ANALYSIS ON THE MINIMAL REPRESENTATION OF $O(p,q)$ – II. BRANCHING LAWS

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Abstract. This is a second paper in a series devoted to the minimal unitary representation of $O(p,q)$. By explicit methods from conformal geometry of pseudo Riemannian manifolds, we find the branching law corresponding to restricting the minimal unitary representation to natural symmetric subgroups. In the case of purely discrete spectrum we obtain the full spectrum and give an explicit Parseval-Plancherel formula, and in the general case we construct an infinite discrete spectrum.

Contents

Introduction

This is the second in a series of papers devoted to the analysis of the minimal representation $\varpi^{p,q}$ of $O(p,q)$. We refer to [23] for a general introduction; also the numbering of the sections is continued from that paper, and we shall refer back to sections there. However, the present paper may be read independently from [23], and and its main object is to study the branching law for the minimal unitary representation $\varpi^{p,q}$ from **analytic** and **geometric** point of view. Namely, we shall find by explicit means, coming from conformal geometry, the restriction of $\varpi^{p,q}$ with respect to the symmetric pair

$$(G, G') = (O(p,q), O(p',q') \times O(p'',q'')).$$

If one of the factors in G' is compact, then the spectrum is discrete (see Theorem 4.2 also for an opposite implication), and we find the explicit branching law; when both factors are non-compact, there will still (generically) be an infinite discrete spectrum, which we also construct (conjecturally almost all of it; see §9.8). We shall see that the (algebraic) situation is similar to the theta-correspondence, where the metaplectic representation is restricted to analogous subgroups.

Let us here state the main results in a little more precise form, referring to sections 8 and 9 for further notation and details.

Theorem A (the branching law for $O(p,q) \downarrow O(p,q') \times O(q'')$; see Theorem 7.1). *If $q'' \geq 1$ and $q' + q'' = q$, then the twisted pull-back $\widetilde{\Phi_1^*}$ of the local conformal map Φ_1 between spheres and hyperboloids gives an explicit irreducible decomposition of the unitary representation $\varpi^{p,q}$ when restricted to $O(p,q') \times O(q'')$:*

$$\widetilde{(\Phi_1)^*} : \varpi^{p,q}|_{O(p,q') \times O(q'')} \xrightarrow{\sim} \sum_{l=0}^{\infty}{}^{\oplus} \pi^{p,q'}_{+,l+\frac{q''}{2}-1} \boxtimes \mathcal{H}^l(\mathbb{R}^{q''}).$$

The representations appearing in the decompositions are in addition to usual spherical harmonics $\mathcal{H}^l(\mathbb{R}^q)$ for compact orthogonal groups $O(q)$, also the representations $\pi^{p,q}_{+,\lambda}$ for non-compact orthogonal groups $O(p,q)$. The latter ones may be thought of as discrete series representations on hyperboloids

$$X(p,q) := \{x = (x', x'') \in \mathbb{R}^{p+q} : |x'|^2 - |x''|^2 = 1\}$$

for $\lambda > 0$ or their analytic continuation for $\lambda \leq 0$; they may be also thought of as cohomologically induced representations from characters of certain θ-stable parabolic subalgebras. The fact that they occur in this branching law gives a different proof of the unitarizability of these modules $\pi^{p,q}_{+,\lambda}$ for $\lambda > -1$, once we know $\varpi^{p,q}$ is unitarizable (cf. Part I, Theorem 3.6.1). It might be interesting to remark that the unitarizability for $\lambda < 0$ (especially, $\lambda = -\frac{1}{2}$ in our setting) does follow neither from a general unitarizability theorem on Zuckerman-Vogan's derived functor modules [31], nor from a general theory of harmonic analysis on semisimple symmetric spaces.

Our intertwining operator $\widetilde{(\Phi_1)^*}$ in Theorem A is derived from a conformal change of coordinate (see §6 for its explanation) and is explicitly written. Therefore, it makes sense to ask also about the relation of unitary inner products between the left-hand and the right-hand side in the branching formula. Here is an answer (see Theorem 8.5): We normalize the inner product $\| \ \|_{\pi^{p,q}_{+,\lambda}}$ (see §8.3) such that for $\lambda > 0$,

$$\|f\|^2_{\pi^{p,q}_{+,\lambda}} = \lambda \|f\|^2_{L^2(X(p,q))}, \qquad \text{for any } f \in (\pi^{p,q}_{+,\lambda})_K.$$

Theorem B (the Parseval-Plancherel formula for $O(p,q) \downarrow O(p,q') \times O(q'')$).
1) *If we develop $F \in \operatorname{Ker} \widetilde{\Delta}_M$ as $F = \sum_l^\infty F_l^{(1)} F_l^{(2)}$ according to the irreducible decomposition in Theorem A, then we have*

$$\|F\|^2_{\varpi^{p,q}} = \sum_{l=0}^{\infty} \|F_l^{(1)}\|^2_{\pi^{p,q'}_{+,l+\frac{q''}{2}-1}} \| F_l^{(2)}\|^2_{L^2(S^{q''-1})}.$$

2) *In particular, if $q'' \geq 3$, then all of $\pi^{p,q'}_{+,l+\frac{q''}{2}-1}$ are discrete series for the hyperboloid $X(p,q')$ and the above formula amounts to*

$$\|F\|^2_{\varpi^{p,q}} = \sum_{l=0}^{\infty} (l + \frac{q''}{2} - 1) \ \|F_l^{(1)}\|^2_{L^2(X(p,q'))} \| F_l^{(2)}\|^2_{L^2(S^{q''-1})}.$$

The formula may be also regarded as an explicit unitarization of the minimal representation $\varpi^{p,q}$ on the "hyperbolic space model" by means of the right side (for an abstract unitarization of $\varpi^{p,q}$, it suffices to choose a single pair (q', q'')). We note that the formula was previously known in the case where $(q', q'') = (0, q)$ (namely, when each summand in the right side is finite dimensional) by Kostant, Binegar-Zierau by a different approach. The formula is new and seems to be particularly interesting even in the special case $q'' = 1$, where the minimal representation $\varpi^{p,q}$ splits into two irreducible summands when restricted to $O(p, q-1) \times O(1)$.

In Theorem 9.1, we consider a more general setting and prove:

Theorem C (discrete spectrum in the restriction $O(p,q) \downarrow O(p',q') \times O(p'',q'')$). *The twisted pull-back of the locally conformal diffeomorphism also constructs*

$$\sum_{\lambda \in A'(p',q') \cap A'(q'',p'')}^{\oplus} \pi_{+,\lambda}^{p',q'} \boxtimes \pi_{-,\lambda}^{p'',q''} \oplus \sum_{\lambda \in A'(q',p') \cap A'(p'',q'')}^{\oplus} \pi_{-,\lambda}^{p',q'} \boxtimes \pi_{+,\lambda}^{p'',q''}$$

as a discrete spectrum in the branching law for the non-compact case.

Even in the special case $(p'',q'') = (0,1)$, our branching formula includes a new and mysterious construction of the minimal representation on the hyperboloid as below (see Corollary 7.2.1): Let $W^{p,r}$ be the set of K-finite vectors ($K = O(p) \times O(r)$) of the kernel of the Yamabe operator

$$\operatorname{Ker} \widetilde{\Delta}_{X(p,r)} = \{f \in C^\infty(X(p,r)) : \Delta_{X(p,r)} f = \frac{1}{4}(p+r-1)(p+r-3)f\},$$

on which the isometry group $O(p,r)$ and the Lie algebra of the conformal group $O(p,r+1)$ act. The following Proposition is a consequence of Theorem 7.2.2 by an elementary linear algebra.

Proposition D. *Let $m > 3$ be odd. There is a long exact sequence*

$$0 \to W^{1,m-1} \xrightarrow{\varphi_1} W^{2,m-2} \xrightarrow{\varphi_2} W^{3,m-3} \xrightarrow{\varphi_3} \cdots \xrightarrow{\varphi_{m-2}} W^{m-1,1} \xrightarrow{\varphi_{m-1}} 0$$

such that $\operatorname{Ker} \varphi_p$ *is isomorphic to* $(\varpi^{p,q})_K$ *for any* (p,q) *such that* $p+q = m+1$.

We note that each representation space $W^{p,q-1}$ is realized on a different space $X(p,q-1)$ whose isometry group $O(p,q-1)$ varies according to p ($1 \le p \le m$). So, one may expect that only the intersections of adjacent groups can act (infinitesimally) on $\operatorname{Ker} \varphi_p$. Nevertheless, a larger group $O(p,q)$ can act on a suitable completion of $\operatorname{Ker} \varphi_p$, giving rise to another construction of the minimal representation on the hyperboloid $X(p,q-1) = O(p,q-1)/O(p-1,q-1)$! We note that $\operatorname{Ker} \varphi_p$ is roughly half the kernel of the Yamabe operator on the hyperboloid (see §7.2 for details).

We briefly indicate the contents of the paper: In section 4 we recall the relevant facts about discretely decomposable restrictions from [16] and [17], and apply the criteria to our present situation. In particular, we calculate the associated variety of $\varpi^{p,q}$ as well as its asymptotic K-support introduced by Kashiwara-Vergne. Theorem 4.2 and Corollary 4.3 clarify the reason why we start with the subgroup $G' = O(p,q') \times O(q'')$ (i.e. $p'' = 0$). Section 5 contains the identification of the representations $\pi_{+,\lambda}^{p,q}$ and $\pi_{-,\lambda}^{p,q}$ of $O(p,q)$ in several ways, namely as: derived functor modules, Dolbeault cohomologies, eigenspaces on hyperboloids, and quotients or subrepresentations of parabolically induced modules. In section 6 we give the main construction of embedding conformally a direct product of hyperboloids into a product of spheres; this gives rise to a canonical intertwining operator between solutions to the so-called Yamabe equation, studied in connection with conformal differential geometry, on conformally related spaces. Applying this principle in section 7 we obtain the branching law in the case where one factor in G' is compact, and in particular when one factor is just $O(1)$. In this case we have Corollary 7.2.1, stating that $\varpi^{p,q}$ restricted to $O(p,q-1)$ is the direct sum of two representations, realized in even respectively odd functions on the hyperboloid for $O(p,q-1)$. Note here the analogy with the metaplectic representation. Also note here Theorem 7.2.2, which gives a mysterious extention of $\varpi^{p,q}$ by $\varpi^{p+1,q-1}$ - both inside the space of solutions to the Yamabe equation on the hyperboloid $X(p,q-1) = O(p,q-1)/O(p-1,q-1)$. We also point out that the representations $\pi^{p,q}_{+,\lambda}$ for $\lambda = 0, -\frac{1}{2}$ are rather exceptional; they are unitary, but outside the usual "fair range" for derived functor modules, see the remarks in section 8.2. Section 8 contains a calculation of spectra of intertwining operators and gives the explicit normalization of a Parseval-Plancherel formula for the branching law, and finally in section 9 we use certain Sobolev estimates to construct an infinite discrete spectrum when both factors in G' are non-compact. We also conjecture the form of the full discrete spectrum (true in the case of a compact factor). It should be interesting to calculate the full Parseval-Plancherel formula in the case of both discrete and continuous spectrum.

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§4. Criterion for discrete decomposable branching laws

4.1. Our object of study is the discrete spectra of the branching law of the restriction $\varpi^{p,q}$ with respect to a symmetric pair $(G,G') = (O(p,q), O(p',q') \times O(p'',q''))$. The aim of this section is to give a necessary and sufficient condition on p', q', p'' and q'' for the branching law to be discretely decomposable.

We start with general notation. Let G be a linear reductive Lie group, and G' its subgroup which is reductive in G. We take a maximal compact subgroup K of G such that $K' := K \cap G'$ is also a maximal compact subgroup. Let $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{p}_0$ be a Cartan decomposition, and $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ its complexification. Accordingly, we have a direct decomposition $\mathfrak{g}^* = \mathfrak{k}^* + \mathfrak{p}^*$ of the dual spaces.

Let $\pi \in \widehat{G}$. We say that the restriction $\pi|_{G'}$ is *G'-admissible* if $\pi|_{G'}$ splits into a direct Hilbert sum of irreducible unitary representations of G' with each multiplicity finite (see [15]). As an algebraic analogue of this notion, we say the underlying $(\mathfrak{g}, K)$-module π_K is *discretely decomposable as a $(\mathfrak{g}', K')$-module*, if π_K is decomposed into an algebraic direct sum of irreducible $(\mathfrak{g}', K')$-modules (see [17]). We note that if the restriction $\pi|_{K'}$ is K'-admissible, then the restriction $\pi|_{G'}$ is also G'-admissible ([15], Theorem 1.2) and the underlying $(\mathfrak{g}, K)$-module π_K is discretely decomposable (see [17], Proposition 1.6). Here are criteria for K'-admissibility and discrete decomosability:

Fact 4.1 (see [16], Theorem 2.9 for (1); [17], Corollary 3.4 for (2)).
1) *If* $\mathrm{AS}_K(\pi) \cap \mathrm{Ad}^*(K)(\mathfrak{k}')^\perp = 0$, *then* π *is* K'*-admissible and also* G'*-admissible.*
2) *If* π_K *is discretely decomposable as a* $(\mathfrak{g}', K')$*-module, then* $\mathrm{pr}_{\mathfrak{g}\to\mathfrak{g}'}(\mathcal{V}_{\mathfrak{g}}(\pi_K)) \subset \mathcal{N}^*_{\mathfrak{g}'}$.

Here, $\mathrm{AS}_K(\pi)$ is the asymptotic cone of

$$\mathrm{Supp}_K(\pi) := \{\text{highest weight of } \tau \in \widehat{K_0} : [\pi|_{K_0} : \tau] \neq 0\}$$

where K_0 is the identity component of K, and $(\mathfrak{k}')^\perp \subset \mathfrak{k}^*$ is the annihilator of $\mathfrak{k}'$. Let $\mathcal{N}^*_{\mathfrak{p}}$ ($\subset \mathfrak{p}^*$) be the nilpotent cone for $\mathfrak{p}$. $\mathcal{V}_{\mathfrak{g}}(\pi_K)$ denotes the associated variety of π_K, which is an $\mathrm{Ad}^*(K_{\mathbb{C}})$-invariant closed subset of $\mathcal{N}^*_{\mathfrak{p}}$. We write the projection $\mathrm{pr}_{\mathfrak{p}\to\mathfrak{p}'} : \mathfrak{p}^* \to \mathfrak{p}'^*$ dual to the inclusion $\mathfrak{p}' \hookrightarrow \mathfrak{p}$.

4.2. Let us consider our setting where $\pi = \varpi^{p,q}$ and $(G,G') = (O(p,q), O(p',q') \times O(p'',q''))$.

Theorem 4.2. *Suppose* $p' + p'' = p$ (≥ 2), $q' + q'' = q$ (≥ 2) *and* $p + q \in 2\mathbb{N}$. *Then the following three conditions on* p', q', p'', q'' *are equivalent:*
i) $\varpi^{p,q}$ *is* K'*-admissible.*

ii) $\varpi_K^{p,q}$ *is discretely decomposable as a* $(\mathfrak{g}', K')$*-module.*
iii) $\min(p', q', p'', q'') = 0$.

The implication (i) $\Rightarrow$ (ii) holds by a general theory as we explained ([17], Proposition 1.6); (ii) $\Rightarrow$ (iii) will be proved in §4.4, and (iii) $\Rightarrow$ (i) in §4.5, by an explicit computation of the asymptotic cone $\mathrm{AS}_K(\varpi^{p,q})$ and the associated variety $\mathcal{V}_{\mathfrak{g}}(\varpi_K^{p,q})$ which are used in Fact 4.1.

Remark. Analogous results to the equivalence (i) $\Leftrightarrow$ (ii) in Theorem 4.2 were first proved in [17], Theorem 4.2 in the setting where (G, G') is any reductive symmetric pair and the representation is any $A_{\mathfrak{q}}(\lambda)$ module in the sense of Zuckerman-Vogan, which may be regarded as "representations attached to elliptic orbits". We note that our representations $\varpi^{p,q}$ are supposed to be attached to nilpotent orbits. We refer [20], Conjecture A to relevant topics.

4.3. The following corollary is a direct consequence of Theorem 4.2, which will be an algebraic background for the proof of the explicit branching law (Theorem 7.1).

Corollary 4.3. *Suppose that* $\min(p', q', p'', q'') = 0$.
1) *The restriction of the unitary representation* $\varpi^{p,q}|_{G'}$ *is also* G'*-admissible.*
2) *The space of* K'*-finite vectors* $\varpi_{K'}^{p,q}$ *coincides with that of* K*-finite vectors* $\varpi_K^{p,q}$.

Proof. See [15], Theorem 1.2 for (1); and [17], Proposition 1.6 for (2). $\square$

A geometric counterpart of Corollary 4.3 (2) is reflected as the removal of singularities of matrix coefficients for the discrete spectra in the analysis that we study in §6; namely, any analytic function defined on an open subset M_+ (see §6 for notation) of M which is a K'-finite vector of a discrete spectrum, extends analytically on M if $p'' = 0$. The reason for this is not only the decay of matrix coefficients but a matching condition of the leading terms for $t \to \pm\infty$. This is not the case for $\min(p', q', p'', q'') > 0$ (see §9).

4.4. Proof of (ii) $\Rightarrow$ (iii) in Theorem 4.2.

We identify $\mathfrak{p}^*$ with $\mathfrak{p}$ via the Killing form, which is in turn identified with $M(p, q; \mathbb{C})$ by

$$M(p, q; \mathbb{C}) \xrightarrow{\sim} \mathfrak{p}, \;\; X \mapsto \begin{pmatrix} O & X \\ {}^tX & O \end{pmatrix}.$$

Then the nilpotent cone $\mathcal{N}_{\mathfrak{p}}^*$ corresponds to the following variety.

$$\{X \in M(p, q; \mathbb{C}) : \text{ both } X\,{}^tX \text{ and } {}^tX X \text{ are nilpotent matrices}\} \tag{4.4.1}$$

We put

$$M_{0,0}(p, q; \mathbb{C}) := \{X \in M(p, q; \mathbb{C}) : X\,{}^tX = O, {}^tX X = O\}.$$

Then $M_{0,0}(p, q; \mathbb{C}) \setminus \{O\}$ is the unique $K_{\mathbb{C}} \simeq O(p, \mathbb{C}) \times O(q, \mathbb{C})$-orbit of dimension $p+q-3$. The associated variety $\mathcal{V}_{\mathfrak{g}}(\varpi_K^{p,q})$ of $\varpi^{p,q}$ is of dimension $p+q-3$, which follows easily from the K-type formula of $\varpi^{p,q}$ (see [23], Theorem 3.6.1). Thus, we have proved:

Lemma 4.4. *The associated variety* $\mathcal{V}_{\mathfrak{g}}(\varpi_K^{p,q})$ *equals* $M_{0,0}(p, q; \mathbb{C})$.

The projection $\mathrm{pr}_{\mathfrak{p} \to \mathfrak{p}'} : \mathfrak{p}^* \to \mathfrak{p}'^*$ is identified with the map

$$\mathrm{pr}_{\mathfrak{p} \to \mathfrak{p}'} : M(p, q; \mathbb{C}) \to M(p', q'; \mathbb{C}) \oplus M(p'', q''; \mathbb{C}), \;\; \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix} \mapsto (X_1, X_4).$$

Suppose $p'p''q'q'' \neq 0$. If we take

$$X := E_{1,1} - E_{p'+1,q'+1} + \sqrt{-1}E_{p'+1,1} + \sqrt{-1}E_{1,p'+1} \in M_{0,0}(p,q;\mathbb{C}),$$

then $\mathrm{pr}_{\mathfrak{p}\to\mathfrak{p}'}(X) = (E_{1,1}, -E_{p'+1,q'+1})$. But $E_{1,1} \notin \mathcal{N}^*_{\mathfrak{o}(p',q')}$ and $-E_{p'+1,q'+1} \notin \mathcal{N}^*_{\mathfrak{o}(p'',q'')}$. Thus, $\mathrm{pr}_{\mathfrak{g}\to\mathfrak{g}'}(X) \notin \mathcal{N}^*_{\mathfrak{g}'}$. It follows from Fact 4.1 (2) that $\varpi^{p,q}_K$ is not discrete decomposable as a $(\mathfrak{g}', K')$-module. Hence (ii) $\Rightarrow$ (iii) in Theorem 4.2 is proved. $\square$

4.5. Proof of (iii) $\Rightarrow$ (i) in Theorem 4.2.

We take an orthogonal complementary subspace $\mathfrak{k}''_0$ of $\mathfrak{k}'_0$ in $\mathfrak{k}_0 \simeq \mathfrak{o}(p) + \mathfrak{o}(q)$. Let $\mathfrak{t}^c_0$ be a Cartan subalgebra of $\mathfrak{k}_0$ such that $\mathfrak{t}''_0 := \mathfrak{t}^c_0 \cap \mathfrak{k}''_0$ is a maximal abelian subspace in $\mathfrak{k}''_0$. We choose a positive system $\Delta^+(\mathfrak{k}, \mathfrak{t}^c)$ which is compatible with a positive system of the restricted root system $\Sigma(\mathfrak{k}, \mathfrak{t}'')$. Then we can find a basis $\{f_i : 1 \le i \le [\frac{p}{2}] + [\frac{q}{2}]\}$ on $\sqrt{-1}\mathfrak{t}^*_0$ such that a positive root system of $\mathfrak{k}$ is given by

$$\begin{aligned}\triangle^+(\mathfrak{k}, \mathfrak{t}^c) =& \{f_i \pm f_j : 1 \le i < j \le [\frac{p}{2}]\}\\ &\cup \{f_i \pm f_j : [\frac{p}{2}] + 1 \le i < j \le [\frac{p}{2}] + [\frac{q}{2}]\}\\ &\cup \Big(\{\pm f_l : 1 \le l \le [\frac{p}{2}]\} \ \ (p\text{:odd}) \Big)\\ &\cup \Big(\{\pm f_l : [\frac{p}{2}] + 1 \le l \le [\frac{p}{2}] + [\frac{q}{2}]\} \ \ (q\text{:odd}) \Big),\end{aligned}$$

and such that

$$\sqrt{-1}(\mathfrak{t}''_0)^* = \sum_{j=1}^{\min(p',p'')} \mathbb{R}f_i + \sum_{j=1}^{\min(q',q'')} \mathbb{R}f_{[\frac{p}{2}]+j} \tag{4.5.1}$$

if we regard $(\mathfrak{t}''_0)^*$ as a subspace of $(\mathfrak{t}^c_0)^*$ by the Killing form.

Suppose $p'q'p''q'' = 0$. Without loss of generality we may and do assume $p'' = 0$, namely, $G' = O(p,q') \times O(q'')$ with $q' + q'' = q$.

Let us first consider the case $p \neq 2$. Then the irreducible $O(p)$-representation $\mathcal{H}^a(\mathbb{R}^p)$ remains irreducible when restricted to $SO(p)$. The corresponding highest weight is given by af_1. It follows from the K-type formula of $\varpi^{p,q}$ (Theorem 3.6.1) that

$$\mathrm{Supp}_K(\varpi^{p,q}) = \{af_1 + bf_{[\frac{p}{2}]+1} : a, b \in \mathbb{N}, a + \frac{p}{2} = b + \frac{q}{2}\}.$$

Therefore we have proved:

$$\mathrm{AS}_K(\varpi^{p,q}) = \mathbb{R}_+(f_1 + f_{[\frac{p}{2}]+1}). \tag{4.5.2}$$

Then $\mathrm{AS}_K(\varpi^{p,q}) \cap \sqrt{-1}(\mathfrak{t}''_0)^* = \{0\}$ from (4.5.1) and (4.5.2), which implies

$$\mathrm{AS}_K(\varpi^{p,q}) \cap \sqrt{-1}\,\mathrm{Ad}^*(K)(\mathfrak{k}'_0)^\perp = \{0\}$$

because $(\mathfrak{t}''_0)^*$ meets any $\mathrm{Ad}^*(K)$-orbit through $(\mathfrak{k}'_0)^\perp$. Therefore, the restriction $\varpi^{p,q}|_{K'}$ is K'-admissible by Fact 4.1 (1).

If $p = 2$, then $\varpi^{p,q}$ splits into two representations (see Remark 3.7.3), say $\varpi^{2,q}_+$ and $\varpi^{2,q}_-$, when restricted to the connected component $SO_0(2,q)$. Likewise, $\mathcal{H}^a(\mathbb{R}^p)$ is a direct sum of two one-dimensional representations when restricted to $SO(2)$ if $a \ge 1$. Then we have

$$\mathrm{AS}_K(\varpi^{2,q}_\pm) = \mathbb{R}_+(\pm f_1 + f_{p+1}).$$

Applying Fact 4.1 (1) to the identity components (G_0, G'_0) of groups (G, G'), we conclude that the restriction $\varpi^{p,q}_\pm|_{K'_0}$ is K'_0-admissible. Hence the restriction $\varpi^{p,q}|_{K'}$ is also K'-admissible. Thus, (iii) $\Rightarrow$ (i) in Theorem 4.2 is proved.

Now the proof of Theorem 4.2 is completed. $\square$

§5. Minimal elliptic representations of $O(p,q)$

5.1. In this section, we introduce a family of irreducible representations of $G = O(p,q)$, denoted by $\pi^{p,q}_{+,\lambda}, \pi^{p,q}_{-,\lambda}$, for $\lambda \in A_0(p,q)$, in three different realizations. These representations are supposed to be attached to minimal elliptic orbits, for $\lambda > 0$ in the sense of the Kirillov-Kostant orbit method. Here, we set

$$A_0(p,q) := \begin{cases} \{\lambda \in \mathbb{Z} + \frac{p+q}{2} : \lambda > -1\} & (p > 1, q \neq 0), \\ \{\lambda \in \mathbb{Z} + \frac{p+q}{2} : \lambda \geq \frac{p}{2} - 1\} & (p > 1, q = 0), \\ \emptyset & (p = 1, q \neq 0) \text{ or } (p = 0), \\ \{-\frac{1}{2}, \frac{1}{2}\} & (p = 1, q = 0). \end{cases} \tag{5.1.1}$$

It seems natural to include the parameter $\lambda = 0, -\frac{1}{2}$ in the definition of $A_0(p,q)$ as above, although $\lambda = -\frac{1}{2}$ is outside the weakly fair range parameter in the sense of Vogan [33]. Cohomologically induced representations for $\lambda = -\frac{1}{2}$ and $\lambda = -1$ will be discussed in details in a subsequent paper. In particular, the case $\lambda = -1$ is of importance in another geometric construction of the minimal representation via Dolbeault cohomology groups (see Part I, Introduction, Theorem B (4)).

5.2. Let $\mathbb{R}^{p,q}$ be the Euclidean space $\mathbb{R}^{p+q}$ equipped with the flat pseudo-Riemannian metric:

$$g_{\mathbb{R}^{p,q}} = {dv_0}^2 + \cdots + {dv_{p-1}}^2 - {dv_p}^2 - \cdots - {dv_{n+1}}^2.$$

We define a hyperboloid by

$$X(p,q) := \{(x,y) \in \mathbb{R}^{p,q} : |x|^2 - |y|^2 = 1\}.$$

We note $X(p,0) \simeq S^{p-1}$ and $X(0,q) = \emptyset$. If $p = 1$, then $X(p,q)$ has two connected components. The group G acts transitively on $X(p,q)$ with isotropy subgroup $O(p-1,q)$ at

$$x^o := {}^t(1, 0, \cdots, 0). \tag{5.2.1}$$

Thus $X(p,q)$ is realized as a homogeneous manifold:

$$X(p,q) \simeq O(p,q)/O(p-1,q).$$

We induce a pseudo-Riemannian metric $g_{X(p,q)}$ on $X(p,q)$ from $\mathbb{R}^{p,q}$ (see [23], §3.2), and write $\Delta_{X(p,q)}$ for the Laplace-Beltrami operator on $X(p,q)$. As in [23], Example 2.2, the Yamabe operator is given by

$$\widetilde{\Delta}_{X(p,q)} = \Delta_{X(p,q)} - \frac{1}{4}(p+q-1)(p+q-3). \tag{5.2.2}$$

For $\lambda \in \mathbb{C}$, we set

$$\begin{aligned} C^\infty_\lambda(X(p,q)) &:= \{f \in C^\infty(X(p,q)) : \Delta_{X(p,q)} f = (-\lambda^2 + \frac{1}{4}(p+q-2)^2) f\} \\ &= \{f \in C^\infty(X(p,q)) : \widetilde{\Delta}_{X(p,q)} f = (-\lambda^2 + \frac{1}{4}) f\}. \end{aligned} \tag{5.2.3}$$

Furthermore, for $\epsilon = \pm$, we write

$$C^\infty_\lambda(X(p,q))_\epsilon := \{f \in C^\infty_\lambda(X(p,q)) : f(-z) = \epsilon f(z),\ z \in X(p,q)\}.$$

Then we have a direct sum decomposition

$$C^\infty_\lambda(X(p,q)) = C^\infty_\lambda(X(p,q))_+ + C^\infty_\lambda(X(p,q))_- \tag{5.2.4}$$

and each space is invariant under left translations of the isometry group G because G commutes with $\Delta_{X(p,q)}$. With the notation in §3.5, we note if $q = 0$, then $C^\infty_\lambda(X(p,0))_{\mathrm{sgn}(-1)^k}$ is finite dimensional and isomorphic to the space of spherical harmonics:

$$\mathcal{H}^k(\mathbb{R}^p) \simeq C^\infty_\lambda(X(p,0)) = C^\infty_\lambda(X(p,0))_{\mathrm{sgn}(-1)^k} \qquad (k := \lambda + \frac{p-2}{2}).$$

<u>**5.3.**</u> Let $G = O(p,q)$ where $p,\ q \geq 1$ and let θ be the Cartan involution corresponding to $K = O(p) \times O(q)$. We extend a Cartan subalgebra $\mathfrak{t}^c_0$ of $\mathfrak{k}_0$ (given in §4.5) to that of $\mathfrak{g}_0$, denoted by $\mathfrak{h}^c_0$. If both p and q are odd, then $\dim \mathfrak{h}^c_0 = \dim \mathfrak{t}^c_0 + 1$; otherwise $\mathfrak{h}^c_0 = \mathfrak{t}^c_0$. The complexification of $\mathfrak{h}^c_0$ is denoted by $\mathfrak{h}^c$.

We can take a basis $\{f_i : 1 \leq i \leq [\frac{p+q}{2}]\}$ of $(\mathfrak{h}^c)^*$ (see §4.5; by a little abuse of notation if both p and q are odd) such that the root systems of $\mathfrak{g}$ is given by

$$\begin{aligned}\triangle(\mathfrak{g},\mathfrak{h}^c) =& \{\pm(f_i \pm f_j) : 1 \leq i < j \leq [\frac{p+q}{2}]\} \\ & \cup \left(\{\pm f_l : 1 \leq l \leq [\frac{p+q}{2}]\}\ \ (p+q\text{:odd})\right).\end{aligned}$$

Let $\{H_i\} \subset \mathfrak{h}^c$ be the dual basis for $\{f_i\} \subset (\mathfrak{h}^c)^*$. Set $\mathfrak{t} := \mathbb{C}H_1$ $(\subset \mathfrak{t}^c \subset \mathfrak{h}^c)$. Then the centralizer L of $\mathfrak{t}$ in G is isomorphic to $SO(2) \times O(p-2,q)$. Let $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ be a θ-stable parabolic subalgebra of $\mathfrak{g}$ with nilpotent radical $\mathfrak{u}$ given by,

$$\triangle(\mathfrak{u},\mathfrak{h}^c) := \{f_1 \pm f_j : 2 \leq j \leq [\frac{p+q}{2}]\} \cup (\{\pm f_1\}\ \ (p+q\text{:odd})\ ),$$

and with a Levi part $\mathfrak{l} = \mathfrak{l}_0 \otimes \mathbb{C}$ given by

$$\mathfrak{l}_0 \equiv Lie(L) \simeq \mathfrak{o}(2) + \mathfrak{o}(p-2,q).$$

Any character of the Lie algebra $\mathfrak{l}_0$ (or any complex character of $\mathfrak{l}$) is determined by its restriction to $\mathfrak{h}^c_0$. So, we shall write $\mathbb{C}_\nu$ for the character of the Lie algebra $\mathfrak{l}_0$ whose restriction to $\mathfrak{h}^c$ is $\nu \in (\mathfrak{h}^c)^*$. With this notation, the character of L acting on $\wedge^{\dim \mathfrak{u}}\mathfrak{u}$ is written as $\mathbb{C}_{2\rho(\mathfrak{u})}$ where

$$\rho(\mathfrak{u}) := (\frac{p+q}{2} - 1)f_1. \tag{5.3.1}$$

The homogeneous manifold G/L carries a G-invariant complex structure with canonical bundle $\wedge^{\mathrm{top}}T^*G/L \simeq G \times_L \mathbb{C}_{2\rho(\mathfrak{u})}$. As an algebraic analogue of a Dolbeault cohomology of a G-equivariant holomorphic vector bundle over a complex manifold G/L, Zuckerman introduced the cohomological parabolic induction $\mathcal{R}^j_\mathfrak{q} \equiv \left(\mathcal{R}^\mathfrak{g}_\mathfrak{q}\right)^j$ $(j \in \mathbb{N})$, which is a covariant functor from the category of metaplectic $(\mathfrak{l},(L\cap K)^\sim)$-modules to that of $(\mathfrak{g},K)$-modules. Here, $\widetilde{L}$ is a metaplectic covering of L defined by the character of L acting on

$\wedge^{\dim \mathfrak{u}}\mathfrak{u} \simeq \mathbb{C}_{2\rho(\mathfrak{u})}$. In this paper, we follow the normalization in [32], Definition 6.20 which is different from the one in [30] by a 'ρ-shift'.

The character $\mathbb{C}_{\lambda f_1}$ of $\mathfrak{l}_0$ lifts to a metaplectic $(\mathfrak{l}, (L \cap K)^{\sim})$-module if and only if $\lambda \in \mathbb{Z} + \frac{p+q}{2}$. In particular, we can define $(\mathfrak{g}, K)$-modules $\mathcal{R}^j_{\mathfrak{q}}(\mathbb{C}_{\lambda f_1})$ for $\lambda \in A_0(p,q)$. The $\mathcal{Z}(\mathfrak{g})$-infinitesimal character of $\mathcal{R}^j_{\mathfrak{q}}(\mathbb{C}_{\lambda f_1})$ is given by

$$(\lambda, \frac{p+q}{2} - 2, \frac{p+q}{2} - 3, \dots, \frac{p+q}{2} - [\frac{p+q}{2}]) \in (\mathfrak{h}^c)^*$$

in the Harish-Chandra parametrization if it is non-zero. In the sense of Vogan [33], we have

$$\begin{aligned} &\mathbb{C}_{\lambda f_1} \text{ is in the good range} && \Leftrightarrow\ \lambda > \frac{p+q}{2} - 2, \\ &\mathbb{C}_{\lambda f_1} \text{ is in the weakly fair range} && \Leftrightarrow\ \lambda \geq 0. \end{aligned}$$

We note that $\mathcal{R}^j_{\mathfrak{q}}(\mathbb{C}_{\lambda f_1}) = 0$ if $j \neq p-2$ and if $\lambda \in A_0(p,q)$. This follows from a general result in [31] for $\lambda \geq 0$; and [14] for $\lambda = -\frac{1}{2}$.

5.4. For $b \in \mathbb{Z}$, we define an algebraic direct sum of $K = O(p) \times O(q)$-modules by

$$\Xi(K : b) \equiv \Xi(O(p) \times O(q) : b) := \bigoplus_{\substack{m,n \in \mathbb{N} \\ m-n \geq b \\ m-n \equiv b \mod 2}} \mathcal{H}^m(\mathbb{R}^p) \boxtimes \mathcal{H}^n(\mathbb{R}^q). \tag{5.4.1}$$

For $\lambda \in A_0(p,q)$, we put

$$b \equiv b(\lambda, p, q) := \lambda - \frac{p}{2} + \frac{q}{2} + 1 \in \mathbb{Z}, \tag{5.4.2}$$

$$\epsilon \equiv \epsilon(\lambda, p, q) := (-1)^b. \tag{5.4.3}$$

We define the line bundle $\mathcal{L}_n$ over G/L by the character nf_1 of L (see §5.3).

Here is a summary for different realizations of the representation $\pi^{p,q}_{+,\lambda}$:

Fact 5.4. *Let $p, q \in \mathbb{N}$ $(p > 1)$.*
1) *For any $\lambda \in A_0(p,q)$, each of the following 5 conditions defines uniquely a $(\mathfrak{g}, K)$-module, which are mutually isomorphic. We shall denote it by $(\pi^{p,q}_{+,\lambda})_K$. The $(\mathfrak{g}, K)$-module $(\pi^{p,q}_{+,\lambda})_K$ is non-zero and irreducible.*
i) *A subrepresentation of the degenerate principal representation* $\mathrm{Ind}^G_{P_{\max}}(\epsilon \otimes \mathbb{C}_\lambda)$ *(see §3.7) with K-type $\Xi(K : b)$.*
i)′ *A quotient of* $\mathrm{Ind}^G_{P_{\max}}(\epsilon \otimes \mathbb{C}_{-\lambda})$ *with K-type $\Xi(K : b)$.*
ii) *A subrepresentation of $C^\infty_\lambda(X(p,q))_K$ with K-type $\Xi(K : b)$.*
iii) *The underlying $(\mathfrak{g}, K)$-module of the Dolbeault cohomology group* $H^{p-2}_{\bar{\partial}}(G/L, \mathcal{L}_{(\lambda + \frac{p+q-2}{2})})_K$.
iii)′ *The Zuckerman-Vogan derived functor module $\mathcal{R}^{p-2}_{\mathfrak{q}}(\mathbb{C}_{\lambda f_1})$.*
2) *In the realization of* (iii), *if $f \in (\pi^{p,q}_{+,\lambda})_K$, then there exists an analytic function $a \in C^\infty(S^{p-1} \times S^{q-1})$ such that*

$$f(\omega \cosh t, \eta \sinh t) = a(\omega, \eta) e^{-(\lambda+\rho)t}(1 + te^{-2t}O(1)) \quad \text{as } t \to \infty$$

Here, we put $\rho = \frac{p+q-2}{2}$.

For details, we refer, for example, to [11] for (i) and (i)$'$; to [27] for (ii) and also for a relation with (i) (under some parity assumption on eigenspaces); to [15], §6 (see also [14]) for (iii)$' \Leftrightarrow$ (ii); and to [34] for (iii) $\Leftrightarrow$ (iii)$'$. The second statement follows from a general theory of the boundary value problem with regular singularities; or also follows from a classical asymptotic formula of hypergeometric functions (see (8.3.1)) in our specific setting.

Remark. 1) By definition, (i) and (i)$'$ make sense for $p > 1$ and $q > 0$; and others for $p > 1$ and $q \geq 0$.
2) Each of the realization (i), (i)$'$, (ii), and (iii) also gives a globalization of $\pi^{p,q}_{+,\lambda}$, namely, a continuous representation of G on a topological vector space. Because all of $(\pi^{p,q}_{+,\lambda})_K$ ($\lambda \in A_0(p,q)$) are unitarizable we may and do take the globalization $\pi^{p,q}_{+,\lambda}$ to be the unitary representation of G.
3) If $\lambda > 0$ and $\lambda \in A_0(p,q)$, then the realization (ii) of $\pi^{p,q}_{+,\lambda}$ gives a discrete series representation for $X(p,q)$. Conversely,

$$\{\pi^{p,q}_{+,\lambda} : \lambda \in A_0(p,q), \lambda > 0\}$$

exhausts the set of discrete series representations for $X(p,q)$.

If $(p,q) = (1,0)$, then $O(p,q) \simeq O(1)$ and it is convenient to define representations of $O(1)$ by

$$\pi^{1,0}_{+,\lambda} = \begin{cases} \mathbf{1} & (\lambda = -\frac{1}{2}), \\ \mathrm{sgn} & (\lambda = \frac{1}{2}), \\ 0 & (\text{otherwise}). \end{cases}$$

As we defined $\pi^{p,q}_{+,\lambda}$ in Fact 5.4, we can also define an irreducible unitary representation, denoted by $\pi^{p,q}_{-,\lambda}$, for $\lambda \in A_0(q,p)$ such that the underlying $(\mathfrak{g}, K)$-module has the following K-type

$$\bigoplus_{\substack{m,n \in \mathbb{N} \\ m-n \leq -\lambda + \frac{q}{2} - \frac{p}{2} - 1 \\ m-n \equiv -\lambda + \frac{q}{2} - \frac{p}{2} - 1 \mod 2}} \mathcal{H}^m(\mathbb{R}^p) \boxtimes \mathcal{H}^n(\mathbb{R}^q).$$

Similarly to $\pi^{p,q}_{+,\lambda}$, the representations $\pi^{p,q}_{-,\lambda}$ are realized in function spaces on another hyperboloid $O(p,q)/O(p,q-1)$.

In order to understand the notation here, we remark:

i) $\pi^{p,q}_{-,\lambda} \in \widehat{O(p,q)}$ corresponds to the representation $\pi^{q,p}_{+,\lambda} \in \widehat{O(q,p)}$ if we identify $O(p,q)$ with $O(q,p)$.

ii) $\pi^{p,0}_{+,\lambda} \simeq \mathcal{H}^k(\mathbb{R}^p)$, where $k = \lambda - \frac{p-2}{2}$ and $p \geq 1, k \in \mathbb{N}$.

5.5. The case $\lambda = \pm\frac{1}{2}$ is delicate, which happens when $p+q \in 2\mathbb{N}+1$.

First, we assume $p+q \in 2\mathbb{N}+1$. By using the equivalent realizations of $\pi^{p,q}_{+,\lambda}$ in Fact 5.4 and by the classification of the composition series of the most degenerate principal series representation $\mathrm{Ind}^G_{P^{\max}}(\epsilon \otimes \mathbb{C}_\lambda)$ (see [11]), we have non-splitting short exact sequences of $(\mathfrak{g}, K)$-modules:

$$0 \to (\pi^{p,q}_{-,-\frac{1}{2}})_K \to \mathrm{Ind}^G_{P^{\max}}((-1)^{\frac{p-q+1}{2}} \otimes \mathbb{C}_{-\frac{1}{2}}) \to (\pi^{p,q}_{+,\frac{1}{2}})_K \to 0, \tag{5.5.1}$$

$$0 \to (\pi^{p,q}_{+,-\frac{1}{2}})_K \to \mathrm{Ind}^G_{P^{\max}}((-1)^{\frac{p-q-1}{2}} \otimes \mathbb{C}_{-\frac{1}{2}}) \to (\pi^{p,q}_{-,\frac{1}{2}})_K \to 0. \tag{5.5.2}$$

Because $\pi^{p,q}_{+,\lambda}$ $(\lambda \in A_0(p,q))$ is self-dual, the dual $(\mathfrak{g}, K)$-modules of (5.5.1) and (5.5.2) give the following non-splitting short exact sequences of $(\mathfrak{g}, K)$-modules:

$$0 \to (\pi^{p,q}_{+,\frac{1}{2}})_K \to \operatorname{Ind}^G_{P^{\max}}((-1)^{\frac{p-q+1}{2}} \otimes \mathbb{C}_{\frac{1}{2}}) \to (\pi^{p,q}_{-,-\frac{1}{2}})_K \to 0, \tag{5.5.3}$$

$$0 \to (\pi^{p,q}_{-,\frac{1}{2}})_K \to \operatorname{Ind}^G_{P^{\max}}((-1)^{\frac{p-q-1}{2}} \otimes \mathbb{C}_{\frac{1}{2}}) \to (\pi^{p,q}_{+,-\frac{1}{2}})_K \to 0. \tag{5.5.4}$$

Next, we assume $p + q \in 2\mathbb{N}$. Then, $\varpi^{p,q}$ is realized as a subrepresentation of some degenerate principal series (see [23], Lemma 3.7.2). More precisely, we have non-splitting short exact sequences of $(\mathfrak{g}, K)$-modules

$$0 \to \varpi^{p,q}_K \to \operatorname{Ind}^G_{P^{\max}}((-1)^{\frac{p-q}{2}} \otimes \mathbb{C}_{-1}) \to \big((\pi^{p,q}_{-,1})_K \oplus (\pi^{p,q}_{+,1})_K\big) \to 0, \tag{5.5.5}$$

$$0 \to \big((\pi^{p,q}_{-,1})_K \oplus (\pi^{p,q}_{+,1})_K\big) \to \operatorname{Ind}^G_{P^{\max}}((-1)^{\frac{p-q}{2}} \otimes \mathbb{C}_{1}) \to \varpi^{p,q}_K \to 0, \tag{5.5.6}$$

and an isomorphism of $(\mathfrak{g}, K)$-modules:

$$\operatorname{Ind}^G_{P^{\max}}((-1)^{\frac{p-q+2}{2}} \otimes \mathbb{C}_0) \simeq (\pi^{p,q}_{-,0})_K \oplus (\pi^{p,q}_{+,0})_K. \tag{5.5.7}$$

These results will be used in another realization of the unipotent representation $\varpi^{p,q}$, namely, as a submodule of the Dolbeault cohomology group in a subsequent paper (cf. Part 1, Introduction, Theorem B (4)).

§6. Conformal embedding of the hyperboloid

This section prepares the geometric setup which will be used in §7 and §9 for the branching problem of $\varpi^{p,q}|_{G'}$. Throughout this section, we shall use the following notation:

$$|x|^2 := |x'|^2 + |x''|^2 = \sum_{i=1}^{p'} (x'_i)^2 + \sum_{j=1}^{p''} (x''_j)^2, \text{ for } x := (x', x'') \in \mathbb{R}^{p'+p''} = \mathbb{R}^p,$$
$$|y|^2 := |y'|^2 + |y''|^2 = \sum_{i=1}^{q'} (y'_i)^2 + \sum_{j=1}^{q''} (y''_j)^2, \text{ for } y := (y', y'') \in \mathbb{R}^{q'+q''} = \mathbb{R}^q.$$

6.1. We define two open subsets of $\mathbb{R}^{p+q}$ by

$$\mathbb{R}^{p'+p'',q'+q''}_+ := \{(x,y) = ((x',x''),(y',y'')) \in \mathbb{R}^{p'+p'',q'+q''} : |x'| > |y'|\},$$
$$\mathbb{R}^{p'+p'',q'+q''}_- := \{(x,y) = ((x',x''),(y',y'')) \in \mathbb{R}^{p'+p'',q'+q''} : |x'| < |y'|\}.$$

Then the disjoint union $\mathbb{R}^{p'+p'',q'+q''}_+ \cup \mathbb{R}^{p'+p'',q'+q''}_-$ is open dense in $\mathbb{R}^{p+q}$. Let us consider the intersection of $\mathbb{R}^{p'+p'',q'+q''}_\pm$ with the submanifolds given in §3.2:

$$M \subset \Xi \subset \mathbb{R}^{p,q}.$$

Then, we define two open subsets of $M \simeq S^{p-1} \times S^{q-1}$ by

$$M_\pm := M \cap \mathbb{R}^{p'+p'',q'+q''}_\pm. \tag{6.1.1}$$

Likewise, we define two open subsets of the cone Ξ by

(6.1.2) $$\Xi_\pm := \Xi \cap \mathbb{R}_\pm^{p'+p'',q'+q''}.$$

We notice that if $(x,y) = ((x',x''),(y',y'')) \in \Xi$ then

$$|x'| > |y'| \iff |x''| < |y''|$$

because $|x'|^2 + |x''|^2 = |y'|^2 + |y''|^2$. The following statement is immediate from definition:

(6.1.3) $$\Xi_+ = \emptyset \;\Leftrightarrow\; M_+ = \emptyset \;\Leftrightarrow\; p'q'' = 0.$$

(6.1.4) $$\Xi_- = \emptyset \;\Leftrightarrow\; M_- = \emptyset \;\Leftrightarrow\; p''q' = 0.$$

6.2. We embed the direct product of hyperboloids

$$X(p',q') \times X(q'',p'') = \{((x',y'),(y'',x'')) : |x'|^2 - |y'|^2 = |y''|^2 - |x''|^2 = 1\}.$$

into Ξ_+ $(\subset \mathbb{R}^{p,q})$ by the map

(6.2.1) $$X(p',q') \times X(q'',p'') \hookrightarrow \Xi_+,\ ((x',y'),(y'',x'')) \mapsto (x',x'',y',y'').$$

The image is transversal to rays (see [23], §3.3 for definition) and the induced pseudo-Riemannian metric $g_{X(p',q')\times X(q'',p'')}$ on $X(p',q') \times X(q'',p'')$ has signature $(p'-1,q') + (p'',q''-1) = (p-1,q-1)$. With the notation in §5.2, we have

$$g_{X(p',q')\times X(q'',p'')} = g_{X(p',q')} \oplus (-g_{X(q'',p'')}).$$

We note that if $p'' = q' = 0$, then $X(p',q') \times X(q'',p'')$ is diffeomorphic to $S^{p-1} \times S^{q-1}$, and $g_{X(p',0)\times X(q'',0)}$ is nothing but the pseudo-Riemannian metric $g_{S^{p-1}\times S^{q-1}}$ of signature $(p-1,q-1)$ (see [23], §3.3).

By the same computation as in (3.4.1), we have the relationship among the Yamabe operators on hyperboloids (see also (5.2.2)) by

(6.2.2) $$\widetilde{\Delta}_{X(p',q')\times X(q'',p'')} = \widetilde{\Delta}_{X(p',q')} - \widetilde{\Delta}_{X(q',p'')}.$$

We denote by Φ_1 the composition of (6.2.1) and the projection $\Phi\colon \Xi \to M$ (see [23], (3.2.4)), namely,

(6.2.3) $$\Phi_1\colon X(p',q') \times X(q'',p'') \hookrightarrow M,\ ((x',y'),(y'',x'')) \mapsto \left(\frac{(x',x'')}{|x|}, \frac{(y',y'')}{|y|}\right).$$

Lemma 6.2. 1) *The map $\Phi_1\colon X(p',q') \times X(q'',p'') \to M$ is a diffeomorphism onto M_+. The inverse map $\Phi_1^{-1}\colon\ M_+ \to X(p',q') \times X(q'',p'')$ is given by the formula:*

(6.2.4) $$((u',u''),(v',v'')) \mapsto \left(\frac{(u',v')}{\sqrt{|u'|^2-|v'|^2}}, \frac{(v'',u'')}{\sqrt{|v''|^2-|u''|^2}}\right).$$

2) *Φ_1 is a conformal map with conformal factor $|x|^{-1} = |y|^{-1}$, where $x = (x',x'') \in \mathbb{R}^{p'+p''}$ and $y = (y',y'') \in \mathbb{R}^{q'+q''}$. Namely, we have*

$$\Phi_1^*(g_{S^{p-1}\times S^{q-1}}) = \frac{1}{|x|^2} g_{X(p',q')\times X(q'',p'')}.$$

Proof. The first statement is straightforward in light of the formula

$$|u'|^2 - |v'|^2 = |v''|^2 - |u''|^2 > 0$$

for $(u,v) = ((u',u''),(v',v'')) \in M_+ \subset S^{p-1} \times S^{q-1}$.
The second statement is a special case of Lemma 3.3. $\square$

6.3. Now, the conformal diffeomorphism $\Phi_1\colon X(p',q') \times X(q'',p'') \xrightarrow{\sim} M_+$ establishes a bijection of the kernels of the Yamabe operators owing to Proposition 2.6:

Lemma 6.3. *$\widetilde{\Phi_1^*}$ gives a bijection from* $\operatorname{Ker}\widetilde{\Delta}_{M_+}$ *onto* $\operatorname{Ker}\widetilde{\Delta}_{X(p',q')\times X(q'',p'')}$.

Here, the twisted pull-backs $\widetilde{\Phi_1^*}$ and $\widetilde{(\Phi_1^{-1})^*}$ (see Definition 2.3), namely,

$$\widetilde{\Phi_1^*}\ :\quad C^\infty(M_+) \to C^\infty(X(p',q')\times X(q'',p'')), \tag{6.3.1}$$

$$\widetilde{(\Phi_1^{-1})^*}\colon C^\infty(X(p',q')\times X(q'',p'')) \to C^\infty(M_+), \tag{6.3.2}$$

are given by the formulae

$$(\widetilde{\Phi_1^*}F)(x',y',y'',x'') := (|x'|^2+|x''|^2)^{-\frac{p+q-4}{4}} F\left(\frac{(x',x'')}{\sqrt{|x'|^2+|x''|^2}}, \frac{(y',y'')}{\sqrt{|y'|^2+|y''|^2}}\right),$$

$$(\widetilde{(\Phi_1^{-1})^*}f)(u',u'',v',v'') := (|u'|^2-|v'|^2)^{-\frac{p+q-4}{4}} f\left(\frac{(u',v')}{\sqrt{|u'|^2-|v'|^2}}, \frac{(u'',v'')}{\sqrt{|v''|^2-|u''|^2}}\right),$$

respectively. We remark that $\widetilde{(\Phi_1^{-1})^*} = (\widetilde{\Phi_1^*})^{-1}$.

6.4. Similarly to §6.2, we consider another embedding

$$X(q',p')\times X(p'',q'') \hookrightarrow \Xi_-, \quad ((y',x'),(x'',y'')) \mapsto (x',x'',y',y''). \tag{6.4.1}$$

The composition of (6.4.1) and the projection $\Phi\colon \Xi\to M$ is denoted by

$$\Phi_2\colon X(q',p')\times X(p'',q'') \hookrightarrow M,\ ((y',x'),(x'',y'')) \mapsto \left(\frac{(x',x'')}{|x|}, \frac{(y',y'')}{|y|}\right). \tag{6.4.2}$$

Obviously, results analogous to Lemma 6.2 and Lemma 6.3 hold for Φ_2. For example, here is a lemma parallel to Lemma 6.2:

Lemma 6.4. *The map $\Phi_2\colon X(q',p')\times X(p'',q'')\to M_-$ is a conformal diffeomorphism onto M_-. The inverse map $\Phi_2^{-1}\colon M_-\to X(q',p')\times X(p'',q'')$ is given by*

$$((u',u''),(v',v''))\mapsto \left(\frac{(v',u')}{\sqrt{|v'|^2-|u'|^2}}, \frac{(u'',v'')}{\sqrt{|u''|^2-|v''|^2}}\right).$$

§7. Explicit branching formula (discrete decomposable case)

If one of p',q',p'' or q'' is zero, then the restriction $\varpi^{p,q}|_{G'}$ is decomposed discretely into irreducible representations of $G' = O(p',q')\times O(p'',q'')$ as we saw in §4. In this case, we can determine the branching laws of $\varpi^{p,q}|_{G'}$ as follows:

Theorem 7.1. *Let $p+q\in 2\mathbb{N}$. If $q''\ge 1$ and $q'+q''=q$, then we have an irreducible decomposition of the unitary representation $\varpi^{p,q}$ when restricted to $O(p,q')\times O(q'')$:*

$$\begin{aligned} \varpi^{p,q}|_{O(p,q')\times O(q'')} &\simeq \sum_{l=0}^{\infty}{}^{\oplus}\, \pi^{p,q'}_{+,l+\frac{q''}{2}-1} \boxtimes \pi^{0,q''}_{-,l+\frac{q''}{2}-1} \\ &\simeq \sum_{l=0}^{\infty}{}^{\oplus}\, \pi^{p,q'}_{+,l+\frac{q''}{2}-1} \boxtimes \mathcal{H}^l(\mathbb{R}^{q''}). \end{aligned} \tag{7.1.1}$$

We shall prove Theorem 7.1 in §7.5 after we prepare an algebraic lemma in §7.3 and a geometric lemma in §7.4.

Remark. The formula in Theorem 7.1 is nothing but a K-type formula (see Theorem 3.6.1) when $q' = 0$.

7.2. The branching law (7.1.1) is an infinite direct sum for $q'' > 1$. This subsection treats the case $q'' = 1$, which is particularly interesting, because the branching formula consists of only two irreducible representations (we recall $\mathcal{H}^l(\mathbb{R}^1) \neq 0$ if and only if $l = 0, 1$). For simplicity, we shall assume $q \geq 3$ in §7.2.

It follows from Theorem 7.1 with $q' = 1$ that

$$\begin{aligned} \varpi^{p,q}|_{O(p,q-1)\times O(1)} &\simeq \left(\pi^{p,q-1}_{+,-\frac{1}{2}} \boxtimes \pi^{0,1}_{-,-\frac{1}{2}}\right) \oplus \left(\pi^{p,q-1}_{+,\frac{1}{2}} \boxtimes \pi^{0,1}_{-,\frac{1}{2}}\right) \\ &\simeq \left(\pi^{p,q-1}_{+,-\frac{1}{2}} \boxtimes \mathbf{1}\right) \oplus \left(\pi^{p,q-1}_{+,\frac{1}{2}} \boxtimes \mathrm{sgn}\right). \end{aligned} \tag{7.2.1}$$

This means that $\varpi^{p,q}_K$ can be realized in a subspace of $C^\infty_{\frac{1}{2}}(X(p, q-1))$, namely, the kernel of the Yamabe operator $\widetilde{\Delta}_{X(p,q-1)}$ (see (5.2.3)).

More precisely, according to the direct sum decomposition (see (5.2.4)), we have

$$\operatorname{Ker} \widetilde{\Delta}_{X(p,q-1)} \equiv C^\infty_{\frac{1}{2}}(X(p, q-1)) = C^\infty_{\frac{1}{2}}(X(p, q-1))_+ + C^\infty_{\frac{1}{2}}(X(p, q-1))_-. \tag{7.2.2}$$

We recall that the central element $-I_{p+q} \in G$ acts on $\varpi^{p,q}$ with scalar δ, where

$$\delta := (-1)^{\frac{p-q}{2}}. \tag{7.2.3}$$

In view of the composition series of eigenspaces on the hyperboloid (see [27] for the case $\delta = +$; similar for $\delta = -$), we have non-splitting exact sequences of Harish-Chandra modules of $O(p, q-1)$:

$$0 \to (\pi^{p,q-1}_{+,-\frac{1}{2}})_K \to \left(C^\infty_{\frac{1}{2}}(X(p, q-1))_\delta\right)_K \to (\pi^{p,q-1}_{-,\frac{1}{2}})_K \to 0, \tag{7.2.4}$$

$$0 \to (\pi^{p,q-1}_{+,\frac{1}{2}})_K \to \left(C^\infty_{\frac{1}{2}}(X(p, q-1))_{-\delta}\right)_K \to (\pi^{p,q-1}_{-,-\frac{1}{2}})_K \to 0, \tag{7.2.5}$$

Here is a realization of $\varpi^{p,q}$ in a subspace of the kernel of the Yamabe operator on the hyperboloid $X(p, q-1)$, on which $O(p, q)$ acts as meromorphic conformal transformations.

Corollary 7.2.1. *Let $W_\pm$ be the unique non-trivial subrepresentation of $O(p, q-1)$ in $(\operatorname{Ker} \widetilde{\Delta}_{X(p,q-1)})_\pm \equiv C^\infty_{\frac{1}{2}}(X(p, q-1))_\pm$. Each of the underlying $(\mathfrak{g}, K)$-module is infinitesimally unitarizable, and we denote the resulting unitary representation by $\overline{W_\pm}$. Then, the irreducible unitary representation $\varpi^{p,q}$ of $O(p, q)$ is realized on the direct sum $\overline{W_+} + \overline{W_-}$.*

We note that $\overline{W_{-\delta}} \subset L^2(X(p, q-1))$ and $\overline{W_\delta} \not\subset L^2(X(p, q-1))$ where $\delta = (-1)^{\frac{p-q}{2}}$.

It is interesting to note that the Laplacian $\Delta_{X(p,q-1)}$ acts on a discrete series $\pi^{p,q-1}_{+,\lambda}$ for the hyperboloid $X(p, q-1)$ as a scalar $-\lambda^2 + \frac{1}{4}(p+q-3)^2$, which attains the maximum when $\lambda = \frac{1}{2}$ if $p + (q-1) \in 2\mathbb{N} + 1$.

Taking the direct sum of two exact sequences (7.2.4) and (7.2.5), we have the following:

Theorem 7.2.2. *There is a non-split exact sequence of Harish-Chandra modules for $O(p-1,q)$.*

$$0 \to (\varpi^{p,q})_K \to (\operatorname{Ker} \widetilde{\Delta}_{X(p,q-1)})_K \to (\varpi^{p+1,q-1})_K \to 0. \tag{7.2.6}$$

It is a mysterious phenomenon in (7.2.6) that $\varpi^{p,q}$ extends to a representation of $O(p,q)$ and $(\varpi^{p+1,q-1})_K$ to that of $O(p+1,q-1)$. So, different real forms of $O(p+q,\mathbb{C})$ act on subquotients of the kernel of the Yamabe operator on the hyperboloid $X(p,q-1) = O(p,q-1)/O(p-1,q-1)$!

Here, we remark that $\operatorname{Ker} \widetilde{\Delta}_{X(p,q-1)} \cap L^2(X(p,q-1)) \neq \{0\}$ if and only if $p+q \in 2\mathbb{Z}$, by the classification of discrete series for the hyperboloid $X(p,q-1)$ for $p>1$.

7.3. By Theorem 4.2, the restriction $\varpi^{p,q}|_{K'}$ is K'-admissible, where $K' \simeq O(p)\times O(q')\times O(q'')$. Let us first find the K'-structure of $\varpi^{p,q}$. We recall a classical branching law with respect to $(O(q), O(q')\times O(q''))$ $(q = q'+q'', q' \geq 1, q'' \geq 1)$:

$$\mathcal{H}^n(\mathbb{R}^q) \simeq \bigoplus_{\substack{k,l\in\mathbb{N}\\ k+l\leq n\\ k+l\equiv n}} \mathcal{H}^k(\mathbb{R}^{q'}) \boxtimes \mathcal{H}^l(\mathbb{R}^{q''}). \tag{7.3.1}$$

Then we have isomorphisms as K'-modules:

$$\begin{aligned}
&\bigoplus_{\substack{m,n\in\mathbb{N}\\ m-n=b\\ m-n\equiv b \mod 2}} \mathcal{H}^m(\mathbb{R}^p) \boxtimes \mathcal{H}^n(\mathbb{R}^q)|_{O(p)\times O(q')\times O(q'')}\\
&\simeq \bigoplus_{\substack{m,n\in\mathbb{N}\\ m-n=b\\ m-n\equiv b \mod 2}} \bigoplus_{\substack{k,l\in\mathbb{N}\\ k+l\leq n\\ k+l\equiv n \mod 2}} \mathcal{H}^m(\mathbb{R}^p) \boxtimes \mathcal{H}^k(\mathbb{R}^{q'}) \boxtimes \mathcal{H}^l(\mathbb{R}^{q''})\\
&\simeq \bigoplus_{l\in\mathbb{N}} \bigoplus_{\substack{m,k\in\mathbb{N}\\ m-b\geq k+l\\ m-b\equiv k+l \mod 2}} \mathcal{H}^m(\mathbb{R}^p) \boxtimes \mathcal{H}^k(\mathbb{R}^{q'}) \boxtimes \mathcal{H}^l(\mathbb{R}^{q''})\\
&\simeq \bigoplus_{l\in\mathbb{N}} \Xi(O(p)\times O(q') : b+l) \boxtimes \mathcal{H}^l(\mathbb{R}^{q''})
\end{aligned}$$

In view of Theorem 3.6.1, we have proved:

Lemma 7.3. *We have an isomorphism of K'-modules:*

$$\varpi_K^{p,q} \simeq \bigoplus_{l\in\mathbb{N}} \Xi(O(p)\times O(q') : \frac{q-p}{2}+l) \boxtimes \mathcal{H}^l(\mathbb{R}^{q''}) \tag{7.3.2}$$

7.4. By Theorem 4.2 and [17], Lemma 1.3, the underlying $(\mathfrak{g},K)$-module $\varpi_K^{p,q}$ is decomposed into an algebraic direct sum of irreducible $(\mathfrak{g}',K')$-modules:

$$\varpi_K^{p,q} \simeq \bigoplus_{\tau} m_\tau \tau \simeq \bigoplus_{\tau_1,\tau_2} m_{\tau_1,\tau_2} \tau_1 \boxtimes \tau_2, \tag{7.4.1}$$

where $m_\tau \in \mathbb{N}$ and τ runs over irreducible $(\mathfrak{g}',K')$-modules or equivalently, τ_1 runs over irreducible $(\mathfrak{g}'_1,K'_1)$-modules (with obvious notation for $G'_1 := O(p,q')$) and τ_2 runs over irreducible $O(q'')$-modules. It follows from Lemma 7.3 that for each l there exists a $(\mathfrak{g}'_1,K'_1)$-module W_l which is a direct sum of irreducible $(\mathfrak{g}'_1,K'_1)$-module such that W_l is isomorphic to $\Xi(O(p)\times O(q') : \frac{q-p}{2}+l)$ as K'_1-modules. Let us prove that W_l is in fact irreducible as a $(\mathfrak{g}'_1,K'_1)$-module.

Lemma 7.4. *W_l is realized in a subspace of $C^\infty_\lambda(X(p,q'))$ with $\lambda = l + \frac{q''}{2} - 1$.*

Proof. In our conformal construction of $\varpi^{p,q}$ in §3, we recall that each K-finite vector of $\varpi^{p,q}$ is an analytic function satisfying the Yamabe equation on $M \simeq S^{p-1} \times S^{q-1}$. By using the conformal diffeomorphism $\Phi_1 \colon X(p,q') \times S^{q''-1} \to M_+$, an open dense subset of M (see (6.1.4) with $p'' = 0$), we can realize $W_l \times \mathcal{H}^l(\mathbb{R}^{q''})$ in the space of smooth functions on $X(p,q') \times S^{q''-1}$ satisfying the Yamabe equation by the following diagram:

$$\begin{array}{ccccccc}
\mathcal{A}(M) & \underset{\text{dense}}{\subset} & C^\infty(M) & \underset{\text{restriction}}{\hookrightarrow} & C^\infty(M_+) & \underset{\Phi_1^*}{\overset{\sim}{\to}} & C^\infty(X(p,q') \times S^{q''-1}) \\
\cup & & \cup & & \cup & & \cup \\
W_l \boxtimes \mathcal{H}^l(\mathbb{R}^{q''}) \subset \varpi^{p,q}_K & \underset{\text{dense}}{\subset} & \operatorname{Ker} \widetilde{\Delta}_M & \underset{\text{restriction}}{\hookrightarrow} & \operatorname{Ker} \widetilde{\Delta}_{M_+} & \underset{\Phi_1^*}{\overset{\sim}{\to}} & \operatorname{Ker} \widetilde{\Delta}_{X(p,q') \times S^{q''-1}}.
\end{array}$$

Because $\widetilde{\Delta}_{X(p,q') \times S^{q''-1}} = \widetilde{\Delta}_{X(p',q')} - \widetilde{\Delta}_{S^{q''-1}}$ (see (6.2.2)) acts on $W_l \boxtimes \mathcal{H}^l(\mathbb{R}^{q''})$ as 0, and because $\widetilde{\Delta}_{S^{q''-1}}$ acts on $\mathcal{H}^l(\mathbb{R}^{q''})$ as a scalar $\frac{1}{4} - (l + \frac{q''}{2} - 1)^2$ (see (3.5.1)), we conclude that $\widetilde{\Delta}_{X(p',q')}$ acts on W_l as the same scalar. Hence, Lemma is proved. $\square$

__7.5.__ Let us complete the proof of Theorem 7.1. It follows from Lemma 7.4 together with the K_1'-structure of W_l in §7.3 that W_l is irreducible and isomorphic to $(\pi^{p,q'}_{+,l+\frac{q''-2}{2}})_{K_1'}$ as $(\mathfrak{g}_1', K_1')$-module (see Fact 5.4). Therefore, we have an isomorphism of $(\mathfrak{g}', K')$-modules

$$\varpi^{p,q}_K \simeq \bigoplus_{l=0}^{\infty} (\pi^{p,q'}_{+,l+\frac{q''}{2}-1})_{K_1'} \boxtimes \mathcal{H}^l(\mathbb{R}^{q''}) \quad \text{(algebraic direct sum).}$$

Taking the closure in the Hilbert space, we have (7.1.1). Hence Theorem 7.1 is proved. $\square$

__7.6.__ So far, we have not used the irreducibility of $\varpi^{p,q}$ in the branching law. Although the irreducibility of $\varpi^{p,q}$ $(p,q) \neq (2,2)$ is known [2], we can give a new and simple proof for it, as an application of the branching formulae in Theorem 7.1.

Theorem 7.6. *Let $p, q \geq 2$, $p + q \in 2\mathbb{Z}$ and $(p,q) \neq (2,2)$. Then, $\varpi^{p,q}$ is an irreducible representation of $O(p,q)$.*

Proof. Suppose $W \neq \{0\}$ is a closed invariant subspace of the unitary representation $(\varpi^{p,q}, \overline{V^{p,q}})$. We want to prove $W = \overline{V^{p,q}}$.

Without loss of generality (p and q play a symmetric role), we may assume $q \geq 3$, and fix $q' \geq 1, q'' \geq 2$ such that $q' + q'' = q$. We write $\overline{V^{p,q}} = \sum_{l=0}^{\infty}{}^{\oplus} V_l$ according to the irreducible decomposition (7.1.1) of $G' = G_1' \times G_2' := O(p,q') \times O(q'')$.

Because W is non-zero, W contains a K-type of the form $\mathcal{H}^a(\mathbb{R}^p) \boxtimes \mathcal{H}^b(\mathbb{R}^q)$, which we fix once for all. In view of the branching formula (7.3.1), $\mathcal{H}^b(\mathbb{R}^q)$ contains a non-zero $O(q'')$-fixed vector. That is,

$$V_0 := \overline{V^{p,q}}^{G_2'} \supset W^{G_2'} \supset \mathcal{H}^a(\mathbb{R}^p) \boxtimes \mathcal{H}^b(\mathbb{R}^q)^{G_2'} \neq \{0\}$$

Because $G_1' = O(p,q')$ acts on V_0 by $\pi^{p,q'}_{+,\frac{q''}{2}-1}$ (see Theorem 7.1) which is irreducible, and because $V_0 \supset W^{G_2'}$ is stable under the action of G_1', we conclude that $V_0 = W^{G_2'}$. Hence, we have proved

$$W \supset V_0. \tag{7.6.1}$$

If $W \neq \overline{V^{p,q}}$, the orthogonal complement $W^\perp \supset V_0$ by the same argument. This would contradict to $W \cap W^\perp = \{0\}$. Therefore, W must coincides with $\overline{V^{p,q}}$. Hence, $\varpi^{p,q}$ is irreducible. □

§8. Inner product on $\varpi^{p,q}$ and the Parseval-Plancherel formula

In this section, we prove the Parseval-Plancherel type formula for our discrete decomposable branching law given in Theorem 7.1. Our main result in this section is Theorem 8.5.

8.1. The subsections §8.1 ∼ §8.3 review some results and notation from [21] without proof. We set

$$\begin{aligned} A_+(p,q) &:= A_0(p,q) \cap \{\lambda \in \mathbb{R} : \lambda > 0\}, \\ \Lambda_{+-}(\lambda) &:= \{(\lambda', \lambda'') \in A_+(p', q') \times A_+(q'', p'') : \lambda' - \lambda'' - \lambda - 1 \in 2\mathbb{N}\}, \\ \Lambda_{++}(\lambda) &:= \{(\lambda', \lambda'') \in A_+(p', q') \times A_+(p'', q'') : \lambda - \lambda' - \lambda'' - 1 \in 2\mathbb{N}\}. \end{aligned}$$

We recall the Jacobi function

$$\varphi_{i\lambda}^{(\lambda'',\lambda')}(t) := {}_2F_1\left(\frac{\lambda'+\lambda''+1-\lambda}{2}, \frac{\lambda'+\lambda''+1+\lambda}{2}; \lambda''+1; -\sinh^2 t\right),$$

which satisfies

$$\varphi_{i\lambda}^{(\lambda'',\lambda')}(0) = 1,$$

$$\left(t(1-t)\frac{d^2}{dt^2} + (c - (a+b+1)t)\frac{d}{dt} - ab\right)\varphi_{i\lambda}(t) = 0$$

with

$$a := \frac{\lambda'+\lambda''+1-\lambda}{2}, \quad b := \frac{\lambda'+\lambda''+1+\lambda}{2}, \quad c := \lambda''+1.$$

We set meromorphic functions of three variables λ, λ' and λ'' by

$$\begin{aligned} V_{+-,\lambda}^{(\lambda',\lambda'')} &:= \frac{(\Gamma(\lambda''+1))^2\ \Gamma(\frac{\lambda'-\lambda''+\lambda+1}{2})\ \Gamma(\frac{\lambda'-\lambda''-\lambda+1}{2})}{2\lambda\ \Gamma(\frac{\lambda'+\lambda''+\lambda+1}{2})\ \Gamma(\frac{\lambda'+\lambda''-\lambda+1}{2})}, \\ V_{++,\lambda}^{(\lambda',\lambda'')} &:= \frac{(\Gamma(\lambda''+1))^2\ \Gamma(\frac{-\lambda'-\lambda''+\lambda+1}{2})\ \Gamma(\frac{\lambda'-\lambda''+\lambda+1}{2})}{2\lambda\ \Gamma(\frac{-\lambda'+\lambda''+\lambda+1}{2})\ \Gamma(\frac{\lambda'+\lambda''+\lambda+1}{2})}. \end{aligned}$$

Then we have

$$\begin{aligned} &\int_0^\infty |\varphi_{i\lambda}^{(\lambda'',\lambda')}(t)|^2 (\cosh t)^{2\lambda'+1} (\sinh t)^{2\lambda''+1}\, dt = V_{+-,\lambda}^{(\lambda',\lambda'')} && \text{for } (\lambda',\lambda'') \in \Lambda_{+-}(\lambda), \\ &\int_0^{\frac{\pi}{2}} |\varphi_{i\lambda}^{(\lambda'',\lambda')}(i\theta)|^2 (\cos\theta)^{2\lambda'+1} (\sin\theta)^{2\lambda''+1}\, d\theta = V_{++,\lambda}^{(\lambda',\lambda'')} && \text{for } (\lambda',\lambda'') \in \Lambda_{++}(\lambda). \end{aligned}$$

8.2. We put

$$M \equiv M(\lambda,\lambda',\lambda'') := (-1)^{\frac{\lambda'-\lambda''-\lambda-1}{2}} \frac{\Gamma(\frac{\lambda'+\lambda''-\lambda+1}{2})\Gamma(\lambda+1)}{\Gamma(\frac{\lambda'-\lambda''+\lambda+1}{2})\Gamma(\lambda''+1)}. \tag{8.2.1}$$

Lemma 8.2 (triangular relation of the Jacobi function). *If* $\cot\theta = \cosh t$ $(0 < \theta < \frac{\pi}{2}, 0 < t)$, *then*

$$\varphi_{i\lambda'}^{(\lambda,\lambda'')}(i\theta) = M(\cosh t)^{\lambda+\lambda'+\lambda''+1}\varphi_{i\lambda}^{(\lambda'',\lambda')}(t).$$

Furthermore, we have

$$\lambda' V_{++,\lambda'}^{(\lambda'',\lambda)} = M^2 \lambda V_{+-,\lambda}^{(\lambda',\lambda'')}. \tag{8.2.2}$$

8.3. Unitarizaition of $\pi_{+,\lambda}^{p,q}$ $(\lambda \in A_0(p,q))$.

In this subsection, we give an explicit unitary inner product on $\pi_{+,\lambda}^{p,q}$ for $\lambda \in A_0(p,q)$. In view of Fact 5.4, the $(\mathfrak{g}, K)$-module $(\pi_{+,\lambda}^{p,q})_K$ is realized in $C_\lambda^\infty(X(p,q))_K$ with K-types $\Xi(K:b)$ where $b = \lambda - \frac{p}{2} + \frac{q}{2} + 1$. Suppose $f \in C_\lambda^\infty(X(p,q))_K$ belongs to the K-type $\mathcal{H}^m(\mathbb{R}^p) \boxtimes \mathcal{H}^n(\mathbb{R}^q) \in \Xi(K:b)$. Then f is of the form

$$f(\omega\cosh t, \eta\sinh t) = h_m(\omega)h_n(\eta)(\cosh t)^m(\sinh t)^n\varphi_{i\lambda}^{(n+\frac{q}{2}-1, m+\frac{p}{2}-1)}(t), \tag{8.3.1}$$

where $h_m \in \mathcal{H}^m(\mathbb{R}^p)$, $h_n \in \mathcal{H}^n(\mathbb{R}^q)$, $\omega \in S^{p-1}, \eta \in S^{q-1}$, and $t > 0$. We put

$$\|f\|^2_{\pi_{+,\lambda}^{p,q}} := \|h_m\|^2_{L^2(S^{p-1})}\ \|h_n\|^2_{L^2(S^{q-1})}\ \lambda V_{+-,\lambda}^{(m+\frac{p-2}{2}, n+\frac{q-2}{2})}. \tag{8.3.2}$$

This norm defines an inner product on $(\pi_{+,\lambda}^{p,q})_K$ $(\lambda \in A_0(p,q))$ and one defines an irreducible unitary representation of $G = O(p,q)$ on its Hilbert completion. There is an obvious inner product for $\lambda > 0$, because $(\pi_{+,\lambda}^{p,q})_K \subset L^2(X(p,q))$. The relation between our norm $\| \ \|_{\pi_{+,\lambda}^{p,q}}$ and L^2-norm $\| \ \|_{L^2(X(p,q))}$ is given by

$$\|f\|^2_{\pi_{+,\lambda}^{p,q}} = \lambda\|f\|^2_{L^2(X(p,q))}, \qquad \text{for any } f \in (\pi_{+,\lambda}^{p,q})_K, \tag{8.3.3}$$

if $\lambda > 0$. For $\lambda \in A_0(p,q)$ such that $\lambda \le 0$ (this can happen if $\lambda = 0, -\frac{1}{2}$), $\pi_{+,\lambda}^{p,q}$ is also unitarizable by this inner product $(\ , \ )_{\pi_{+,\lambda}^{p,q}}$ given in (8.3.2), which is still positive definite.

Remark 8.3 (Unitarity). As we explained, all of $(\pi_{+,\lambda}^{p,q})_K$ are unitarizable for $\lambda \in A_0(p,q)$. We explain four different approaches:

1) If $\lambda > 0$, then $(\pi_{+,\lambda}^{p,q})_K$ is unitarizable because of the realization in $L^2(X(p,q))$.
2) If $\lambda \ge 0$, then $(\pi_{+,\lambda}^{p,q})_K$ is unitarizable because of the realization of Zuckerman-Vogan's derived functor module $\mathcal{R}_{\mathfrak{q}}^{p-2}(\mathbb{C}_\lambda)$ with the parameter λ in the weakly fair range [31].

   We note that the case $\lambda = -\frac{1}{2}$ is not treated in the above two methods. However,
3) Use the classification of unitarizable subquotients of $\mathrm{Ind}_{P_{\max}}^G(\epsilon \otimes \mathbb{C}_\lambda)$ given in [11], §3.

At last, here is a new proof of the unitarizability of $(\pi_{+,\lambda}^{p,q})_K$ for all $\lambda \in A_0(p,q)$. The idea is to use our branching formula Theorem 7.1, for which the proof does not use the unitarizability of $(\pi_{+,\lambda}^{p,q})_K$:

4) All of $(\pi_{+,\lambda}^{p,q})_K$ are unitarizable because they appear as discrete spectra in the branching law of a unitary representation $\varpi^{p,q+c}$ of a larger group $O(p,q+c)$ to $O(p,q) \times O(c)$ for some $c > 0$ (see Theorem 7.1). For this purpose, $c \le 3$ will do.

8.4. We notice that the map $\Phi_1\colon X(p,q') \times S^{q''-1} \to M_+$ (6.2.3) (in the case $p'' = 0$) is given by

$$((\omega\cosh t, \eta'\sinh t), \eta'') \mapsto (\omega, (\eta'\cos\theta, \eta''\sin\theta)),$$

where θ and t satisfy $\cot\theta = \cosh t$. Suppose $f \in C^\infty(X(p,q') \times S^{q''-1})$ belongs to

$$\pi^{p,q'}_{+,l+\frac{q''}{2}-1} \boxtimes \mathcal{H}^l(\mathbb{R}^{q''})$$

as an $O(p') \times O(q'')$-module, and furthermore to $\mathcal{H}^m(\mathbb{R}^p) \boxtimes \mathcal{H}^n(\mathbb{R}^{q'})$ as an $O(p) \times O(q')$-module in the first factor. Then f is of the form:

$$f((\omega\cosh t, \eta'\sinh t), \eta'') = h_m(\omega)\ h_k(\eta')\ h_l(\eta'')(\cosh t)^m(\sinh t)^k \varphi^{(k+\frac{q'}{2}-1, m+\frac{p}{2}-1)}_{i(l+\frac{q''}{2}-1)}(t)$$

where $h_m \in \mathcal{H}^m(\mathbb{R}^p)$, $h_k \in \mathcal{H}^k(\mathbb{R}^{q'})$, $h_l \in \mathcal{H}^l(\mathbb{R}^{q''})$, $\omega \in S^{p-1}, \eta' \in S^{q'-1}, \eta'' \in S^{q''-1}$, and $t > 0$.

Lemma 8.4. *The twisted pull-back* $\widetilde{(\Phi_1^{-1})^*}\colon C^\infty(X(p,q') \times S^{q''-1}) \to C^\infty(M_+)$ *(see (6.3.2) for definition) is given by the formula:*

$$(8.4.1)\quad (\widetilde{(\Phi_1^{-1})^*} f)(\omega, (\eta'\cos\theta, \eta''\sin\theta)) = M^{-1} h_m(\omega)\ h_k(\eta')\ h_l(\eta'')\ (\cos\theta)^k\ (\sin\theta)^l\ \varphi^{(l+\frac{q''}{2}-1, k+\frac{q'}{2}-1)}_{i(n+\frac{q}{2}-1)}(i\theta).$$

Proof. In view of the definition, we have

$$(\widetilde{(\Phi_1^{-1})^*} f)(\omega, (\eta'\cos\theta, \eta''\sin\theta)) = (\cosh t)^{\frac{p+q-4}{2}} f((\omega\cosh t, \eta'\sinh t), \eta'').$$

Then (8.4.1) follows from Lemma 8.2 and from $m + \frac{p}{2} = n + \frac{q}{2}$. $\square$

8.5. Let $G' = O(p,q') \times O(q'')$. We define an inner product on the irreducible constituent $\pi^{p,q'}_{+,\lambda} \boxtimes \pi^{0,q''}_{-,\lambda}$ in the branching law $\varpi^{p,q}|_{G'}$ (see Theorem 7.1) for each $\lambda = l + \frac{q''}{2} - 1$ ($l \in \mathbb{N}$) by

$$\|f\| := \|f_1\|_{\pi^{p,q'}_{+,\lambda}}\ \|f_2\|_{L^2(S^{q''-1})}$$

for $f = f_1\ f_2$ with $f_1 \in \pi^{p,q'}_{+,\lambda}$ and $f_2 \in \pi^{0,q''}_{-,\lambda} \simeq \mathcal{H}^l(\mathbb{R}^{q''})$.

Because we use an explicit map to prove the branching law, the generalized Parseval-Plancherel formula makes sense.

Theorem 8.5. 1) *If we develop* $F \in \operatorname{Ker}\widetilde{\Delta}_M$ *as* $F = \sum_l^\infty F_l^{(1)} F_l^{(2)}$ *according to the irreducible decomposition*

$$(8.5.1)\qquad \widetilde{(\Phi_1)^*}\colon \varpi^{p,q}|_{O(p,q')\times O(q'')} \xrightarrow{\sim} \sum_{l=0}^{\infty}{}^{\oplus}\, \pi^{p,q'}_{+,l+\frac{q''}{2}-1} \boxtimes \mathcal{H}^l(\mathbb{R}^{q''})$$

then we have

$$(8.5.2)\qquad \|F\|^2_{\varpi^{p,q}} = \sum_{l=0}^{\infty} \|F_l^{(1)}\|^2_{\pi^{p,q'}_{+,l+\frac{q''}{2}-1}}\ \|\ F_l^{(2)}\|^2_{L^2(S^{q''-1})}.$$

2) *In particular, if* $q'' \geq 3$*, then all of* $\pi^{p,q'}_{+,l+\frac{q''}{2}-1}$ *are discrete series for the hyperboloid* $X(p,q')$ *and*

$$\|F\|^2_{\varpi^{p,q}} = \sum_{l=0}^{\infty}(l + \frac{q''}{2} - 1)\ \|F_l^{(1)}\|^2_{L^2(X(p,q'))}\|\ F_l^{(2)}\|^2_{L^2(S^{q''-1})}. \tag{8.5.3}$$

Remark. The formula (8.5.3) coincides with the Kostant-Binegar-Zierau formula (see §3) in the special case where $q' = 0$ (namely, where G' is compact).

Proof. We write $\lambda := l + \frac{q''}{2} - 1$, $\lambda' := m + \frac{p}{2} - 1 = n + \frac{q}{2} - 1$ and $\lambda'' := k + \frac{q'}{2} - 1$. If F is of the form of the right side of (8.4.1), then

$$\begin{aligned}\frac{\|\widetilde{(\Phi_1)^* F}\|^2_{\pi^{p,q'}_{+,\lambda}\boxtimes\mathcal{H}^l(\mathbb{R}^{q''})}}{\|F\|^2_{\varpi^{p,q}}} &= \frac{M^2\ \|h_m\|^2_{L^2(S^{p-1})}\ \|h_k\|^2_{L^2(S^{q'-1})}\ \|h_l\|^2_{L^2(S^{q''-1})}\ \lambda V^{(\lambda',\lambda'')}_{+-,\lambda}}{\|h_m\|^2_{L^2(S^{p-1})}\ \|h_k\|^2_{L^2(S^{q'-1})}\ \|h_l\|^2_{L^2(S^{q''-1})}\ \lambda' V^{(\lambda'',\lambda)}_{++,\lambda'}} \\ &= 1\end{aligned}$$

because of (8.2.1). Hence the first statement is proved. The second statement follows from (8.3.2). $\square$

§9. Construction of discrete spectra in the branching laws

9.1. In §7, we determined explicitly the branching law $\varpi^{p,q}|_{G'}$ of the minimal unipotent representation $\varpi^{p,q} \in \widehat{G}$ where $G' = O(p,q') \times O(q'')$. The resulting branching law has no continuous spectrum (see Theorem 4.2 and Theorem 7.1). In this section, we treat a more general case where continuous spectrum may appear, namely, the branching law with respect to the semisimple symmetric pair

$$(G, G') = (O(p,q), O(p',q') \times O(p'',q'')),$$

where $p' + p'' = p\ (\geq 2)$, $q' + q'' = q\ (\geq 2)$, $p + q \in 2\mathbb{N}$, and $(p,q) \neq (2,2)$.

We shall construct explicitly discrete spectra by using the conformal geometry.

Retain the notation in §5.1. We set

$$A'(p,q) := A_0(p,q) \cap \{\lambda \in \mathbb{R} : \lambda > 1\}. \tag{9.1.1}$$

Theorem 9.1. *The restriction of the unitary representation* $\varpi^{p,q}|_{G'}$ *contains*

$$\sideset{}{^\oplus}\sum_{\lambda \in A'(p',q') \cap A'(q'',p'')} \pi^{p',q'}_{+,\lambda} \boxtimes \pi^{p'',q''}_{-,\lambda} \oplus \sideset{}{^\oplus}\sum_{\lambda \in A'(q',p') \cap A'(p'',q'')} \pi^{p',q'}_{-,\lambda} \boxtimes \pi^{p'',q''}_{+,\lambda}$$

as a discrete summand. Here, $\pi^{p',q'}_{+,\lambda} \boxtimes \pi^{p'',q''}_{-,\lambda} \in \widehat{G'}$ *is the outer tensor product of* $\pi^{p',q'}_{+,\lambda} \in \widehat{O(p',q')}$ *and* $\pi^{p'',q''}_{-,\lambda} \in \widehat{O(q'',p'')}$*.*

We have already established the full branching law $\varpi^{p,q}|_{G'}$ if $p'q'p''q'' = 0$. Thus, the main part of this section will be devoted to the proof of Theorem 9.1 when $p'q'p''q'' \neq 0$. We shall give some remarks of Theorem 9.1 at the end of this subsection.

9.2. Let $K' = O(p') \times O(q') \times O(p'') \times O(q'')$. We realize the unitary representation $\varpi^{p,q}$ on the Hilbert space $\overline{V^{p,q}}$. Here we recall the notation of §3.9, briefly as follows:

$$\overline{V^{p,q}} = \operatorname{Ker}(D_p - D_q) \underset{\text{closed}}{\subset} \mathcal{V} = \operatorname{Dom}(D_p) \cap \operatorname{Dom}(D_q) \quad (\underset{\text{dense}}{\subset} L^2(M))$$
$$M = S^{p-1} \times S^{q-1} \underset{\text{dense}}{\supset} M_+ \cup M_-.$$

Different from Corollary 4.3 in the discretely decomposable case ($p'q'p''q'' = 0$), a K'-finite vector of a G'-irreducible summand (i.e. a discrete spectrum) in $\varpi^{p,q}|_{G'}$ is not necessarily a real analytic function on M if $p'q'p''q'' \neq 0$ in our conformal construction of $\varpi^{p,q}$. With this in mind, we extend $\widetilde{(\Phi_1^{-1})^* f} \in C^\infty(M_+)$ (see §6 for notation) to a function, denoted by $T_+ f$, on $S^{p-1} \times S^{q-1}$ as

$$T_+ f := \begin{cases} \widetilde{(\Phi_1^{-1})^* f} & \text{on } M_+, \\ 0 & \text{on } M \setminus M_+. \end{cases} \tag{9.2.1}$$

Here is a key lemma:

Lemma 9.2. *Suppose $\lambda' \in A_0(p', q')$ and $\lambda'' \in A_0(q'', p'')$. Let $f \in C^\infty(X(p', q') \times X(q'', p''))$ be a K'-finite function which belongs to $\pi^{p',q'}_{+,\lambda'} \boxtimes \pi^{p'',q''}_{-,\lambda''}$ (see Fact 5.4 (1) (ii)).*
1) If $\lambda' \geq \frac{1}{2}$ and $\lambda'' \geq \frac{1}{2}$, then $T_+ f \in \mathcal{V}$.
2) If $\lambda' > 1$ and $\lambda'' > 1$, then $Y(T_+ f) \in L^2(M)$ and $YY'(T_+ f) \in L^1(M)$ for any smooth vector field Y, Y' on M.
3) If $\lambda' = \lambda'' > 1$ then $T_+ f \in \overline{V^{p,q}}$.

Before proving Lemma 9.2, we first show that Lemma 9.2 implies Theorem 9.1. In fact, Lemma 9.2 (3) constructs a non-zero $(\mathfrak{g}', K')$-homomorphism

$$T_+ \colon (\pi^{p',q'}_{+,\lambda})_{K_1'} \boxtimes (\pi^{p'',q''}_{-,\lambda})_{K_2'} \to \varpi^{p,q},$$

for $\lambda \in A'(p', q') \cap A'(q'', p'')$. Then T_+ is injective because $(\pi^{p',q'}_{+,\lambda})_{K_1'} \boxtimes (\pi^{p'',q''}_{-,\lambda})_{K_2'}$ is irreducible. Since $\varpi^{p,q}$ is a unitary representation of G, T_+ extends to an isometry of unitary representations of G':

$$\pi^{p',q'}_{+,\lambda} \boxtimes \pi^{p'',q''}_{-,\lambda} \to \varpi^{p,q},$$

by taking the closure with respect to the inner product induced from $\varpi^{p,q}$. This proves Theorem 9.1 for the irreducible representation $\pi^{p',q'}_{+,\lambda} \boxtimes \pi^{p'',q''}_{-,\lambda}$ that appears in the first summand. The second summand is constructed similarly by using the conformal diffeomorphism (see Lemma 6.4)

$$\Phi_2 \colon X(q', p') \times X(p'', q'') \xrightarrow{\sim} M_- \ (\subset M).$$

Hence, the proof of Theorem 9.1 is completed by assuming Lemma 9.2.

9.3.

Remark 9.3. There exist $\lambda' \in A_0(p', q')$ and $\lambda'' \in A_0(q'', p'')$ satisfying $\lambda' = \lambda'' > 1$ if and only if $p' + q' \equiv p'' + q'' \mod 2$ and $p' \geq 2, q'' \geq 2$. This implies $p + q \in 2\mathbb{N}$, $p \geq 2$ and $q \geq 2$, and $V^{p,q}$ is non-zero (see §3). Of course, Lemma 9.2 (3) also implies $V^{p,q} \neq \{0\}$.

9.4. It follows from Lemma 3.8.1 that the first statement of Lemma 9.2 is proved if $Y(T_+) \in L^{2-\epsilon}$ for any $\epsilon > 0$ and for any smooth vector field Y on M. Therefore, both of (1) and (2) of Lemma 9.2 are proved by studying the asymptotic behaviour of $T_+ f$ near the boundary of M_+ in M. This asymptotic estimate is studied below.

Any K'-finite vector f is a finite linear combination of the form $f_1 f_2$, where

$$\begin{cases} f_1 \in C^\infty(X(p',q')) \text{ is an } O(p') \times O(q')\text{-finite vector that belongs to } \pi^{p',q'}_{+,\lambda'}, \\ f_2 \in C^\infty(X(q'',p'')) \text{ is an } O(p'') \times O(q'')\text{-finite vector that belongs to } \pi^{p'',q''}_{-,\lambda''}. \end{cases}$$

In order to prove Lemma 9.2, we may and do assume f is of the form $f_1 f_2$. Then we have

$$(9.4.1) \quad (T_+(f_1 f_2))(u,v) = (|u'|^2 - |v'|^2)_+^{-\frac{p+q-4}{4}} f_1\left(\frac{(u',v')}{\sqrt{|u'|^2-|v'|^2}}\right) f_2\left(\frac{(u'',v'')}{\sqrt{|u''|^2-|v''|^2}}\right).$$

Here, we have used the following notation:

$$r_+^\nu := \begin{cases} r^\nu & (r > 0), \\ 0 & (r \le 0). \end{cases}$$

9.5. In order to analyze the asymptotic behaviour of $T_+(f_1 f_2)$ (see (9.4.1)) near the boundary of M_+, we consider a change of variables on $S^{p-1} \times S^{q-1}$ by the surjective map

$$\begin{aligned} (S^1 \times S^{p'-1} \times S^{p''-1}) \times (S^1 \times S^{q'-1} \times S^{q''-1}) &\to S^{p-1} \times S^{q-1}, \\ (e^{i\theta}, \omega', \omega''), (e^{i\varphi}, \eta', \eta'') &\mapsto (u, \quad v) \end{aligned}$$

defined by

$$(9.5.1) \qquad (u,v) \equiv (u',u'',v',v'') := (\omega' \cos\theta, \omega'' \sin\theta, \eta' \cos\varphi, \eta'' \sin\varphi).$$

Because $|u'|^2 - |v'|^2 = |\omega' \cos\theta|^2 - |\eta' \cos\varphi|^2 = |\cos\theta|^2 - |\cos\varphi|^2$, $M_\pm$ defined in (6.1.1) is rewritten as

$$\begin{aligned} M_+ &= \{(\omega' \cos\theta, \omega'' \sin\theta, \eta' \cos\varphi, \eta'' \sin\varphi) : |\cos\theta| > |\cos\varphi|\} \\ M_- &= \{(\omega' \cos\theta, \omega'' \sin\theta, \eta' \cos\varphi, \eta'' \sin\varphi) : |\cos\theta| < |\cos\varphi|\} \end{aligned}$$

Here is an elementary computation corresponding to the change of variables (9.5.1):

Lemma 9.5. 1) *The volume element $du\,dv$ on $S^{p-1} \times S^{q-1}$ is given by*

$$du\,dv = |\cos\theta|^{p'-1}\,|\sin\theta|^{p''-1}\,|\cos\varphi|^{q'-1}\,|\sin\varphi|^{q''-1}\,d\theta\,d\varphi\,d\omega'\,d\omega''\,d\eta'\,d\eta'',$$

where $d\omega'$ is the volume element on $S^{p'-1}$ and so on.
2) *Any smooth vector field on $S^{p-1} \times S^{q-1}$ is a linear combination of*

$$\frac{1}{\cos\theta} X', \frac{1}{\sin\theta} X'', \frac{1}{\cos\varphi} Y', \frac{1}{\sin\varphi} Y'', \frac{\partial}{\partial\varphi}, \frac{\partial}{\partial\theta},$$

whose coefficients are smooth functions of $(\omega', \omega'', \eta', \eta'', \theta, \varphi)$. *Here,* X', X'', Y' *and* Y'' *are smooth vector fields on* $S^{p'-1}, S^{p''-1}, S^{q'-1}$ *and* $S^{q''-1}$, *respectively.*

Fact 5.4 (2) describes the asymptotic behaviour of K-finite functions that belong to discrete series representations for a hyperboloid. Applying it to f_1 with respect to the coordinate (9.5.1), we find $a_1 \in C^\infty(S^{p'-1} \times S^{q'-1})$ and $h_1 \in C^\infty(\mathbb{R})$ such that

$$f_1\left(\frac{(u',v')}{\sqrt{|u'|^2-|v'|^2}}\right) = a_1(\omega',\eta')\left(\frac{\cos^2\theta+\cos^2\varphi}{\cos^2\theta-\cos^2\varphi}\right)_+^{-\frac{2\lambda'+p'+q'-2}{4}} h_1\left(\sqrt{\frac{\cos^2\theta-\cos^2\varphi}{\cos^2\theta+\cos^2\varphi}}\right)$$

Likewise, there exist $a_2 \in C^\infty(S^{q''-1} \times S^{p''-1})$ and $h_2 \in C^\infty(\mathbb{R})$ such that

$$f_2\left(\frac{(v'',u'')}{\sqrt{|v''|^2-|u''|^2}}\right) = a_2(\eta'',\omega'')\left(\frac{\sin^2\theta+\sin^2\varphi}{\cos^2\theta-\cos^2\varphi}\right)_+^{-\frac{2\lambda''+p''+q''-2}{4}} h_2\left(\sqrt{\frac{\cos^2\theta-\cos^2\varphi}{\sin^2\theta+\sin^2\varphi}}\right)$$

We treat the boundary ∂M_+ of $M_+ \subset M$ locally in the following three cases:
Case 1) $\cos^2\theta - \cos^2\varphi = 0$, $(\cos\theta, \cos\varphi) \neq (0,0)$, $(\sin\theta, \sin\varphi) \neq (0,0)$.
Case 2) $\cos\theta = \cos\varphi = 0$.
Case 3) $\sin\theta = \sin\varphi = 0$.

We note that Case (2) or (3) happens only when $p'q'p''q'' \neq 0$.

9.6. (Case 1): In this subsection, we consider a generic part of the boundary ∂M_+ corresponding to Case 1. In a local coordinate $(\omega', \omega'', \eta', \eta'', \theta, \varphi) \in S^{p'-1} \times S^{p''-1} \times S^{q'-1} \times S^{q''-1} \times \mathbb{R}^2$, $T_+(f_1 f_2))(u,v)$ is written as

$$A\ (\cos^2\theta-\cos^2\varphi)_+^{-\frac{p+q-4}{4}+\frac{2\lambda'+p'+q'-2}{4}+\frac{2\lambda''+p''+q''-2}{4}} = A\ (\cos^2\theta-\cos^2\varphi)_+^{\frac{\lambda'+\lambda''}{2}}.$$

Here, A is a smooth function of variables $\omega', \omega'', \eta', \eta'', (\cos^2\theta - \cos^2\varphi)_+^{\frac{1}{2}}$. Therefore, by using Lemma 9.5, we have:

$$\begin{aligned}
\lambda'+\lambda'' > -1 &\quad\Rightarrow\quad T_+ f \in L^2_{\mathrm{loc}}, \\
\lambda'+\lambda'' \geq 1 &\quad\Rightarrow\quad Y(T_+ f) \in L^{2-\epsilon}_{\mathrm{loc}} \qquad \text{for any } \epsilon > 0, Y \in \mathfrak{X}(M), \\
\lambda'+\lambda'' > 2 &\quad\Rightarrow\quad Y_1 Y_2 (T_+ f) \in L^1_{\mathrm{loc}} \qquad \text{for any } Y_1, Y_2 \in \mathfrak{X}(M),
\end{aligned}$$

in a neighbourhood of the boundary point of M_+ for Case (1).

9.7. (Case 2): In this subsection we consider a neighbourhood of $(\omega', \omega'', \eta', \eta'', \theta, \varphi)$ satisfying the condition of Case 2. We take a polar coordinate for $(\cos\theta, \cos\varphi) \risingdotseq (0,0)$ as

$$\begin{cases} \cos\theta &= r\cos\psi, \\ \cos\varphi &= r\sin\psi. \end{cases}$$

The composition to (9.5.1) yields a new coordinate on $S^{p-1} \times S^{q-1}$ given by

$$(u,v) = (u',u'',v',v'') = (\omega' r\cos\psi, \omega''\sqrt{1-r^2\cos^2\psi}, \eta' r\sin\psi, \eta''\sqrt{1-r^2\sin^2\psi}),$$

where $\omega' \in S^{p'-1}$, $\omega'' \in S^{p''-1}$, $\eta' \in S^{q'-1}$, $\eta'' \in S^{q''-1}$, $r \geq 0$, and $\psi \in \mathbb{R}$. Then our interest is in a neighbourhood of $r = 0$. In this coordinate, we have

$$M_+ = \{r \neq 0, \cos 2\psi > 0\}$$

The Jacobian matrix of the transform $(\theta, \varphi) \to (r, \psi)$ is given by

$$\begin{pmatrix} \frac{\partial r}{\partial\theta} & \frac{\partial r}{\partial\varphi} \\ \frac{\partial\psi}{\partial\theta} & \frac{\partial\psi}{\partial\varphi} \end{pmatrix} = \begin{pmatrix} -\cos\psi\sin\theta & -\sin\psi\sin\varphi \\ \frac{1}{r}\sin\psi\sin\theta & \frac{-1}{r}\cos\psi\sin\varphi \end{pmatrix}.$$

Lemma 9.7. 1) *The standard measure on $S^{p-1}\times S^{q-1}$ is locally represented as*

$$\text{smooth function of } (r,\psi,\omega',\eta',\omega'',\eta'')\times r^{p'+q'-1}\,drd\psi d\omega' d\eta' d\omega'' d\eta''.$$

2) *Any smooth vector field on $S^{p-1}\times S^{q-1}$ near $r=0$ is a linear combination of*

$$\frac{1}{r\cos\psi}X', X'', \frac{1}{r\sin\psi}Y', Y'', \frac{\partial}{\partial r}, \frac{1}{r}\frac{\partial}{\partial\psi},$$

whose coefficients are smooth functions of $(\omega',\omega'',\eta',\eta'',r,\psi)$. Here, X',X'',Y' and Y'' are smooth vector fields on $S^{p'-1},S^{p''-1},S^{q'-1}$ and $S^{q''-1}$, respectively.

By noting the relations

$$\frac{\cos^2\theta+\cos^2\varphi}{\cos^2\theta-\cos^2\varphi}=\frac{1}{\cos 2\psi},\quad \frac{\sin^2\varphi+\sin^2\theta}{\sin^2\varphi-\sin^2\theta}=\frac{2-r^2}{r^2\cos 2\psi},$$

$T_+(f_1f_2)(u,v)$ is locally written as

$$\begin{aligned}&B\ (r^2\cos 2\psi)^{-\frac{p+q-4}{4}}(\cos 2\psi)_+^{\frac{2\lambda'+p'+q'-2}{4}}(r^2\cos 2\psi)_+^{\frac{2\lambda''+p''+q''-2}{4}}\\ =&B\ r_+^{\frac{2\lambda''-p'-q'+2}{2}}(\cos 2\psi)_+^{\frac{\lambda'+\lambda''}{2}},\end{aligned}$$

where B is a smooth function of variables $\omega',\omega'',\eta',\eta'',r_+,(\cos 2\psi)_+^{\frac{1}{2}}$. Therefore, by using Lemma 9.7, we have:

$$\begin{array}{llll} 2\lambda''>-1, & \lambda'+\lambda''>-1 & \Rightarrow\ T_+f\in L^2_{\mathrm{loc}}, & \\ 2\lambda''\geq 0, & \lambda'+\lambda''\geq 1 & \Rightarrow\ Y(T_+f)\in L^{2-\epsilon}_{\mathrm{loc}} & \text{for any } \ \epsilon>0, Y\in\mathfrak{X}(M),\\ 2\lambda''>2-p'-q', & \lambda'+\lambda''>2 & \Rightarrow\ Y_1Y_2(T_+f)\in L^1_{\mathrm{loc}} & \text{for any } \ Y_1,Y_2\in\mathfrak{X}(M), \end{array}$$

in a neighbourhood of the boundary ∂M_+ for Case (2).

The asymptotic estimate for Case (3) parallels to that of Case (2).

Remark 9.7. Assume $\lambda'=\lambda''\in\mathbb{Z}+\frac{p'+q'}{2}$. Then $r_+^{\frac{2\lambda''-p'-q'+2}{2}}$ is bounded near $r=0$ if and only if $\lambda'\geq\frac{p'+q'}{2}-1$, equivalently, $\lambda'>\frac{p'+q'}{2}-2$, which means that $\mathbb{C}_{\lambda'}$ is in the good range with respect to the θ-stable parabolic subalgebra defined by $\mathbb{C}_{\lambda'}$ in the sense of Vogan.

9.8. We end with some remarks and conjectures, primarily concerning the precise form of the discrete spectrum and also the continuous spectrum (where it would be very interesting to develop the complete Plancherel formula, given our explicit intertwining operator).

1) (multiplicity free property)

Each irreducible component in Theorem 9.1 occurs as multiplicity free. If $p=2$, then $\varpi^{p,q}$ is the direct sum of an irreducible highest weight module and a lowest one. It was proved in [19] that the multiplicity in the full Plancherel formula (both discrete and continuous spectrum) is at most one, in the branching law of any highest weight module of scalar type with respect to any symmetric pair.

2) (at most finitely many discrete spectrum, and full discrete spectrum)

If p', q', p'' and q'' satisfy

$$\min(p', q'') \leq 1 \text{ and } \min(q', p'') \leq 1, \tag{9.8.1}$$

then the parameter set in Theorem 9.1 is empty, namely,

$$A'(p', q') \cap A'(q'', p'') = A'(q', p') \cap A'(p'', q'') = \emptyset.$$

We conjecture that there are at most finitely many discrete spectra in the branching law $\varpi^{p,q}|_{G'}$ if (9.8.1) holds. We further conjecture that the full discrete spectrum is as in Theorem 9.1 with $A'(p, q)$ replaced by $A_0(p, q)$ everywhere.

3) (no discrete spectrum)

Furthermore, if we exclude the case such as $G' = O(p, q-1) \times O(1)$ (see §7.2), namely, if

$$\min(p', q'') \leq 1,\ \min(q', p'') \leq 1, p' + q' > 1,\ \text{ and } p'' + q'' > 1, \tag{9.8.2}$$

then

$$A_0(p', q') \cap A_0(q'', p'') = A_0(q', p') \cap A_0(p'', q'') = \emptyset.$$

It is likely that there is no discrete spectra in the branching law $\varpi^{p,q}|_{G'}$ if (9.8.2) is satisfied.

We note that the condition (9.8.2) is equivalent to that at least one of $X(p', q')$ or $X(q'', p'')$ is a non-compact Riemannian symmetric space and at least one of $X(q', p')$ or $X(p'', q'')$ is a non-compact Riemannian symmetric space.

4) (discretely decomposable case)

The opposite extremal case is when

$$\min(p', p'', q', q'') = 0. \tag{9.8.3}$$

As we have proved in Theorem 4.2, the restriction $\varpi^{p,q}|_{G'}$ is discretely decomposable without any continuous spectrum. We have obtained the full branching formula in Theorem 7.1 by using Theorem 4.2 and the K-type formula of $\varpi^{p,q}$.

If we employ only the method in this section to the special case (9.8.3), then we do not have to consider Cases (2) and (3) in §9.7. Then, Theorem 9.1 exhausts all discrete spectra in Theorem 7.1 in most cases, but there are some few exceptions. To be precise, we consider the case $p'' = 0$ without loss of generality. Then, in view of Theorem 7.1, the right side of Theorem 9.1 exhausts all discrete spectra if $q'' \geq 5$; while at most two of $(\mathfrak{g}', K')$-modules $\pi^{p,q'}_{+,\lambda} \boxtimes \pi^{0,q''}_{-,\lambda}$ are missing in Theorem 9.1 if $q' \leq 4$. The precise missing parameters in the case $p'' = 0$ and $q' = 0$ are: $\lambda = \pm\frac{1}{2}$ ($q'' = 1$); $\lambda = 0, 1$ ($q'' = 2$); $\lambda = \frac{1}{2}$ ($q'' = 3$); and $\lambda = 1$ ($q'' = 4$). In order to to cover all missing parameter by the purely geometric method of this section, we should notice a specific feature in the case $p'' = 0$:

i) M_+ is a dense subspace of M.

ii) Any real analytic functions on M_+ satisfying the Yamabe equation corresponding to K'-finite vectors of the $(\mathfrak{g}', K')$-module with the above missing parameter extend to real analytic functions on M (see Corollary 4.3 (2)).

5) (explicit continuous spectrum)

We conjecture that

$$L^2\text{-}\operatorname{Ind}_{P_1^{\max}}^{G_1'}(\epsilon \otimes \mathbb{C}_{\sqrt{-1}\lambda}) \boxtimes L^2\text{-}\operatorname{Ind}_{P_2^{\max}}^{G_2'}(\epsilon \otimes \mathbb{C}_{\sqrt{-1}\lambda}) \quad (\lambda \in \mathbb{R})$$

is a continuous spectrum with multiplicity free if $\min(p', p'', q', q'') > 0$.

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