Conformal geometry and global solutions to the Yamabe equations on classical pseudo-Riemannian manifolds

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§0. Introduction

This article is based on three lectures "Conformal geometry and analysis on minimal representations" delivered at the 22nd Winter School, GEOMETRY AND PHYSICS, Czech Republic, Šumava Mountains, January 12–19, 2002.

We shall illustrate a recent progress on the interaction of conformal geometry with unitary representations of Lie groups and global analysis of ultra-hyperbolic differential equations.

The lectures are organized as follows:

Lecture 1 (§1 and §2). For any pseudo-Riemannian manifold (M, g_M), we construct naturally a representation ϖ of the conformal group $G := \mathrm{Conf}(M, g_M)$ on the solution space $\mathrm{Ker}\,\widetilde{\Delta}_M$ of the Yamabe equation (see Theorem 1.7). Let $H := \mathrm{Isom}(M, g_M)$ be the group of isometries. Then H is naturally a subgroup of G. A new line of investigation about the representation $(\varpi, \mathrm{Ker}\,\widetilde{\Delta}_M)$ is initiated by raising the following Problems A $\sim$ D (see §1.8):

Problem A (non-vanishing). When is $\mathrm{Ker}\,\widetilde{\Delta}_M \neq \{0\}$?
Problem B (irreducibility). Is $(\varpi, \mathrm{Ker}\,\widetilde{\Delta}_M)$ irreducible as a representation of G ?
Problem C (unitarization). Find a G-invariant inner product on $\mathrm{Ker}\,\widetilde{\Delta}_M$ (if exists).
Problem D (branching law). Decompose the representation ϖ into irreducibles of H.

Then, we examine these problems in a special case (see Theorem 2.4):

$M \simeq S^p \times S^q$,
g_M is the standard pseudo-Riemannian metric of signature (p, q),
$\mathrm{Conf}(M, g_M) \simeq O(p+1, q+1)$.

Materials in §1 and §2 are taken from [20].

Lecture 2 (§3). Our object of the second lecture is the space of global solutions of the ultra-hyperbolic operator

$$\square_{\mathbb{R}^{p,q}} = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial y_1^2} - \cdots - \frac{\partial^2}{\partial y_q^2} \quad \text{on } \mathbb{R}^{p+q},$$

which equals the Yamabe operator on the flat pseudo-Riemannian manifold $\mathbb{R}^{p,q}$. Motivated by recent results on "minimal" unitary representations of semisimple Lie groups, we construct in a natural way a "Hilbert space" of global solutions on $\mathbb{R}^{p+q}$ such that the (meromorphic) conformal group $O(p+1,q+1)$ acts as a **unitary** representation under the assumption that $p+q>2$ is even and $p,q \geq 1$. Main results of the second lecture are Theorems 3.4 and 3.6, which present **explicitly** the inner product. The interesting property about this inner product is its large invariance group; even translational invariance amounts to a remarkable "conservation law" in the case of the Minkowski space ($p=1$). By heuristic argument based on Sato's hyperfunctions as boundary values of holomorphic functions, we try to explain key ingredients of the inner product such as an explicit form of the Green kernel and an integral expression over non-characteristic hyperplanes. This treatment will play a complementary role to [22], where we gave a proof of Theorems 3.4 and 3.6 in a rigorous and rather different way.

Lecture 3 (§4 and §5). The third lecture focuses on branching laws, especially, from the **conformal** group to the **isometry** subgroup. A key observation here is that if (M_1, g_{M_1}), $(M_2, g_{M_2}), \cdots$ are conformally equivalent pseudo-Riemannian manifolds, then the distinguished representations of the (mutually isomorphic) conformal groups are isomorphic to one another. On the other hand, they are realized in different geometric models, and moreover, the subgroups of isometries vary. Thus, the branching laws to various isometry subgroups give a clue to understand the distinguished representation of the conformal group. We review a general framework of branching laws with emphasis on discretely decomposable cases, and then examine some concrete examples arising from pseudo-hyperbolic spaces. Materials of §4 are taken from [13, 14, 15, 17]. Main results of §5 are proved in (or readily deduced from) [21]. The presentation of §5 differs from [21] in the point that we put more emphasis here on Problems A $\sim$ D for the solution space $\operatorname{Ker} \widetilde{\Delta}_{X_{p,q}}$ on the pseudo-hyperbolic spaces $X_{p,q}$ (see Theorem 5.4), so that the readers can have an overview on the current status of Problems A $\sim$ D for the classical pseudo-Riemannian manifolds $S^p \times S^q$, $\mathbb{R}^{p,q}$ and $X_{p,q}$, respectively in §2, §3 and §5 in this exposition.

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§1. Conformal geometry and distinguished representations.

1.1. Let (M, g_M) be a pseudo-Riemannian manifold, and $\operatorname{Conf}(M, g_M)$ denote the group of conformal transformations of (M, g_M). The aim of this section is to construct a distinguished representation ϖ of the group $\operatorname{Conf}(M, g_M)$ on the solution space $\operatorname{Ker} \widetilde{\Delta}_M$ of the Yamabe equation, and to pin down some fundamental questions on $\operatorname{Ker} \widetilde{\Delta}_M$ from a view point of representation theory.

1.2. Let (M, g_M) and (N, g_N) be pseudo-Riemannian manifolds.

Definition 1.2 A local diffeomorphism $\Phi : M \to N$ is called *conformal* if there exists a positive function $\Omega \in C^\infty(M)$ such that

$$\Phi^*(g_{N,\Phi(x)}) = \Omega(x)^2 g_{M,x} \quad \text{for any } x \in M. \tag{1.2.1}$$

We note that Φ is an *isometry* if and only if $\Omega \equiv 1$.

1.3. For a pseudo-Riemannian manifold (M, g_M), we write

$$\begin{aligned}
&\Delta_M : \text{the Laplace-Beltram operator,}\\
&K_M : \text{the scalar curvature of } M,\\
&\widetilde{\Delta}_M = \Delta_M - \frac{n-2}{4(n-1)} K_M : \text{the } \textbf{Yamabe operator}.
\end{aligned}$$

Here, the scalar curvature K_M is a smooth function on M by definition, and it also acts on $C^\infty(M)$ as a multiplication in the Yamabe operator.

Example 1.3. We equip $\mathbb{R}^n$ and S^n with standard Riemannian metrics. Then

1) For $\mathbb{R}^n$; $K_{\mathbb{R}^n} \equiv 0$ and $\widetilde{\Delta}_{\mathbb{R}^n} = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}$.

2) For S^n; $K_{S^n} \equiv n(n-1)$ and $\widetilde{\Delta}_{S^n} = \Delta_{S^n} - \frac{1}{4} n(n-2)$.

Similar formulas will be given in (3.1.1) and (5.1.1) for pseudo-Riemannian manifolds $\mathbb{R}^{p,q}$ and $X_{p,q}$, of constant sectional curvature, respectively.

1.4. Suppose $\Phi : M \to N$ is a smooth map. We write $\Phi^* : C^\infty(N) \to C^\infty(M)$ for its pull-back. It is easy to see that if Φ is an isometry then

$$\Phi^* \circ \Delta_N = \Delta_M \circ \Phi^*. \tag{1.4.1}$$

A generalization of (1.4.1) is given in [25] for a conformal map Φ, namely,

$$\Omega^{\frac{n+2}{2}} \Phi^* \circ \widetilde{\Delta}_N = \widetilde{\Delta}_M \circ \Omega^{\frac{n-2}{2}} \Phi^*. \tag{1.4.2}$$

Here Ω^λ is regarded as a multiplication operator on $C^\infty(M)$. In particular, we have

Corollary 1.4. Suppose Φ is surjective and conformal. Then, for $f \in C^\infty(N)$,

$$\widetilde{\Delta}_N f = 0 \;\Leftrightarrow\; \widetilde{\Delta}_M(\Omega^{\frac{n-2}{2}}(\Phi^* f)) = 0.$$

In the case $n = 2$, we have $\widetilde{\Delta}_M = \Delta_M$ and $\widetilde{\Delta}_N = \Delta_N$. Then Corollary 1.4 means:

$$\Delta_N f = 0 \;\Leftrightarrow\; \Delta_M(\Phi^* f) = 0.$$

This corresponds to a well-known fact on complex analysis of one variable: *if f is harmonic and Φ is holomorphic, then the composition $f \circ \Phi$ is harmonic.*

1.6. Suppose a group G acts conformally on a pseudo-Riemannian manifold (M, g_M). This means that there exists a positive-valued function Ω on the direct product manifold $G \times M$ such that

$$L_h^*(g_{M, L_h \cdot x}) = \Omega(h, x)^2 g_{M,x} \quad \text{for any } x \in M \text{ and } h \in G,$$

where we write the action of h as $L_h : M \to M$.

Fix $\lambda \in \mathbb{C}$, and we define a linear map $\varpi_\lambda(h^{-1}) : C^\infty(M) \to C^\infty(M)$ by

$$(\varpi_\lambda(h^{-1}) f)(x) := \Omega(h, x)^\lambda f(L_h \cdot x).$$

Since $L : G \to \mathrm{Diffeo}(M)$ is a group homomorphism, the conformal factor Ω satisfies a cocycle condition

$$\Omega(h_1 h_2, x) = \Omega(h_1, L_{h_2} x)\ \Omega(h_2, x) \quad (h_1, h_2 \in G, x \in M),$$

and in turn, the map $\varpi_\lambda : G \to GL(C^\infty(M))$ becomes a group homomorphism. Hence, we have formed a family of representations $(\varpi_\lambda, C^\infty(M))$ of G with parameter $\lambda \in \mathbb{C}$.

Remark. If G acts on M as isometries, then $\Omega \equiv 1$ and consequently the representation ϖ_λ does not depend on the complex parameter λ.

Now, the formula (1.4.2) is interpreted in terms of representation theory as follows:

Lemma 1.6. The Yamabe operator $\widetilde{\Delta}_M : C^\infty(M) \to C^\infty(M)$ intertwines $\varpi_{\frac{n-2}{2}}$ and $\varpi_{\frac{n+2}{2}}$. Namely, we have

$$\varpi_{\frac{n+2}{2}}(h) \circ \widetilde{\Delta}_M = \widetilde{\Delta}_M \circ \varpi_{\frac{n-2}{2}}(h) \quad \text{for any } h \in G. \tag{1.6.1}$$

<u>1.7.</u> Lemma 1.6 leads us to construct a representation of G on the space of solutions:

Theorem 1.7 (see [20, Theorem 2.5]). $\operatorname{Ker} \widetilde{\Delta}_M$ is a representation of G via $\varpi_{\frac{n-2}{2}}$.

We shall write $(\varpi, \operatorname{Ker} \widetilde{\Delta}_M)$ for this representation. As we shall see in §2, it can happen that $\operatorname{Ker} \widetilde{\Delta}_M$ is **infinite dimensional** even if M is compact in the case where the metric g_M is indefinite. The point of Theorem 1.7 is that our construction relies only on the conformal structure of a pseudo-Riemannian manifold.

For a pseudo-Riemannian manifold (M, g_M), we write

$$\begin{aligned} G &:= \operatorname{Conf}(M, g_M), \ \text{the conformal group,} \\ H &:= \operatorname{Isom}(M, g_M), \ \text{the isometry group.} \end{aligned}$$

Naturally we have $H \subset G$. Then, we ask:

Problem 1.8. Understand $\operatorname{Ker} \widetilde{\Delta}_M$ by means of the groups G and H.

In particular, we focus on the following questions:

Problem A (non-vanishing). When is $\operatorname{Ker} \widetilde{\Delta}_M \neq \{0\}$?
Problem B (irreducibility). Is $(\varpi, \operatorname{Ker} \widetilde{\Delta}_M)$ irreducible as a representation of G ?
Problem C (unitarization). Find a G-invariant inner product on $\operatorname{Ker} \widetilde{\Delta}_M$ (if exists).
Problem D (branching law). Decompose the representation ϖ into irreducibles of H.

Problems A $\sim$ C depend only on the conformal structure, while Problem D involves the pseudo-Riemannian structure of M as well. As we shall see in §2 and §5, the knowledge on branching laws (Problem D) is sometimes useful in solving Problems A and B. Conversely, if $\operatorname{Ker} \widetilde{\Delta}_M$ is non-zero and irreducible (Problems A and B), then an invariant inner product (Problem C) is unique up to a scalar multiple by Schur's lemma. It seems a challenging problem to find such an inner product in an **intrinsic** way of conformal geometry if it exists. We also mention that the description of the inner product (Problem C) according to the branching law (Problem D) gives a **Parseval-Plancherel** type theorem.

So far, we have been able to find the invariant inner products explicitly only for some classical pseudo-Riemannian manifolds such as $S^p \times S^q$, $\mathbb{R}^{p,q}$, and pseudo-Riemannian hyperbolic spaces $X_{p,q}$ or their product manifolds with S^r (see §2, §3 and §5).

§2. Minimal unitary representation of $O(p+1, q+1)$.

<u>2.1.</u> This section gives an answer to Problems A $\sim$ D for the direct product manifold $M = S^p \times S^q$ endowed with a pseudo-Riemannian metric g_M of signature (p, q).

2.2. We set up some notation. Let

$$\begin{aligned}\mathbb{R}^{p+1,q+1} &:= \text{the Euclidean space } \mathbb{R}^{(p+1)+(q+1)} \text{ equipped with} \\ &\qquad \text{the pseudo-Riemannian metric } ds^2 = dx_1^2 + \cdots + dx_p^2 - dy_1^2 - \cdots - dy_q^2, \\ \Xi &:= \{(x,y) \in \mathbb{R}^{(p+1)+(q+1)} : |x| = |y| \neq 0\}, \\ M &:= \{(x,y) \in \mathbb{R}^{(p+1)+(q+1)} : |x| = |y| = 1\} \simeq S^p \times S^q.\end{aligned}$$

Then we have natural embeddings:

$$M \subset \Xi \subset \mathbb{R}^{p+1,q+1}$$

with each codimension one. Induced from $\mathbb{R}^{p+1,q+1}$, we have a pseudo-Riemanniam metric

$$g_M = g_{S^p} \oplus (-g_{S^q})$$

on $M \simeq S^p \times S^q$ with signature (p,q). We define a positive-valued function by

$$\nu : \Xi \to \mathbb{R}, \quad (x,y) \mapsto |x|.$$

We note that the multiplicative group $\mathbb{R}_{>0}$ acts on Ξ by dilations, and the corresponding quotient space $\Xi/\mathbb{R}_{>0}$ is naturally diffeomorphic to M. Since the indefinite orthogonal group $O(p+1,q+1)$ acts linearly on $\mathbb{R}^{p+1,q+1}$ and stabilizes Ξ, it also acts on the quotient space $\Xi/\mathbb{R}_{>0} \simeq M$. This action will be denoted by L. Then, for $h \in O(p+1,q+1)$, we have

$$L_h \cdot z = \frac{h \cdot z}{\nu(h \cdot z)} \quad \text{for } z = (x,y) \in M.$$

An elementary computation shows

$$L_h^*(g_{M, L_h \cdot z}) = \frac{1}{\nu(h \cdot z)^2} g_{M,z} \quad (z \in M).$$

Hence, $O(p+1,q+1)$ acts conformally on $M \simeq S^p \times S^q$. It is known (see for example [11, Chapter IV]) that any conformal transformation on (M, g_M) is given in this form, namely, we have

$$G := \mathrm{Conf}(M, g_M) \simeq O(p+1,q+1). \tag{2.2.1}$$

Then, the group of isometries of M is given by

$$K := \mathrm{Isom}(M, g_M) \simeq O(p+1) \times O(q+1). \tag{2.2.2}$$

We note that K is a maximal compact subgroup of G.

2.3. (Spherical harmonics).

We review classical results on spherical harmonics. At the end of this subsection, we introduce a pseudo-differential operator $(\frac{1}{4} - \widetilde{\Delta}_{S^n})^{\frac{1}{4}}$ on the sphere S^n.

We recall from Example 1.3 (2) that

$$\widetilde{\Delta}_{S^n} = \Delta_{S^n} - \frac{1}{4} n(n-2)$$

is the Yamabe operator on S^n. For each $l \in \mathbb{N}$, we define the space of spherical harmonics of degree l by

$$\begin{aligned}\mathcal{H}^l(\mathbb{R}^{n+1}) :=& \{f \in C^\infty(S^n) : \Delta_{S^n} f = -l(l+n-1)f\} \\ =& \{f \in C^\infty(S^n) : (\frac{1}{4} - \widetilde{\Delta}_{S^n})f = (l + \frac{n-1}{2})^2 f\}.\end{aligned}$$

Then $O(n+1)$ acts irreducibly on $\mathcal{H}^l(\mathbb{R}^{n+1})$ for each l, and we have a direct sum decomposition of Hilbert spaces:

$$L^2(S^n) \simeq \sum_{l=0}^{\infty}{}^{\oplus} \mathcal{H}^l(\mathbb{R}^{n+1}).$$

It follows that $\frac{1}{4} - \widetilde{\Delta}_{S^n}$ is a non-negative self-adjoint operator on $L^2(S^n)$. Hence, we can define a non-negative self-adjoint operator

$$D_n := (\frac{1}{4} - \widetilde{\Delta}_{S^n})^{\frac{1}{4}}$$

on $L^2(S^n)$ with the domain of definition

$$\mathrm{Dom}(D_n) := \{F = \sum_{l=0}^{\infty} F_l : \sum_{l=0}^{\infty} (l + \frac{n-1}{2}) \|F_l\|^2_{L^2(S^n)} < \infty\}.$$

<u>**2.4.**</u> We return our setting $M \simeq S^p \times S^q$. We note that $C^\infty(M)$ contains

$$\bigoplus_{a,b \in \mathbb{N}} \mathcal{H}^a(\mathbb{R}^{p+1}) \otimes \mathcal{H}^b(\mathbb{R}^{q+1})$$

as a dense subspace. On the other hand, the Yamabe operator $\widetilde{\Delta}_M$ has a decomposition formula:

$$\widetilde{\Delta}_M = \widetilde{\Delta}_{S^p} - \widetilde{\Delta}_{S^q} \tag{2.4.1}$$

(this is easy but non-trivial). Therefore, for $F \in C^\infty(M)$, the following three conditions are equivalent:

$$F \in \mathrm{Ker}\, \widetilde{\Delta}_M \ \Leftrightarrow\ \widetilde{\Delta}_{S^p} F = \widetilde{\Delta}_{S^q} F \ \Leftrightarrow\ D_p F = D_q F.$$

In particular, for $a, b \in \mathbb{N}$, we have

$$\mathcal{H}^a(\mathbb{R}^{p+1}) \otimes \mathcal{H}^b(\mathbb{R}^{q+1}) \subset \mathrm{Ker}\, \widetilde{\Delta}_M \ \Leftrightarrow\ a + \frac{p-1}{2} = b + \frac{q-1}{2}. \tag{2.4.2}$$

This observation solves Problems A and D as below. We note that the right-hand side of (2.4.2) holds only if $p+q$ is even. Here, we collect the answers to Problems A $\sim$ D for the pseudo-Riemannian manifold $S^p \times S^q$:

Theorem 2.4 (Kostant [23], Binegar-Zierau [1], Kobayashi-Ørsted [20]). Let $M = S^p \times S^q$ be endowed with the pseudo-Riemannian metric $g_{S^p} \oplus (-g_{S^q})$. Suppose $p, q \geq 1$ and $p+q > 2$.

1) (non-vanishing) The solution space $\mathrm{Ker}\, \widetilde{\Delta}_M \neq 0$ if and only if $p+q$ is even.

From now, we shall assume that $p+q$ is even.

2) (irreducibility) $\mathrm{Ker}\, \widetilde{\Delta}_M$ is an irreducible representation of $\mathrm{Conf}(M, g_M)$.

3) (unitarization) We introduce an inner product $(\ ,\ )_M$ on $\operatorname{Ker}\widetilde{\Delta}_M$ with the norm $||\ ||_M$ given by

$$||F||_M := ||D_pF||_{L^2(M)} \quad (= ||D_qF||_{L^2(M)}). \tag{2.4.3}$$

Then, this inner product is invariant under the conformal group $\operatorname{Conf}(M, g_M)$.

4) (branching law; $\operatorname{Conf}(M, g_M) \downarrow \operatorname{Isom}(M, g_M)$) $\operatorname{Ker}\widetilde{\Delta}_M$ contains

$$\bigoplus_{\substack{a,b\in\mathbb{N} \\ a+\frac{p-1}{2}=b+\frac{q-1}{2}}} \mathcal{H}^a(\mathbb{R}^{p+1}) \otimes \mathcal{H}^b(\mathbb{R}^{q+1}) \qquad \text{(an algebraic direct sum)} \tag{2.4.4}$$

as a dense subspace.

__2.5.__ Here are some few comments on Theorem 2.4.

It follows from Theorem 2.4 (2) and (3) that we can extend the representation $(\varpi, \operatorname{Ker}\widetilde{\Delta}_M)$ to an **irreducible unitary** representation of $G = \operatorname{Conf}(M, g_M)$ on the completion of the pre-Hilbert space $(\operatorname{Ker}\widetilde{\Delta}_M, (\ ,\ )_M)$. The resulting irreducible unitary representation of the group $G \simeq O(p+1, q+1)$ will be denoted by $\varpi^{p+1,q+1}$.

As we shall review a general theory in §4.3, the algebraic direct sum (2.4.4) has a structure of a $(\mathfrak{g}, K)$-module, for which we write $(\varpi^{p+1,q+1})_{K\text{-finite}}$. Here, $\mathfrak{g}$ is the Lie algebra of G. Since the isometry group $K \simeq O(p+1) \times O(q+1)$ acts irreducibly on each summand of (2.4.4), Theorem 2.4 (4) may be regarded as a branching law from $G = \operatorname{Conf}(M, g_M)$ to $K = \operatorname{Isom}(M, g_M)$.

As for the relations of Theorem 2.4 (3) and (4), it is useful to write the inner product $(\ ,\ )_M$ according to the branching law: if $F = \sum_{a,b} F_{a,b} \in \operatorname{Ker}\widetilde{\Delta}_M$ then the formula (2.4.3) is equivalent to the following Parseval-Plancherel type theorem:

$$||F||_M^2 = \sum_{\substack{a,b\in\mathbb{N}, \\ a+\frac{p-1}{2}=b+\frac{q-1}{2}}} (b + \frac{q-1}{2})||F_{a,b}||^2_{L^2(M)}. \tag{2.5.1}$$

This formula will be generalized in Theorem 5.6 in the case where isometry groups are not necessarily compact.

__2.6.__ Let us mention briefly the ideas of the proof of Theorem 2.4.

As we have already discussed, our approach to the branching law (Problem D) relies on the decomposition formula (2.4.1) of the Yamabe operator. Then the non-vanishing condition (Problem A) is an immediate corollary.

The irreducibility (Problem B) is more difficult. In §5, we shall find other branching laws of the same representation $\varpi^{p+1,q+1}$ to various non-compact subgroups (Theorem 5.6). They are **discretely decomposable**, and informative enough to conclude that the original representation $\varpi^{p+1,q+1}$ is irreducible. This is an idea adopted in [21, Theorem 7.6]. Another proof for the irreducibility given in [1] is based on a careful computation of the action of the Lie algebra $\mathfrak{g}$ on $(\varpi^{p+1,q+1})_{K\text{-finite}}$ (see also [7], [27] for similar computations in a closely related setting).

For the unitarization, there are two proofs known (see [1],[20]), but we do not mention here even the ideas. Instead, we shall discuss more details on the unitarization problem for the solution space on the flat space $\mathbb{R}^{p,q}$ in §3.

__2.7.__ The representation $\varpi^{p+1,q+1}$ is so called a minimal unitary representation of $O(p+1, q+1)$ for $p + q > 4$, and is interesting by its own right in representation theory of semisimple Lie groups. The Weil representation (sometimes called the oscillator representation or the Segal-Shale-Weil representation) yields another example of minimal unitary representations. There has been a very extensive litereature on the Weil representation, while our representation $\varpi^{p+1,q+1}$ has been paid attention only quite recently since Kostant's work on $SO(4,4)$ [23]. Our approach to the representation $\varpi^{p+1,q+1}$ is based on conformal geometry. There are also other (more algebraic) approaches (e.g. [1], [8], [7], [23]) to the same representation $\varpi^{p+1,q+1}$. This representation will be constructed also in §3 and §5 on locally conformally equivalent pseudo-Riemannian manifolds whose isometry groups are **non-compact**.

§3 Canonical Hilbert space of ultra-hyperbolic solutions

3.1. In this section, we consider the following ultra-hyperbolic operator:

$$\square_{\mathbb{R}^{p,q}} = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial y_1^2} - \cdots - \frac{\partial^2}{\partial y_q^2} \tag{3.1.1}$$

on $\mathbb{R}^n = \mathbb{R}^{p+q}$ $(p, q \geq 1)$. It coincides with the Yamabe operator $\widetilde{\Delta}_{\mathbb{R}^{p,q}}$ on $\mathbb{R}^{p,q}$, because the curvature tensor vanishes on $\mathbb{R}^{p,q}$.

For $\mathbb{R}^{p,q}$, the conformal group is not larger than the isometry group, namely,

$$\mathrm{Conf}(\mathbb{R}^{p,q}) = \mathrm{Isom}(\mathbb{R}^{p,q}) \simeq O(p,q) \ltimes \mathbb{R}^{p,q}.$$

As we shall see in §3.7, we have a conformal compactification:

$$\mathbb{R}^{p,q} \hookrightarrow (S^p \times S^q)/\mathbb{Z}_2 \tag{3.1.2}$$

through which $O(p+1, q+1)$ acts **meromorphically** and conformally on $\mathbb{R}^{p,q}$. Instead of $\mathrm{Conf}(\mathbb{R}^{p,q})$, our analysis on $\mathrm{Ker}\,\square_{\mathbb{R}^{p,q}}$ will be studied by the larger group, $\mathrm{Conf}^{\mathrm{m}}(\mathbb{R}^{p,q})$, of meromorphic conformal transformations. Our main concern in this section is with Problem C in this sense, namely,

Question 3.1. Find an $O(p+1, q+1)$-invariant **inner pruduct** on $\mathrm{Ker}\,\square_{\mathbb{R}^{p,q}}$ (if exists).

There are two points here, in comparison to the case $S^p \times S^q$ treated in §2:

1) In Theorem 2.4, the unitarization of $\mathrm{Ker}\,\widetilde{\Delta}_M$ was given by the use of the pseudo-differential operator $(\frac{1}{4} - \widetilde{\Delta}_{S^p})^{\frac{1}{4}}$. However, since a pseudo-differential operator is not a local operator, it is another problem to find its counterpart on $\mathbb{R}^{p,q}$ even though there is a conformal embedding $\mathbb{R}^{p,q} \hookrightarrow S^p \times S^q$.

2) We have seen in Theorem 1.7 that the kernel of the Yamabe operator is preserved by the conformal group. However, this is not the case if we allow the transformation to be meromorphic. A price to pay is to replace $\mathrm{Ker}\,\square_{\mathbb{R}^{p,q}}$ by its suitable subspace in Theorem 1.7 (and in Question 3.1).

With these points in mind, we shall give answers to Question 3.1 in two (apparently, very different) ways: by the use of the Green function (Theorem 3.4), and by an integration over a non-characteristic hyperplane (Theorem 3.6).

Throughout this section, we shall assume that

$$n = p + q$$

is even and > 2 and that $p, q \geq 1$.

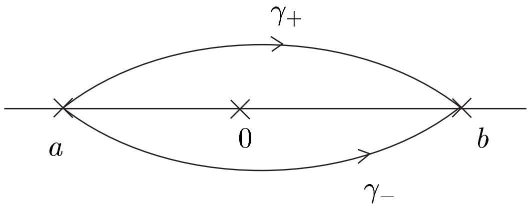
3.2. (Hyperfunction of one variable).

This subsection presents a quick review on Sato's hyperfunctions of one variable from [26]. The underlying idea becomes a hint to find the Green function (§3.4) and also an "intrinsic" inner product (§3.6).

Fist of all, we recall Cauchy's integral formula

$$\frac{1}{2\pi i}\int_C \frac{1}{z}dz = 1,$$

where C is a contour around the origin 0. We divide C into two parts $C = -\gamma_+ + \gamma_-$ as below (here $a < 0 < b$).



Letting $\gamma_\pm$ be close to the real axis, we have

$$\frac{1}{2\pi i}\int_a^b \left(\frac{-1}{x+i0} + \frac{1}{x-i0}\right)dx = 1, \tag{3.2.1}$$

where we define $\frac{1}{x \pm i0}$ as a limit in the sense of Schwartz's distributions $\mathcal{S}'(\mathbb{R})$:

$$\frac{1}{x \pm i0} := \lim_{y \downarrow 0} \frac{1}{x \pm iy}.$$

Conversely, one may regard that the distribution $\frac{1}{x \pm i0}$ "extends" holomorphically to the half plane $\{z \in \mathbb{C} : \pm \operatorname{Im} z > 0\}$. Its singularity at $x = 0$ is canceled by the multiplication of x:

$$x \cdot \frac{1}{x \pm i0} = 1 \quad \text{in } \mathcal{S}'(\mathbb{R}). \tag{3.2.2}$$

More precisely, the difference of $\frac{1}{x+i0}$ and $\frac{1}{x-i0}$ is given by the following formula:

Lemma 3.2. (Dirac's delta)

$$\delta(x) = \frac{1}{2\pi i}\left(\frac{-1}{x+i0} + \frac{1}{x-i0}\right) \tag{3.2.3}$$

Proof of Lemma 3.2. It is enough to show, for any test function $\varphi \in C_0^\infty(\mathbb{R})$,

$$\frac{1}{2\pi i}\int_a^b \varphi(x)\left(\frac{-1}{x+i0} + \frac{1}{x-i0}\right)dx = \varphi(0). \tag{3.2.4}$$

The equation (3.2.4) holds for a constant function φ because of Cauchy's integral formula (3.2.1). It follows from (3.2.2) that the equation (3.2.4) also holds if $\varphi(x)$ is of the form $\varphi(x) = x\psi(x)$ ($\psi \in C_0^\infty(\mathbb{R})$). Hence (3.2.4) holds for any $\varphi \in C_0^\infty(\mathbb{R})$, in view of the expression

$$\varphi(x) = \varphi(0) + x\psi(x), \quad \psi(x) := \frac{\varphi(x) - \varphi(0)}{x}.$$

Thus, Lemma 3.2 is proved. □

For any compactly supported distribution $f \in \mathcal{E}'(\mathbb{R})$, we define distributions $f_\pm$ by

$$f_\pm := \frac{1}{2\pi i}\frac{\mp 1}{x \pm i0} * f.$$

Then $f_\pm(x)$ extends holomorphically on a half-plane $\{z \in \mathbb{C} : \pm \operatorname{Im} z > 0\}$ and we have

$$f = \delta * f = f_+ + f_- \tag{3.2.5}$$

by Lemma 3.2. Hence, f has an expression as the sum of "boundary values" of holomorphic functions. Likewise, any distribution f (possibly, with non-compact support) is expressed by the boundary values of holomorphic functions ([26]).

<u>3.3.</u> (Fundamental solution and Green function of $\square_{\mathbb{R}^{p,q}}$).

In this subsection, we explain a heuristic idea to find the invariant inner product (given in Theorem 3.4 below).

We recall from §3.2, the following two equations of distributions of one variable:

$$\delta(x) = \frac{1}{2\pi i}\left(\frac{-1}{x+i0} + \frac{1}{x-i0}\right), \tag{3.3.1}$$

$$x \cdot \frac{1}{x-i0} = 1. \tag{3.3.2}$$

Then the "substitution" of the quadratic form

$$Q(\xi, \eta) := \xi_1^2 + \cdots + \xi_p^2 - \eta_1^2 - \cdots - \eta_q^2$$

into (3.3.1) and (3.3.2) would result in the following equations in $\mathcal{S}'(\mathbb{R}^n)$:

$$\delta(Q) = \frac{1}{2\pi i}\left(-(Q+i0)^{-1} + (Q-i0)^{-1}\right) \tag{3.3.3}$$

$$Q \cdot (Q-i0)^{-1} = 1. \tag{3.3.4}$$

Here, following [3], we have defined distributions $(Q \pm i0)^\lambda$ as a meromorphic continuation of

$$\lim_{\varepsilon \downarrow 0} (Q \pm i\varepsilon R)^\lambda,$$

where R is a positive definite quadratic form. We note that the limit does not depend on the choice of R. In the left-hand side of (3.3.3), we define $\delta(Q)$ to be the measure $d\mu$ supported on the cone

$$C := \{\zeta \in \mathbb{R}^n : Q(\zeta) = 0\}$$

such that

$$dQ d\mu = d\xi_1 \cdots d\xi_p d\eta_1 \cdots d\eta_q.$$

We write down the measure $d\mu$ on C explicitly by

$$\int_C \phi d\mu = \frac{1}{2}\int_0^\infty \int_{S^p}\int_{S^q} \phi(s\omega, s\eta) s^{p+q-3} ds d\omega d\eta$$

for a test function ϕ. Then $\delta(Q)$ becomes a distribution on $\mathbb{R}^n$ if $n = p+q > 2$. (We note that a general theory by using the wave front set is not enough to justify the "substitution" of Q into $\delta(x)$.)

Now, we take the Fourier inverse $\mathcal{F}^{-1}$ of (3.3.3) and (3.3.4) to obtain two equations in $\mathcal{S}'(\mathbb{R}^n)$:

$$E_0 = \frac{1}{2\pi i}(\overline{E} - E) \tag{3.3.5}$$

$$\Box_{\mathbb{R}^{p,q}} E = \delta(x) \tag{3.3.6}$$

where we put

$$E_0 := \mathcal{F}^{-1}(\delta(Q)) \tag{3.3.7}$$

$$\begin{aligned} E &:= -\mathcal{F}^{-1}((Q-i0)^{-1}) \\ &= \frac{-\Gamma(\frac{n}{2}-1)e^{\pi i q}}{4\pi^{\frac{n}{2}}}(x_1^2+\cdots+x_p^2-y_1^2-\cdots-y_q^2+i0)^{1-\frac{n}{2}}. \end{aligned} \tag{3.3.8}$$

Then (3.3.6) means that E is a **fundamental solution** of the ultra-hyperbolic equation $\Box_{\mathbb{R}^{p,q}} f = 0$. An explicit formula of E_0 (the Green function) can be written by using (3.3.5) and (3.3.8) (see also a recent paper [6] of Hörmander on the Green function E_0).

3.4. (Integral expression of ultra-hyperbolic solutions).

It follows from the formulas (3.3.5) and (3.3.6) that

$$\Box_{\mathbb{R}^{p,q}} E_0 = \frac{1}{2\pi i}(-\Box_{\mathbb{R}^{p,q}} E + \Box_{\mathbb{R}^{p,q}} \overline{E}) = \frac{1}{2\pi i}(-\delta(x)+\delta(x)) = 0. \tag{3.4.1}$$

Then, $E_0 * \varphi$ solves the ultra-hyperbolic equation for any $\varphi \in C_0^\infty(\mathbb{R}^n)$ because

$$\Box_{\mathbb{R}^{p,q}}(E_0 * \varphi) = (\Box_{\mathbb{R}^{p,q}} E_0) * \varphi = 0.$$

Therefore, if we define the integral transformation against the **Green kernel** E_0:

$$S : C_0^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n), \quad \varphi \mapsto E_0 * \varphi,$$

then its image constructs global solutions on $\mathbb{R}^n$.

On the image of S, denoted by $\operatorname{Image} S$, we define a sesqui-linear form $(\ ,\ )_S$ by

$$(f_1, f_2)_S := \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} E_0(x-y)\varphi_1(x)\overline{\varphi_2(y)}dxdy \tag{3.4.2}$$

for $f_i = S(\varphi_i) \in \operatorname{Image} S$ $(i = 1, 2)$. The integral (3.4.2) converges because $\varphi_i \in C_0^\infty(\mathbb{R}^n)$, and do not depend on the choice of φ_i but only on f_i.

Theorem 3.4 (Unitarization of ultrahyperbolic solutions; [22, Theorem 4.7]).

1) $\operatorname{Image} S \subset \operatorname{Ker} \Box_{\mathbb{R}^{p,q}}$.
2) $(\ ,\ )_S$ is positive definite on $\operatorname{Image} S$.
3) $O(p+1, q+1)$ acts unitarily on the completion of the pre-Hilbert space $\operatorname{Image} S$.

Sketch of Proof: We have already explained the proof of (1).
(2) corresponds to the positivity of the measure $\delta(Q) = \mathcal{F}(E_0)$ on $\mathbb{R}^n$ (see (3.3.7)). In fact, the inner product $(\ ,\ )_S$ is proved to be equivalent (up to scalar) to the L^2-inner product of functions on the cone C via the Fourier transform $\mathcal{F}$ ([22, Theorem 4.9]).
(3) corresponds to the intertwining property (cf. (1.6.1)):

$$\varpi_{\frac{n-2}{2}}(h) \circ S = S \circ \varpi_{\frac{n+2}{2}}(h) \quad (h \in G). \tag{3.4.3}$$

(We have written (3.4.3) with a little abuse of notation. In fact, neither $C_0^\infty(\mathbb{R}^n)$ nor $C^\infty(\mathbb{R}^n)$ is preserved under ϖ_λ because the action of $O(p+1, q+1)$ on $\mathbb{R}^n = \mathbb{R}^{p+q}$ is meromorphic.) We shall explain briefly why the intertwining property (3.4.3) leads to the conformal invariance of the inner product $(\ ,\ )_S$, namely,

$$(\varpi_{\frac{n-2}{2}}(h^{-1})f_1, \varpi_{\frac{n-2}{2}}(h^{-1})f_2)_S = (f_1, f_2)_S \quad \text{for any } h \in H \tag{3.4.4}$$

in the following subsection. □

3.5. Suppose we are given a general pseudo-Riemannian manifold M of dimension n and an abstract self-adjoint operator S on $L^2(M)$ satisfying (3.4.3). On $\operatorname{Image} S$, we define a sesquilinear form $(\ ,\ )_S$ by

$$(f_1, f_2)_S := (f_1, \varphi_2)_{L^2(M)} = (\varphi_1, f_2)_{L^2(M)}$$

for $f_i = S(\varphi_i)$ $(i = 1, 2)$. Here, the second equation follows from $S = S^*$. Then,

$$\begin{aligned}
\text{the left-hand side of (3.4.4)} =& (\varpi_{\frac{n-2}{2}}(h^{-1})f_1, S\varpi_{\frac{n+2}{2}}(h^{-1})\varphi_2)_S \\
=& \int_M \Omega(h, x)^{\frac{n-2}{2}} f_1(L_h x) \Omega(h, x)^{\frac{n+2}{2}} \overline{\varphi_2(L_h x)} dx \\
=& \int_M f_1(L_h x) \overline{\varphi_2(L_h x)} (L_h^* dx) \\
=& \int_M f_1(x) \overline{\varphi_2(x)} dx \\
=& \text{the right-hand side of (3.4.4).}
\end{aligned}$$

This explains that (3.4.3) formally implies (3.4.4), namely, the conformal invariance of the inner product.

Remark 3.5. In [22], we used the naive ideas of §3.3 and §3.5 just as a guiding principle to Theorem 3.4 (2) and (3), and took a different approach for an actual proof (with a little aid of representation theory) in order to avoid some technical difficulties.

3.6. (Intrinsic inner product).

The inner product in Theorem 3.4 gives an answer to Question 3.1, but it relies on the integral expression of ultra-hyperbolic solutions. In this subsection, we ask furthermore:

Question. Write the invariant inner product on the solution space **directly** without the integral expression of solutions.

An idea for this is to use the Cauchy data (or its variant) of the solution on a non-characteristic hyperplane, say $x_1 = 0$, as follows: Let $f \in \operatorname{Ker} \square_{\mathbb{R}^{p,q}}$. We regard f as a hyperfunction of the first variable, and as in (3.2.5) we write

$$f(x_1, x_2, \cdots, y_q) = f_+(x_1, x_2, \cdots, y_q) + f_-(x_1, x_2, \cdots, y_q), \tag{3.6.1}$$

where $f_\pm(x_1, x_2, \cdots, x_p, y_1, \cdots, y_q)$ extend holomorphically with respect to x_1 in some regions of $\pm \operatorname{Im} z_1 > 0$. We define a sesqui-linear form $(\ ,\ )_W$ as the polarization of the following quadratic form:

$$(f, f)_W := \frac{1}{\sqrt{-1}} \int_{\mathbb{R}^{n-1}} (f_+ \overline{\frac{\partial f_+}{\partial x_1}} - f_- \overline{\frac{\partial f_-}{\partial x_1}})|_{x_1=0} dx_2 \cdots dx_p dy_1 \cdots dy_q. \tag{3.6.2}$$

It is easy to see that the right-hand side of (3.6.2) is independent of the decomposition (3.6.1). Surprisingly, the following equation holds:

$$4\pi(\ ,\ )_W = (\ ,\ )_S \qquad \text{on } \operatorname{Image} S. \tag{3.6.3}$$

Hence, we have:

Theorem 3.6 ([22, Theorem 6.2]). The sesqui-linear form $(\ ,\ )_W$ is positive definite on $\operatorname{Image} S$ and invariant under the meromorphic and conformal actions of $O(p+1, q+1)$.

If we replace the hyperplane $x_1 = 0$ by another one $y_1 = 0$ in (3.6.2), then the resulting inner product is still the same. Since any non-characteristic hyperplane in $\mathbb{R}^{p,q}$ is conjugate to either $x_1 = 0$ or $y_1 = 0$ by the isometry group $\operatorname{Isom}(\mathbb{R}^{p,q}) \simeq O(p,q) \ltimes \mathbb{R}^{p,q}$, Theorem 3.6 implies that the right-hand side of (3.6.2) does not change for any non-characteristic hyperplane. Thus, even the isometry invariance of $(\ ,\ )_W$ is not trivial. Furthermore, a special case of the translation invariance amounts to a remarkable "conservation law" in the Minkowski space for the wave equation (namely, the case $p = 1$) (see [28], [22, §6.7]). It is interesting that the inner product enjoys the invariance of $\operatorname{Conf}^{\mathrm{m}}(\mathbb{R}^{p,q})$, which is much larger than $\operatorname{Conf}(\mathbb{R}^{p,q}) = \operatorname{Isom}(\mathbb{R}^{p,q}) \simeq O(p,q) \ltimes \mathbb{R}^{p+q}$.

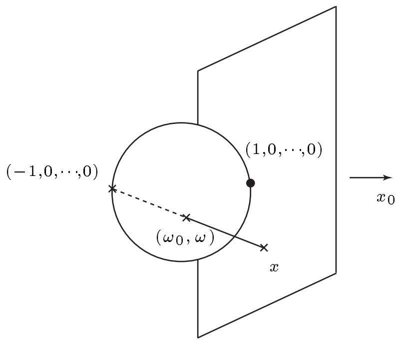
<u>3.7.</u> (Stereographic projection and conformal embedding).

For the convenience of the readers, we shall compare the results of §2 (the case $S^p \times S^q$) and §3 (the case $\mathbb{R}^{p,q}$) in §3.7 and §3.8.

1) Riemannian case

First, we recall a usual stereographic projection, which gives a conformal diffeomorphism between an open dense subset of the standard Riemannian sphere S^n and the flat Riemannian manifold $\mathbb{R}^n$:

$$\begin{array}{ccc} \frac{2}{1+\omega_0}\omega & \longleftarrow\!\shortmid & (\omega_0, \omega) \\ \Cap & & \Cap \\ \mathbb{R}^n & \simeq & S^n \setminus \{(-1, 0, \cdots, 0)\} \\ \Cup & & \Cup \\ x & \longmapsto & \frac{1}{1+\frac{|x|^2}{4}} \left(1 - \frac{|x|^2}{4}, x\right) \end{array}$$



2) General case (pseudo-Riemannian case)

Next, we generalize the above map to an embedding of the flat pseudo-Riemannian manifold $\mathbb{R}^{p,q}$ into the pseudo-Riemannan manifold $S^p \times S^q$ as follows:

$$\Psi : \mathbb{R}^{p,q} \hookrightarrow S^p \times S^q, \quad (x, y) \mapsto \frac{1}{\tau(x,y)}(1 - \frac{|x|^2 - |y|^2}{4}, x, y, 1 + \frac{|x|^2 - |y|^2}{4}).$$

Here, $S^p \times S^q$ is regarded as a submanifold of $\mathbb{R}^{1+p+q+1}$, and we put

$$\tau(x,y) := \sqrt{\left(1+\left(\frac{|x|+|y|}{2}\right)^2\right)\left(1+\left(\frac{|x|-|y|}{2}\right)^2\right)}.$$

Then, Ψ is a conformal map in view of the formula:

$$\Psi^* g_{S^p\times S^q} = \frac{1}{\tau(x,y)^2} g_{\mathbb{R}^{p,q}}.$$

The image of Ψ is roughly a half of $S^p \times S^q$, namely, $\{(u,v) \in S^p \times S^q : u_0 + v_q > 0\}$. We write $(S^p \times S^q)/\mathbb{Z}_2$ for the quotient manifold of $S^p \times S^q$ with respect to the equivalence relation in $S^p \times S^q$:

$$(u,v) \sim (-u,-v).$$

Then Ψ induces a **conformal compactification**:

$$\mathbb{R}^{p,q} \hookrightarrow (S^p \times S^q)/\mathbb{Z}_2.$$

Then the group $O(p+1,q+1)$ acts meromorphically on $\mathbb{R}^{p,q}$ as an open dense subset of $(S^p \times S^q)/\mathbb{Z}_2$. We note that $-I = \mathrm{diag}(-1,\cdots,-1) \in O(p+1,q+1)$ acts identically on $(S^p \times S^q)/\mathbb{Z}_2$. Thus, the quotient group $O(p+1,q+1)/\{\pm I\}$ acts effectively on $\mathbb{R}^{p,q}$, and in fact we have

$$\mathrm{Conf}^{\mathrm{m}}(\mathbb{R}^{p,q}) \simeq O(p+1,q+1)/\{\pm I\}.$$

<u>**3.8.**</u> Let us compare the results of §2 and §3. First, we summarize conformal groups, isometry groups and Yamabe operators on $\mathbb{R}^{p,q}$ and $S^p \times S^q$, respectively:

$$\begin{array}{ccc}
(\S 3) & & (\S 2) \\
\mathbb{R}^{p,q} & \underset{\text{conformal}}{\overset{\Psi}{\hookrightarrow}} & M = S^p \times S^q \\
\Box_{\mathbb{R}^{p,q}} = \displaystyle\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2} - \sum_{j=1}^{q} \frac{\partial^2}{\partial y_j^2} & & \widetilde{\Delta}_M = \Delta_{S^p} - \Delta_{S^q} - \dfrac{(p+q-2)(p-q)}{4} \\
\mathrm{Conf}^{\mathrm{m}}(\mathbb{R}^{p,q}) \simeq O(p+1,q+1)/\{\pm I\} & & O(p+1,q+1) \simeq \mathrm{Conf}(M) \\
\cup \qquad\qquad \cup & & \cup \qquad\qquad \cup \\
\mathrm{Isom}(\mathbb{R}^{p,q}) \simeq O(p,q) \ltimes \mathbb{R}^{p+q} & \not\simeq & O(p+1) \times O(q+1) \simeq \mathrm{Isom}(M).
\end{array}$$

We define the **twisted pull-back** of the conformal map Ψ by

$$\widetilde{\Psi}^* : C^\infty(M) \to C^\infty(\mathbb{R}^{p+q}), \qquad F \mapsto \tau^{-\frac{p+q-2}{2}} \Psi^* F.$$

Then $\widetilde{\Psi}^*$ sends $\mathrm{Ker}\, \widetilde{\Delta}_M$ into $\mathrm{Ker}\, \Box_{\mathbb{R}^{p,q}}$ by Corollary 1.4. In the following diagram

$$\begin{array}{ccccc}
C^\infty(M) & \xrightarrow{\widetilde{\Psi}^*} & C^\infty(\mathbb{R}^{p+q}) & & \\
\cup & & \cup & & \\
\mathrm{Ker}\, \widetilde{\Delta}_M & \xrightarrow{\widetilde{\Psi}^*} & \mathrm{Ker}\, \Box_{\mathbb{R}^{p,q}} & \xleftarrow{S} & C_0^\infty(\mathbb{R}^{p+q}),
\end{array}$$

we can prove

$$\operatorname{Image} S \subset \widetilde{\Psi}^*(\operatorname{Ker} \widetilde{\Delta}_M).$$

In other words, if $\varphi \in C_0^\infty(\mathbb{R}^{p+q})$ then $S(\varphi) \in C^\infty(\mathbb{R}^{p+q})$ extends to a smooth solution on $S^p \times S^q$. The inner products $(\ ,\ )_M$ in §2, $(\ ,\ )_S$ and $(\ ,\ )_W$ in §3 have the following relations:

Theorem 3.8. In the above setting, if

$$f_i = S_i(\varphi_i) = \widetilde{\Psi}^*(F_i)\ (\in \operatorname{Ker} \square_{\mathbb{R}^{p,q}}),$$

for $\varphi_i \in C_0^\infty(\mathbb{R}^{p+q})$, $F_i \in \operatorname{Ker} \widetilde{\Delta}_M$ $(i = 1, 2)$, then we have

$$2^{4-p-q}(F_1, F_2)_M = 4\pi (f_1, f_2)_W = (f_1, f_2)_S.$$

The proof of Theorem 3.8 is given in [22, §4 and §6].

§4. (Discretely decomposable) branching laws

This section gives a short survey of general results on branching laws of unitary representations with emphasis on discretely decomposable cases (see [4, 12, 13, 14, 15, 16, 17, 18, 24] for some of recent results). In §5, we shall find discretely decomposable branching laws from the conformal group to various isometry subgroups.

4.1. We consider a general setting :

$$G' \subset G \xrightarrow{\pi} GL_{\mathbb{C}}(\mathcal{H}),$$

where G is a group, G' is its subgroup and $(\pi, \mathcal{H})$ is a unitary representation of G.

The restriction of π to G' is no more irreducible in general. By a **branching law**, we mean the irreducible decomposition of the restriction $\pi|_{G'}$:

$$\pi|_{G'} \simeq \int_{\widehat{G'}}^{\oplus} m_\pi(\tau) \tau \, d\mu(\tau)$$

defined on a direct integral of Hilbert spaces against a measure $d\mu$ on the unitary dual $\widehat{G'}$. Here,

$$m_\pi : \widehat{G'} \to \mathbb{N} \cup \{\infty\}$$

is a measurable function, which stands for the multiplicity. Such an irreducible decomposition is unique if G' is a reductive Lie group (more generally, if G' is of type I in the sense of von-Neumann algebras).

4.2. In this subsection, we give a simple example of branching laws of unitary representations. Let $G = SL(2, \mathbb{R})$ and $\mathcal{H} = L^2(\mathbb{R})$. We define a unitary representation π on $\mathcal{H}$ by

$$\pi(g) : L^2(\mathbb{R}) \to L^2(\mathbb{R}), \quad f(x) \mapsto \frac{1}{|cx+d|} f(\frac{ax+b}{cx+d})$$

for $g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$. We consider three subgroups N, A and K.

$$G \supset G' := \begin{cases} N := \{\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in \mathbb{R}\} & \simeq \mathbb{R}, \\ A := \{\begin{pmatrix} e^x & 0 \\ 0 & e^{-x} \end{pmatrix} : x \in \mathbb{R}\} & \simeq \mathbb{R}, \\ K := \{\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} : \theta \in \mathbb{R}/2\pi\mathbb{Z}\} & \simeq S^1. \end{cases}$$

Then the unitary dual of N, A and K is parametrized by $\mathbb{R}$, $\mathbb{R}$ and $\mathbb{Z}$, respectively, as usual.

Exercise: Find branching laws of the restrictions $\pi|_N$, $\pi|_A$ and $\pi|_K$.
Answers: There exist unitary equivalent maps:

$$\pi|_N \simeq \int_{\mathbb{R}}^{\oplus} e^{ix\xi} d\xi \quad \text{(no discrete spectrum)}, \tag{4.2.1}$$

$$\pi|_A \simeq \int_{\mathbb{R}}^{\oplus} 2e^{ix\xi} d\xi \quad \text{(no discrete spectrum)}, \tag{4.2.2}$$

$$\pi|_K \simeq \sum_{n \in 2\mathbb{Z}}^{\oplus} e^{in\theta} \quad \text{(discretely decomposable)}. \tag{4.2.3}$$

Sketch of Proof:

1) The restriction $\pi|_N$ is nothing but the regular representation of the abelian Lie group $\mathbb{R}$ on $L^2(\mathbb{R})$. Therefore, the branching law (4.2.1) is obtained by means of the Fourier transform.

2) Correspondingly to the decomposition $\mathbb{R} = \mathbb{R}_{>0} \cup \{0\} \cup \mathbb{R}_{<0}$, the Hilbert space $L^2(\mathbb{R})$ splits as
$$L^2(\mathbb{R}) \simeq L^2(\mathbb{R}_{>0}) \oplus L^2(\mathbb{R}_{<0}). \tag{4.2.4}$$
Each component is unitarily equivalent to the regular representation of the abelian Lie group $A \simeq \mathbb{R}$. Hence, we have the branching law (4.2.2), especially, the multiplicity of $e^{ix\xi}$ is two.

3) By using the isomorphism (up to scalar) between Hilbert spaces:
$$L^2(\mathbb{R}) \simeq L^2(S^1), \quad f(x) \mapsto \frac{f(\tan\frac{\psi}{2})}{|\cos\frac{\psi}{2}|},$$
the branching law (4.2.3) reduces to the Fourier series expansion of $L^2(S^1)$.

4.3. (K-finite vectors and $(\mathfrak{g}, K)$-modules).
Generalizing the feature of Theorem 2.4 (4) or the branching law (4.2.3) of the last case, we introduce the following notion:

Definition 4.3.1 ([13, §1]). In the setting of §4.1, we say the restriction $\pi|_{G'}$ is G'**-admissible** if it decomposes discretely and the multiplicity $m_\pi(\tau)$ is finite for any $\tau \in \widehat{G'}$.

Let G be a real reductive linear Lie group, and K a maximal compact subgroup of G. Here is Harish-Chandra's fundamental result on admissible restrictions to maximal compact subgroups.

Theorem 4.3.2 ([5]). For any $\pi \in \widehat{G}$, the restriction $\pi|_K$ is K-admissible.

For the proof, we refer also to [30, Theorem 3.4.1].

We return the previous example of $G = SL(2, \mathbb{R})$. In (4.2.3), instead of the Hilbert space

$$L^2(S^1) = \sum_{n\in\mathbb{Z}}^{\oplus} \mathbb{C}e^{in\theta} \quad \text{(discrete sum of Hilbert spaces)},$$

on which the group $G = SL(2, \mathbb{R})$ acts continuously, one may consider its dense subspace

$$L^2(S^1)_{K\text{-finite}} := \bigoplus_{n\in\mathbb{Z}} \mathbb{C}e^{in\theta} \quad \text{(algebraic direct sum)},$$

on which both the Lie algebra $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{R})$ and a maximal compact subgroup $K = SO(2)$ act in a compatible way (so called a $(\mathfrak{g}, K)$**-module**). Here, we note that $L^2(S^1)$ is not preserved by the differential action of the Lie algebra $\mathfrak{g}$. Neither does $L^2(S^1)_{K\text{-finite}}$ by the group G.

More generally, let π be a continuous representation of G defined on a Fréchet space $\mathcal{H}$. We assume that π has the following property (cf. Definition 4.3.1):

$$m_\pi(\tau) = \dim \operatorname{Hom}_K(\tau, \pi|_K) < \infty \quad \text{for any } \tau \in \widehat{K}.$$

Then one can define a differential action of the Lie algebra $\mathfrak{g}$ on a dense subspace of $\mathcal{H}$,

$$\mathcal{H}_{K\text{-finite}} := \{v \in \mathcal{H} : \dim_{\mathbb{C}} \mathbb{C}\text{-span}\{\pi(k)v : k \in K\} < \infty\},$$

so that the $(\mathfrak{g}, K)$-module structure, denoted by $\pi_{K\text{-finite}}$, is naturally defined on $\mathcal{H}_{K\text{-finite}}$.

The notion of $(\mathfrak{g}, K)$-modules has been a powerful **algebraic** tool of unitary representation theory of real reductive Lie groups. The point here is:

1) No topology is specified.

2) Representation theoretic properties (e.g. composition series) are not lost in passing from G-modules to $(\mathfrak{g}, K)$-modules.

We refer to [10], [30], and references therein for the theory of $(\mathfrak{g}, K)$-modules, which is built on the K-admissibility of the restriction $\pi|_K$.

4.4. For a non-compact subgroup G' and for a unitary representation π of G, we ask:

Problem ([13]). When does the restriction $\pi|_{G'}$ become G'-admissible ?

The branching law of the restriction $\pi|_{G'}$ sometimes contains continuous spectrum as in the examples (4.2.1) and (4.2.2), and consequently is not G'-admissible. Here is a sufficient condition for the restriction $\pi|_{G'}$ to be G'-admissible:

Theorem 4.4 ([14, Theorem 2.9]). Let $G \supset G'$ be a pair of reductive linear Lie groups, and $\pi \in \widehat{G}$. If

$$\mathrm{Cone}(G') \cap \mathrm{AS}_K(\pi) = \{0\}, \tag{4.4.1}$$

then $\pi|_{G'}$ is G'-admissible.

Here, $\mathrm{Cone}(G')$ and $\mathrm{AS}_K(\pi)$ are closed cone in $\mathbb{R}^l$ determined respectively by G' and π, where l is the rank of K. We refer to [16], Definition 4.2 for the definition of $\mathrm{Cone}(G')$; and [14, §2.7] for that of the **asymptotic K-support** $\mathrm{AS}_K(\pi)$ which was introduced by Kashiwara and Vergne [9]. A main tool for Theorem 4.4 is the micro-local study of characters by using the wave front set (or the singularity spectrum).

The following very special cases may help us to understand Theorem 4.4.

$$\mathrm{Cone}(G') = \{0\} \;\Leftrightarrow\; G' \supset K. \tag{4.4.2}$$

$$\mathrm{AS}_K(\pi) = \{0\} \;\Leftrightarrow\; \dim \pi < \infty. \tag{4.4.3}$$

In particular, if $G' = K$ then the assumption (4.4.1) is automatically fulfilled. The conclusion of Theorem 4.4 in this case corresponds to Harish-Chandra's theorem (Theorem 4.3.2).

4.5. The criterion in Theorem 4.4 works in many important cases, as is illustrated by the following:

Example 4.5 ([13], see also [16, Proposition 4.3.2 and Theorem 4.10]).

1) $\mathrm{Cone}(G')$ is a linear subspace (modulo the Weyl group) if (G, G') is a symmetric pair.

2) $\mathrm{AS}_K(\pi)$ is computable if π is a discrete series representation, etc.

Here, we say that (G, G') is a *symmetric pair* if there is an involutive automorphism σ of the group G such that G' is an open subgroup of $G^\sigma := \{g \in G : \sigma g = g\}$. For example, $(GL(n,\mathbb{R}), O(p, n-p))$ is a symmetric pair. (To see this, we put $\sigma(g) := I_{p,n-p}\, {}^t g^{-1}\, I_{p,n-p}$, where $I_{p,n-p} = \mathrm{diag}(1, \cdots, 1, -1, \cdots, -1) \in GL(n,\mathbb{R})$.)

4.6. Suppose that there is a G-invariant measure on a homogeneous space G/G'. Let $L^2(G/G')$ be the Hilbert space consisting of square integrable functions on G/G'. Then we can define naturally a unitary representation of G on $L^2(G/G')$ by the pull-back of functions.

Let $\pi \in \widehat{G}$. We say that π is a *discrete series representation* for G/G' if

$$\mathrm{Hom}_G(\pi, L^2(G/G')) \neq \{0\}.$$

In relation to Theorem 4.4, we have the following **exclusive law** of discrete spectrum for the **restriction** and the **induction**:

Theorem 4.6 ([15]). Let G be a non-compact real reductive linear Lie group, (G, G') a symmetric pair, and $\pi \in \widehat{G}$. If $\mathrm{Cone}(G') \cap \mathrm{AS}_K(\pi) = \{0\}$, then $\mathrm{Hom}_G(\pi, L^2(G/G')) = \{0\}$, namely, π is not a discrete series representation for G/G'.

4.7. (Applications of discretely decomposable restrictions).

Let $(\pi, \mathcal{H})$ be an irreducible unitary representation of G, and G' a subgroup of G. As the K-admissibility theorem of Harish-Chandra (Theorem 4.3.2) has given a foundation of algebraic representation theory of $(\mathfrak{g}, K)$-modules, we expect new methods and objects may arise from the **G'-admissibility** of the restrictions π with respect to **non-compact** subgroups G'. Here is some listing of related topics:

Representation theory

(1) Study of representations of a subgroup G' as irreducible summands of $\pi|_{G'}$.
   a) (π: Weil representation). Construction of highest weight modules by the theta correspondence (Howe, Kashiwara-Vergne, Adams $\cdots$).
   b) (π: other "minimal" representations). Construction of "small" representations (Gross, Wallach, $\cdots$).

(2) Study of representations of G by means of the restrictions to subgroups G'.
   a) (G': compact). Theory of $(\mathfrak{g}, K)$-modules (Lepowsky, Harish-Chandra, $\cdots$).
   b) (G': non-compact). Study of algebraic properties of "small" representations (Kobayashi, Ørsted, Li, Lee, Loke, $\cdots$).

(3) Branching laws of their own right.

Non-commutative Harmonic Analysis

(4) Construction of (new) discrete series representations for (non-symmetric and non-Riemannian) homogeneous spaces G/H.

Number Theory

(5) Topology of modular varieties in Clifford-Klein forms of Riemannian symmetric spaces.

Let us mention briefly the role of discrete decomposability in (1) $\sim$ (3). Discrete decomposability (or more strongly, the G'-admissibility) of the restriction $\pi|_{G'}$ makes the construction easier in (1), gives a useful clue to (2), and enables us to approach (3) by purely algebraic methods. We refer to [16] for a short survey on this.

As a simple example of (4), the complete classification of discrete series representations for non-symmetric homogeneous spaces such as $G_2(\mathbb{R})/SL(3,\mathbb{R})$ and $G_2(\mathbb{R})/SU(2,1)$ is given by the branching laws $SO(4,3) \downarrow G_2(\mathbb{R})$ (see [13, Theorem 6.4]).

We refer to [17] for more details on (4) and (5) and references therein.

4.8. As one may see from the criterion in Theorem 4.4, the restriction $\pi|_{G'}$ tends to be discretely decomposable if $\mathrm{AS}_K(\pi)$ is "small". Loosely, this is the case if the (infinite dimensional) representation π is "small". It turns out that the representations of conformal groups constructed in §1 are often "small". In fact, the following inclusive relations

$$\operatorname{Ker} \widetilde{\Delta}_M \subset C^\infty(M) \quad \text{and} \quad \mathrm{Conf}(M) \supset \mathrm{Isom}(M)$$

may give a feeling on the "size" of representation spaces compared to groups:

$$\text{size of } \frac{\operatorname{Ker} \widetilde{\Delta}_M}{\operatorname{Conf}(M)} \ll \text{size of } \frac{C^\infty(M)}{\operatorname{Isom}(M)},$$

namely, the representation space $\operatorname{Ker} \widetilde{\Delta}_M$ with respect to the group $\operatorname{Conf}(M)$ is much "smaller" if we compare a usual representation of $\operatorname{Isom}(M)$ on $C^\infty(M)$. For example, $\operatorname{AS}_K(\pi)$ is one dimensional if π is the representation constructed in §2. More precisely, we have:

Example 4.8 (cf. [14, Example 3.4]). Let $p+q$ be even, and $p, q \geq 2$.

1) There is a vector v such that $\operatorname{AS}_K(\varpi^{p,q}) = \mathbb{R}_+ \cdot v$. (Clearly, this is a non-zero minimal cone.)

2) Let $G' = O(p', q') \times O(p'', q'')$ be a natural subgroup of G with $p' + p'' = p$ and $q' + q'' = q$. Then

$$\operatorname{AS}_K(\varpi^{p,q}) \cap \operatorname{Cone}(G') = \{0\} \Leftrightarrow \min(p', p'', q', q'') = 0.$$

In particular, the restriction $\varpi^{p,q}|_{O(p,q') \times O(q'')}$ is discretely decomposable, and Theorem 4.6 says that the representation $\varpi^{p,q}$ cannot be captured as discrete spectrum in $L^2(O(p,q)/(O(p,q') \times O(q'')))$ for any q' and q'' with $q' + q'' = q$.

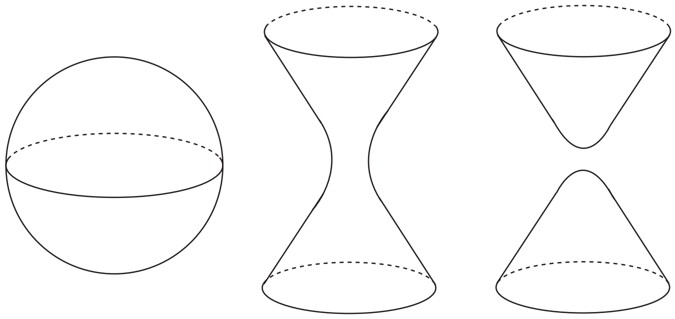
"Small" represenations are currently one of the most mysterious part of unitary representation theory of reductive Lie groups. Theorem 4.4 suggests that such representations fit well into the framework of admissible restrictions to non-compact subgroups.

§5. Yamabe operator on pseudo-hyperbolic spaces

<u>**5.1.**</u> So far, we have been examined the solution space of the Yamabe operator on $S^p \times S^q$ and $\mathbb{R}^{p,q}$. As a third example, we consider the *pseudo-hyperbolic space*:

$$X_{p,q} := \{(x, y) \in \mathbb{R}^{p+q} : |x|^2 - |y|^2 = 1\} \simeq O(p,q)/O(p-1,q).$$

Here is an example for the case $p + q = 3$:



$X_{3,0}$ $X_{2,1}$ $X_{1,2}$

The pseudo-Riemannian metric on $X_{p,q}$ induced from $\mathbb{R}^{p,q}$ is of signature $(p-1, q)$. With this pseudo-Riemannian metric, $X_{p,q}$ has a constant sectional curvature 1, and the Yamabe operator amounts to:

$$\widetilde{\Delta}_{X_{p,q}} = \Delta_{X_{p,q}} - \frac{1}{4}(p+q-1)(p+q-3). \tag{5.1.1}$$

Let $L^2(X_{p,q})$ be the Hilbert space of square integrable functions on $X_{p,q}$ with respect to the natural volume element. For $\lambda \in \mathbb{C}$, we put

$$V_\lambda^{p,q} := \{f \in L^2(X_{p,q}) : \widetilde{\Delta}_{X_{p,q}} f = (\frac{1}{4} - \lambda^2) f\}.$$

Since $V_\lambda^{p,q} = V_{-\lambda}^{p,q}$, we may and do assume $\operatorname{Re} \lambda \geq 0$ without loss of generality. The classification of discrete series representations for the pseudo-hyperbolic space $X_{p,q} \simeq O(p,q)/O(p-1,q)$ is well-known:

Proposition 5.1 (cf. [2], [27]).

1) $(p = 1)$ $V_\lambda^{p,q} = \{0\}$ for any $\lambda \in \mathbb{C}$.
2) $(p \neq 1)$ $V_\lambda^{p,q} \neq \{0\} \Leftrightarrow \lambda \in \frac{p+q}{2} + 2\mathbb{Z}$ and $\lambda \neq 0$.
   Furthermore, the isometry group $\operatorname{Isom}(X_{p,q}) \simeq O(p,q)$ acts irreducibly on each $V_\lambda^{p,q}$, for $\lambda \in \frac{p+q}{2} + 2\mathbb{Z}$. This representation will be denoted by $\pi_{+,\lambda}^{p,q}$.

There exist also irreducible unitary representations $\pi_{+,0}^{p,q}$ ($p+q$:even) and $\pi_{+,-\frac{1}{2}}^{p,q}$ ($p+q$:odd) of $O(p,q)$ which are not realized in $L^2(X_{p,q})$ but enjoy similar algebraic properties (see [21, §5.4] for a rigorous definition).

Likewise, one can define irreducible unitary representations, denoted by $\pi_{-,\lambda}^{p,q}$, of $O(p,q)$ realized on a pseudo-hyperbolic space

$$X_{p,q}^- := \{(x,y) \in \mathbb{R}^{p+q} : |x|^2 - |y|^2 = -1\} \simeq O(p,q)/O(p,q-1)$$

for $\lambda \in \frac{p+q}{2} + 2\mathbb{Z}$ if $q > 1$.

5.2. Let us consider Problems A $\sim$ D in §1.8 for the solution space $\operatorname{Ker} \widetilde{\Delta}_{X_{p,q}}$ on the pseudo-hyperbolic space $X_{p,q}$. An interesting case is where $p+q$ is odd and $p > 1$, which we shall explain here. Then, we note that $V_{\frac{1}{2}}^{p,q} = \operatorname{Ker} \widetilde{\Delta}_{X_{p,q}} \cap L^2(X_{p,q}) \neq \{0\}$.

On $\operatorname{Ker} \widetilde{\Delta}_{X_{p,q}}$, there is not only the action $\operatorname{Isom}(X_{p,q}) \simeq O(p,q)$, but also the infinitesimal action of $\mathfrak{o}(p, q+1)$, which is the Lie algebra of the group $\operatorname{Conf}^{\mathrm{m}}(X_{p,q})$ of **meromorphic conformal** transformations. We shall treat $\operatorname{Ker} \widetilde{\Delta}_{X_{p,q}}$ as a module of the pair $(\mathfrak{o}(p,q+1), O(p,q))$, corresponding to the Lie groups:

$$\operatorname{Conf}^{\mathrm{m}}(X_{p,q}) \supset \operatorname{Conf}(X_{p,q}) = \operatorname{Isom}(X_{p,q}).$$

5.3. We put $K = O(p) \times O(q)$ and define a subspace of $C^\infty(X_{p,q})$ by

$$W^{p,q} := (\operatorname{Ker} \widetilde{\Delta}_{X_{p,q}})_{K\text{-finite}} = \{f \in \operatorname{Ker} \widetilde{\Delta}_{X_{p,q}} : \dim_{\mathbb{C}} \mathbb{C}\text{-span}\{k \cdot f : k \in K\} < \infty\}.$$

Then $W^{p,q}$ is dense in $\operatorname{Ker} \widetilde{\Delta}_{X_{p,q}}$. The Lie algebra $\mathfrak{o}(p,q+1)$ acts linearly on $W^{p,q}$, and so does its complexified Lie algebra $\mathfrak{o}(p+q+1, \mathbb{C})$.

Theorem 5.3 ([21, Proposition D]). Let m be odd. Then there exists a long exact sequence of $\mathfrak{o}(m+1, \mathbb{C})$-modules:

$$0 \longrightarrow W^{1,m-1} \xrightarrow{\varphi_1} W^{2,m-2} \xrightarrow{\varphi_2} \cdots \xrightarrow{\varphi_{m-2}} W^{m-1,1} \longrightarrow 0 \tag{5.3.1}$$

such that we have isomorphisms as $\mathfrak{o}(m+1,\mathbb{C})$-modules:

$$\operatorname{Ker}\varphi_p \simeq (\varpi^{p,q+1})_{K\text{-finite}} \tag{5.3.2}$$
$$\operatorname{Coker}\varphi_{p-1} \simeq (\varpi^{p+1,q})_{K\text{-finite}} \tag{5.3.3}$$

for any (p,q) with $p+q=m$. Here, we recall §2.4 for the notation of the right-hand sides of (5.3.2) and (5.3.3).

One can regard Theorem 5.3 as another (new) construction of the "minimal" representations $\varpi^{a,b}$ of indefinite orthogonal groups $O(a,b)$ with $a+b=m+1$. An interesting feature of Theorem 5.3 is that there is **no** conformal local diffeomorphism among pseudo-hyperbolic spaces $X_{a,b}$ of different signature, however, there **exist** Lie algebra homomorphisms among the solution spaces $\operatorname{Ker}\widetilde{\Delta}_{X_{a,b}}$.

<u>**5.4.**</u> Let $p+q$ be odd, and $p,q>1$. It follows from Theorem 5.3 that Problem B has a negative answer, namely, the irreducibility fails. This is so, even if we allow meromorphic transformations. Nevertheless, $W^{p,q}$ is nearly irreducible, and its socle filtration is not very complicated. Here is a precise statement on the module structure of $W^{p,q}=(\operatorname{Ker}\widetilde{\Delta}_{X_{p,q}})_{K\text{-finite}}$:

Theorem 5.4.

$$(\operatorname{Ker}\widetilde{\Delta}_{X_{p,q}})_{K\text{-finite}} \simeq \begin{array}{|c|}\hline (\varpi^{p+1,q})_{K\text{-finite}} \\ \hline (\varpi^{p,q+1})_{K\text{-finite}} \\ \hline \end{array} \qquad \text{as } \mathfrak{o}(p+q+1,\mathbb{C})\text{-modules,}$$

$$\simeq \begin{array}{|c|}\hline (\pi^{p,q}_{-,\frac{1}{2}})_{K\text{-finite}} \\ \hline (\pi^{p,q}_{+,-\frac{1}{2}})_{K\text{-finite}} \\ \hline \end{array} \oplus \begin{array}{|c|}\hline (\pi^{p,q}_{-,-\frac{1}{2}})_{K\text{-finite}} \\ \hline (\pi^{p,q}_{+,\frac{1}{2}})_{K\text{-finite}} \\ \hline \end{array} \qquad \text{as } \mathfrak{o}(p+q,\mathbb{C})\text{-modules.}$$

We give some few comments on Theorem 5.4:

1) The first formula in Theorem 5.4 means that $(\varpi^{p,q+1})_{K\text{-finite}}$ (respectively, $(\varpi^{p+1,q})_{K\text{-finite}}$) is isomorphic to the unique submodule (respectively, quotient) of $(\operatorname{Ker}\widetilde{\Delta}_{X_{p,q}})_{K\text{-finite}}$ as $\mathfrak{o}(p+q+1,\mathbb{C})$-modules. Thus, we have a non-splitting exact sequence

$$0 \longrightarrow (\varpi^{p,q+1})_{K\text{-finite}} \longrightarrow (\operatorname{Ker}\widetilde{\Delta}_{X_{p,q}})_{K\text{-finite}} \longrightarrow (\varpi^{p+1,q})_{K\text{-finite}} \longrightarrow 0 \tag{5.4.1}$$

of $\mathfrak{o}(p+q+1,\mathbb{C})$-modules. In particular, the above socle filtration gives even a finer structure than what we have asked in Problem B.

2) The solution space $\operatorname{Ker}\widetilde{\Delta}_{X_{p,q}}$ (or its dense subspace) is not unitarizable as a representation of the group $\operatorname{Conf}^{\mathrm{m}}(X_{p,q})$ because the exact sequence (5.4.1) does not split. This gives a negative answer to Problem C (unitarizability), too. On the other hand, the isomorphism (5.3.2) implies that roughly a half of the solution space $\operatorname{Ker}\widetilde{\Delta}_{X_{p,q}}$, namely, the submodule $\operatorname{Ker}\varphi_p$ is unitarizable as a representation of $\operatorname{Conf}^{\mathrm{m}}(X_{p,q})$.

3) The second formula in Theorem 5.4 is regarded as branching laws with respect to:

$$\operatorname{Conf}^{\mathrm{m}}(X_{p,q}) \supset \operatorname{Isom}(X_{p,q}).$$

Thus, Theorem 5.4 contains answers (and even more) to Problems A $\sim$ D for $X_{p,q}$ with $p+q$ odd.

4) In view of the second formula in Theorem 5.4, $V_{\frac{1}{2}}^{p,q} = \operatorname{Ker} \widetilde{\Delta}_{X_{p,q}} \cap L^2(X_{p,q})$ corresponds to only one piece $(\pi_{+,\frac{1}{2}}^{p,q})_{K\text{-finite}}$ among four irreducible subquotients of $(\operatorname{Ker} \widetilde{\Delta}_{X_{p,q}})_{K\text{-finite}}$ as $\mathfrak{o}(p+q,\mathbb{C})$-modules. Since $\operatorname{Ker} \varphi_p \simeq (\varpi^{p,q+1})_{K\text{-finite}}$ decomposes into the diret sum

$$(\pi_{+,-\frac{1}{2}}^{p,q})_{K\text{-finite}} \oplus (\pi_{+,\frac{1}{2}}^{p,q})_{K\text{-finite}}$$

as $\mathfrak{o}(p,q)$-modules, the $O(p,q+1)$-invariant inner product on $\operatorname{Ker} \varphi_p$ must be a scalar multiple of the L^2-inner product on $X_{p,q}$ when restricted to the second factor $(\pi_{+,\frac{1}{2}}^{p,q})_{K\text{-finite}}$, namely, $(V_{\frac{1}{2}}^{p,q})_{K\text{-finite}}$. The inner product when restricted to the first factor $(\pi_{+,-\frac{1}{2}}^{p,q})_{K\text{-finite}}$ is not the L^2-inner product and is more mysterious.

We shall consider the branching law in a more general setting in §5.5 and §5.6.

<u>**5.5.**</u> (Conformal compactification).

Let $q = q' + q''$ (the previous case will correspond to $q'' = 1$). We define a map Φ by:

$$\Phi : X_{p,q'} \times S^{q''-1} \hookrightarrow S^{p-1} \times S^{q-1}, \ ((x,y'),y'') \mapsto (\frac{x}{|x|}, \frac{(y',y'')}{|x|}). \tag{5.5.1}$$

Then Φ is injective with open dense image. We induce pseudo-Riemannian structures on

$$\begin{aligned} Y &:= X_{p,q'} \times S^{q''-1} \\ M &:= S^{p-1} \times S^{q-1} \end{aligned} \tag{5.5.2}$$

from $\mathbb{R}^{p,q'} \times \mathbb{R}^{0,q''}$ and $\mathbb{R}^{p,q}$, respectively. Then, M is a special case of Y by putting $q' = 0$. It follows from

$$\Phi^* g_M = \frac{1}{|x|^2} g_Y$$

that Φ is a comformal map. In particular, if we define a twisted pull-back $\widetilde{\Phi}^* : C^\infty(M) \to C^\infty(M)$ by

$$(\widetilde{\Phi}^* f)(x) := |x|^{-\frac{p+q-4}{2}} f(\Phi(x))$$

then $\widetilde{\Phi}^*$ sends $\operatorname{Ker} \widetilde{\Delta}_M$ injectively into $\operatorname{Ker} \widetilde{\Delta}_Y$. We shall explain it by means of the isometry group Y below. Since $\operatorname{Ker} \widetilde{\Delta}_M$ extends to an irreducible unitary representation of $O(p,q)$ (see §2), we are curious about its counterpart for $\operatorname{Ker} \widetilde{\Delta}_Y$. Let us explain it by means of the irometry group of Y below.

<u>**5.6.**</u> We summarize the settings for Y and M:

$$\begin{array}{ccccccc} & & \mathrm{Conf}^{\mathrm{m}}(Y) & \simeq & O(p,q) & \simeq & \mathrm{Conf}(M) \\ & & \cup & & \cup & & \cup \\ \mathrm{Conf}(Y) & \underset{(q'\neq 0)}{=} & \mathrm{Isom}(Y) & \simeq & O(p,q') \times O(q'') & \underset{(q'\neq 0)}{\not\simeq} & \mathrm{Isom}(M) \end{array}$$

Theorem 5.6. Let $p+q$ (>4) be even, $q = q' + q''$, and $p, q \geq 2$. Then the twisted pull-back $\widetilde{\Phi}^*$ of the conformal embedding $\Phi : Y \to M$ induces the following:

1) (Discrete branching law; $\mathrm{Conf}^{\mathrm{m}}(Y) \downarrow \mathrm{Isom}(Y)$).

$$\varpi^{p,q}|_{O(p,q')\times O(q'')} \simeq \sum_{l=0}^{\infty}{}^{\oplus} \pi^{p,q'}_{l+\frac{q''}{2}-1} \otimes \mathcal{H}^l(\mathbb{R}^{q''}) \qquad \text{(discrete direct sum)} \tag{5.6.1}$$

2) (Parseval-Plancherel theorem). For $F \in \operatorname{Ker} \widetilde{\Delta}_M$, we decompose $\widetilde{\Phi}^* F$ as $\sum_l F_l$ according to (5.6.1). Then we have:

$$||F||^2_M = \sum_{l=0}^{\infty} (l + \frac{q''}{2} - 1)||F_l||^2_{L^2(Y)} \tag{5.6.2}$$

Here $||\ ||_M$ is the norm corresponding to the invariant inner product $(\ ,\ )_M$ defined in §2.

Remark.

1) There are a few cases where $l + \frac{q''}{2} - 1 \leq 0$. Then we can give a suitable interpretation of $(l + \frac{q''}{2} - 1)||F_l||^2_{L^2(Y)}$ that justifies (5.6.2) by using analytic continuation.

2) In the case $q' = 0$, we have $Y \simeq S^{p-1} \times S^{q-1}$ and the above theorem coincides with Theorem 2.4 (3) and (4) (here, we need to change from (p,q) to $(p+1, q+1)$).

3) In the case $q'' = 1$, we have $Y \simeq X_{p,q-1} \times S^0$, namely, Y consists of two copies of $X_{p,q-1}$. Then Theorem 5.6 asserts that

$$\varpi^{p,q}|_{O(p,q-1)} \simeq \pi^{p,q-1}_{-\frac{1}{2}} \oplus \pi^{p,q-1}_{\frac{1}{2}}$$

because $\mathcal{H}^l(\mathbb{R}^1) = 0$ for $l \geq 2$. This is the case treated in Theorem 5.4 (here, we need to change from q to $q+1$).

<u>5.7.</u> (Open problems).

We end this section with some open problems for interested readers.

Let us consider a generalization of the setting (5.5.2) by putting

$$Y = X_{p',q'} \times X_{q'',p''},$$

where $p' + p'' = p$ and $q' + q'' = q$. Then we have a conformal embedding

$$\Psi : Y \hookrightarrow M.$$

The setting of §5.1 corresponds to the case where $p'' = 0$ and $p' = p$. Then, the twisted pull-back $\widetilde{\Psi}^*$ is similarly defined, and sends $\operatorname{Ker} \widetilde{\Delta}_M$ injectively into $\operatorname{Ker} \widetilde{\Delta}_Y$.

Open problem 5.7.

1) Find explicitly branching laws of the representation $\varpi^{p,q}$ of $O(p,q)$ when restricted to

$$\mathrm{Isom}(Y) \simeq O(p',q') \times O(p'',q'').$$

2) Find explicitly the inner product on (a subspace of) $\operatorname{Ker} \widetilde{\Delta}_Y$ which is $O(p,q)$-invariant.

As was stated in Theorem 5.6, this has been solved when $\min(p',q',p'',q'') = 0$. For general p', p'', q' and q'', the branching law from $\mathrm{Conf}^{\mathrm{m}}(Y)$ to $\mathrm{Isom}(Y)$ must contain continuous spectrum (cf. Example 4.7 (2)) and involve more analytic aspects.

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