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Branching Problems of Unitary Representations

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Abstract

The irreducible decomposition of a unitary representation often contains continuous spectrum when restricted to a non-compact subgroup. The author singles out a nice class of branching problems where each irreducible summand occurs discretely with finite multiplicity (admissible restrictions). Basic theory and new perspectives of admissible restrictions are presented from both analytic and algebraic view points. We also discuss some applications of admissible restrictions to modular varieties and L^p-harmonic analysis.

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1. Introduction

Let π be an irreducible unitary representation of a group G. A **branching law** is the irreducible decomposition of π when restricted to a subgroup G':

$$\pi|_{G'} \simeq \int_{\widehat{G'}}^{\oplus} m_\pi(\tau)\tau \, d\mu(\tau) \qquad \text{(a direct integral).} \tag{1.1}$$

Such a decomposition is unique, for example, if G' is a reductive Lie group, and the **multiplicity** $m_\pi : \widehat{G'} \to \mathbb{N} \cup \{\infty\}$ makes sense as a measurable function on the unitary dual $\widehat{G'}$.

Special cases of **branching problems** include (or reduce to) the followings: Clebsch-Gordan coefficients, Littlewood-Richardson rules, decomposition of tensor product representations, character formulas, Blattner formulas, Plancherel theorems for homogeneous spaces, description of breaking symmetries in quantum mechanics, theta-lifting in automorphic forms, etc. The restriction of unitary representations serves also as a method to study discontinuous groups for non-Riemannian homogeneous spaces (e.g. [Mg, Oh]).

Our interest is in the branching problems for (non-compact) reductive Lie groups $G \supset G'$. In this generality, there is no known algorithm to find branching

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laws. Even worse, branching laws usually contain both discrete and continuous spectrum with possibly infinite multiplicities (the multiplicity is infinite, for example, in the decomposition of the tensor product of two principal series representations of $SL(n, \mathbb{C})$ for $n \geq 3$, [Ge-Gr]).

The author introduced the notion of **admissible restrictions** and **infinitesimal discrete decomposability** in [Ko$_5$] and [Ko$_9$], respectively, seeking for a good framework of branching problems, in which we could expect especially a simple and detailed study of branching laws, which in turn might become powerful methods in other fields as well where restrictions of representations naturally arise.

The criterion in Theorem B indicates that there is a fairly rich examples of admissible restrictions; some are known and the others are new. In this framework, a number of explicit branching laws have been newly found (e.g. [D-Vs, Gr-W$_{1,2}$, Hu-P-S, Ko$_{1,3,4,8}$, Ko-Ø$_{1,2}$, Li$_2$, Lo$_{1,2}$, X]). The point here is that branching problems become accessible by algebraic techniques if there is no continuous spectrum.

The first half of this article surveys briefly a general theory of admissible restrictions both from analytic and algebraic view points (§2, §3). For the simplicity of exposition, we restrict ourselves to unitary representations, although a part of the theory can be generalized to non-unitary representations. The second half discusses some applications of discretely decomposable restrictions. The topics range from representation theory itself (§4) to some other fields such as L^p-analysis on non-symmetric homogeneous spaces (§5) and topology of modular varieties (§6).

2. Admissible Restrictions to Subgroups

Let G' be a subgroup of G, and $\pi \in \widehat{G}$. In light of (1.1), we introduce:

Definition 2.1. We say the restriction $\pi|_{G'}$ is **G'-admissible** if it decomposes **discretely** and the multiplicity $m_\pi(\tau)$ is **finite** for any $\tau \in \widehat{G'}$.

One can easily prove the following assertion:

Theorem A ([Ko$_5$, Theorem 1.2]). *Let $G \supset G' \supset G''$ be a chain of groups, and $\pi \in \widehat{G}$. If the restriction $\pi|_{G''}$ is G''-admissible, then $\pi|_{G'}$ is G'-admissible.*

Throughout this article, we shall treat the setting as below:

Definition 2.2. We say (G, G') is a **pair of reductive Lie groups** if
1) G is a real reductive linear Lie group or its finite cover, and
2) G' is a closed subgroup, and is reductive in G.

Then, we shall fix maximal compact subgroups $K \supset K'$ of $G \supset G'$, respectively.

A typical example is a **reductive symmetric pair** (G, G'), by which we mean that G is as above and that G' is an open subgroup of the set G^σ of the fixed points of an involutive automorphism σ of G. For example, $(G, G') = (GL(n, \mathbb{C}), GL(n, \mathbb{R}))$, $(SL(n, \mathbb{R}), SO(p, n-p))$ are the cases.

Let (G, G') be a pair of reductive Lie groups. Here are previously known examples of **admissible restrictions**:

Example 2.3. The restriction $\pi|_{G'}$ is G'-admissible in the following cases:

1) (Harish-Chandra's admissibility theorem) $\pi \in \widehat{G}$ is arbitrary and $G' = K$.
2) (Howe, [Ho$_1$]) π is the Segal-Shale-Weil representation of the metaplectic group G, and its subgroup $G' = G'_1 G'_2$ forms a dual pair with G'_1 compact.

In these examples, either the subgroup G' or the representation π is very special, namely, G' is compact or π has a highest weight. Surprisingly, without such assumptions, it can happen that the restriction $\pi|_{G'}$ is G'-admissible. The following criterion asserts that the "balance" of G' and π is crucial to the G'-admissibility.

Theorem B (criterion for admissible restrictions, [Ko$_7$]**).** *Let $G \supset G'$ be a pair of reductive Lie groups, and $\pi \in \widehat{G}$. If*

$$\mathrm{Cone}(G') \cap \mathrm{AS}_K(\pi) = \{0\}, \tag{2.1}$$

then the restriction $\pi|_{K'}$ is K'-admissible. In particular, the restriction $\pi|_{G'}$ is G'-admissible, namely, decomposes discretely with finite multiplicity.

A main tool of the proof of Theorem B is the microlocal study of characters by using the singularity spectrum of hyperfunctions. The idea goes back to Atiyah, Howe, Kashiwara and Vergne [A, Ho$_2$, Ks-Vr] in the late '70s. The novelty of Theorem B is to establish a framework of **admissible restrictions** with a number of new examples of interest, which rely on a deeper understanding of the unitary dual developed largely in the '80s (see [Kn-Vo] and references therein).

Let us briefly explain the notation used in Theorem B. We write $\mathfrak{k}'_0 \subset \mathfrak{k}_0$ for the Lie algebras of $K' \subset K$, respectively. Take a Cartan subalgebra $\mathfrak{t}_0$ of $\mathfrak{k}_0$ and fix a positive system $\Delta^+(\mathfrak{k}, \mathfrak{t})$ with dominant Weyl chamber $\sqrt{-1}(\mathfrak{t}_0^*)_+$. Then, $\mathrm{AS}_K(\pi)$ is the **asymptotic K-support** of π ([Ks-Vr]), and $\mathrm{Cone}(G')$ is defined as

$$\mathrm{Cone}(G') := \sqrt{-1}((\mathfrak{t}_0^*)_+ \cap \mathrm{Ad}^*(K)({\mathfrak{k}'_0}^{\perp})). \tag{2.2}$$

By definition, both $\mathrm{AS}_K(\pi)$ and $\mathrm{Cone}(G')$ are closed cones in $\sqrt{-1}\mathfrak{t}_0^*$.

Example 2.4. If $G' = K$, then the assumption (2.1) is automatically fulfilled because $\mathrm{Cone}(G') = \{0\}$. The conclusion of Theorem B in this special case is nothing but **Harish-Chandra's admissibility theorem** (Example 2.3 (1)).

To apply Theorem B for non-compact G', we rewrite the assumption (2.1) more explicitly in specific settings. On the part $\mathrm{Cone}(G')$, we mention:

Example 2.5. $\mathrm{Cone}(G')$ is a linear subspace $\sqrt{-1}(\mathfrak{t}_0^*)^{-\sigma}$ (modulo the Weyl group) if (G, G') is a reductive symmetric pair given by an involution σ. Here, we have chosen a Cartan subalgebra $\mathfrak{t}_0$ to be maximally σ-split.

On the part $\mathrm{AS}_K(\pi)$, let us consider a unitary representation π_λ which is "attached to" an elliptic coadjoint orbit $\mathcal{O}_\lambda := \mathrm{Ad}^*(G)\lambda$, in the orbit philosophy due to Kirillov-Kostant. This representation is a unitarization of a Zuckerman-Vogan module $A_{\mathfrak{q}}(\lambda)$ after some ρ-shift, and can be realized in the Dolbeault cohomology group on $\mathcal{O}_\lambda$ by the results of Schmid and Wong. (Here, we adopt the same polarization and normalization as in a survey [Ko$_4$, §2], for the **geometric quantization** $\mathcal{O}_\lambda \Rightarrow \pi_\lambda$.) We note that $\pi_\lambda \in \widehat{G}$ for "most" λ. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the complexification of a Cartan decomposition of the Lie algebra $\mathfrak{g}_0$ of G. We set

$$\Delta_\lambda^+(\mathfrak{p}) := \{\alpha \in \Delta(\mathfrak{p}, \mathfrak{t}) : \langle \lambda, \alpha \rangle > 0\}, \qquad \text{for } \lambda \in \sqrt{-1}\mathfrak{t}_0^*.$$

The original proof (see [Ko$_5$]) of the next theorem was based on an algebraic method without using microlocal analysis. Theorem B gives a simple and alternative proof.

Theorem C ([Ko$_5$]). *Let $\pi_\lambda \in \widehat{G}$ be attached to an elliptic coadjoint orbit $\mathcal{O}_\lambda$. If*

$$\mathbb{R}_{\geq 0}\text{-span}\, \Delta_\lambda^+(\mathfrak{p}) \cap \mathrm{Cone}(G') = \{0\}, \tag{2.2}$$

then the restriction $\pi_\lambda|_{G'}$ is G'-admissible.

Let us illustrate Theorem C in Examples 2.6 and 2.7 for non-compact G'. For this, we note that a maximal compact subgroup K is sometimes of the form $K_1 \times K_2$ (locally). This is the case if G/K is a Hermitian symmetric space (e.g. $G = Sp(n, \mathbb{R}), SO^*(2n), SU(p, q)$). It is also the case if $G = O(p, q)$, $Sp(p, q)$, etc.

Example 2.6 ($K \simeq K_1 \times K_2$)**.** Suppose K is (locally) isomorphic to the direct product group $K_1 \times K_2$. Then, *the restriction $\pi_\lambda|_{G'}$ is G'-admissible if $\lambda|_{\mathfrak{t}\cap\mathfrak{k}_2} = 0$ and $G' \supset K_1$*. So does the restriction $\pi|_{G'}$ if π is any subquotient of a coherent continuation of π_λ. This case was a prototype of G'-admissible restrictions $\pi|_{G'}$ (where G' is non-compact and π is a non-highest weight module) proved in 1988 by the author ([Ko$_1$; Ko$_2$, Proposition 4.1.3]), and was later generalized to Theorems B and C. Special cases include:

(1) $K_1 \simeq \mathbb{T}$, then π is a unitary highest weight module. The admissibility of the restrictions $\pi|_{G'}$ in this case had been already known in '70s (see Martens [Mt], Jakobsen-Vergne [J-Vr]).
(2) $K_1 \simeq SU(2)$, then π_λ is a quaternionic discrete series. Admissible restrictions $\pi|_{G'}$ in this case are especially studied by Gross and Wallach [Gr-W$_1$] in '90s.
(3) $K_1 \simeq O(q), U(q), Sp(q)$. Explicit branching laws of the restriction $\pi_\lambda|_{G'}$ for singular λ are given in [Ko$_3$, Part I] with respect to the vertical inclusions of the diagram below (see also [Ko$_1$, Ko$_5$] for branching laws to horizontal inclusions).

$$\begin{array}{ccccc}
O(4p,4q) & \supset & U(2p,2q) & \supset & Sp(p,q) \\
\cup & & \cup & & \cup \\
O(4r)\times O(4p-4r,4q) & \supset & U(2r)\times U(2p-2r,2q) & \supset & Sp(r)\times Sp(p-r,q)
\end{array}$$

Example 2.7 (conformal group)**.** There are 18 series of irreducible unitary representations of $G := U(2,2)$ with regular integral infinitesimal characters. Among them, 12 series (about "67% " !) are G'-admissible when restricted to $G' := Sp(1,1)$.

The assumption in Theorem B is in fact necessary. By using the technique of symplectic geometry, the author proved the converse statement of Theorem B:

Theorem D ([Ko_{13}])**.** *Let $G \supset G'$ be a pair of reductive Lie groups, and $\pi \in \widehat{G}$. If the restriction $\pi|_{K'}$ is K'-admissible, then* $\mathrm{Cone}(G') \cap \mathrm{AS}_K(\pi) = \{0\}$.

3. Infinitesimal Discrete Decomposability

The definition of admissible restrictions (Definition 2.1) is "analytic", namely, based on the direct integral decomposition (1.1) of unitary representations. Next, we consider discrete decomposable restrictions by a purely algebraic approach.

Definition 3.1 ([Ko_9, Definition 1.1])**.** Let $\mathfrak{g}$ be a Lie algebra. We say a $\mathfrak{g}$-module X is **discretely decomposable** if there is an increasing sequence of $\mathfrak{g}$-submodules of finite length:

$$X = \bigcup_{m=0}^{\infty} X_m, \quad X_0 \subset X_1 \subset X_2 \subset \cdots . \tag{3.1}$$

We note that $\dim X_m = \infty$ in most cases below.

Next, consider the restriction of group representations.

Definition 3.2. Let $G \supset G'$ be a pair of reductive Lie groups, and $\pi \in \widehat{G}$. We say that the restriction $\pi|_{G'}$ is **infinitesimally discretely decomposable** if the underlying $(\mathfrak{g}, K)$-module π_K is discretely decomposable as a $\mathfrak{g}'$-module.

The terminology "discretely decomposable" is named after the following fact:

Theorem E ([Ko_9])**.** *Let (G, G') be a pair of reductive Lie groups, and π_K the underlying $(\mathfrak{g}, K)$-module of $\pi \in \widehat{G}$. Then* (i) *and* (ii) *are equivalent:*

i) *The restriction $\pi|_{G'}$ is infinitesimally discretely decomposable.*
ii) *The $(\mathfrak{g}, K)$-module π_K has a* **discrete branching law** *in the sense that π_K is isomorphic to an algebraic direct sum of irreducible $(\mathfrak{g}', K')$-modules.*

Moreover, the following theorem holds:

Theorem F (infinitesimal $\Rightarrow$ Hilbert space decomposition; [Ko_{11}]**).** *Let $\pi \in \widehat{G}$. If the restriction $\pi|_{G'}$ is infinitesimally discretely decomposable, then the restriction $\pi|_{G'}$ decomposes without continuous spectrum:*

$$\pi|_{G'} \simeq \sum_{\tau \in \widehat{G'}}^{\oplus} m_\pi(\tau)\tau \quad \textit{(a discrete direct sum of Hilbert spaces).} \tag{3.2}$$

At this stage, the multiplicity $m_\pi(\tau) := \dim \mathrm{Hom}_{G'}(\tau, \pi|_{G'})$ can be infinite.

However, for a reductive symmetric pair (G, G'), it is likely that the multiplicity of discrete spectrum is finite under the following assumptions, respectively.

(3.3) π is a discrete series representation for G.

(3.4) The restriction $\pi|_{G'}$ is infinitesimally discretely decomposable.

Conjecture 3.3 (Wallach, [X]). *$m_\pi(\tau) < \infty$ for any $\tau \in \widehat{G'}$ if (3.3) holds.*

Conjecture 3.4 ([Ko_{11}, Conjecture C]). *$m_\pi(\tau) < \infty$ for any $\tau \in \widehat{G'}$ if (3.4) holds.*

We note that Conjecture 3.4 for compact G' corresponds to Harish-Chandra's admissibility theorem. A first affirmative result for general non-compact G' was given in [Ko_9], which asserts that Conjecture 3.4 holds if π is attached to an elliptic coadjoint orbit. A special case of this assertion is:

Theorem G ([Ko_9]). *$m_\pi(\tau) < \infty$ for any $\tau \in \widehat{G'}$ if both (3.3) and (3.4) hold.*

In particular, Wallach's Conjecture 3.3 holds in the discretely decomposable case. We note that an analogous finite-multiplicity statement fails if **continuous spectrum** occurs in the restriction $\pi|_{G'}$ for a reductive symmetric pair (G, G'):

Counter Example 3.5 ([Ko_{11}]). *$m_\pi(\tau)$ can be ∞ if neither (3.3) nor (3.4) holds.*

Recently, I was informed by Huang and Vogan that they proved Conjecture 3.4 for any π [Hu-Vo].

A key step of Theorem G is to deduce the K'-admissibility of the restriction $\pi|_{K'}$ from the discreteness assumption (3.4), for which we employ Theorem H below. Let us explain it briefly. We write $\mathcal{V}_{\mathfrak{g}}(\pi)$ for the **associated variety** of the underlying $(\mathfrak{g}, K)$-module of π (see [Vo]), which is an algebraic variety contained in the **nilpotent cone** of $\mathfrak{g}^*$. Let $\mathrm{pr}_{\mathfrak{g}\to\mathfrak{g}'} : \mathfrak{g}^* \to (\mathfrak{g}')^*$ be the projection corresponding to $\mathfrak{g}' \subset \mathfrak{g}$. Here is a necessary condition for infinitesimal discrete decomposability:

Theorem H (criterion for discrete decomposability [Ko_9, Corollary 3.4]**).** *Let $\pi \in \widehat{G}$. If the restriction $\pi|_{G'}$ is infinitesimally discretely decomposable, then $\mathrm{pr}_{\mathfrak{g}\to\mathfrak{g}'}(\mathcal{V}_{\mathfrak{g}}(\pi))$ is contained in the nilpotent cone of $(\mathfrak{g}')^*$.*

We end this section with a useful information on irreducible summands.

Theorem I (size of irreducible summands, [Ko_9]**).** *Let $\pi \in \widehat{G}$. If the restriction $\pi|_{G'}$ is infinitesimally discretely decomposable, then any irreducible summand has the same associated variety, especially, the same Gelfand-Kirillov dimension.*

Here is a special case of Theorem I:

Example 3.6 (highest weight modules, [N-Oc-T]). Let G be the metaplectic group, and $G' = G_1'G_2'$ is a dual pair with G_1' compact. Let $\theta(\sigma)$ be an irreducible unitary highest weight module of G_2' obtained as the **theta-correspondence** of $\sigma \in \widehat{G_1'}$. Then the associated variety of $\theta(\sigma)$ does not depend on σ, but only on G_1'.

An analogous statement to Theorem I fails if there exists continuous spectrum in the branching law $\pi|_{G'}$ (see [Ko_{11}] for counter examples).

4. Applications to Representation Theory

So far, we have explained basic theory of discretely decomposable restrictions of unitary representations for reductive Lie groups $G \supset G'$. Now, we ask what discrete decomposability can do for representation theory. Let us clarify advantages of admissible restrictions, from which the following applications (and some more) have been brought out and seem to be promising furthermore.

1) Study of $\widehat{G'}$ as irreducible summands of $\pi|_{G'}$.
2) Study of $\widehat{G}$ by means of the restrictions to subgroups G'.
3) Branching laws of their own right.

4.1. From the view point of the study of $\widehat{G'}$ (smaller group), one of advantages of admissible restrictions is that each irreducible summand of the branching law $\pi|_{G'}$ gives an explicit construction of an element of $\widehat{G'}$.

Historically, an early success of this idea (in '70s and '80s) was the construction of irreducible highest weight modules (Howe, Kashiwara-Vergne, Adams, $\cdots$). A large part of these modules can be constructed as irreducible summands of discrete branching laws of the Weil representation (see Examples 2.3 (2) and 3.6).

This idea works also for non-highest weight modules. As one can observe from the criterion in Theorem B, the restriction $\pi|_{G'}$ tends to be discretely decomposable, if $\mathrm{AS}_K(\pi)$ is "small". In particular, if π is a **minimal representation** in the sense that its annihilator is the Joseph ideal, then a result of Vogan implies that $\mathrm{AS}_K(\pi)$ is one dimensional. Thus, there is a good possibility of finding subgroups G' such that $\pi|_{G'}$ is G'-admissible. This idea was used to construct "small" representations of subgroups G' by Gross-Wallach [Gr-W$_1$]. In the same line, discretely decomposable branching laws for non-compact G' are used also in the theory of automorphic forms for exceptional groups by J-S. Li [Li$_2$].

4.2. From the view point of the study of $\widehat{G}$ (larger group), one of advantages of admissible restrictions is to give a clue to a detailed study of representations of G by means of discrete branching laws.

Needless to say, an early success in this direction is the theory of $(\mathfrak{g}, K)$-modules (Lepowsky, Harish-Chandra, $\cdots$). The theory relies heavily on Harish-Chandra's admissibility theorem (Example 2.3 (1)) on the restriction of π to K.

Instead of a maximal compact subgroup K, this idea applied to a non-compact subgroup G' still works, especially in the study of "small" representations of G. In particular, this approach makes sense if the K-type structure is complicated but the G'-type structure is less complicated. Successful examples in this direction include:

1) To determine an explicit condition on λ such that a Zuckerman-Vogan module $A_{\mathfrak{q}}(\lambda)$ is non-zero, where we concern with the parameter λ outside the good range. In the setting of Example 2.6 (3), the author found in [Ko$_2$] a combinatorial formula on K_1-types of $A_{\mathfrak{q}}(\lambda)$ and determined explicitly when $A_{\mathfrak{q}}(\lambda) \neq 0$. The point here is that the computation of K-types of $A_{\mathfrak{q}}(\lambda)$ is too complicated to carry out because a lot of cancellation occurs in the generalized Blattner formula, while K_1-type formula (or G'-type formula for some non-compact subgroup G') behaves much simpler in this case.
2) To study a fine structure of standard representations. For example, Lee and Loke [Le-Lo] determined the Jordan-Hölder series and the unitarizability

of subquotients of certain degenerate non-unitary principal series representations π, by using G'-admissible restrictions for some non-compact reductive subgroup G'. Their method works successfully even in the case where K-type multiplicity of π is not one.

4.3. From the view point of finding explicit branching law, an advantage of admissible restrictions is that one can employ algebraic techniques because of the lack of continuous spectrum. A number of explicit branching laws are newly found (e.g. [D-Vs, Gr-W$_{1,2}$, Hu-P-S, Ko$_{1,3,4,8}$, Ko-Ø$_{1,2}$, Li$_2$, Lo$_{1,2}$, X]) in the context of admissible restrictions to non-compact reductive subgroups. A mysterious feature is that "different series" of irreducible representations may appear in discretely decomopsable branching laws (see [Ko$_5$, p.184] for a precise meaning), although all of them have the same Gelfand-Kirillov dimensions (Theorem I).

5. New Discrete Series for Homogeneous Spaces

Let $G \supset H$ be a pair of reductive Lie groups. Then, there is a G-invariant Borel measure on the homogeneous space G/H, and one can define naturally a unitary representation of G on the Hilbert space $L^2(G/H)$.

Definition 5.1. We say π is a **discrete series representation** for G/H, if $\pi \in \widehat{G}$ is realized as a subrepresentation of $L^2(G/H)$.

A discrete series representation corresponds to a discrete spectrum in the Plancherel formula for the homogeneous space G/H. One of basic problems in non-commutative harmonic analysis is:

Problem 5.2. 1) *Find a condition on the pair of groups (G, H) such that there exists a discrete series representation for the homogeneous space G/H.*
2) *If exist, construct discrete series representations.*

Even the first question has not found a final answer in the generality that (G, H) is a pair of reductive Lie groups. Here are some known cases:

Example 5.3. Flensted-Jensen, Matsuki and Oshima proved in '80s that discrete series representations for a reductive symmetric space G/H exist if and only if

$$\operatorname{rank} G/H = \operatorname{rank} K/(H \cap K). \tag{5.1}$$

This is a generalization of Harish-Chandra's condition, $\operatorname{rank} G = \operatorname{rank} K$, for a group manifold $G \times G/\operatorname{diag}(G) \simeq G$ ([FJ, Mk-Os]).

Our strategy to attack Problem 5.2 for more general (non-symmetric) homogeneous spaces G/H consists of two steps:

1) To embed G/H into a larger homogeneous space $\widetilde{G}/\widetilde{H}$, on which harmonic analysis is well-understood (e.g. symmetric spaces).
2) To take functions belonging to a discrete series representation $\mathcal{H}$ ($\hookrightarrow L^2(\widetilde{G}/\widetilde{H})$), and to restrict them with respect to a submanifold G/H ($\hookrightarrow \widetilde{G}/\widetilde{H}$).

If G/H is "generic", namely, a **principal orbit** in $\widetilde{G}/\widetilde{H}$ in the sense of Richardson, then it is readily seen that discrete spectrum of the branching law $\pi|_G$ gives a discrete series for G/H ([Ko$_{10}$, §8]; see also [Hu, Ko$_{1,5}$, Li$_1$] for concrete examples).

However, some other interesting homogeneous spaces G/H occur as non-principal orbits on $\widetilde{G}/\widetilde{H}$, where the above strategy does not work in general. A remedy for this is to impose the **admissibility of the restriction** of π, which justifies the restriction of L^p-functions to submanifolds, and then gives rise to many non-symmetric homogeneous spaces that admit discrete series representations. For example, let us consider the case where $G = \widetilde{G}^\tau$ and $H = \widetilde{G}^\sigma$ for commuting involutive automorphisms τ and σ of $\widetilde{G}$ such that $\widetilde{G}/\widetilde{H}$ satisfies (5.1). Then by using Theorem C and an asymptotic estimate of invariant measures [Ko$_6$], we have:

Theorem J (discrete series for non-symmetric spaces, [Ko$_{10}$]**).** *Assume that there is $w \in W_\sigma$ such that*

$$\mathbb{R}_+\text{-span}\,\Delta^+(\mathfrak{p})_{\sigma,w} \cap \sqrt{-1}(\mathfrak{t}_0^*)^{-\tau} = \{0\}. \tag{5.2}$$

Then there exist infinitely many discrete series representations for any homogeneous space of G that goes through $x\widetilde{H} \in \widetilde{G}/\widetilde{H}$ for any $x \in \widetilde{K}$.

We refer to [Ko$_{10}$, Theorem 5.1] for definitions of a finite group W_σ and $\Delta^+(\mathfrak{p})_{\sigma,w}$. The point here is that the condition (5.2) can be easily checked.

For instance, if $G \simeq Sp(2n,\mathbb{R}) \simeq \widetilde{G}/\widetilde{H}$ (a group manifold), then Theorem J implies that there exist discrete series on all homogeneous spaces of the form:

$$G/H = Sp(2n,\mathbb{R})/(Sp(n_0,\mathbb{C}) \times GL(n_1,\mathbb{C}) \times \cdots \times GL(n_k,\mathbb{C})), \quad (\textstyle\sum n_i = n).$$

The choice of x in Theorem J corresponds to the partition $(n_0, n_1, \ldots, n_k)$. We note that the above G/H is a symmetric space if and only if $n_1 = n_2 = \cdots = n_k = 0$.

The restriction of unitary representations gives new methods even for symmetric spaces where harmonic analysis has a long history of research. Let us state two results that are proved by the theory of discretely decomposable restrictions.

Theorem K (holomorphic discrete series for symmetric spaces). *Suppose G/H is a non-compact irreducible symmetric space. Then* (i) *and* (ii) *are equivalent:*

i) *There exist unitary highest weight representations of G that can be realized as subrepresentations of $L^2(G/H)$.*
ii) *G/K is Hermitian symmetric and $H/(H \cap K)$ is its totally real submanifold.*

This theorem in the group manifold case is a restatement of Harish-Chandra's well-known result. The implication (ii) $\Rightarrow$ (i) was previously obtained by a different geometric approach ('Olafsson-Ørsted [Ol-Ø]). Our proof uses a general theory of discretely decomposable restrictions, especially, Theorems B, H and J.

Theorem L (exclusive law of discrete spectrum for restriction and induction). *Let G/G' be a non-compact irreducible symmetric space, and $\pi \in \widehat{G}$. Then both (1) and (2) cannot occur simultaneously.*

1) *The restriction $\pi|_{G'}$ is infinitesimally discretely decomposable.*
2) *π is a discrete series representation for the homogeneous space G/G'.*

We illustrate Theorems K and L by $G = SL(2, \mathbb{R})$. In this special case, the examples below are well-known results on harmonic analysis, however, the point here is that we can give a new proof of them by a simple idea coming from restrictions of unitary representations.

Example 5.4. 1) Holomorphic discrete series exist for $G/H = SL(2, \mathbb{R})/SO(1, 1)$ (a hyperboloid of one sheet). This is explained by Theorem K because the geodesic $H/(H \cap K)$ is obviously totally real in the Poincaré disk $G/K = SL(2, \mathbb{R})/SO(2)$.
2) There is no discrete series for the Poincaré disk $G/K = SL(2, \mathbb{R})/SO(2)$. This fact is explained by Theorem L because any representation of G is obviously discretely decomposable when restricted to a compact K.

6. Modular Varieties, Vanishing Theorem

Retain the setting as in Definition 2.2. Let $\Gamma' \subset \Gamma$ be cocompact torsion-free discrete subgroups of $G' \subset G$, respectively. For simplicity, let G' be a semisimple Lie group without compact factors. Then, both of the double cosets $X := \Gamma\backslash G/K$ and $Y := \Gamma'\backslash G'/K'$ are compact, orientable, locally Riemannian symmetric spaces. Then, the inclusion $G' \hookrightarrow G$ induces a natural map $\iota : Y \to X$. The image $\iota(Y)$ defines a totally geodesic submanifold in X. Consider the induced homomorphism of the homology groups of degree $m := \dim Y$,

$$\iota_* : H_m(Y; \mathbb{Z}) \to H_m(X; \mathbb{Z}).$$

The **modular symbol** is defined to be the image $\iota_*[Y] \in H_m(X; \mathbb{Z})$ of the fundamental class $[Y] \in H_m(Y; \mathbb{Z})$. Though its definition is simple, the understanding of modular symbols is highly non-trivial.

Let us first recall some results of Matsushima-Murakami and Borel-Wallach on the de Rham cohomology group $H^*(X; \mathbb{C})$ summarized as:

$$H^*(X; \mathbb{C}) = \bigoplus_{\pi \in \widehat{G}} H^*(X)_\pi, \quad H^*(X)_\pi := \mathrm{Hom}_G(\pi, L^2(\Gamma\backslash G)) \otimes H^*(\mathfrak{g}, K; \pi_K). \tag{6.1}$$

The above result describes the topology of a single X by means of representation theory. For the topology of the pair (Y, X), we need restrictions of representations:

Theorem M (vanishing theorem for modular symbols, [Ko-Od]**).** *If*

$$\mathrm{AS}_K(\pi) \cap \mathrm{Cone}(G') = \{0\}, \quad \pi \neq \mathbf{1},$$

then the modular symbol $\iota_*[Y]$ is annihilated by the π-component $H^m(X)_\pi$ in the perfect paring $H^m(X;\mathbb{C}) \times H_m(X;\mathbb{C}) \to \mathbb{C}$.

Theorem M determines, for example, the middle Hodge components of totally real modular symbols of compact Clifford-Klein forms of type IV domains.

The discreteness of irreducible decompositions plays a crucial role both in Matsushima-Murakami's formula (6.1) and in a vanishing theorem for modular varieties (Theorem M). In the former $L^2(\Gamma\backslash G)$ is G-admissible (Gelfand and Piateski-Shapiro), while the restriction $\pi|_{G'}$ is G'-admissible (cf. Theorem B) in the latter.

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