(x) & (x \in \mathrm{dom}\, f), \\ +\infty & (x \notin \mathrm{dom}\, f), \end{cases}$$

also satisfies (QM$_\mathrm{w}$).

Proof. The proof is easy from Theorem 3.10. □

Remark 3.23. The following example shows that the statement of Theorem 3.22 cannot be strengthened by replacing (QM$_\mathrm{w}$) with (QM), even if f and g are linear functions.

Let $V = \{a, b, c, d\}$. Define a function $r : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ as

$$\mathrm{dom}\, r = \{x \in \{0,1\}^V \mid x_a + x_b + x_c + x_d = 2\}, \qquad r(x) = \frac{x_a + x_b + 3x_c + 3x_d}{2x_a + x_c + x_d} \quad (x \in \mathrm{dom}\, r).$$

Let $x = (1, 1, 0, 0)$, $y = (0, 0, 1, 1)$, and $u = b \in \mathrm{supp}^+(x - y)$. Then, for $v \in \mathrm{supp}^-(x - y) = \{c, d\}$ we have

$$4/3 = f(x - \chi_u + \chi_v) > f(x) = 1, \quad 4 = f(y + \chi_u - \chi_v) > f(y) = 3.$$

Therefore, (QM) does not hold for r. □

3.4 Characterization of Quasi M-convexity by Local Exchange Properties

An M-convex function is known to be characterized by a localized version of the property (M-EXC):

> **(M-EXC-loc)** $\forall x, y \in \mathrm{dom}\, f$ with $||x-y||_1 = 4$, $\forall u \in \mathrm{supp}^+(x-y)$, $\exists v \in \mathrm{supp}^-(x-y)$ such that
>
> $$f(x) + f(y) \geq f(x - \chi_u + \chi_v) + f(y + \chi_u - \chi_v).$$

Theorem 3.24 ([13, Th. 3.1], [21, Th. 2]). *Let $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ be a function such that* $\mathrm{dom}\, f$ *is a nonempty set with* (Q-EXC$_\mathrm{w}$). *Then,* (M-EXC) $\Longleftrightarrow$ (M-EXC-loc).

As a corollary, we also have a characterization of an M-convex set by a local exchange property:

> **(B-EXC-loc)** $\forall x, y \in \mathrm{dom}\, f$ with $||x-y||_1 = 4$, $\forall u \in \mathrm{supp}^+(x-y)$, $\exists v \in \mathrm{supp}^-(x-y)$ such that $x - \chi_u + \chi_v \in B$ and $y + \chi_u - \chi_v \in B$.

Theorem 3.25. *Let* $B \subseteq \mathbf{Z}^V$ *be a set with* (Q-EXC$_\mathrm{w}$). *Then,* (B-EXC) $\Longleftrightarrow$ (B-EXC-loc).

We show that semistrict quasi M-convexities can be characterized also by the localized version of (SSQM) and (SSQM$_\mathrm{w}$).

(SSQM-loc) $\forall x, y \in \mathrm{dom}\, f$ with $||x-y||_1 = 4$, $\forall u \in \mathrm{supp}^+(x-y)$, $\exists v \in \mathrm{supp}^-(x-y)$ such that

$$\text{(i) } \Delta f(x; v, u) \geq 0 \Longrightarrow \Delta f(y; u, v) \leq 0, \qquad \text{and}$$
$$\text{(ii) } \Delta f(y; u, v) \geq 0 \Longrightarrow \Delta f(x; v, u) \leq 0.$$

(SSQM$_\mathrm{w}$-loc) $\forall x, y \in \mathrm{dom}\, f$ with $||x-y||_1 = 4$, $\exists u \in \mathrm{supp}^+(x-y)$, $\exists v \in \mathrm{supp}^-(x-y)$ such that

$$\text{(i) } \Delta f(x; v, u) \geq 0 \Longrightarrow \Delta f(y; u, v) \leq 0, \qquad \text{and}$$
$$\text{(ii) } \Delta f(y; u, v) \geq 0 \Longrightarrow \Delta f(x; v, u) \leq 0.$$

Theorem 3.26. *Let* $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ *be a function such that* $\mathrm{dom}\, f$ *satisfies* (Q-EXC$_\mathrm{w}$). *Then,*
(i) (SSQM) $\Longleftrightarrow$ (SSQM-loc). (ii) (SSQM$_\mathrm{w}$) $\Longleftrightarrow$ (SSQM$_\mathrm{w}$-loc).

Proof. For both (i) and (ii), the "$\Longrightarrow$" parts are obvious.

["$\Longleftarrow$" part of (i)] Assume, to the contrary, that (SSQM) does not hold for some $x, y \in \mathrm{dom}\, f$ and $u_* \in \mathrm{supp}^+(x-y)$. We also assume that (x, y) minimizes the value $||x-y||_1$ of all such pairs. Note that $||x-y||_1 \geq 6$ and $x(V) = y(V)$ by Lemma 3.9 (i).

Claim 1: There exists $u_0 \in \mathrm{supp}^+(x-y)$ such that $y + \chi_{u_0} - \chi_v \in \mathrm{dom}\, f$ for some $v \in \mathrm{supp}^-(x-y)$. Moreover, if $x(u_*) - y(u_*) = 1$ then we can assume $u_0 \neq u_*$.

[Proof of Claim 1] By Lemma 3.9 (ii), $\mathrm{dom}\, f$ satisfies (Q-EXC$_{\mathrm{w}+}$). Applying (Q-EXC$_{\mathrm{w}+}$) to y and x, there exist $u_1 \in \mathrm{supp}^+(x-y)$ and $v_1 \in \mathrm{supp}^-(x-y)$ with $y_1 = y + \chi_{u_1} - \chi_{v_1} \in \mathrm{dom}\, f$. Hence, the former part of Claim 1 holds. In the following, we assume $x(u_*) - y(u_*) = 1$ and show the latter part of Claim 1. If $u_1 \neq u_*$ then we are done. Thus, we assume $u_1 = u_*$. Since $||x-y_1||_1 \geq 4$, we can again apply (Q-EXC$_{\mathrm{w}+}$) to y_1 and x to obtain $u_2 \in \mathrm{supp}^+(x-y_1) = \mathrm{supp}^+(x-y) \setminus \{u_*\}$ and $v_2 \in \mathrm{supp}^-(x-y_1) \subseteq \mathrm{supp}^-(x-y)$ with $y_2 = y_1 + \chi_{u_2} - \chi_{v_2} \in \mathrm{dom}\, f$. Then, we apply (SSQM-loc) to y_2, y, and $u_* \in \mathrm{supp}^+(y_2-y)$ to obtain some $v \in \mathrm{supp}^-(y_2-y) = \{v_1, v_2\}$ such that if $\Delta f(y; u_*, v) \geq 0$ then $\Delta f(y_2; v, u_*) \leq 0$. By the choice of x and y we have $\Delta f(y; u_*, v) \geq 0$, from which follow $\Delta f(y_2; v, u_*) \leq 0$. Hence, $y_2 + \chi_v - \chi_{u_*} = y + \chi_{u_2} - \chi_{v'} \in \mathrm{dom}\, f$ for some $v' \in \{v_1, v_2\}$. [End of Claim 1]

We can divide the set $\mathrm{supp}^-(x-y)$ into three sets $S^-_{>>}$, $S^-_{>=}$, and $S^-_{=>}$, where

$$S^-_{>>} = \{v \in \mathrm{supp}^-(x-y) \mid \Delta f(x; v, u_*) > 0, \Delta f(y; u_*, v) > 0\},$$
$$S^-_{>=} = \{v \in \mathrm{supp}^-(x-y) \mid \Delta f(x; v, u_*) > 0, \Delta f(y; u_*, v) = 0\},$$
$$S^-_{=>} = \{v \in \mathrm{supp}^-(x-y) \mid \Delta f(x; v, u_*) = 0, \Delta f(y; u_*, v) > 0\}.$$

Then, we define $v_0 \in \mathrm{supp}^-(x-y)$ as follows: if

$$\min\{f(y+\chi_{u_0}-\chi_v) \mid v \in S^-_{>=}\} < \min\{f(y+\chi_{u_0}-\chi_v) \mid v \in S^-_{>>} \cup S^-_{=>}\},$$

then let v_0 be any element in $\arg\min\{f(y+\chi_{u_0}-\chi_v) \mid v \in S^-_{>=}\}$, and otherwise let v_0 be any element in $\arg\min\{f(y+\chi_{u_0}-\chi_v) \mid v \in S^-_{>>} \cup S^-_{=>}\}$. Put $y' = y+\chi_{u_0}-\chi_{v_0}$. The, $y' \in \mathrm{dom}\, f$ by Claim 1.

Claim 2:

$$\Delta f(y'; u_*, v) \geq 0 \qquad (\forall v \in \mathrm{supp}^-(x-y')), \tag{3.6}$$

$$\Delta f(y'; u_*, v) > 0 \qquad (\forall v \in \mathrm{supp}^-(x-y') \cap S^-_{=>}). \tag{3.7}$$

[Proof of Claim 2] For $v \in \mathrm{supp}^-(x-y')$, put

$$y'' = y' + \chi_{u_*} - \chi_v = y + \chi_{u_0} + \chi_{u_*} - \chi_{v_0} - \chi_v.$$

We may assume $y'' \in \mathrm{dom}\, f$, since otherwise the claim holds obviously. Applying (SSQM-loc) to y'', y, and $u_* \in \mathrm{supp}^+(y''-y)$, we have

$$\Delta f(y''; v', u_*) \geq 0 \Longrightarrow \Delta f(y; u_*, v') \leq 0, \tag{3.8}$$

$$\Delta f(y; u_*, v') \geq 0 \Longrightarrow \Delta f(y''; v', u_*) \leq 0 \tag{3.9}$$

for some $v' \in \{v_0, v\}$. Since $\Delta f(y; u_*, v') \geq 0$, (3.9) implies that

$$f(y'') \geq f(y''+\chi_{v'}-\chi_{u_*}) = f(y+\chi_{u_0}-\chi_{v_0}-\chi_v+\chi_{v'}) \geq f(y+\chi_{u_0}-\chi_{v_0}) = f(y'). \tag{3.10}$$

This proves (3.6).

Next, we assume $v \in S^-_{=>}$. If $\Delta f(y; u_*, v') > 0$, then (3.8) implies that the first inequality in (3.10) holds with strict inequality, i.e., (3.7) holds. Hence, we assume

$$\Delta f(y; u_*, v') = 0, \tag{3.11}$$

which implies $v' = v_0 \in S^-_{>=}$ since $v \in S^-_{=>}$. Due to the choice of v_0, we have

$$f(y') < f(y+\chi_{u_0}-\chi_v). \tag{3.12}$$

By (3.11) and (3.9), we have

$$f(y'') \geq f(y''+\chi_{v_0}-\chi_{u_*}) = f(y+\chi_{u_0}-\chi_v). \tag{3.13}$$

From (3.12) and (3.13) follows (3.7). [End of Claim 2]

Since $u_* \in \mathrm{supp}^+(x-y')$ and $||x-y'||_1 < ||x-y||_1$, Claim 2 contradicts the choice of x and y. This concludes the proof.

["$\Longleftarrow$" part of (ii)] We show ($\mathrm{SSQM_w}$) for $x, y \in \mathrm{dom}\, f$ by induction on the value $||x-y||_1$. We may assume that $||x-y||_1 > 4$ and

$$\Delta f(x; v, u) \geq 0, \quad \Delta f(y; u, v) \geq 0 \qquad (\forall u \in \mathrm{supp}^+(x-y), \forall v \in \mathrm{supp}^-(x-y)), \tag{3.14}$$

since otherwise the claim holds immediately. We are to show $\Delta f(x; v, u) = \Delta f(y; u, v) = 0$ for some $u \in \mathrm{supp}^+(x-y)$ and $v \in \mathrm{supp}^-(x-y)$.

Claim 1: Let y' be any vector in $[x \wedge y, x \vee y] \cap \mathrm{dom}\, f$. Put $k = ||x-y'||_1$ and $x_0 = x$. For $i = 1, 2, \cdots, k$, we define $x_i \in \mathbf{Z}^V$ by

$$x_i \in \arg\min\{f(x_{i-1} - \chi_u + \chi_v) \mid u \in \mathrm{supp}^+(x_{i-1} - y'),\ v \in \mathrm{supp}^-(x_{i-1} - y')\}. \tag{3.15}$$

Then, we have $x_i \in [x \wedge y, x \vee y] \cap \mathrm{dom}\, f$ and $f(x_i) \geq f(x_{i-1})$ $(i = 1, \cdots, k)$. In particular, we have $f(y') \geq f(x)$.

[Proof of Claim 1] From ($\mathrm{Q\text{-}EXC_w}$) for $\mathrm{dom}\, f$ follows $x_i \in [x \wedge y, x \vee y] \cap \mathrm{dom}\, f$ $(i = 1, \cdots, k)$.

We show the inequality $f(x_{i+1}) \geq f(x_i)$ by induction on i. The inequality $f(x_1) \geq f(x_0)$ follows from (3.14) and the fact that $\mathrm{supp}^+(x_0 - y') \subseteq \mathrm{supp}^+(x-y)$, $\mathrm{supp}^-(x_0 - y') \subseteq \mathrm{supp}^-(x-y)$. We then suppose $i \geq 1$. Since $||x_{i-1} - x_{i+1}||_1 = 4$, we can apply ($\mathrm{SSQM_w}$-loc) to x_{i-1} and x_{i+1} to obtain some $u \in \mathrm{supp}^+(x_{i-1} - x_{i+1})$ and $v \in \mathrm{supp}^-(x_{i-1} - x_{i+1})$ such that if $\Delta f(x_{i-1}; v, u) \geq 0$ then $\Delta f(x_{i+1}; u, v) \leq 0$. By the inductive hypothesis and the choice of x_i, we have $\Delta f(x_{i-1}; v, u) \geq f(x_i) - f(x_{i-1}) \geq 0$. Hence, we have $f(x_{i+1}) \geq f(x_{i+1} + \chi_u - \chi_v) \geq f(x_i)$. [End of Claim 1]

In a similar way, we can show that $f(y') \geq f(y)$ $(\forall y' \in [x \wedge y, x \vee y])$. Hence, we have $f(x) = f(y)$.

We define a set $S_0 \subseteq \mathbf{Z}^V$ by

$$S_0 = \{x' \in [x \wedge y, x \vee y] \cap \mathrm{dom}\, f \mid f(x') = f(x) = f(y)\}.$$

Claim 2: S_0 satisfies (B-EXC).

[Proof of Claim 2] From Theorem 3.25 it suffices to show ($\mathrm{Q\text{-}EXC_w}$) and (B-EXC-loc) for S_0.

We first prove ($\mathrm{Q\text{-}EXC_w}$) for $\tilde{x}, \tilde{y} \in S_0$ with $||\tilde{x} - \tilde{y}||_1 < ||x-y||_1$. By the inductive hypothesis, we can apply ($\mathrm{SSQM_w}$) to $\tilde{x}$ and $\tilde{y}$ to obtain (a) $\Delta f(\tilde{x}; v, u) < 0$, (b) $\Delta f(\tilde{y}; u, v) < 0$, or (c) $\Delta f(\tilde{x}; v, u) = \Delta f(\tilde{y}; u, v) = 0$ for some $u \in \mathrm{supp}^+(\tilde{x} - \tilde{y})$ and $v \in \mathrm{supp}^-(\tilde{x} - \tilde{y})$. Since $\tilde{x} - \chi_u + \chi_v, \tilde{y} + \chi_u - \chi_v \in [x \wedge y, x \vee y]$, we have $\Delta f(\tilde{x}; v, u) \geq 0$ and $\Delta f(\tilde{y}; u, v) \geq 0$. Therefore (c) must hold, i.e., $\tilde{x} - \chi_u + \chi_v \in S_0$ and $\tilde{y} + \chi_u - \chi_v \in S_0$. This fact also yields (B-EXC-loc) for $\tilde{x}, \tilde{y} \in S_0$ with $||\tilde{x} - \tilde{y}||_1 = 4$.

We next prove ($\mathrm{Q\text{-}EXC_w}$) for $\tilde{x}, \tilde{y} \in S_0$ with $||\tilde{x} - \tilde{y}||_1 = ||x-y||_1$. Then, we have $\{\tilde{x}, \tilde{y}\} = \{x, y\}$. For $y' = y$ and $i = 0, 1, \cdots, ||x-y||_1$, define $x_i \in \mathbf{Z}^V$ by (3.15). By Claim 1, we have $x_i \in S_0$ for each $i = 0, 1, \cdots, k$. Hence, ($\mathrm{Q\text{-}EXC_w}$) holds for $x, y \in S_0$. [End of Claim 2]

Applying (B-EXC) to x, y, we have $x - \chi_u + \chi_v \in S_0$ and $y + \chi_u - \chi_v \in S_0$ for some $u \in \mathrm{supp}^+(x-y)$ and $v \in \mathrm{supp}^-(x-y)$. Hence follows $\Delta f(x; v, u) = \Delta f(y; u, v) = 0$. $\square$

Remark 3.27. The localized version of (QM) does not characterize (QM) in general. Let $f : \mathbf{Z}^2 \to \mathbf{R} \cup \{+\infty\}$ be a function such that

$$\mathrm{dom}\, f = \{(0,0),(1,-1),(2,-2),(3,-3)\}, \quad f(0,0) = f(3,-3) = 0,\ f(1,-1) = f(2,-2) = 1.$$

Then, $\mathrm{dom}\, f$ satisfies (Q-EXC$_\mathrm{w}$), and (QM) holds for any $x, y \in \mathrm{dom}\, f$ with $||x-y||_1 = 4$. However, (QM) does not hold for $x = (0,0)$ and $y = (3,-3)$. □

4 Minimization of Quasi M-convex Functions

In this section, we use the following weaker properties than (SSQM) and (SSQM$_\mathrm{w}$):

(SSQM$^{\neq}$) $\forall x, y \in \mathrm{dom}\, f$ with $f(x) \neq f(y)$, $\forall u \in \mathrm{supp}^+(x-y)$, $\exists v \in \mathrm{supp}^-(x-y)$ such that

$$\begin{aligned}&\text{(i) } \Delta f(x; v, u) \geq 0 \Longrightarrow \Delta f(y; u, v) \leq 0, \qquad \text{and}\\ &\text{(ii) } \Delta f(y; u, v) \geq 0 \Longrightarrow \Delta f(x; v, u) \leq 0.\end{aligned}$$

(SSQM$^{\neq}_\mathrm{w}$) $\forall x, y \in \mathrm{dom}\, f$ with $f(x) \neq f(y)$, $\exists u \in \mathrm{supp}^+(x-y)$, $\exists v \in \mathrm{supp}^-(x-y)$ such that

$$\begin{aligned}&\text{(i) } \Delta f(x; v, u) \geq 0 \Longrightarrow \Delta f(y; u, v) \leq 0, \qquad \text{and}\\ &\text{(ii) } \Delta f(y; u, v) \geq 0 \Longrightarrow \Delta f(x; v, u) \leq 0.\end{aligned}$$

These properties imply neither (SSQM) nor (SSQM$_\mathrm{w}$); an example is $f : \mathbf{Z}^2 \to \mathbf{R} \cup \{+\infty\}$ with $\mathrm{dom}\, f = \{(0,0),(1,-1),(2,-2)\}$ given by $f(0,0) = f(2,-2) = 0$ and $f(1,-1) = 1$. Note that f does not satisfy (QM$_\mathrm{w}$).

The property (SSQM$^{\neq}_\mathrm{w}$) is equivalent to each of the following two conditions:

$$\max\{f(x), f(y)\} > \min_{\substack{u \in \mathrm{supp}^+(x-y)\\ v \in \mathrm{supp}^-(x-y)}} \{f(x - \chi_u + \chi_v), f(y + \chi_u - \chi_v)\}$$

$$(\forall x, y \in \mathrm{dom}\, f \text{ with } f(x) \neq f(y)), \tag{4.1}$$

$$f(x) > \min_{\substack{u \in \mathrm{supp}^+(x-y)\\ v \in \mathrm{supp}^-(x-y)}} f(x - \chi_u + \chi_v) \qquad (\forall x, y \in \mathrm{dom}\, f \text{ with } f(x) > f(y)). \tag{4.2}$$

The condition (4.2) is also considered in in [18].

Theorem 4.1. *For $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$, (SSQM$^{\neq}_\mathrm{w}$) $\iff$ (4.1) $\iff$ (4.2).*

Proof. The proof is quite similar to that for Theorem 3.8. See Appendix for details. □

4.1 Properties of Minimizers of Quasi M-convex Functions

Global minimality of quasi M-convex functions is characterized by local minimality.

Theorem 4.2. *Let* $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ *and* $x \in \mathrm{dom}\, f$.
(i) *Assume* $(\mathrm{QM_w})$ *for* f. *Then,* $\Delta f(x; v, u) > 0$ $(\forall u, v \in V,\ u \neq v)$ $\iff$ $f(x) < f(y)$ $(\forall y \in \mathbf{Z}^V \setminus \{x\})$.
(ii) *Assume* $(\mathrm{SSQM}^{\neq}_{\mathrm{w}})$ *for* f. *Then,* $\Delta f(x; v, u) \geq 0$ $(\forall u, v \in V)$ $\iff$ $f(x) \leq f(y)$ $(\forall y \in \mathbf{Z}^V)$.

Proof. We show the "$\Longrightarrow$" part of (ii) only. The "$\Longleftarrow$" part of (ii) is easy to prove, and the proof of (i) can be done in a similar way as that of (ii) by using Theorem 3.8.

Assume, to the contrary, that there exists some $y \in \mathrm{dom}\, f$ such that $f(y) < f(x)$. By Theorem 4.1, f satisfies (4.2), which implies that there exist some $u' \in \mathrm{supp}^+(x - y)$ and $v' \in \mathrm{supp}^-(x - y)$ such that $\Delta f(x; v', u') < 0$, a contradiction to the assumption for x. □

If f satisfies $(\mathrm{SSQM}^{\neq})$, then any vector in $\mathrm{dom}\, f$ can be easily separated from some minimizer of f (cf. [20, Th. 2.2, Cor. 2.3]). This property will be used as a basis of the domain reduction method in Section 4.2.

Theorem 4.3. *Let* $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ *be a function with* $(\mathrm{SSQM}^{\neq})$. *Assume* $\arg\min f \neq \emptyset$.
(i) *For* $x \in \mathrm{dom}\, f$ *and* $v \in V$, *let* $u \in V$ *be such that* $f(x - \chi_u + \chi_v) = \min_{s \in V} f(x - \chi_s + \chi_v)$. *Then, there exists* $x_* \in \arg\min f$ *with* $x_*(u) \leq x(u) - 1 + \chi_v(u)$.
(ii) *For* $x \in \mathrm{dom}\, f$ *and* $u \in V$, *let* $v \in V$ *be such that* $f(x - \chi_u + \chi_v) = \min_{t \in V} f(x - \chi_u + \chi_t)$. *Then, there exists* $x_* \in \arg\min f$ *with* $x_*(v) \geq x(v) - \chi_u(v) + 1$.
(iii) *For* $x \in \mathrm{dom}\, f \setminus \arg\min f$, *let* $u, v \in V$ *be such that* $f(x - \chi_u + \chi_v) = \min_{s,t \in V} f(x - \chi_s + \chi_t)$. *Then, there exists* $x_* \in \arg\min f$ *with* $x_*(u) \leq x(u) - 1$ *and* $x_*(v) \geq x(v) + 1$.

Proof. (i): Put $x' = x - \chi_u + \chi_v$. We may assume $x' \notin \arg\min f$, since otherwise the claim holds immediately. Assume, to the contrary, that there is no $x \in \arg\min f$ with $x(u) \leq x'(u)$. Let $x_* \in \arg\min f$ minimize $x_*(u)$. Then, we have $x_*(u) > x'(u)$. Since $f(x_*) \neq f(x')$, we can apply $(\mathrm{SSQM}^{\neq})$ to x_*, x', and u to obtain some $w \in \mathrm{supp}^-(x_* - x')$ such that if $\Delta f(x_*; w, u) > 0$ then $\Delta f(x'; u, w) < 0$. Due to the choice of x_*, we have $\Delta f(x_*; w, u) > 0$. Hence, $f(x') > f(x' + \chi_u - \chi_w) = f(x - \chi_w + \chi_v)$ holds, a contradiction to the definition of $u \in V$.

(ii): The proof is similar to that for (i) and therefore omitted.

(iii): Put $x' = x - \chi_u + \chi_v$ $(\neq x)$. We may assume $x' \notin \arg\min f$, since otherwise the claim holds immediately. By (i), there exists some $x_* \in \arg\min f$ such that $x_*(u) \leq x(u) - 1$, and we may assume that x_* maximize $x_*(v)$ among all such vectors. To the contrary assume $x_*(v) < x'(v)$. Since $f(x_*) \neq f(x')$, we can apply $(\mathrm{SSQM}^{\neq})$ to x', x_*, and $v \in \mathrm{supp}^+(x' - x_*)$ to obtain some $w \in \mathrm{supp}^-(x' - x_*)$ satisfying at least one of the following:

$$\text{(a) } \Delta f(x'; w, v) < 0, \quad \text{(b) } \Delta f(x_*; v, w) < 0, \quad \text{(c) } \Delta f(x'; w, v) = \Delta f(x_*; v, w) = 0.$$

Due to the choice of $u, v \in V$, we have $\Delta f(x'; w, v) \geq 0$ since $x' - \chi_v + \chi_w = x - \chi_u + \chi_w$. We also have $\Delta f(x_*; v, w) \geq 0$ since $x_* \in \arg\min f$. Therefore, we have (c), which implies $x_* + \chi_v - \chi_w \in \arg\min f$, a contradiction to the choice of x_*. □

Remark 4.4. The following example shows that the statements in Theorem 4.3 do not hold even if a function satisfies the property ($\mathrm{SSQM_w}$) (and not (SSQM)).

Let $V = \{a, b, c, d, e, g\}$, and $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ be a function defined as follows:

$$\begin{aligned} \mathrm{dom}\, f = \{&(1,1,1,0,0,0), (0,1,1,1,0,0), (0,0,1,1,1,0), (0,0,0,1,1,1), \\ &(1,0,1,1,0,0), (1,0,0,1,0,1)\}, \end{aligned}$$
$$f(1,1,1,0,0,0) = 0,\ f(0,1,1,1,0,0) = 1,\ f(0,0,1,1,1,0) = 2,$$
$$f(0,0,0,1,1,1) = 3,\ f(1,0,1,1,0,0) = 4,\ f(1,0,0,1,0,1) = 5.$$

We can easily check that f satisfies ($\mathrm{SSQM_w}$). Put $x = (1,0,0,1,0,1)$, $v = e$, $u = a$, and $x' = x - \chi_u + \chi_v = (0,0,0,1,1,1)$. Then, we have $f(x') = \min_{s \in V} f(x - \chi_s + \chi_v)$. The unique minimizer $x_* = (1,1,1,0,0,0)$ of f, however, does not satisfy the inequality $x_*(u) \leq x'(u)$ since $x_*(u) = 1$ and $x'(u) = 0$. Hence, the statement (i) does not hold for f. We can show in the similar way that the statement (ii) does not hold for this f. □

We apply the scaling technique to the minimization of quasi M-convex functions in Section 4.2. Let $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ be a semistrictly quasi M-convex function. For $x_0 \in \mathrm{dom}\, f$ and $\alpha \in \mathbf{Z}_{++}$, we define $f_\alpha : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ by

$$f_\alpha(x) = f(x_0 + \alpha x) \qquad (x \in \mathbf{Z}^V). \tag{4.3}$$

Remark 4.5. The following example shows that a function f_α defined by (4.3) is not quasi M-convex in general, even if f is an M-convex function.

Define $B \subseteq \mathbf{Z}^4$ by

$$\begin{aligned} B = \{x \in \mathbf{Z}^4 \mid \textstyle\sum_{i=1}^4 x_i = 4,\ 0 \leq x_i \leq 2\ (i = 1,2,3,4)\} \\ \setminus \{(2,0,2,0), (0,2,0,2), (0,2,2,0), (2,0,0,2)\} \end{aligned}$$

and $f : \mathbf{Z}^4 \to \mathbf{R} \cup \{+\infty\}$ by $f = \delta_B$. Since B is an M-convex set, f is an M-convex function. For $x_0 = \mathbf{0}$ and $\alpha = 2$, define $f_\alpha : \mathbf{Z}^4 \to \mathbf{R} \cup \{+\infty\}$ by (4.3). Then, f_α is the indicator function of the set $\{(1,1,0,0), (0,0,1,1)\}$, which does not satisfy ($\mathrm{Q\text{-}EXC_w}$). Hence, f_α is not quasi M-convex. □

Let $y \in \mathrm{dom}\, f_\alpha$ be a local minimum of f_α, i.e., $x_\alpha = x_0 + \alpha y$ satisfies

$$f(x_\alpha) \leq f(x_\alpha + \alpha(\chi_v - \chi_u)) \qquad (\forall u, v \in V). \tag{4.4}$$

The following theorem shows that a global minimum of a semistrictly M-convex function exists in the neighborhood of x_α. This generalizes [8, Th. 4.1].

Theorem 4.6. *Let $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ be a function with* (SSQM$^{\neq}$)*, and $\alpha \in \mathbf{Z}_{++}$. Suppose that $x_\alpha \in \mathrm{dom}\, f$ satisfies (4.4). Then,* $\arg\min f \neq \emptyset$ *and there exists some $x_* \in \arg\min f$ such that*

$$|x_\alpha(v) - x_*(v)| \leq (n-1)(\alpha - 1) \qquad (v \in V). \tag{4.5}$$

Proof. It suffices to show that for any $\gamma \in \mathbf{R}$ with $\gamma > \inf f$, there exists some $x_* \in \mathrm{dom}\, f$ satisfying $f(x_*) \leq \gamma$ and (4.5). Let $x_* \in \mathrm{dom}\, f$ satisfy $f(x_*) \leq \gamma$, and suppose that x_* minimizes the value $||x_* - x_\alpha||_1$ among all such vectors. In the following, we fix $v \in V$ and prove $x_\alpha(v) - x_*(v) \leq (n-1)(\alpha-1)$. The inequality $x_*(v) - x_\alpha(v) \leq (n-1)(\alpha-1)$ can be shown similarly.

We may assume $x_\alpha(v) > x_*(v)$. Put

$$S = \left\{ x_\alpha - \lambda\chi_v + \sum_{w \in \mathrm{supp}^-(x_\alpha - x_*)} \mu_w \chi_w \in \mathrm{dom}\, f \;\middle|\; \begin{array}{c} 0 \leq \lambda \leq x_\alpha(v) - x_*(v), \\ 0 \leq \mu_w \leq x_*(w) - x_\alpha(w) \ (w \in \mathrm{supp}^-(x_\alpha - x_*)) \end{array} \right\}$$

Claim 1: For $y \in \arg\min\{f(y') \mid y' \in S\}$, we have $y(v) = x_*(v)$.

[Proof of Claim 1] Assume, to the contrary, that $y(v) > x_*(v)$. Since $||y - x_\alpha||_1 < ||x_* - x_\alpha||_1$, we have $f(y) > f(x_*)$. By (SSQM$^{\neq}$) applied to y, x_*, and $v \in \mathrm{supp}^+(y - x_*) \subseteq \mathrm{supp}^+(x_\alpha - x_*)$, we have some $w \in \mathrm{supp}^-(y - x_*) \subseteq \mathrm{supp}^-(x_\alpha - x_*)$ such that if $\Delta f(x_*; v, w) > 0$ then $\Delta f(y; w, v) < 0$. By the choice of x_*, we have $\Delta f(x_*; v, w) > 0$ since $||(x_* + \chi_v - \chi_w) - x_\alpha||_1 < ||x_* - x_\alpha||_1$. Therefore, $f(y - \chi_v + \chi_w) < f(y)$, which is a contradiction since $y - \chi_v + \chi_w \in S$.

[End of Claim 1]

Let $\tilde{y} = x_\alpha - \tilde{\lambda}\chi_v + \sum\{\tilde{\mu}_w \chi_w \mid w \in \mathrm{supp}^-(x_\alpha - x_*)\} \in \arg\min\{f(y') \mid y' \in S\}$.

Claim 2: For any $w \in \mathrm{supp}^-(x_\alpha - x_*)$ with $\tilde{\mu}_w > 0$ and $\mu \in [0, \tilde{\mu}_w - 1]$, we have

$$x_\alpha - \mu(\chi_v - \chi_w) \in \mathrm{dom}\, f, \qquad f(x_\alpha - (\mu+1)(\chi_v - \chi_w)) < f(x_\alpha - \mu(\chi_v - \chi_w)). \tag{4.6}$$

[Proof of Claim 2] For $\mu \in [0, \tilde{\mu}_w - 1]$, put $x' = x_\alpha - \mu(\chi_v - \chi_w)$ and suppose $x' \in \mathrm{dom}\, f$. Note that $x' \in S$ and $x'(v) > x_*(v)$. Therefore, Claim 1 yields $f(x') > f(\tilde{y})$. Since $\mathrm{supp}^-(\tilde{y} - x') = \{v\}$, (SSQM$^{\neq}$) applied to $\tilde{y}$, x', and $w \in \mathrm{supp}^+(\tilde{y} - x')$ implies that if $\Delta f(\tilde{y}; v, w) > 0$ then $\Delta f(x'; w, v) < 0$. By Claim 1, we have $\Delta f(\tilde{y}; v, w) > 0$, from which (4.6) follows.

[End of Claim 2]

Claim 2 and (4.4) imply $\tilde{\mu}_w \leq \alpha - 1$ for $w \in \mathrm{supp}^-(x_\alpha - x_*)$. Thus,

$$x_\alpha(v) - x_*(v) = x_\alpha(v) - \tilde{y}(v) = \sum_{w \in \mathrm{supp}^-(x_\alpha - x_*)} \tilde{\mu}_w \leq (n-1)(\alpha-1),$$

where the second equality is by Lemma 3.9 (i). □

4.2 Algorithms

Let $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ be a function such that $\mathrm{dom}\, f$ is a nonempty bounded set, and put

$$L = \max\{|x(v) - y(v)| \mid x, y \in \mathrm{dom}\, f,\ v \in V\}.$$

Assume (SSQM$^{\neq}_{\mathrm{w}}$) for f. Then, Theorem 4.2 immediately leads to the following algorithm.

Algorithm DESCENT_M

Step 0: Let x be any vector in $\mathrm{dom}\, f$.

Step 1: If $f(x) = \min_{s,t \in V} f(x - \chi_s + \chi_t)$ then stop. [x is a minimizer of f.]

Step 2: Find $u, v \in V$ with $f(x - \chi_u + \chi_v) < f(x)$.

Step 3: Set $x := x - \chi_u + \chi_v$. Go to Step 1. □

Algorithm DESCENT_M terminates in at most $|\mathrm{dom}\, f| \leq (L+1)^{n-1}$ iterations since it generates a distinct x in each iteration.

To the end of this section we assume (SSQM$^{\neq}$) for f. Based on Theorem 4.6, we apply the scaling technique to Algorithm DESCENT_M to obtain a faster algorithm.

Algorithm SCALING_DESCENT_M

Step 0: Let x be any vector in $\mathrm{dom}\, f$. Put $\alpha := 2^{\lceil \log_2 L \rceil}$, $B := \mathrm{dom}\, f$.

Step 1:

Step 1-1: If $f(x) = \min\{f(x - \alpha(\chi_s - \chi_t)) \mid s, t \in V,\ x - \alpha(\chi_s - \chi_t) \in B\}$, then go to Step 2.

Step 1-2: Find $u, v \in V$ with $x - \alpha(\chi_u - \chi_v) \in B$ satisfying $f(x - \alpha(\chi_u - \chi_v)) < f(x)$.

Step 1-3: Set $x := x - \alpha(\chi_u - \chi_v)$. Go to Step 1-1.

Step 2: If $\alpha = 1$ then stop. [x is a minimizer of f.]

Step 3: Put $B := B \cap \{y \in \mathbf{Z}^V \mid |y(v) - x(v)| \leq (n-1)(\alpha - 1)\ (v \in V)\}$ and $\alpha := \alpha/2$. Go to Step 1. □

The number of scaling phases is $\lceil \log_2 L \rceil$, and each scaling phase terminates in $(4n)^{n-1}$ iterations. Therefore, Algorithm SCALING_DESCENT_M runs in $(4n)^{n-1} \lceil \log_2 L \rceil$ iterations.

We then propose another elaboration of Algorithm DESCENT_M. Note that the algorithm STEEPEST_DESCENT_M reduces the set B iteratively in Step 3 by exploiting Theorem 4.3 (iii).

Algorithm STEEPEST_DESCENT_M

Step 0: Let x be any vector in $\mathrm{dom}\, f$. Set $B := \mathrm{dom}\, f$.

Step 1: If $f(x) = \min_{s,t \in V} f(x - \chi_s + \chi_t)$ then stop. [x is a minimizer of f.]

Step 2: Find $u, v \in V$ with $x - \chi_u + \chi_v \in B$ satisfying

$$f(x - \chi_u + \chi_v) = \min\{f(x - \chi_s + \chi_t) \mid s, t \in V,\ x - \chi_s + \chi_t \in B\}. \tag{4.7}$$

Step 3: Set $x := x - \chi_u + \chi_v$ and

$$B := B \cap \{y \in \mathbf{Z}^V \mid y(u) \leq x(u) - 1,\ y(v) \geq x(v) + 1\}. \tag{4.8}$$

Go to Step 1. □

By Theorem 4.3 (iii), the set B always contains a minimizer of f. Hence, STEEPEST_DESCENT_M finds a minimizer of f. To analyze the number of iterations, we consider the value $\sum_{w\in V}\{u_B(w)-l_B(w)\}$, where $l_B(w)=\min_{y\in B} y(w)$ and $u_B(w)=\max_{y\in B} y(w)$ $(w\in V)$. This value is bounded by nL and decreases at least by two in each iteration. Therefore, STEEPEST_DESCENT_M terminates in $\mathrm{O}(nL)$ iterations. In particular, if $\mathrm{dom}\, f\subseteq\{0,1\}^V$ then the number of iterations is $\mathrm{O}(n^2)$.

It is shown in [20] that the minimization of an M-convex function can be done in polynomial time by the domain reduction method explained below. We show that the domain reduction method also works for the minimization of a function with (SSQM$^{\neq}$) if its effective domain is a bounded M-convex set.

Given a bounded M-convex set $B\subseteq \mathbf{Z}^V$, we define the set $N_B\subseteq B$ by $N_B=\{y\in B \mid l'_B\leq y\leq u'_B\}$, where

$$l'_B(w)=\left\lfloor\left(1-\frac{1}{n}\right)l_B(w)+\frac{1}{n}u_B(w)\right\rfloor,\quad u'_B(w)=\left\lceil\frac{1}{n}l_B(w)+\left(1-\frac{1}{n}\right)u_B(w)\right\rceil\quad (w\in V).$$

Lemma 4.7 ([20, Th. 2.4]). *N_B is a (nonempty) M-convex set.*

The next algorithm maintains a set B ($\subseteq \mathrm{dom}\, f$) which is an M-convex set containing a minimizer of f. It reduces B iteratively by exploiting Theorem 4.3 (iii) and finally finds a minimizer.

Algorithm DOMAIN_REDUCTION
Step 0: Set $B:=\mathrm{dom}\, f$.
Step 1: Find a vector $x\in N_B$.
Step 2: If $f(x)=\min\limits_{s,t\in V} f(x-\chi_s+\chi_t)$ then stop. [x is a minimizer of f.]
Step 3: Find $u,v\in V$ with $x-\chi_u+\chi_v\in B$ satisfying (4.7).
Step 4: Set B by (4.8). Go to Step 1. □

We analyze the number of iterations of DOMAIN_REDUCTION. Denote by B_i the set B in the i-th iteration, and let $l_i(w)=l_{B_i}(w)$, $u_i(w)=u_{B_i}(w)$ $(w\in V)$. It is clear that $u_i(w)-l_i(w)$ is nonincreasing w.r.t. i. Furthermore, we have the following property:

Lemma 4.8 ([20, Lemma 3.1]). *$u_{i+1}(w)-l_{i+1}(w)<(1-1/n)\{u_i(w)-l_i(w)\}$ for $w\in\{u,v\}$, where $u,v\in V$ are the elements found in Step 3.*

This lemma implies that Algorithm DOMAIN_REDUCTION terminates in $\mathrm{O}(n^2\log L)$ iterations.

We now consider the time complexity of each step. Steps 2, 3, and 4 can be done in $\mathrm{O}(n^2)$ time. In Step 1, we use the exchange capacity to compute the values $l_B(w)$ and $u_B(w)$ and to find a vector in N_B. For any $w\in V$, the values $l_B(w)$ and $u_B(w)$ can be computed by evaluating

Table 2: Possible sign patterns of $g(p \wedge q) - g(p)$ and $g(p \vee q) - g(q)$ in submodular inequality

$g(p \wedge q) - g(p) \setminus g(p \vee q) - g(q)$	$-$	0	$+$
$-$	○	○	○
0	○	○	×
$+$	○	×	×

○ $\cdots$ possible, × $\cdots$ impossible

the exchange capacity at most n times, provided that a vector in B is given [6, Th. 3.27]. A vector in N_B can be found by evaluating the exchange capacity at most n^2 times, provided that a vector in B is given [20, Th. 2.5]. The exchange capacity can be computed in $\mathrm{O}(\log L)$ time by binary search. Hence, Step 1 requires $\mathrm{O}(n^2 \log L)$ time.

Theorem 4.9. *Suppose that* $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ *satisfies* (SSQM$^{\neq}$) *and that* $\mathrm{dom}\, f$ *is a bounded M-convex set. If a vector in* $\mathrm{dom}\, f$ *is given, Algorithm* DOMAIN_REDUCTION *finds a minimizer of* f *in* $\mathrm{O}(n^4(\log L)^2)$ *time.*

5 Quasi L-convex and Submodular Functions

5.1 Definition of Quasi L-convex and Submodular Functions

To extend the concept of L-convexity to quasi L-convexity, we relax the submodularity condition (SBM) while keeping in mind the possible sign patterns of the values $g(p \wedge q) - g(p)$ and $g(p \vee q) - g(q)$. Table 2 shows the possible sign patterns of those values.

Let $g : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ be a function. We call g *quasi-submodular* if it satisfies (QSB):

(QSB) For all $p, q \in \mathbf{Z}^V$ we have $g(p \wedge q) \leq g(p)$ or $g(p \vee q) \leq g(q)$,

and call g *quasi L-convex* if $\mathrm{dom}\, g \neq \emptyset$ and it satisfies (QSB) and (TRF). Since p and q are interchangeable, (QSB) implies $g(p \wedge q) \leq g(q)$ or $g(p \vee q) \leq g(p)$. Similarly, we call g *semistrictly quasi-submodular* if it satisfies the following property:

(SSQSB) For all $p, q \in \mathbf{Z}^V$ we have both (i) and (ii):
(i) $g(p \vee q) \geq g(q) \Longrightarrow g(p \wedge q) \leq g(p)$, and (ii) $g(p \wedge q) \geq g(p) \Longrightarrow g(p \vee q) \leq g(q)$,

and call g *semistrictly quasi L-convex* if $\mathrm{dom}\, g \neq \emptyset$ and it satisfies (SSQSB) and (TRF).

We also consider weaker properties than (QSB) and (SSQSB) by keeping in mind the possible sign patterns of the four values $g(p \wedge q) - g(p)$, $g(p \wedge q) - g(q)$, $g(p \vee q) - g(p)$, and $g(p \vee q) - g(q)$.

($\mathbf{QSB_w}$) For any $p, q \in \mathrm{dom}\, g$, we have $\max\{g(p), g(q)\} \geq \min\{g(p\wedge q), g(p\vee q)\}$.

($\mathbf{SSQSB_w}$) For any $p, q \in \mathrm{dom}\, g$, we have either of (i) and (ii):
(i) $\max\{g(p), g(q)\} > \min\{g(p\wedge q), g(p\vee q)\}$, (ii) $g(p) = g(q) = g(p\wedge q) = g(p\vee q)$.

The property ($\mathrm{SSQSB_w}$) says that either (i) at least one of the values $g(p\wedge q)-g(p)$, $g(p\wedge q)-g(q)$, $g(p\vee q)-g(p)$, and $g(p\vee q)-g(q)$ is negative or (ii) all the four values are equal to zero. Similarly, ($\mathrm{QSB_w}$) says that at least one of the four values is nonpositive.

The set version of quasi-submodularity can be obtained by translating the property (QSB) for the indicator function $\delta_D : \mathbf{Z}^V \to \{0, +\infty\}$ of a set $D \subseteq \mathbf{Z}^V$ in terms of D.

(QDL) $p, q \in D \Longrightarrow p \wedge q \in D$ or $p \vee q \in D$.

The following properties for $D \subseteq \mathbf{Z}^V$ can be shown easily:

- (QDL) for $D \iff$ (QSB) for $\delta_D \iff$ ($\mathrm{QSB_w}$)for δ_D,
- (DL) for $D \iff$ (SSQSB) for $\delta_D \iff$ ($\mathrm{SSQSB_w}$) for δ_D,
- (TRS) for $D \iff$ (TRF) for δ_D.

We show some examples of quasi L-convex/submodular functions below.

Example 5.1. Let $\varphi : \mathbf{Z} \to \mathbf{R}\cup\{+\infty\}$. We define $g : \mathbf{Z}^2 \to \mathbf{R}\cup\{+\infty\}$ by $g(p_1, p_2) = \varphi(p_1-p_2)$ ($(p_1, p_2) \in \mathbf{Z}^2$). Then, g satisfies (TRF) with $r = 0$. Moreover, g satisfies (QSB) (or ($\mathrm{QSB_w}$)) if and only if φ is quasiconvex, and g satisfies (SSQSB) (or ($\mathrm{SSQSB_w}$)) if and only if φ is semistrictly quasiconvex.

Proof. [($\mathrm{QSB_w}$) for $g \Longrightarrow$ quasiconvexity for φ] For any $\alpha_1, \alpha_2 \in \mathbf{Z}$ with $\alpha_1 < \alpha_2$, we have

$$\begin{aligned} \max\{\varphi(\alpha_1), \varphi(\alpha_2)\} &= \max\{g(\alpha_1, 0), g(\alpha_2 - 1, -1)\} \\ &\geq \min\{g(\alpha_1, -1), g(\alpha_2 - 1, 0)\} = \min\{\varphi(\alpha_1 + 1), \varphi(\alpha_2 - 1)\} \end{aligned}$$

by ($\mathrm{QSB_w}$). This implies the quasiconvexity of φ by Theorem 2.2 (i).

[quasiconvexity for $\varphi \Longrightarrow$ (QSB) for g] Let $p, q \in \mathbf{Z}^2$, and we may assume $p_1 > q_1$ and $p_2 < q_2$. Since $q_1 - q_2 < p_1 - q_2 < p_1 - p_2$, we have

$$\max\{g(p_1, p_2), g(q_1, q_2)\} = \max\{\varphi(p_1 - p_2), \varphi(q_1 - q_2)\} \geq \varphi(p_1 - q_2) = g(p_1, q_2).$$

We can prove the statement "g satisfies($\mathrm{SSQSB_w}$) $\iff$ φ is semistrictly quasiconvex" in the similar way by using Theorem 2.2 (ii). □

Example 5.2. Let $g : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ be a submodular function, and $\varphi : \mathbf{R} \to \mathbf{R} \cup \{+\infty\}$ be a nondecreasing function. We define the function $\tilde{g} : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ by

$$\tilde{g}(p) = \begin{cases} \varphi(g(p)) & (p \in \mathrm{dom}\, g), \\ +\infty & (p \notin \mathrm{dom}\, g). \end{cases} \tag{5.1}$$

Then, $\tilde{g}$ satisfies (QSB). Furthermore, if φ is strictly increasing, then $\tilde{g}$ satisfies (SSQSB). Note that if g satisfies (TRF) then $\tilde{g}$ also does. $\square$

Example 5.3. Let $D \subseteq \mathbf{Z}^V$ satisfy (DL), and $x \in \mathbf{R}^V$, $\alpha \in \mathbf{R}$. Then, the set $S = \{p \in D \mid \langle p, x\rangle \leq \alpha\}$ satisfies (QDL). Moreover, the function $g : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ with $\mathrm{dom}\, g = S$ defined by $g(p) = \langle p, x\rangle$ $(p \in S)$ satisfies (SSQSB) and (TRF) with $r = p(V)$. These properties are obvious from the equation $\langle p, x\rangle + \langle q, x\rangle = \langle p \wedge q, x\rangle + \langle p \vee q, x\rangle$. $\square$

Remark 5.4. The concept of (semistrict) quasi submodularity/L-convexity can be naturally extended to functions $g : S \to T$ with $S \subseteq \mathbf{Z}^V$ and a totally ordered set T with total order $\preceq$. For example, the property (SSQSB) is rewritten for such functions as follows:

> For any $p, q \in S$, we have both (i) and (ii):
> (i) if $p \vee q \notin S$, or $p \vee q \in S$ and $g(p \vee q) \succeq g(q)$, then $p \wedge q \in S$ and $g(p \wedge q) \preceq g(p)$, and
> (ii) if $p \wedge q \notin S$, or $p \wedge q \in S$ and $g(p \wedge q) \succeq g(p)$, then $p \vee q \in S$ and $g(p \vee q) \preceq g(q)$.

It is easy to see that the properties of (semistrictly) quasi submodular/L-convex functions shown in this paper still hold true. For simplicity and convenience, however, we assume in this paper that the codomain of a function is $\mathbf{R} \cup \{+\infty\}$. $\square$

Example 5.5. Suppose that $V = \{0, 1, \cdots, n\}$ $(n \geq 1)$. Let $a : V \setminus \{0\} \to \mathbf{Z} \cup \{-\infty\}$, $b : V \setminus \{0\} \to \mathbf{Z} \cup \{+\infty\}$ satisfy $a(v) \leq b(v)$ $(v \in V \setminus \{0\})$. For $i \in V \setminus \{0\}$, let $f_i : [a(i), b(i)] \to \mathbf{R}$ be a semistrictly quasiconvex function. We define $g : [a, b] \to \mathbf{R}^{V \setminus \{0\}}$ by

$$g(p) = (g_i(p(i) - p(0)) \mid i \in V \setminus \{0\}) \quad (x \in [a, b]),$$

where the total order $\preceq$ on the codomain $\mathbf{R}^{V \setminus \{0\}}$ of g is given by the lexicographic order. Then, g satisfies (TRF) with $r = 0$ and (SSQSB) in the extended sense (see Remark 5.4).

Proof. We show (SSQSB) for g only. Let $p, q \in [a, b]$. Then, $p \wedge q, p \vee q \in [a, b]$, and at least one of (a), (b) or (c) holds for each $i \in V \setminus \{0\}$ (see Example 5.1):

> (a) $g_i(p(i) \wedge q(i) - p(0) \wedge q(0)) < g_i(p(i) - p(0))$,
> (b) $g_i(p(i) \vee q(i) - p(0) \vee q(0)) < g_i(q(i) - q(0))$,
> (c) $g_i(p(i) \wedge q(i) - p(0) \wedge q(0)) = g_i(p(i) - p(0))$ and $g_i(p(i) \vee q(i) - p(0) \vee q(0)) = g_i(q(i) - q(0))$.

If (c) holds for all $i \in V \setminus \{0\}$, then we have $g(p \vee q) = g(q)$ and $g(p \wedge q) = g(p)$, implying (SSQSB). Otherwise, let $i_* \in V \setminus \{0\}$ be the minimum element satisfying (a) or (b). Then, we have $g(p \vee q) \prec g(q)$ or $g(p \wedge q) \prec g(p)$ since (c) holds for all $i \in \{1, 2, \cdots, i_* - 1\}$. $\square$

The relationship among various quasi-submodularity is summarized as follows.

Theorem 5.6. *For a function $g : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$, we have*

$$\begin{array}{ccccc} \text{(SBM)} & \Longrightarrow & \text{(SSQSB)} & \Longrightarrow & \text{(QSB)} \\ & & \Downarrow & & \Downarrow \\ & & (\text{SSQSB}_{\text{w}}) & \Longrightarrow & (\text{QSB}_{\text{w}}). \end{array}$$

Remark 5.7. It is easy to see that (DL) $\Longrightarrow$ (QDL), but the converse does not hold in general, even under the condition (TRS). For example, the set

$$\{(p_1+\lambda, p_2+\lambda, \lambda) \in \mathbf{Z}^3 \mid \lambda \in \mathbf{Z},\ (p_1,p_2) \text{ is either } (0,0),(1,0), \text{ or } (0,1)\}$$

satisfies (QDL) and (TRS) and not (DL). This fact shows that the implication "(QSB) $\Longrightarrow$ (SSQSB$_{\text{w}}$)" does not hold necessarily, even under the condition (TRF).

We present some examples to show that implications not mentioned in Theorem 5.6 do not hold in general, even if functions are assumed to satisfy (TRF). For $i = 1, 2$, let $g_i : \mathbf{Z}^2 \to \mathbf{R} \cup \{+\infty\}$ be a function with $\mathrm{dom}\, g_i = \{0,1\}^2$ such that

$$g_i(0,0) = 1, \quad g_i(1,0) = 2, \quad g_i(1,1) = 4 \quad (i = 1,2), \quad g_1(0,1) = 0, \quad g_2(0,1) = 2,$$

and define $\tilde{g}_i : \mathbf{Z}^3 \to \mathbf{R} \cup \{+\infty\}$ by $\tilde{g}_i(p_1,p_2,p_3) = g_i(p_1 - p_3, p_2 - p_3)$ $((p_1,p_2,p_3) \in \mathbf{Z}^3)$. Then, $\tilde{g}_1$ satisfies (SSQSB$_{\text{w}}$) and (TRF) and not (QSB), and $\tilde{g}_2$ satisfies (SSQSB) and (TRF) and not (SBM). □

Due to the definitions of quasi L-convexity/submodularity, most of the properties of quasi-submodular functions can be naturally restated in terms of quasi L-convex functions, and vice versa. In the following sections, we state properties mainly in terms of quasi-submodular functions and omit those for quasi L-convex functions whenever the restatements are immediate.

5.2 Level Sets of Quasi L-convex and Submodular Functions

We show various properties of level sets of quasi L-convex/submodular functions.

The following two theorems claim that level sets of quasi-submodular functions have nice properties such as (DL) and (QDL). Furthermore, the weaker version of quasi-submodularity (QSB$_{\text{w}}$) for functions can be characterized by the property (QDL) of level sets.

Theorem 5.8. *A function $g : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ satisfies* (QSB$_{\text{w}}$) *if and only if the level set* $\mathrm{L}(g,\alpha)$ *satisfies* (QDL) *for every $\alpha \in \mathbf{R} \cup \{+\infty\}$. In particular, if g satisfies* (QSB$_{\text{w}}$)*, then* $\mathrm{dom}\, g$ *and* $\arg\min g$ *satisfy* (QDL).

Proof. We show the "if" part only. Let $p, q \in \mathrm{dom}\, g$, and put $\alpha = \max\{g(p), g(q)\}$. Since the level set $\mathrm{L}(g,\alpha)$ satisfies (QDL), we have either $p \wedge q \in \mathrm{L}(g,\alpha)$ or $p \vee q \in \mathrm{L}(g,\alpha)$, implying $\max\{g(p), g(q)\} \geq \min\{g(p \wedge q), g(p \vee q)\}$. □

Theorem 5.9. *Let $g : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$, and suppose that the level set* $\mathrm{L}(g, \alpha)$ *satisfies* (DL) *for every $\alpha \in \mathbf{R} \cup \{+\infty\}$. Then, g satisfies* (QSB).

Proof. Let $p, q \in \mathrm{dom}\, g$ and assume $g(p) \geq g(q)$. Since the level set $\mathrm{L}(g, g(p))$ satisfies (DL) and contains p and q, we have $p \wedge q, p \vee q \in \mathrm{L}(g, g(p))$, implying $g(p \wedge q) \leq g(p)$. □

Remark 5.10. A function does not necessarily satisfy ($\mathrm{SSQSB_w}$) even if every level set satisfies (DL), as shown in the following example. Let $g : \mathbf{Z}^2 \to \mathbf{R} \cup \{+\infty\}$ be a function given by

$$\mathrm{dom}\, g = \{p \in \mathbf{Z}^V \mid p_1 - p_2 \text{ is either of 0, 1, and 2}\},$$
$$g(p_1, p_2) = 0 \text{ if } p_1 - p_2 = 0, \qquad g(p_1, p_2) = 0 \text{ if } p_1 - p_2 \in \{1, 2\}.$$

Every level set of g satisfies (DL), but ($\mathrm{SSQSB_w}$) does not hold for g. □

Theorem 5.11. *If $g : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ satisfies* ($\mathrm{SSQSB_w}$), *then* $\arg\min g$ *satisfies* (DL).

A submodular function over integer lattice can be characterized by using level sets of functions perturbed by linear functions. Recall the definition of the function $g[x] : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ in (2.1).

Theorem 5.12 ([10, Th. 10]). *A function $g : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ satisfies* (SBM) *if and only if for all $x \in \mathbf{R}^V$ and $\alpha \in \mathbf{R}$ the level set* $\mathrm{L}(g[x], \alpha)$ *satisfies* (QDL).

Proof. The "only if" part follows from Theorem 5.8 and submodularity of $g[x]$. We prove the "if" part. Let $p, q \in \mathrm{dom}\, g$. Since $p, q \in \mathrm{L}(g, \max\{g(p), g(q)\})$, we have either $p \wedge q \in \mathrm{L}(g, \max\{g(p), g(q)\}) \subseteq \mathrm{dom}\, g$ or $p \vee q \in \mathrm{L}(g, \max\{g(p), g(q)\}) \subseteq \mathrm{dom}\, g$. Assume, w.l.o.g., that $p \wedge q \in \mathrm{dom}\, g$ and $p \wedge q \neq p, q$. For any $\varepsilon > 0$, we can choose some $x \in \mathbf{R}^V$ and $\alpha \in \mathbf{R}$ such that $\alpha = g[x](p) = g[x](q) = g[x](p \wedge q) - \varepsilon$. By (QDL) for $\mathrm{L}(g[x], \alpha)$, we have $p \vee q \in \mathrm{L}(g[x], \alpha)$. This implies that

$$g[x](p) + g[x](q) = 2\alpha \geq g[x](p \wedge q) + g[x](p \vee q) - \varepsilon.$$

Since ε can be chosen arbitrarily, we have

$$g[x](p) + g[x](q) \geq g[x](p \wedge q) + g[x](p \vee q),$$

which is equivalent to the submodular inequality $g(p) + g(q) \geq g(p \wedge q) + g(p \vee q)$. □

Combining Theorems 5.8 and 5.12, we see the following:

Corollary 5.13. *Let $g : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$. Then,*

$$g \text{ satisfies (SBM)} \iff \forall x \in \mathbf{R}^V,\ g[x] \text{ satisfies (QSB)} \iff \forall x \in \mathbf{R}^V,\ g[x] \text{ satisfies } (\mathrm{QSB_w}).$$

5.3 Operations for Quasi L-convex and Submodular Functions

The class of (semistrictly) quasi L-convex/submodular functions is closed under several fundamental operations.

Let $g : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ be a function. For any subset $U \subseteq V$, we define $g^U : \mathbf{Z}^U \to \mathbf{R} \cup \{\pm\infty\}$ by

$$g^U(p) = \inf\{g(p, q) \mid q \in \mathbf{Z}^{V \setminus U}\} \qquad (p \in \mathbf{Z}^U).$$

Theorem 5.14. *Let* $(*\mathrm{QSB}_*)$ *be one of the properties* (QSB), $(\mathrm{QSB_w})$, (SSQSB), *and* $(\mathrm{SSQSB_w})$, *and* $g : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ *be a function with the property* $(*\mathrm{QSB}_*)$.
(i) *For any* $a \in \mathbf{Z}^V$, $\beta \in \mathbf{Z}$, *and* $\nu > 0$, *the function* $\nu \cdot g(a + \beta p)$ *satisfies* $(*\mathrm{QM}_*)$ *as a function in* x.
(ii) *For any* $U \subseteq V$, *the function* $g^U : \mathbf{Z}^U \to \mathbf{R} \cup \{\pm\infty\}$ *satisfies* $(*\mathrm{QSB}_*)$ *if* $g^U > -\infty$.
(iii) *For any* $a : V \to \mathbf{Z} \cup \{-\infty\}$ *and* $b : V \to \mathbf{Z} \cup \{+\infty\}$ *with* $a \leq b$, *the function* $g_a^b : \mathbf{Z}^V \to \mathbf{Z} \cup \{+\infty\}$ *defined by (3.5) satisfies* $(*\mathrm{QSB}_*)$.
(iv) *Let* $g_i : \mathbf{Z}^{V_i} \to \mathbf{R}_{++} \cup \{+\infty\}$ $(i = 1, 2)$ *be functions with* $(*\mathrm{QSB}_*)$. *Then, the function* $g : \mathbf{Z}^{V_1} \times \mathbf{Z}^{V_2} \to \mathbf{R}_{++} \cup \{+\infty\}$ *defined by* $g(p_1, p_2) = g_1(p_1)g_2(p_2)$ $(p_i \in \mathbf{Z}^{V_i}, i = 1, 2)$ *satisfies* $(*\mathrm{QSB}_*)$.

Proof. Proof is similar to that for Theorem 3.18 and therefore omitted. □

Remark 5.15. The class of (semistrictly) quasi-submodular functions is not closed under addition; in particular, it is not closed under the addition of a linear function.

For $i = 1, 2$, let $g_i : \mathbf{Z}^2 \to \mathbf{Z} \cup \{+\infty\}$ be functions such that

$$\begin{aligned} &\mathrm{dom}\, g_1 = \{(0,0), (1,0), (0,1)\}, \qquad g_1(0,0) = 0,\ g_1(1,0) = g_1(0,1) = 1, \\ &g_2(p_1, p_2) = -2(p_1 + p_2) \qquad ((p_1, p_2) \in \mathbf{Z}^2). \end{aligned}$$

It is easy to see that g_1 satisfies (SSQSB) (and not (SBM)), and that g_2 is linear. The sum $g = g_1 + g_2$, however, does not even satisfy $(\mathrm{QSB_w})$ since $g(1, 0) = g(0, 1) = -1 < 0 = \min\{g(0, 0), g(1, 1)\}$. □

Theorem 5.16. *For* $g : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ *and* $\varphi : \mathbf{R} \to \mathbf{R} \cup \{+\infty\}$, *define* $\tilde{g} : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ *by (5.1).*
(i) *Suppose that* φ *is nondecreasing. If* g *satisfies* (QSB) *(resp.* $(\mathrm{QSB_w})$*), then* $\tilde{g}$ *also satisfies* (QSB) *(resp.* $(\mathrm{QSB_w})$*).*
(ii) *Suppose that* φ *is strictly increasing. If* g *satisfies* (SSQSB) *(resp.* $(\mathrm{SSQSB_w})$*), then* $\tilde{g}$ *also satisfies* (SSQSB) *(resp.* $(\mathrm{SSQSB_w})$*).*

Remark 5.17. In [10], a semistrictly quasi-submodular function $\tilde{g} : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ is called *submodularizable* if there exists some strictly increasing function $\psi : \mathbf{R} \to \mathbf{R}$ such that the function $g : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ with $\mathrm{dom}\, g = \mathrm{dom}\, \tilde{g}$ defined by $g(p) = \psi(\tilde{g}(p))$ $(p \in \mathrm{dom}\, g)$ is submodular as a function in $p \in \mathbf{Z}^V$; in other words, a function $\tilde{g}$ is submodularizable if and only if it is represented in the form (5.1) with a submodular function $g : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ and a strictly increasing function $\varphi : \mathbf{R} \to \mathbf{R}$. The following example, which is essentially equivalent to Example 1 in [10], shows that a (semistrictly) quasi-submodular function is not necessarily given as the form (5.1) with a submodular function g and a nondecreasing function φ.

Let us consider the function $\tilde{g} : \mathbf{Z}^5 \to \mathbf{R} \cup \{+\infty\}$ given by

$$\mathrm{dom}\, \tilde{g} = \{p \in \mathbf{Z}^5 \mid 0 \le p_1 - p_5 \le p_2 - p_5 \le p_3 - p_5 \le 1,\ 0 \le p_4 - p_5 \le 1\},$$
$$\tilde{g}(p) = \begin{cases} 0 & \text{if } (p_1 - p_5, p_2 - p_5, p_3 - p_5, p_4 - p_5) = (1,1,0,1), \\ 1 & \text{if } (p_1 - p_5, p_2 - p_5, p_3 - p_5, p_4 - p_5) = (1,0,0,1), \\ 2 & \text{if } (p_1 - p_5, p_2 - p_5, p_3 - p_5, p_4 - p_5) \in \{(0,0,0,1), (1,1,1,1)\}, \\ 3 & \text{if } (p_1 - p_5, p_2 - p_5, p_3 - p_5, p_4 - p_5) \in \{(1,0,0,0), (1,1,0,0)\}, \\ 4 & \text{if } (p_1 - p_5, p_2 - p_5, p_3 - p_5, p_4 - p_5) \in \{(0,0,0,0), (1,1,1,0)\}. \end{cases}$$

Then, $\tilde{g}$ satisfies (SSQSB) and (TRF) with $r = 0$. Suppose that $\tilde{g}$ is given in the form (5.1) in terms of an L-convex function $g : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ and a strictly increasing function $\varphi : \mathbf{R} \to \mathbf{R} \cup \{+\infty\}$. Then, the submodularity of g implies that

$$g(0,0,0,1,0) + g(1,0,0,0,0) \ge g(0,0,0,0,0) + g(1,0,0,1,0),$$
$$g(1,1,0,1,0) + g(1,1,1,0,0) \ge g(1,1,0,0,0) + g(1,1,1,1,0),$$

whereas we have

$$g(0,0,0,1,0) = g(1,1,1,1,0), \quad g(1,0,0,0,0) = g(1,1,0,0,0),$$
$$g(1,1,0,1,0) < g(1,0,0,1,0), \quad g(1,1,1,0,0) = g(0,0,0,0,0),$$

since φ is strictly increasing. Hence, we have a contradiction. □

Theorem 5.18. *Let $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ and $g : \mathbf{Z}^V \to \mathbf{R} \cup \{-\infty\}$ be functions such that $g(p) > 0$ for all $p \in \mathrm{dom}\, f$. Suppose that the function $f(\cdot) - \alpha g(\cdot)$ satisfies* (QSB$_\mathrm{w}$) *for all $\alpha \in \mathbf{R} \cup \{+\infty\}$. Then, the function $r : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ given by*

$$r(p) = \begin{cases} f(p)/g(p) & (p \in \mathrm{dom}\, f), \\ +\infty & (p \notin \mathrm{dom}\, f), \end{cases}$$

also satisfies (QSB$_\mathrm{w}$). *In particular, if f and $-g$ satisfy* (SBM), *then r satisfies* (QSB$_\mathrm{w}$).

Proof. The proof is clear from Theorem 5.8. □

Remark 5.19. The following example shows that the statement of Theorem 5.18 cannot be strengthened by replacing (QSB$_\mathrm{w}$) with (QSB), even if f and g are affine functions.

Define a function $r : \mathbf{Z}^3 \to \mathbf{R} \cup \{+\infty\}$ as

$$\mathrm{dom}\, r = \{p \in \mathbf{Z}^3 \mid 0 \leq p_i - p_3 \leq 1 \ (i = 1, 2)\}, \qquad r(p) = \frac{4p_1 - p_2 - p_3}{2 - p_2 + p_3} \quad (p \in \mathrm{dom}\, r).$$

The function r, however, does not satisfy (QSB) since

$$g(1, 0, 0) = 2, \quad g(0, 1, 0) = -1, \quad g(0, 0, 0) = 0, \quad g(1, 1, 0) = 3.$$

$\square$

6 Minimization of Quasi L-convex Functions

In this section, we consider the minimization of quasi L-convex functions. To the end of this section we assume $r = 0$ in (TRF) since otherwise quasi L-convex functions have no minimizer. Under this assumption, the minimization of a function $g : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ is equivalent to the minimization of $g_0 : \mathbf{Z}^{V \setminus \{v_0\}} \to \mathbf{R} \cup \{+\infty\}$ which is defined as

$$g_0(p') = g(0, p') \qquad ((0, p') \in \mathbf{Z} \times \mathbf{Z}^{V \setminus \{v_0\}}) \tag{6.1}$$

with an element $v_0 \in V$.

6.1 Properties of Minimizers of Quasi L-convex Functions

Global minimality of quasi L-convex functions is characterized by local minimality.

Lemma 6.1. *Let $g : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ satisfy* (TRF) *with $r = 0$.*
(i) *Assume* ($\mathrm{QSB_w}$) *for g. Then, for all $p, q \in \mathbf{Z}^V$ and $\lambda \in \mathbf{Z}$ we have*

$$\max\{g(p), g(q)\} \geq \min\{g(p \vee (q - \lambda \mathbf{1})), g((p + \lambda \mathbf{1}) \wedge q)\}. \tag{6.2}$$

In particular, for all $p, q \in \mathrm{dom}\, g$ and $\lambda \in [0, \lambda_1 - \lambda_2]$ we have

$$\max\{g(p), g(q)\} \geq \min\{g(p + \lambda \chi_X), g(q - \lambda \chi_X)\}, \tag{6.3}$$

where $X \subseteq V$, $\lambda_1 \in \mathbf{Z}$, and $\lambda_2 \in \mathbf{Z} \cup \{-\infty\}$ are defined by

$$X = \arg\max_{v \in V}\{q(v) - p(v)\}, \quad \lambda_1 = \max_{v \in V}\{q(v) - p(v)\}, \quad \lambda_2 = \max_{v \in V \setminus X}\{q(v) - p(v)\}. \tag{6.4}$$

(ii) *Assume* (QSB) *for g. Then, for all $p, q \in \mathbf{Z}^V$ with $g(p) \neq g(q)$ and $\lambda \in \mathbf{Z}$ we have the inequality (6.2) with strict inequality. In particular, for all $p, q \in \mathrm{dom}\, g$ with $g(p) \neq g(q)$ and*

$\lambda \in [0, \lambda_1 - \lambda_2]$ *we have (6.3) with strict inequality.*
(iii) *Assume* (SSQSB) *for* g. *Then, for all* $p, q \in \mathbf{Z}^V$ *and* $\lambda \in \mathbf{Z}$ *we have the following properties:*

$$g(p \vee (q - \lambda \mathbf{1})) \geq g(p) \Longrightarrow g((p + \lambda \mathbf{1}) \wedge q) \leq g(q),$$
$$g((p + \lambda \mathbf{1}) \wedge q) \geq g(q) \Longrightarrow g(p \vee (q - \lambda \mathbf{1})) \leq g(p).$$

In particular, for all $p, q \in \mathrm{dom}\, g$ *and* $\lambda \in [0, \lambda_1 - \lambda_2]$ *we have*

$$g(p + \lambda \chi_X) \geq g(p) \Longrightarrow g(q - \lambda \chi_X) \leq g(q), \quad g(q - \lambda \chi_X) \geq g(q) \Longrightarrow g(p + \lambda \chi_X) \leq g(p), \tag{6.5}$$

where $\lambda_1 \in \mathbf{Z}$, $\lambda_2 \in \mathbf{Z} \cup \{-\infty\}$, *and* $X \subseteq V$ *are given by (6.4).*

Proof. We show the proof of (i) only. Proofs of (ii) and (iii) are similar to that for (i).

The inequality (6.2) can be shown as follows:

$$\begin{aligned}
\text{LHS of (6.2)} &= \max\{g(p), g(q - \lambda \mathbf{1})\} \\
&\geq \min\{g(p \vee (q - \lambda \mathbf{1})), g(p \wedge (q - \lambda \mathbf{1}))\} \\
&= \min\{g(p \vee (q - \lambda \mathbf{1})), g(p \wedge (q - \lambda \mathbf{1}) + \lambda \mathbf{1})\} \;=\; \text{RHS of (6.2)}.
\end{aligned}$$

The inequality (6.3) is obvious from (6.2) since

$$p \vee \{q - (\lambda_1 - \lambda)\mathbf{1}\} = p + \lambda \chi_X, \quad (p + (\lambda_1 - \lambda)\mathbf{1}) \wedge q = q - \lambda \chi_X \qquad (\forall \lambda \in [0, \lambda_1 - \lambda_2]).$$

□

Theorem 6.2. *Let* $g : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ *be a function satisfying* (TRF) *with* $r = 0$, *and* $p \in \mathrm{dom}\, g$.
(i) *Assume* ($\mathrm{QSB_w}$) *for* g. *Then,* $g(p) < g(q)$ *for all* $q \in \mathbf{Z}^V$ *such that* $q - p$ *is not a multiple of* $\mathbf{1}$ *if and only if* $g(p) < g(p + \chi_X)$ *for all* $X \subseteq V$ *with* $X \notin \{\emptyset, V\}$.
(ii) *Assume* ($\mathrm{SSQSB_w}$) *for* g. *Then,* $g(p) \leq g(q)$ $(\forall q \in \mathbf{Z}^V) \iff g(p) \leq g(p + \chi_X)$ $(\forall X \subseteq V)$.

Proof. We prove the "if" part of (i) by contradiction. Suppose that $g(q) \leq g(p)$ holds for some $q \in \mathrm{dom}\, g$ such that $q - p$ is not a multiple of $\mathbf{1}$. We may assume $q \geq p$ by (TRF) for g, and also assume that q minimizes the value $\max_{v \in V}\{q(v) - p(v)\}$ among all such vectors. Put $X = \arg\max_{v \in V}\{q(v) - p(v)\}$, where $X \neq V$. By applying Lemma 6.1 to p and q, we obtain

$$g(p) = \max\{g(p), g(q)\} \geq \min\{g(p + \chi_X), g(q - \chi_X)\}.$$

Due to the choice of q, we have $g(p) < g(q - \chi_X)$. Hence, $g(p) \geq g(p + \chi_X)$ follows, a contradiction to the strict local minimality of p.

The "only if" part of (i) is obvious, and (ii) can be shown similarly by using Lemma 6.1 (ii). □

Corollary 6.3. *For a function* $g : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ *satisfying* (TRF) *with* $r = 0$*, define* $g_0 : \mathbf{Z}^{V \setminus \{v_0\}} \to \mathbf{R} \cup \{+\infty\}$ *by (6.1). Let* $p \in \mathrm{dom}\, g_0$*.*
(i) *Assume* ($\mathrm{QSB_w}$) *for* g*. Then,* $g_0(p) < g_0(q)$ $(\forall q \in \mathbf{Z}^{V\setminus\{v_0\}} \setminus \{p\}) \iff g_0(p) < g_0(p \pm \chi_X)$ $(\emptyset \neq \forall X \subseteq V)$*.*
(ii) *Assume* ($\mathrm{SSQSB_w}$) *for* g*. Then,* $g_0(p) \leq g_0(q)$ $(\forall q \in \mathbf{Z}^{V\setminus\{v_0\}}) \iff g_0(p) \leq g_0(p \pm \chi_X)$ $(\forall X \subseteq V)$*.*

Remark 6.4. We see from its proof that the statement of Theorem 6.2 (i) holds even if ($\mathrm{QSB_w}$) is replaced with the following weaker condition:

For any distinct $p, q \in \mathrm{dom}\, g$, (i) or (ii) holds:
(i) $\min\{g(p+\chi_X), g(q-\chi_X)\} \leq \max\{g(p), g(q)\}$ for $X = \arg\max_{v\in V}\{q(v)-p(v)\}$,
(ii) $\min\{g(p-\chi_X), g(q+\chi_X)\} \leq \max\{g(p), g(q)\}$ for $X = \arg\min_{v\in V}\{q(v)-p(v)\}$.

This property is strictly weaker than ($\mathrm{QSB_w}$) under (TRF), as shown in the following example.

Let $g : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ be a function such that

$$g(p_1,p_2,p_3) = \begin{cases} 0 & ((p_1-p_3, p_2-p_3) = (0,1)), \\ 1 & ((p_1-p_3, p_2-p_3) \text{ is either } (1,0),(2,0), \text{ or } (1,1)), \\ +\infty & (\text{otherwise}). \end{cases}$$

It is easy to see that the function g satisfies the property above. For $p = (2,0,0), q = (0,1,0) \in \mathrm{dom}\, g$, neither $p \wedge q$ nor $p \vee q$ is contained in $\mathrm{dom}\, g$, i.e., $\mathrm{dom}\, g$ does not satisfy (QDL). Therefore, g does not satisfy ($\mathrm{QSB_w}$) by Theorem 5.8. $\square$

Remark 6.5. We see from its proof that the statement of Theorem 6.2 (ii) holds even if ($\mathrm{SSQSB_w}$) is replaced with the following weaker condition:

For any $p, q \in \mathrm{dom}\, g$ with $g(p) \neq g(q)$, (i) or (ii) holds:
(i) $\min\{g(p+\chi_X), g(q-\chi_X)\} < \max\{g(p), g(q)\}$ for $X = \arg\max_{v\in V}\{q(v)-p(v)\}$,
(ii) $\min\{g(p-\chi_X), g(q+\chi_X)\} < \max\{g(p), g(q)\}$ for $X = \arg\min_{v\in V}\{q(v)-p(v)\}$.

This property is strictly weaker than ($\mathrm{SSQSB_w}$) under (TRF), as shown in the following example.

Let $D \subseteq \mathbf{Z}^V$ be any set which satisfies (TRF) and not (QDL), and consider its indicator function $\delta_D : \mathbf{Z}^V \to \{0, \infty\}$. Since $\delta_D(p) = \delta_D(q)$ for any $p, q \in \mathrm{dom}\, \delta_D$, the function δ_D satisfies the property above. However, δ_D does not satisfy ($\mathrm{QSB_w}$). $\square$

We apply the scaling technique to the minimization of quasi L-convex functions in Section 6.2. Let $g : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ be a semistrictly quasi L-convex function. For $p_0 \in \mathrm{dom}\, g$ and $\alpha \in \mathbf{Z}_{++}$, we define $g_\alpha : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ by $g_\alpha(x) = g(p_0 + \alpha p)$ $(p \in \mathbf{Z}^V)$. We here consider the relationship between a global minimum of g and a local minimum of g_α. Let $q \in \mathrm{dom}\, g_\alpha$ be a local minimum of g_α, i.e., $p_\alpha = p_0 + \alpha q$ satisfies

$$g(p_\alpha) \le g(p_\alpha + \alpha\chi_X) \qquad (\forall X \subseteq V). \tag{6.6}$$

The following theorem shows that a global minimum of a semistrictly L-convex function exists in the neighborhood of p_α. This generalizes an observation in [9].

Theorem 6.6. *Let* $g : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ *be a function satisfying* (SSQSB) *and* (TRF) *with* $r = 0$, *and* $\alpha \in \mathbf{Z}_{++}$. *Suppose that* $p_\alpha \in \mathrm{dom}\, g$ *satisfies (6.6). Then,* $\arg\min g \neq \emptyset$ *and there exists some* $q_* \in \arg\min g$ *with*

$$p_\alpha \le q_* \le \ p_\alpha + (n-1)(\alpha - 1)\mathbf{1}. \tag{6.7}$$

Proof. It suffices to show that for any $\gamma \in \mathbf{R}$ with $\gamma > \inf g$, there exists some $q_* \in \mathrm{dom}\, g$ satisfying $g(q_*) \le \gamma$ and (6.7). Assume, w.l.o.g., $p_\alpha = \mathbf{0}$. By (TRF) for g, there exists some $q_* \in \mathrm{dom}\, g$ such that $g(q_*) \le \gamma$ and $q_* \ge \mathbf{0}$. We assume that q_* is minimal (w.r.t. the partial order $\ge$) among all such vectors. This assumption implies $q_*(v) = 0$ for some $v \in V$, i.e., $\mathrm{supp}^+(q_*) \neq V$, and

$$g(q_* - \chi_X) > g(q_*) \qquad (\forall X \subseteq \mathrm{supp}^+(q_*)). \tag{6.8}$$

Then, there exist some $X_i \subseteq \mathrm{supp}^+(q_*)$ $(i = 1, \cdots, k)$ and $\{\mu_i\}_{i=1}^k \subseteq \mathbf{Z}_{++}$ $(0 \le k \le n-1)$ such that

$$\emptyset \subset X_1 \subset X_2 \subset \cdots \subset X_k \subset V, \qquad q_* = \sum_{i=1}^k \mu_i \chi_{X_i}.$$

Claim 1: For any $j = 1, \cdots, k$ and $\mu \in [0, \mu_j - 1]$, we have

$$g(\sum_{i=1}^{j-1} \mu_i\chi_{X_i} + \mu\chi_{X_j}) > g(\sum_{i=1}^{j-1} \mu_i\chi_{X_i} + (\mu+1)\chi_{X_j}).$$

[Proof of Claim 1] Put $p = \sum_{i=1}^{j-1} \mu_i\chi_{X_i} + \mu\chi_{X_j}$ and suppose $p \in \mathrm{dom}\, g$. Then, $\arg\max_{v\in V}\{q_*(v) - p(v)\} = X_j$. Since $X_j \subseteq \mathrm{supp}^+(q_*)$, we have $g(q_* - \chi_{X_j}) > g(q_*)$ by (6.8). This fact, together with (6.5), yields $g(p + \chi_{X_j}) < g(p)$. [End of Claim 1]

Claim 2 $g(\mu\chi_{X_j}) > g((\mu+1)\chi_{X_j})$ holds for any $j = 1, \cdots, k$ and $\mu \in [0, \mu_j - 1]$.

[Proof of Claim 2] Suppose $\mu\chi_{X_j} \in \mathrm{dom}\, g$. Put $p = \sum_{i=1}^{j} \mu_i\chi_{X_i}$ and $q = \mu\chi_{X_j}$. Then, $\arg\max_{v\in V}\{q(v) - p(v)\} = V \setminus X_j$. Since $g(p + \chi_{V\setminus X_j}) = g(p - \chi_{X_j}) > g(p)$ by Claim 1, (6.5) implies that $g(q) > g(q - \chi_{V\setminus X_j}) = g(q + \chi_{X_j})$. [End of Claim 2]

From Claim 2 and (6.6) follows $\mu_i < \alpha$ for $i = 1, 2, \cdots, k$. Hence, we have

$$\mathbf{0} \leq q_* \leq (\alpha - 1) \sum_{i=1}^{k} \chi_{X_i} \leq (n-1)(\alpha - 1)\mathbf{1}.$$

□

Corollary 6.7. *Given a function* $g : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ *satisfying* (SSQSB) *and* (TRF) *with* $r = 0$, *define* $g_0 : \mathbf{Z}^{V \setminus \{v_0\}} \to \mathbf{R} \cup \{+\infty\}$ *by (6.1). Let* $\alpha \in \mathbf{Z}_{++}$. *Suppose that* $p_\alpha \in \mathrm{dom}\, g_0$ *satisfies* $g_0(p_\alpha) \leq g_0(p_\alpha \pm \alpha\chi_X)$ $(\forall X \subseteq V)$. *Then, there exists some* $q_* \in \arg\min g_0$ *such that*

$$|q_*(v) - p_\alpha(v)| \leq (n-1)(\alpha - 1) \qquad (v \in V).$$

6.2 Algorithms

Let $g : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ satisfy (SSQSB$_\mathrm{w}$) and (TRF) with $r = 0$, and define $g_0 : \mathbf{Z}^{V \setminus \{v_0\}} \to \mathbf{R} \cup \{+\infty\}$ by (6.1). In the following, we explain two minimization algorithms for g_0.

By Corollary 6.3, we can find a minimizer of g_0 by a descent method.

Algorithm DESCENT_L
Step 0: Let p be any vector in $\mathrm{dom}\, g_0$.
Step 1: If $g_0(p) = \min\{g_0(p \pm \chi_X) \mid X \subseteq V\}$ then stop. [p is a minimizer]
Step 2: Find $X \subseteq V$ and $\lambda \in \{1, -1\}$ such that $g_0(p + \lambda\chi_X) < g_0(p)$.
Step 3: Set $p := p + \lambda\chi_X$. Go to Step 1. □

If $\mathrm{dom}\, g_0$ is bounded, the algorithm DESCENT_L terminates in at most $|\mathrm{dom}\, g_0| \leq K^{n-1}$ iterations, where K is given by

$$K = \max\{|p(v) - q(v)| \mid p, q \in \mathrm{dom}\, g_0,\ v \in V\}.$$

We further assume (SSQSB) for g. Based on Corollary 6.7, we apply the scaling technique to Algorithm DESCENT_L to obtain a faster algorithm.

Algorithm SCALING_DESCENT_L
Step 0: Put $\alpha := 2^{\lceil \log_2 K \rceil}$, $D := \mathrm{dom}\, g_0$. Let p_* be any vector in $\mathrm{dom}\, g_0$.
Step 1: Find $q \in \mathbf{Z}^{V \setminus \{v_0\}}$ such that $p_* + \alpha q \in D$ and

$$g_0(p_* + \alpha q) = \min\{p_* + \alpha q' \mid q' \in \mathbf{Z}^{V \setminus \{v_0\}},\ p_* + \alpha q' \in D\}.$$

Step 2: If $\alpha = 1$ then stop. [$p_* + \alpha q$ is a minimizer of g_0.]
Step 3: Put $p_* := p_* + \alpha q$, $D := D \cap \{p \in \mathbf{Z}^V \mid |p(v) - p_*(v)| \leq (n-1)(\alpha - 1)\ (v \in V)\}$, and $\alpha := \alpha/2$. Go to Step 1. □

The number of scaling phases is $\lceil \log_2 K \rceil$. Therefore, if we could perform Step 1 in each iteration in polynomial time, Algorithm SCALING_DESCENT_L would run in polynomial time. Unfortunately, we do not know yet such a polynomial-time algorithm for Step 1.

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A Proofs

A.1 Proof of Theorem 2.2

Proof of (i): The "$\Longrightarrow$" part is easy to see. Hence, we show the "$\Longleftarrow$" part only. Let $\alpha_1, \alpha_2 \in \mathrm{dom}\,\varphi$ with $\alpha_1 < \alpha_2$, where we assume $\varphi(\alpha_1) \geq \varphi(\alpha_2)$. We show the claim by induction on the value $\alpha_2 - \alpha_1$. From the assumption, we have (i) $\varphi(\alpha_1 + 1) \leq \varphi(\alpha_1)$ or (ii) $\varphi(\alpha_2 - 1) \leq \varphi(\alpha_1)$. If (i) holds, then the inductive assumption implies $\varphi(\beta) \leq \max\{\varphi(\alpha_1 + 1), \varphi(\alpha_2)\}$ $(\alpha_1 + 1 < \forall\beta < \alpha_2)$. If (ii) holds, then the inductive assumption implies $\varphi(\beta) \leq \max\{\varphi(\alpha_1), \varphi(\alpha_2 - 1)\}$ $(\alpha_1 < \forall\beta < \alpha_2 - 1)$. Therefore, we have (2.2).

Proof of (ii): Proof is quite similar to that for (i). The "$\Longrightarrow$" part is easy to see. Hence, we show the "$\Longleftarrow$" part only. Let $\alpha_1, \alpha_2 \in \mathrm{dom}\,\varphi$ with $\alpha_1 < \alpha_2$ and $\varphi(\alpha_1) \neq \varphi(\alpha_2)$, where we assume $\varphi(\alpha_1) > \varphi(\alpha_2)$. We show the claim by induction on the value $\alpha_2 - \alpha_1$. From the assumption, we have (i) $\varphi(\alpha_1+1) < \varphi(\alpha_1)$ or (ii) $\varphi(\alpha_2-1) < \varphi(\alpha_1)$. If (i) holds, then the quasiconvexity of φ implies

$$\varphi(\beta) \le \max\{\varphi(\alpha_1+1), \varphi(\alpha_2)\} < \varphi(\alpha_1) \qquad (\alpha_1+1 < \forall\beta < \alpha_2).$$

If (ii) holds, then the inductive assumption implies

$$\varphi(\beta) < \max\{\varphi(\alpha_1), \varphi(\alpha_2-1)\} = \varphi(\alpha_1) \qquad (\alpha_1 < \forall\beta < \alpha_2-1).$$

Therefore, we have (2.3).

Proof of (iii): ["$\Longrightarrow$" part] Let $\alpha_1, \alpha_2 \in \operatorname{dom}\varphi$ with $\alpha_1 < \alpha_2$. We may assume that $\alpha_2-\alpha_1 \ge 2$. It suffices to show that we have $\varphi(\alpha_1+1) < \varphi(\alpha_1)$, $\varphi(\alpha_2-1) < \varphi(\alpha_2)$, or $\varphi(\alpha_1+1) = \varphi(\alpha_1)$ and $\varphi(\alpha_2-1) = \varphi(\alpha_2)$.

(Case 1: $\varphi(\alpha_1) \ne \varphi(\alpha_2)$) We have

$$\varphi(\alpha_1+1) < \max\{\varphi(\alpha_1), \varphi(\alpha_2)\}, \quad \varphi(\alpha_2-1) < \max\{\varphi(\alpha_1), \varphi(\alpha_2)\},$$

from which we have $\varphi(\alpha_1+1) < \varphi(\alpha_1)$ or $\varphi(\alpha_2-1) < \varphi(\alpha_2)$.

(Case 2: $\varphi(\alpha_1) = \varphi(\alpha_2)$) The quasiconvexity of φ implies

$$\varphi(\alpha_1+1) \le \max\{\varphi(\alpha_1), \varphi(\alpha_2)\} = \varphi(\alpha_1), \quad \varphi(\alpha_2-1) \le \max\{\varphi(\alpha_1), \varphi(\alpha_2)\} = \varphi(\alpha_2),$$

from which the claim follows.

["$\Longleftarrow$" part] The assumption for φ and the property (i) immediately yield the quasiconvexity of φ. It suffices from (ii) to prove that

$$\begin{aligned}&\min\{\varphi(\alpha_1+1), \varphi(\alpha_2-1)\} < \max\{\varphi(\alpha_1), \varphi(\alpha_2)\} \\ &\qquad\qquad (\forall\alpha_1, \alpha_2 \in \operatorname{dom}\varphi \text{ with } \alpha_1 < \alpha_2,\ \varphi(\alpha_1) \ne \varphi(\alpha_2)).\end{aligned} \tag{A.9}$$

We assume $\varphi(\alpha_1) > \varphi(\alpha_2)$), and prove $\varphi(\alpha_1+1) < \varphi(\alpha_1)$ by induction on the value $\alpha_2-\alpha_1$. From the assumption, we have $\varphi(\alpha_1+1) < \varphi(\alpha_1)$ or $\varphi(\alpha_2-1) \le \varphi(\alpha_2)$. If the latter holds, then the inductive hypothesis yields $\varphi(\alpha_1+1) < \varphi(\alpha_1)$. Hence, we have (A.9).

A.2 Proof of Theorem 4.1

It is easy to see that (4.2) implies both (SSQM$^{\ne}_{\mathrm{w}}$) and (4.1). Hence, we prove "(SSQM$^{\ne}_{\mathrm{w}}$) $\Longrightarrow$ (4.2)" and "(4.1) $\Longrightarrow$ (4.2)" below.

Suppose that $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ satisfies (SSQM$^{\ne}_{\mathrm{w}}$) or (4.1). Let $x, y \in \operatorname{dom} f$ be vectors such that $f(x) > f(y)$. We show by induction on the value $||x-y||_1$ that there exist some $u \in \operatorname{supp}^+(x-y)$ and $v \in \operatorname{supp}^-(x-y)$ such that $\Delta f(x; v, u) < 0$. We may assume $||x-y||_1 > 2$, since otherwise the claim holds obviously.

Suppose that f satisfies (SSQM$^{\neq}_{\mathrm{w}}$). Then, there exist some $u \in \mathrm{supp}^+(x-y)$ and $v \in \mathrm{supp}^-(x-y)$ such that $\Delta f(x;v,u) < 0$ or $\Delta f(y;u,v) \leq 0$. If the latter holds, then we have $f(x) > f(y')$ for $y' = y + \chi_u - \chi_v$ and $||x-y'||_1 < ||x-y||_1$. Hence, the inductive hypothesis yields $\Delta f(x;v',u') < 0$ for some $u' \in \mathrm{supp}^+(x-y') \subseteq \mathrm{supp}^+(x-y)$ and $v' \in \mathrm{supp}^-(x-y') \subseteq \mathrm{supp}^-(x-y)$.

We next suppose that f satisfies (4.1). Then, there exist some $u \in \mathrm{supp}^+(x-y)$ and $v \in \mathrm{supp}^-(x-y)$ such that $\Delta f(x;v,u) < 0$ or $f(y+\chi_u-\chi_v) < f(x)$. If the latter holds, then we have $f(x) > f(y')$ for $y' = y + \chi_u - \chi_v$ and $||x-y'||_1 < ||x-y||_1$. Hence, the inductive hypothesis yields $\Delta f(x;v',u') < 0$ for some $u' \in \mathrm{supp}^+(x-y') \subseteq \mathrm{supp}^+(x-y)$ and $v' \in \mathrm{supp}^-(x-y') \subseteq \mathrm{supp}^-(x-y)$.