INTRODUCTION TO ACTIONS OF DISCRETE GROUPS ON PSEUDO-RIEMANNIAN HOMOGENEOUS MANIFOLDS

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This article is based on two lectures on "Discontinuous groups for **non-Riemannian** homogeneous spaces" delivered at the 2000 Twente Conference on Lie Groups. The topic has developed quite rapidly since late 1980s, by various new methods including theory of actions of non-compact Lie groups, discrete groups, characteristic classes, ergodic theory, symplectic geometry, unitary representation theory (especially, the restriction to subgroups) and so on.

Such diversity of methods has pushed forward the topic actively by stimulating interactions of different fields of mathematics on one hand, but on the other hand it might make beginners feel difficult to access it. The purpose of the lectures was to give an accessible exposition by clarifying the current status of some of central problems in this topic.

For interested readers, we suggest [IW01b], [Ko90], [Ko96a], [Ko00b], [La96], [Ma00] for more extensive surveys, examples and different view points on related topics.

I am grateful to the organizers for the opportunity to participate in this conference.

§1. Basic problems of discontinuous groups for G/H

1.1. (Group language) Our group setting is the triple of groups

$$\Gamma \subset G \supset H$$

where G is a Lie group, Γ is a discrete subgroup and H is a closed subgroup. Then we have a natural commutative diagram of quotient maps:

$$\begin{tikzcd} & G \arrow[dl, "\pi"'] \arrow[dr] & \\ \Gamma\backslash G \arrow[dr] & & G/H \arrow[dl, "p"] \\ & \Gamma\backslash G/H & \end{tikzcd}$$

The first three spaces G, $\Gamma\backslash G$ and G/H have a natural C^ω-manifold structure, and π is a local diffeomorphism. However, the double coset space $\Gamma\backslash G/H$ may fail to

be Hausdorff in the quotient topology if H is non-compact. (Failure can occur even though Γ acts freely. See [Ko90], Example 1.) With definition in §2.1, the right requirement is:
(1.1) The left action of Γ on G/H is properly discontinuous and free.

If (1.1) is satisfied, then the double coset space $\Gamma\backslash G/H$ is Hausdorff and has a C^ω manifold structure such that $p\colon G/H \to \Gamma\backslash G/H$ is a local diffeomorphism..

Definition 1.1. We say that $\Gamma\backslash G/H$ is a Clifford-Klein form of G/H, and Γ is a **discontinuous group** for G/H, if (1.1) is satisfied.

If H is compact, the following two conditions are equivalent:
i) (property of a group) Γ is discrete.
ii) (property of an action) The action of Γ on G/H is properly discontinuous.

In other words, the requirement (ii) is not important as far as we treat compact H and discrete Γ. Our main concern is with **non-compact** H, and the understanding of (ii) becomes crucial. This will be the main theme of §2.

1.2. (Geometric language) A Clifford-Klein form $\Gamma\backslash G/H$ enjoys any G-invariant geometric structure on G/H through a covering map $p\colon G/H \to \Gamma\backslash G/H$. Conversely, let us reconstruct a Clifford-Klein form from a manifold with some geometric structure.

Suppose we are given a manifold M with geometric structure $\mathcal{T}$. Here, by geometric structure, we have in our mind a Riemannian [(or more generally) pseudo-Riemannian, complex, symplectic, ...] structure.

Let $\widetilde{M}$ be a universal covering manifold of M. Through the covering map $p\colon \widetilde{M} \to M$, the geometric structure $\mathcal{T}$ is pulled back to $\widetilde{M}$. We fix a point o in $\widetilde{M}$ and write $\bar{o} := p(o)$. We define two subgroups Γ and H of G by

$$\begin{aligned} H &:= \{g \in G : g \cdot o = o\} \qquad &&\text{(the isotropy subgroup)},\\ \Gamma &:= \pi_1(M, \bar{o}) &&\text{(the fundamental group)}. \end{aligned}$$

The fundamental group Γ acts on $\widetilde{M}$ as a covering transformation, especially, properly discontinuously, freely and effectively. This action preserves the geometric structure $\mathcal{T}$ on $\widetilde{M}$ by a tautological reason. Hence we can regard Γ as a subgroup of

$$G := \mathrm{Aut}(\widetilde{M}, \mathcal{T}) = \text{the group of diffeomorphisms of } \widetilde{M} \text{ preserving } \mathcal{T}.$$

This information on Γ is much better than just sitting in a huge group $\mathrm{Homeo}(\widetilde{M})$. For instance, if $\mathcal{T}$ is taken to be a pseudo-Riemannian structure then G is nothing but the group of isometries, which is known to become a Lie group. The connection with Clifford-Klein forms is summarized as:

Proposition 1.2. *Let M be a manifold with geometric structure $\mathcal{T}$. Assume that $G := \mathrm{Aut}(\widetilde{M}, \mathcal{T})$ is a Lie group and acts transitively on $\widetilde{M}$. Then, M is naturally diffeomorphic to a Clifford-Klein form $\Gamma\backslash G/H$ by the following commutative diagram:*

$$\begin{array}{ccccc} G/H & \xrightarrow{\sim} & \widetilde{M} & \qquad gH & \mapsto g \cdot o \\ \downarrow & & \downarrow & \qquad \downarrow & \downarrow \\ \Gamma\backslash G/H & \xrightarrow{\sim} & M & \qquad \Gamma gH & \mapsto g \cdot \bar{o}. \end{array}$$

Under these maps, the geometric structure $\mathcal{T}$ on M comes down from a corresponding G-invariant geometric structure on the homogeneous space G/H.

1.3. (Examples)

Example 1.3.1 (Riemann surface)**.** Let M be a Riemann surface, and $\mathcal{T}$ a complex structure. Then $\widetilde{M}$ is one of $\mathcal{H}$ (upper half plane), $\mathbb{C}$ or $\mathbb{P}^1\mathbb{C}$ by the uniformization theorem due to Klein-Poincaré-Koebe. Correspondingly, $G = \mathrm{Aut}(\widetilde{M}, \mathcal{T})$ is isomorphic to $PSL(2,\mathbb{R})$, $\mathrm{Aff}(1,\mathbb{C}) = \mathbb{C}^\times \ltimes \mathbb{C}$, or $PSL(2,\mathbb{C})$, and in all three cases the assumption of Proposition 1.2 is satisfied. Thus, any Riemann surface is represented as a Clifford-Klein form of G/H where $H = S^1$, $\mathbb{C}^\times$ or $\mathbb{C}^\times \ltimes \mathbb{C}$, respectively.

Example 1.3.2 (pseudo-Riemannian spherical space form)**.** Suppose M is a $p+q$ dimensional manifold equipped with a complete pseudo-Riemannian structure $\mathcal{T}$ of signature (p,q) with constant sectional curvature > 0. Then the assumption of Proposition 1.2 is satisfied. If $p \neq 1$, then the group of isometries $G = \mathrm{Aut}(\widetilde{M}, \mathcal{T})$ is isomorphic to the orthogonal group $O(p+1,q)$ with $H \simeq O(p,q)$.

Surprisingly, there is a strong restriction on the global topology of M in Lorentz case:

Fact 1.3.3 (Calabi-Markus phenomenon [CM62])**.** *Assume $q = 1$ and $p \geq 2$ in Example 1.3.2. Then, any such M is non-compact and $\pi_1(M)$ is finite.*

1.4. (basic questions) We are mostly interested in a homogeneous space G/H where $G \supset H$ are reductive linear groups over $\mathbb{R}$. Then, G/H carries a G-invariant pseudo-Riemannian structure. Typical examples are semisimple symmetric pairs such as

$(G,H) = (SL(p+q,\mathbb{R}), SO(p,q)), (O(p_1+p_2, q_1+q_2), O(p_1,q_1) \times O(p_2,q_2))$.

In [Ko88] (see also [Ko89]), we initiated an investigation of discontinuous groups for non-Riemannian homogeneous spaces G/H in this generality, by posing the following problems:

Problem A. *Find a criterion for a discrete subgroup Γ to act properly discontinuously on G/H.*

Problem B. *Determine all possible pairs (G,H) such that G/H admits a compact Clifford-Klein form $\Gamma\backslash G/H$.*

We shall explain a solution to Problem A for a reductive group G in §2. The Calabi-Markus phenomenon (Fact 1.3.3) will be explained as its corollary.

Problem B has been studied particularly actively in the last decade, and many methods have been developed, though it is still an open problem even for spherical space forms. The following is a special case of [Ko00b], Conjecture 4.3 applied to $G/H = O(p+1,q)/O(p,q)$:

Conjecture 1.4. *There exists a* **compact** *spherical space form if and only if the signature (p,q) is in the following list (see Corollary 3.4 for "if" part):*

p	$\mathbb{N}$	0	1	3	7
q	0	$\mathbb{N}$	$2\mathbb{N}$	$4\mathbb{N}$	8

We shall explain known construction and some obstruction of compact Clifford-Klein forms in §3 and §4, respectively.

1.5. (Beyond group theory) Clifford-Klein problems are also related to a traditional question in differential geometry

Question. *How local geometric structure affects the global nature of a manifold?*

This question has been particularly studied in Riemannian geometry, as a relation between curvatures and global topology (we note that curvature tensors determine locally a Riemannian metric).

Since homogeneous manifolds offer various examples of geometric structures (though they are very special in differential geometry), we may expect that answers to Problems A and B give some intuition in non-Riemannian differential geometry beyond group theory.

For example, the Calabi-Markus phenomenon (Corollary 2.6.1) leads us to the following conjecture on a global pseudo-Riemannian geometry ([Ko00b], Conjecture 3.8.2 for details):

Conjecture 1.5. *Let M be a complete pseudo-Riemannian manifold of signature (p,q) with $p \geq q > 0$. Assume the infimum of K_M is positive. Then we conjecture:*
1) *M is always non-compact.*
2) *If $p+q \geq 3$ then the fundamental group $\pi_1(M)$ is a finite group.*

§2. Criterion of properly discontinuous actions

2.1. (Basic notion) Suppose a locally compact group L acts continuously on a Hausdorff, locally compact topological space X. Given a subset S of X, we define a subset of L by

$$L_S := \{\gamma \in L : \gamma S \cap S \neq \emptyset\}.$$

We say the action of L on X is:

(2.1.1) *proper* if L_S is compact for any compact subset S.

(2.1.2) *properly discontinuous* if L_S is finite for any compact subset S.

(2.1.3) *free* if $L_{\{p\}} = \{e\}$ for any $p \in X$.

Remark 2.1. If L is a torsion free discrete group, then (2.1.1) $\Leftrightarrow$ (2.1.2) $\Rightarrow$ (2.1.3).

2.2. (Strategy) Our primary interest in §2 is properly discontinuous and free actions of a discrete subgroup L on a homogeneous space $X = G/H$. In light of Remark 2.1, we shall consider a more general problem, that is, to find a criterion of proper actions by forgetting that L is a discrete group.

In order to study proper actions on G/H, the idea used in [Ko89] was to work inside the group G itself and then to use a representation theory of G, rather than working topologically on a homogeneous space G/H. For this purpose, we note that L acts properly on G/H if and only if

(2.2) $L \cap SHS$ is relatively compact for any compact subset S of G.

Definition 2.2. We write $L \pitchfork H$ if (2.2) holds.

A next observation is that the definition of $L \pitchfork H$ did not use the group structure of L and H. Thus, our setting can be generalized to the triple

$$L \subset G \supset H$$

where L and H are just subsets. Here are two advantages of this formulation:
- We can use the projection which is not a group homomorphism (see (2.5)).
- L and H play a symmetric role. For instance, one can show: $L \pitchfork H \Leftrightarrow H \pitchfork L$.

2.3. ($\sim$ and $\pitchfork$) Now our aim is to understand the relation $L \pitchfork H$. A third idea is to introduce an equivalence relation $\sim$ with the property that $H_1 \sim H_2$ implies

$$L \pitchfork H_1 \Leftrightarrow L \pitchfork H_2 \quad \text{for any subset } L \text{ of } G. \tag{2.3}$$

Here is a definition (see [Ko96b] for basic properties of $\sim$ and $\pitchfork$):

Definition 2.3. We write $L \sim H$ if there exists a compact subset S of G such that $L \subset SHS$ and $H \subset SLS$.

2.4. (Duality) For a subset H of G, the *discontinuous dual* of H is defined as:

$$H^\vee \equiv H^\vee_G := \{L : L \text{ is a subset of } G \text{ such that } L \pitchfork H \text{ in } G\}.$$

As we have already mentioned, the discontinuous dual $H^\vee_G$ is determined by H modulo $\sim$. Conversely, for a real reductive linear group G, we have proved in [Ko96b]:

Theorem A (duality theorem). *The discontinuous dual $H^\vee$ recovers H modulo $\sim$.*

That is, (2.3) $\Leftrightarrow H_1 \sim H_2$.

2.5. (Criterion) We want to find a criterion for $L \pitchfork H$ in a reductive Lie group G. On the other hand, the conditions $\sim$ and $\pitchfork$ are very easy to understand if G is abelian. For example, suppose L and H are subspaces of $G = \mathbb{R}^n$. Then it is easy to see

$$\begin{aligned} L \sim H \quad &\Leftrightarrow \quad L \cap H \neq \{0\}, \\ L \pitchfork H \quad &\Leftrightarrow \quad L \cap H = \{0\}. \end{aligned}$$

Let G be a real reductive group, and K a maximal compact subgroup, $G = K \exp(\mathfrak{a}) K$ a Cartan decomposition, and W_G the Weyl group. Then we define a Cartan projection

$$\nu : G \to \mathfrak{a}/_{\sim W_G}, \quad k_1 \exp(X) k_2 \mapsto X. \tag{2.5}$$

For example, if $G = GL(n, \mathbb{R})$ then $K = O(n)$, $W_G \simeq \mathcal{S}_n$ (the symmetric group), and $\mathfrak{a} \simeq \mathbb{R}^n$ is realized as the set of diagonal matrices. For $g \in G$, the matrix ${}^t g g$ is a positive definite symmetric matrix. Let $\lambda_1 \geq \cdots \geq \lambda_n > 0$ be its eigenvalues. Then the Cartan projection amounts to: $\nu \colon G \to \mathbb{R}^n$, $g \mapsto \frac{1}{2}(\log \lambda_1, \ldots, \log \lambda_n)$.

The Cartan projection will reduce a general problem to the abelian case, as stated in the following criterion which was proved independently by Benoist and Kobayashi:

Theorem 2.5 ([Be96],[Ko96b]). *Let H and L be subsets of a real reductive group G.*
1) *$H \sim L$ in G if and only if $\nu(H) \sim \nu(L)$ in $\mathfrak{a}$.*
2) *$L \pitchfork H$ in G if and only if $\nu(L) \pitchfork \nu(H)$ in $\mathfrak{a}$.*

2.6. (Some applications of the criterion) In the case where H and L are reductive subgroups, Theorem 2.5 was first proved in [Ko89], Theorem 4.1. This special case was enough to prove the criterion of the Calabi-Markus phenomenon for a pseudo-Riemannian homogeneous space in the following general setting ([CM62], [W62], [W64], [Ku81], [Ko89]):

Corollary 2.6.1 (Calabi-Markus phenomenon). *Let $G \supset H$ be reductive linear groups over $\mathbb{R}$. Then the following two conditions are equivalent:*
i) *There is no infinite discontinuous group for G/H.*
ii) $\mathbb{R}\text{-}\operatorname{rank} G = \mathbb{R}\text{-}\operatorname{rank} H$.

Example 2.6.2 (spherical space form). Let $G/H = O(p+1,q)/O(p,q)$. Then (ii) $\Leftrightarrow \min(p+1,q) = \min(p,q) \Leftrightarrow p \geq q$. The case $q = 1$ was already mentioned in Fact 1.3.3.

Theorem 2.5 is also useful in more delicate problems such as deformation of discontinuous groups (§3.5) and actions of free groups (see §4.1).

§3. Construction of compact Clifford-Klein forms

In this section, we shall briefly describe a construction of compact Clifford-Klein forms for G/H where $G \supset H$ are reductive linear Lie groups.

3.1. (H compact case) Let us start with the case where H is compact. Suppose Γ is a discrete subgroup of G. Then $\Gamma\backslash G/H$ is compact if and only if $\Gamma\backslash G$ is compact. Furthermore, such Γ always contains a torsion free discrete subgroup Γ' of finite index by a theorem of Selberg. From Remark 2.1, $\Gamma'\backslash G/H$ becomes a compact Clifford-Klein form. In summary, in order to find compact Clifford-Klein forms of G/H for compact H, we just need to find a co-compact discrete subgroup of G.

For $G = \mathbb{R}^n$, $\Gamma = \mathbb{Z}^n$ is co-compact. Similarly, if $G = SL(n,\mathbb{R})$, $\Gamma = SL(n,\mathbb{Z})$ is not co-compact but is co-volume finite. This example is a special case of arithmetic subgroups, and some other arithmetic subgroups become co-compact. In the early 1960s, Borel constructed co-compact arithmetic subgroups:

Theorem 3.1. *Any linear reductive Lie group G contains a co-compact discrete subgroup.*

Thus, the Riemannian symmetric space G/K admits a compact Clifford-Klein form.

3.2. (H non-compact case) In contrast to §3.1, not all G/H admits a compact Clifford-Klein form if H is non-compact.

We shall explain construction of compact Clifford-Klein forms, first building on Theorem 3.1 (Steps 1 and 2 below), and second by deformations (Step 3).

We start with an elementary observation for an abelian case $(G,H) = (\mathbb{R}^n, \mathbb{R}^k)$ $(n > k)$. It is easy to see that $\Gamma\backslash G/H$ is a compact Clifford-Klein form if $\Gamma \simeq \mathbb{Z}^{n-k}$

meets H only at $\{0\}$. This observation suggests the following construction of a compact Clifford-Klein form $\Gamma\backslash G/H$:

Step 1. Take a connected subgroup L acting properly and co-compactly on G/H.

Step 2. Take a co-compact (and torsion free) discrete subgroup Γ in L.

Obviously, Step 1 for $G = \mathbb{R}^n$ implies that L is a linear subspace such that
(3.2.1)(a) $L \cap H = \{0\}$,
(3.2.1)(b) $\dim L + \dim H = \dim G$.

Step 2 always works for reductive L by Theorem 3.1. How about Step 1 for reductive L? We shall give a criterion for Step 1. We recall that K is a maximal compact subgroup of G. We put

$$d(G) := \dim G - \dim K.$$

For instance, $d(G) = \frac{1}{2}n(n+1)$ if $G = GL(n, \mathbb{R})$; $d(G) = pq$ if $G = O(p, q)$.

Correspondingly to (3.2.1)(a) and (b), we can prove ([Ko89], Theorem 4.7):

Theorem 3.2. *The homogeneous space G/H admits a compact Clifford-Klein form if there is a reductive subgroup L such that*
(3.2.2)(a) $\nu(L) \pitchfork \nu(H)$ *in* $\mathfrak{a}$ *(see Theorem 2.5).*
(3.2.2)(b) $d(L) + d(H) = d(G)$.

The point here is both of (3.2.2)(a) and (b) are easily checked like an abelian case (3.2.1)(a) and (b), under our assumption that both H and L are reductive.

3.3. Let θ be a Cartan involution of G. We recall that if a subgroup H is reductive in G then, after a conjugation if necessary, we may assume that H is θ-stable. In order to find the triple (G, H, L) in Theorem 3.2, the following lemma is useful:

Lemma 3.3. 1) *Let H and L be θ-stable closed subgroups of G with finitely many connected components. Then the following two conditions are equivalent:*
(3.3)(a) $G = HL$.
(3.3)(b) $\dim H + \dim L = \dim G + \dim (H \cap L)$.
2) *Assume $H \cap L$ is compact. Then, (3.3)(b) $\Rightarrow$ (3.2.2)(a) and (b).*

Proof. (1) is proved in [Ko94], Lemma 5.1. (2) is easy.

3.4. (Examples) From Theorem 3.2 9 together with the above observations, we obtain a family of homogeneous spaces G/H (also G/L by symmetry) that admit compact Clifford-Klein forms, by finding a triple (G, H, L) such that $H \cap L$ is compact and satisfies (3.3)(b). Here is a list of triples (G, H, L) taken from [Ko96a]. In each row, we write a pair

$$\text{a) } G/H, \qquad \text{b) } G/L,$$

if the triple (G, H, L) satisfies (3.3)(b). Among them, the cases (2-b) and (3-b) were obtained by Kulkarni [Ku81], and (1-a), (1-b), (2-a), (3-a) were by [Ko89].

Corollary 3.4. *The following homogeneous spaces admit compact Clifford-Klein forms.*
1) a) $SU(2, 2n)/Sp(1, n)$, b) $SU(2, 2n)/U(1, 2n)$, $(n = 1, 2, 3, \dots)$

2) a) $SO(2,2n)/U(1,n)$, b) $SO(2,2n)/SO(1,2n)$, $(n = 1,2,3,\dots)$
3) a) $SO(4,4n)/Sp(1,n)$, b) $SO(4,4n)/SO(3,4n)$, $(n = 1,2,3,\dots)$
4) a) $SO(8,8)/SO(8,7)$, b) $SO(8,8)/Spin(8,1)$,
5) a) $SO(4,4)/SO(4,1) \times SO(3)$, b) $SO(4,4)/Spin(4,3)$,
6) a) $SO(4,3)/SO(4,1) \times SO(2)$, b) $SO(4,3)/G_{2(2)}$.

3.5. (Deformation) Steps 1 and 2 give a construction of compact Clifford-Klein forms $\Gamma \backslash G/H$. This is not the end of story. In some cases, we can proceed furthermore:

Step 3. Deform Γ in G (possibly outside L).

So far, all known examples of compact Clifford-Klein forms have been constructed either by Steps 1-2 or by Step 3.

A rigorous formulation of deformation is a part of the theorem. We do not go into details here (see [Ko00b], §5 and references therein) but try to give some flavor of it.

For an irreducible Riemannian symmetric space G/H (especially, H compact), one cannot "deform" Γ except for the case $\dim G/H = 2$ (the Selberg-Weil rigidity theorem). However, for irreducible pseudo-Riemannian symmetric spaces G/H, an analog of the "rigidity theorem" can fail for arbitrarily higher dimension ([Ko93], Remark 3).

A special example of pseudo-Riemannian symmetric spaces is a semisimple group manifold G/H where $(G,H) = (G' \times G', \mathrm{diag}(G'))$. Step 1 is fulfilled by taking $L := G' \times 1$, and Γ is identified with a co-compact discrete subgroup of G' in Step 2. Then, Step 3 asks a deformation of the Γ-action on $G/H \simeq G'$, from the left to the both-sides.

Goldman [Go85] first obtained such a deformation for a three dimensional Lorentz space form, and called it *non-standard*. In this case, G' is locally isomorphic to $SL(2,\mathbb{R})$. Ghys's [Gh95] studied deformation for $G' \simeq SL(2,\mathbb{C})$, and Salein [Sa99] furthermore for $G' \simeq SL(2,\mathbb{R})$. Let G' be a simple Lie group. Then, it is proved in [Ko98a], Theorem A that $G' \times G'/\mathrm{diag}(G')$ admits a non-trivial deformation for some Γ if and only if the Lie algebra $\mathfrak{g}'$ is either $\mathfrak{so}(n,1)$ or $\mathfrak{su}(n,1)$. We note that $\mathfrak{sl}(2,\mathbb{R}) \simeq \mathfrak{su}(1,1) \simeq \mathfrak{so}(2,1)$, and $\mathfrak{sl}(2,\mathbb{C}) \simeq \mathfrak{so}(3,1)$.

Step 3 consists of two problems.

1) Deform an abstract group Γ in G (say, Γ_ϵ with a deformation parameter ϵ).
2) Find ϵ such that Γ_ϵ acts properly discontinuously on G/H.

An illustrative example is where $(G,H) = (\mathbb{R}^n, \mathbb{R}^{n-1})$ and $L \simeq \mathbb{R}$ such that $L \cap H = \{0\}$. A lattice $\Gamma = \mathbb{Z}\gamma_0$ of L can be deformed (possibly outside L) by replacing the generator γ_0 by $\gamma_0 + \epsilon$. Then, $\Gamma_\epsilon := \mathbb{Z}(\gamma_0 + \epsilon)$ acts properly discontinuously on G/H unless $\gamma_0 + \epsilon \in H$. This is the case if the deformation parameter ϵ is sufficiently near $0 \in \mathbb{R}^n$. An explicit bound on ϵ can be given by the diameter of $\Gamma \backslash L$ (= the norm of γ_0) and the angle of L and H.

Similar quantitative estimate holds for reductive Lie groups (see [Ko98a], Theorem 2.4 for a precise statement). The key tool of the proof is the criterion of proper actions (Theorem 2.5) and the word length of the fundamental group ([Mi77]). In particular, a small deformation of Γ preserves proper discontinuity (this question was raised by Goldman [Go85]). See also [Sa99], [Ze98] for related results.

§4. Obstruction of compact Clifford-Klein forms

In this section, we shall discuss necessary conditions for the existence of compact Clifford-Klein forms. Let us keep our setting that $G \supset H$ are reductive linear groups over $\mathbb{R}$. We shall assume G/H is non-compact, equivalently $d(G) > d(H)$.

<u>4.1.</u> (Infinite discontinuous group) We start with an obvious observation:

Observation 4.1.1. *If G/H is non-compact and $\Gamma\backslash G/H$ is compact, then $\sharp\Gamma = \infty$.*

Hence, we have from the Calabi-Markus phenomenon (Corollary 2.6.1):

Theorem 4.1.2. *A compact Clifford-Klein form of G/H exists only if $\mathbb{R}\text{-}\operatorname{rank} G > \mathbb{R}\text{-}\operatorname{rank} H$.*

For example, there does not exist a compact Clifford-Klein form of G/H if $(G, H) = (GL(p+q, \mathbb{R}), GL(p, \mathbb{R}) \times GL(q, \mathbb{R}))$ because $\mathbb{R}\text{-}\operatorname{rank} G = \mathbb{R}\text{-}\operatorname{rank} H (= p+q)$.

Benoist [Be96] proved a necessary and sufficient condition that G/H admits a discontinuous action of a free non-abelian group. This strengthens Theorem 4.1.2. Some of the latest examples in this direction include:

Example 4.1.3. 1) (Benoist, [Be96]) $SL(2n+1, \mathbb{R})/SL(2n, \mathbb{R})$ does not admit a compact Clifford-Klein form.
2) (Oh-Witte, [OW01b]) Let $G = SL(3, \mathbb{R})$. Then a homogeneous space G/H does not admit a compact Clifford-Klein form unless either G/H or H is compact.

<u>4.2.</u> (Dimension Theorem) The argument in §4.1 is not sharp. Here is another observation:

Observation 4.2.1. *Consider a continuous action of $\Gamma = \mathbb{Z}^m$ on $X = \mathbb{R}^n$. If it is properly discontinuous, then $m < n$. Furthermore, $\Gamma\backslash X$ is compact if and only if $m = n$.*

This is proved by using the virtual cohomological dimension $vcd(\Gamma)$ of an abstract group Γ ([Se71]). Different from $\mathbb{R}^n$, the homogeneous space G/H is not contractible in general but has a vector bundle structure of rank $d(G) - d(H)$ over a compact manifold. Then, by using Serre's spectral sequence, we can prove the following result ([Ko89], Corollary 5.5):

Lemma 4.2.2. *Let Γ be a virtually torsion free subgroup of G.*
1) *If Γ acts properly discontinuously on G/H then $vcd(\Gamma) \leq d(G) - d(H)$.*
2) *Furthermore, $\Gamma\backslash G/H$ is compact if and only if $vcd(\Gamma) = d(G) - d(H)$.*

<u>4.3.</u> (Necessary condition) By an iterative use of Theorem 2.5 and Lemma 4.2.2, we have:

Theorem 4.3 ([Ko92], Theorem 1.5)**.** *G/H does not admit a compact Clifford-Klein form if there exists a reductive subgroup L such that*
(4.3)(a) *$\nu(L) \subset \nu(H)$ (upto $\sim$).*
(4.3)(b) *$d(L) > d(H)$.*

Sketch of Proof. Assume $\Gamma\backslash G/H$ is a compact Clifford-Klein form. Then $vcd(\Gamma) = d(G) - d(H)$. Since $\Gamma \pitchfork H$, we have $\nu(\Gamma) \pitchfork \nu(H)$. Since $\nu(L) \subset \nu(H)$, we have

$\nu(\Gamma) \pitchfork \nu(L)$. Then $\Gamma \pitchfork L$, and therefore $vcd(\Gamma) \le d(G) - d(L)$. This contradicts to (4.3)(b). $\square$

The point of Theorem 4.3 is that two conditions (4.3)(a), (b) are very easily verified for any given (L, H). See also [OW01b] for non-reductive H.

As compact symmetric spaces and flag varieties are very important homogeneous spaces for compact groups, so are reductive symmetric spaces (or pseudo-Riemannian symmetric spaces) and semisimple orbits for reductive groups (especially, they have rich geometric structures, see [Ko98c], §1 and references therein). We examine Theorem 4.3 in these two cases (§4.5, §4.6) as well as some other cases, and also compare them with more recent methods to see to which extent Problem B has been solved (some of them give overlapping examples, and some others cover different parts).

4.4. (Calabi-Markus phenomenon re-examined) A completely different proof of Theorem 4.1.2 can be given as a special case of Theorem 4.3:

Sketch of Proof. Put $L := G$ in Theorem 4.3. $\square$

4.5. (Reductive symmetric spaces) Let us consider reductive symmetric spaces G/H, that is G is a reductive linear Lie group and H is an open subgroup of the fixed point of an involutive automorphism of G. The local classification of irreducible ones was accomplished by Berger [Br57].

Theorem 4.3 applied to symmetric spaces gives ([Ko92], Theorem 1.4):

Theorem 4.5.1. *A reductive symmetric space G/H admits a compact Clifford-Klein form only if its associated pair (G, H^a) is basic in ϵ-family.*

Instead of explaining technical terms "associated pair" and "basic in ϵ-family" ([OS84]), we illustrate Theorem 4.5.1 by an example:

Example 4.5.2. Let $G/H = O(i+j, k+l)/O(i,k) \times O(j,l)$. This is a reductive symmetric space. (It reduces to a pseudo-Riemannian spherical space form if $i = 0$ and $k = 1$.) Without loss of generality, we may assume $i \le j, k, l$. Theorem 4.5.1 means that G/H admits a compact Clifford-Klein form, only if

$$i = 0 \text{ and } 0 < l \le j - k,$$

except for a trivial case where H or G/H is compact. Kobayashi-Ono [KO90] proved some parity condition on j, k, l by another method using Hirzebruch's proportionality principle.

Example 4.5.3. (complex reductive symmetric space) A reductive symmetric space G/H is a complex reductive symmetric space if G and H are furthermore complex Lie groups. The local classification of irreducible ones is equivalent to the classification of real simple Lie algebras, and there are 10 classical series and 22 exceptional ones (É. Cartan 1914). It is likely (in fact, a special case of [Ko00b], Conjecture 4.3) that an irreducible complex symmetric space G/H admits a compact Clifford-Klein form if and only if G/H is locally isomorphic to a group manifold. By using Theorem 4.3 we have proved in [Ko92] that this is the case except for $SO(2n+2, \mathbb{C})/SO(2n+1, \mathbb{C})$, $SL(2n, \mathbb{C})/Sp(n, \mathbb{C})$ and $E_{6,\mathbb{C}}/F_{4,\mathbb{C}}$. Benoist [Be96] proved it for $SO(2n+2,\mathbb{C})/SO(2n+1,\mathbb{C})$ provided n is even by using Theorem 2.5. Other cases remain open.

As in the above examples, Theorem 4.3 is quite useful on Problem B for reductive symmetric spaces, and two other methods [Be96] and [KO90] also produce some new results. As far as I understand, many other methods (e.g. [BL92], [C94], [LMZ95], [LZ95], [Sh00], [Zi94]) do not seem to give any new results for reductive symmetric spaces because of their strong assumptions (loosely, H is required to be "very small").

4.6. (Semisimple orbit) Next, we consider an adjoint orbit $\mathcal{O}_X$ of G through an element X of the Lie algebra. The orbit $\mathcal{O}_X$ is *semisimple* if $\mathrm{ad}(X)$ is semisimple. Then, $\mathcal{O}_X$ has a G-invariant symplectic, pseudo-Riemannian structure. Theorem 4.3 applied to $\mathcal{O}_X$ yields (see [Ko92], Theorem 1.3):

Theorem 4.6 (semisimple orbits). *Let G be simple. A semisimple orbit $\mathcal{O}_X$ admits a compact Clifford-Klein form only if $\mathcal{O}_X$ has a G-invariant complex structure.*

Benoist-Labourie gave a different proof of this fact by using symplectic geometry [BL92].

4.7. $(SL(n,\mathbb{F})/SL(m,\mathbb{F}), m > n > 1)$ Here, $\mathbb{F}=\mathbb{R},\mathbb{C},\mathbb{H}$. This is a non-symmetric space. The typical feature of this example is that the centralizer of H in G contains a "large" semisimple group $SL(n-m,\mathbb{F})$.

1) R. Zimmer [Zi94] proved that $SL(n,\mathbb{R})/SL(m,\mathbb{R})$ does not admit a compact Clifford-Klein form if $n \geq 2m+1$. His method uses superrigidity for cocycles and Ratner's orbit closure theorem.
2) Y. Shalom [Sh00] proved that $SL(n,\mathbb{R})/SL(m,\mathbb{R})$ does not admit a compact Clifford-Klein form if $n \geq 4$ and $m=2$. His method uses unitary representation theory.

Although not mentioned in these papers, these cases (even in stronger forms) were previously obtained as special cases of Theorem 4.3 ([Ko92], Theorem 1.5):

3) $SL(n,\mathbb{F})/SL(m,\mathbb{F})$ does not admit a compact Clifford-Klein form if $n \geq 3[\frac{m+1}{2}]$ $(\mathbb{F}=\mathbb{C},\mathbb{H})$, or if $n > 3[\frac{m+1}{2}]$ $(\mathbb{F}=\mathbb{R})$ ([Ko90], [Ko92]).
   (We note that $2m \geq 3[\frac{m+1}{2}]$ for any $m>1$. Thus, (3) $\Rightarrow$ (1) and (2).)

To see (3), it is enough to put $L := SU(p, n-p;\mathbb{F})$ in Theorem 4.3 where $p := [\frac{m}{2}]$ (see [Ko90], Example 7 for details).

As in the above cases, there are several overlapping examples which can be proved by completely different methods. It would be interesting to examine them, which might suggest future interactions among different fields through Clifford-Klein problems. We refer to [C94], [LZ95], [LMZ95], [Ko96a], [Ko00b] for more examples of this type.

4.8. (Restriction of unitary representations) We finish this article with some relation of Problem B with unitary representation theory.

Definition 4.8.1. The subgroup H is said to be *tempered* in G if there exists a positive function $f \in L^1(H)$ with respect to a left Haar measure on H such that

$|(\pi(h)u,v)| \leq f(h)\|u\|\|v\|$ for any $h \in H$ and any K-finite vectors u, v,

for any unitary representation π of G without G-fixed non-zero vectors.

Theorem 4.8.2 (Margulis, [Ma97]). *If H is noncompact and tempered, then G/H does not admit a compact Clifford-Klein form.*

Oh [O98] obtained many examples of tempered subgroups in simple Lie groups. It turns out that tempered subgroups are relatively "small" in G. By this reason, it seems that Theorem 4.8.2 (at least in the present form) is not strong enough to apply to reductive symmetric spaces G/H. On the other hand, Theorem 4.8.2 produces some new examples for which other methods do not cover:

Example 4.8.3. 1)(Margulis, cf. [O98]) $SL(n,\mathbb{R})/\varphi(SL(2,\mathbb{R}))$ $(n \geq 4)$ does not admit a compact Clifford-Klein form if φ is an irreducible n-dimensional representation of $SL(2,\mathbb{R})$.
2)(Oh-Witte [OW01b]) If $d(H) = 1$ and G is simple, then G/H does not admit a compact Clifford-Klein form.

Margulis's idea [Ma97] is to use the restriction of the unitary representation π of G on $L^2(\Gamma\backslash G)$ to the subgroup H. In an opposite way, consider the restriction of an irreducible unitary representation ϖ of G occurring in $L^2(G/H)$ to the subgroup L (the Zariski closure of Γ, in Step 2 of §3.2). It is mysterious that the restriction $\varpi|_L$ often decomposes without continuous spectrum, yielding discrete decomposable branching laws ([Ko94], [Ko98b], [Ko00a]). The relation between discontinuous groups for non-Riemannian homogeneous spaces and the restriction of unitary representations merits further study, I think.

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