Multiplicity One Theorem in the Orbit Method

In memory of Professor F. Karpelevič

Toshiyuki Kobayashi
Research Institute for Mathematical Sciences
Kyoto University
toshi@kurims.kyoto-u.ac.jp

Salma Nasrin
Graduate School of Mathematical Sciences
University of Tokyo
nasrin@ms.u-tokyo.ac.jp

Abstract

Let $G \supset H$ be Lie groups, $\mathfrak{g} \supset \mathfrak{h}$ their Lie algebras, and $\mathrm{pr} : \mathfrak{g}^* \to \mathfrak{h}^*$ the natural projection. For coadjoint orbits $\mathcal{O}^G \subset \mathfrak{g}^*$ and $\mathcal{O}^H \subset \mathfrak{h}^*$, we denote by $n(\mathcal{O}^G, \mathcal{O}^H)$ the number of H-orbits in the intersection $\mathcal{O}^G \cap \mathrm{pr}^{-1}(\mathcal{O}^H)$. In the spirit of the **orbit method** due to Kirillov and Kostant, one expects that $n(\mathcal{O}^G, \mathcal{O}^H)$ coincides with the multiplicity of $\tau \in \widehat{H}$ occurring in the restriction $\pi|_H$ if $\pi \in \widehat{G}$ is 'attached' to $\mathcal{O}^G$ and $\tau \in \widehat{H}$ is 'attached' to $\mathcal{O}^H$. Such a result is known for nilpotent Lie groups and certain solvable groups, however, very few attempts have been made so far for semisimple Lie groups.

In this paper, we give a sufficient condition on $\mathcal{O}^G$ so that

$$n(\mathcal{O}^G, \mathcal{O}^H) \leq 1 \quad \text{for any coadjoint orbit } \mathcal{O}^H \subset \mathfrak{h}^*,$$

for a semisimple symmetric pair (G, H). Our assumption on $\mathcal{O}^G$ corresponds to a *multiplicity-free* theorem of branching laws of unitary representations obtained recently in [7], [8] by one of the authors.

1 Introduction

The celebrated **Gindikin-Karpelevič formula** on the c-function gives an explicit Plancherel measure for the Riemannian symmetric space G/K of non-compact type. Implicitly important in this formula is the following:

Fact 1.1. The regular representation on $L^2(G/K)$ decomposes into irreducible unitary representations of G with **multiplicity free**.

Let us fix some notation. Suppose G is a non-compact semisimple Lie group with maximal compact subgroup K. We write $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ for the corresponding Cartan decomposition of the Lie algebra $\mathfrak{g}$ of G. We take a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$, and denote by $\Sigma(\mathfrak{g}, \mathfrak{a})$ the restricted root system. We fix a positive root system Σ^+ and write $\mathfrak{a}^*_+$ for the dominant Weyl chamber. Let m_α be the dimension of the root space $\mathfrak{g}(\mathfrak{a}; \alpha)$ for each $\alpha \in \Sigma(\mathfrak{g}, \mathfrak{a})$, and we define $\Sigma_0 := \{\alpha \in \Sigma(\mathfrak{g}, \mathfrak{a}) : \frac{\alpha}{2} \notin \Sigma(\mathfrak{g}, \mathfrak{a})\}$.

Spherical unitary principal series representations of G are parametrized by $\lambda \in \mathfrak{a}^*_+$, which we shall denote by $\pi_\lambda \, (\in \widehat{G})$. Then a qualitative refinement of Fact 1.1 (multiplicity free result) is given by the following direct integral decomposition into irreducible unitary representations (an abstract Plancherel formula):

$$L^2(G/K) \simeq \int^{\oplus}_{\mathfrak{a}^*_+} \pi_\lambda \, d\lambda. \tag{1.1}$$

A further refinement of (1.1) is the **Gindikin-Karpelevič formula** on the c-function ([2], see also [3]),

$$c(\lambda) = c_0 \prod_{\alpha \in \Sigma_0^+} \frac{2^{-\frac{\langle i\lambda, \alpha \rangle}{\langle \alpha, \alpha \rangle}} \Gamma\big(\frac{\langle i\lambda, \alpha \rangle}{\langle \alpha, \alpha \rangle}\big)}{\Gamma\Big(\frac{1}{2}\big(\frac{1}{2}m_\alpha + 1 + \frac{\langle i\lambda, \alpha \rangle}{\langle \alpha, \alpha \rangle}\big)\Big)\Gamma\Big(\frac{1}{2}\big(\frac{1}{2}m_\alpha + m_{2\alpha} + \frac{\langle i\lambda, \alpha \rangle}{\langle \alpha, \alpha \rangle}\big)\Big)},$$

which enriches (1.1) with quantitative result, namely, an explicit Plancherel density for (1.1) with respect to the spherical Fourier transform. Here c_0 is a normalized constant.

On the other hand, one can also enrich Fact 1.1 and (1.1) from another viewpoint, namely, with geometry of coadjoint orbits, motivated by the philosophy of the orbit method due to Kirillov. One way to formulate this is to regard $L^2(G/K)$ as an induced representation (see [9], Example 5). Another way is to use the restriction of some other representation of $\widetilde{G}$ such that $\widetilde{G} \supset G$. We shall take the latter viewpoint, which then leads us to interesting geometric results on coadjoint orbits in much wider settings.

In this paper, we first recall a multiplicity free theorem in the branching problem in §2 ([7], [8]) which contains Fact 1.1 as a special case (for classical groups), and then formulate its (predicted) counterpart in the orbital geometry. We shall see in §3 that it turns out to be true (Theorems A and B), and illustrate them by lower dimensional examples in §4. A detailed proof of Theorems A and B will be given elsewhere.

2 Multiplicity-one decomposition and branching laws

There are several different approaches to prove Fact 1.1 (a multiplicity free result). A classical approach due to Gelfand is based on the commutativity of the convolution algebra $L^1(K\backslash G/K)$.

Another approach is based on the restriction of a representation of an overgroup $\widetilde{G}$. For instance, consider a symmetric pair

$$(\widetilde{G}, G) = (\mathrm{Sp}(n, \mathbb{R}), \mathrm{GL}(n, \mathbb{R})).$$

Then we have a natural embedding $G/K \hookrightarrow \widetilde{G}/\widetilde{K}$, namely,

$$\mathrm{GL}(n, \mathbb{R})/O(n) \hookrightarrow \mathrm{Sp}(n, \mathbb{R})/U(n). \tag{2.1}$$

Via the embedding (2.1), G/K becomes a totally real submanifold in a complex manifold $\widetilde{G}/\widetilde{K}$. Let π be a holomorphic discrete series representation of scalar type of $\widetilde{G}$. Then π is realized in the space of holomorphic sections of a certain holomorphic line bundle over $\widetilde{G}/\widetilde{K}$. The restriction of the representation π to G factors through that of holomorphic sections to a totally real submanifold, and its abstract Plancherel formula coincides with that of $L^2(G/K)$ (see [4], [6], [10]). This representation $\pi|_G$ is essentially known as a *canonical representation* in the case of Vershik-Gelfand-Graev. Thus the multiplicity one property in Fact 1.1 can be formulated in a much more general framework of the **branching laws**, namely, irreducible decompositions of the restrictions of unitary representations to subgroups.

In this direction, one of the authors proved the following theorems (see [7], [8]) (we shall replace the above pair $(\widetilde{G}, G)$ by (G, H)): Suppose G is a semisimple Lie group such that G/K is a Hermitian symmetric space of non-compact type. Then

Theorem 2.1. *Let π be an irreducible unitary highest weight representation of scalar type of G, and (G, H) an arbitrary symmetric pair. Then the restriction $\pi|_H$ decomposes into irreducible representations of H with multiplicity free.*

Theorem 2.2. *Let π_1, π_2 be unitary highest (or lowest) weight representations of scalar type. Then $\pi_1 \otimes \pi_2$ decomposes with multiplicity free.*

3 Multiplicity-one theorem in the orbit method

The object of this paper is to provide a 'predicted' result in the orbit philosophy corresponding to Theorems 2.1 and 2.2.

For this, let us recall an idea of the orbit method in unitary representation theory of Lie groups.

Let $\mathfrak{g}$ be the Lie algebra of G and $\mathfrak{g}^*$ the linear dual of $\mathfrak{g}$. Let us consider the contragradient representation $\mathrm{Ad}^* : G \to \mathrm{GL}(\mathfrak{g}^*)$ of the adjoint representation of G, $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$. This non-unitary finite dimensional representation often has a surprisingly intimate relation with the unitary dual $\widehat{G}$, which consists mostly of infinite dimensional representations.

For example, let us first consider the case where G is a connected and simply connected nilpotent Lie group. Then Kirillov ([5]) proved that the unitary dual $\widehat{G}$ is parametrized by $\mathfrak{g}^*/G$, the set of coadjoint orbits. We shall write the corresponding coadjoint orbit $\mathcal{O}_\pi \subset \mathfrak{g}^*$ for $\pi \in \widehat{G}$. Let H be a subgroup of G. Then the restriction $\pi|_H$ is decomposed into a direct integral of irreducible representations of H

$$\pi|_H \simeq \int_{\widehat{H}}^{\oplus} m_\pi(\tau)\tau d\mu(\tau) \qquad \text{(branching law)}, \tag{3.1}$$

where $d\mu$ is a measure on $\widehat{H}$. Then, Corwin and Greenleaf proved that the multiplicity $m_\pi(\tau)$ in (3.1) is given by the 'mod H' intersection number $n(\mathcal{O}_\pi^G, \mathcal{O}_\tau^H)$ defined as follows:

$$n(\mathcal{O}_\pi^G, \mathcal{O}_\tau^H) := \sharp\Big(\big(\mathcal{O}_\pi^G \cap \mathrm{pr}^{-1}(\mathcal{O}_\tau^H)\big)/H\Big). \tag{3.2}$$

Here, $\mathcal{O}_\pi^G \subset \mathfrak{g}^*$ and $\mathcal{O}_\tau^H \subset \mathfrak{h}^*$ are the coadjoint orbits corresponding to $\pi \in \widehat{G}$ and $\tau \in \widehat{H}$, respectively, under the Kirillov correspondence, and

$$\mathrm{pr} : \mathfrak{g}^* \to \mathfrak{h}^*$$

is the natural projection. $n(\mathcal{O}_\pi^G, \mathcal{O}_\tau^H)$ is sometimes referred as the **Corwin-Greenleaf multiplicity function**

Contrary to nilpotent Lie groups, it has been observed by many specialists that the orbit method does not work very well for a non-compact semisimple Lie group (e.g. [11]); there is no reasonable bijection between $\widehat{G}$ and (a subset of) $\mathfrak{g}^*/G$. Therefore, it is not obvious if an analogous statement of Corwin-Greenleaf's theorem makes sense for a semisimple Lie group G. But, the orbit method still gives a good approximation of the unitary dual $\widehat{G}$. For example, to an 'integral' elliptic coadjoint orbit $\mathcal{O}_\lambda^G = \mathrm{Ad}^*(G)\lambda \subset \mathfrak{g}^*$, one can associate a unitary representation, denoted by π_λ, of G as a generalization of the Borel-Weil-Bott theorem due to Schmid and Wong, combined with a unitarization theorem of Vogan and Wallach. Furthermore, π_λ is nonzero and irreducible for 'most' λ (see [6] for a survey). Namely, to such a coadjoint orbit $\mathcal{O}_\lambda^G$, one can naturally attach an irreducible unitary representation $\pi_\lambda \in \widehat{G}$. In particular, if G/K is Hermitian, associated to an (integral) coadjoint orbit that goes through $([\mathfrak{k}, \mathfrak{k}] + \mathfrak{p})^\perp$ $(\subset \mathfrak{g}^*)$, the corresponding unitary representation is a highest weight module of scalar type.

From now on, we shall identify $\mathfrak{g}^*$ with $\mathfrak{g}$. Then the above coadjoint orbit corresponds to

$$\mathcal{O}_z^G := \mathrm{Ad}(G) \cdot z \subset \mathfrak{g},$$

where z is a central element in $\mathfrak{k}$. We also write $\mathrm{pr} : \mathfrak{g} \to \mathfrak{h}$ for the projection instead of $\mathrm{pr} : \mathfrak{g}^* \to \mathfrak{h}^*$.

Then in the spirit of the Kirillov-Kostant orbit method, Theorem 2.1 predicts that the Corwin-Greenleaf multiplicity function $n(\mathcal{O}_z^G, \mathcal{O}^H)$ is either 0 or 1 for any coadjoint orbit $\mathcal{O}^H$ in $\mathfrak{h}^*$. Since unitary representations only correspond to integral or admissible orbits (even if the orbit method works), this prediction might look a little optimistic. However, it turns out to be true:

Theorem A. *If (G, H) is a symmetric pair, then the intersection*

$$\mathcal{O}_z^G \cap \mathrm{pr}^{-1}(\mathcal{O}^H)$$

is a single H-orbit for any adjoint orbit $\mathcal{O}^H \subset \mathfrak{h}$, whenever it is non-empty.

Correspondingly to Theorem 2.2 in the tensor product ([7], [8]), we also expect a geometric result in the (co)adjoint orbits. Let us consider the projection

$$\mathrm{pr} : \mathfrak{g} \oplus \mathfrak{g} \to \mathfrak{g}, \quad (X, Y) \mapsto \frac{1}{2}(X + Y).$$

Let $\mathcal{O}^{G\times G}_{(z,z)} = \mathrm{Ad}(G \times G)(z, z) \subset \mathfrak{g} \oplus \mathfrak{g}$. Then we have the following

Theorem B. *The intersection*

$$\mathcal{O}^{G\times G}_{(z,z)} \cap \mathrm{pr}^{-1}(\mathcal{O}^G)$$

is a single G-orbit for any adjoint orbit $\mathcal{O}^G \subset \mathfrak{g}$, whenever it is non-empty.

Likewise, the intersection

$$\mathcal{O}^{G\times G}_{(z,-z)} \cap \mathrm{pr}^{-1}(\mathcal{O}^G)$$

is a single G-orbit for any adjoint orbit $\mathcal{O}^G \subset \mathfrak{g}$, whenever it is non-empty.

4 Examples and Remarks

Let us illustrate our main results (Theorems A and B) by a number of examples of lower dimensions.

First, let $G = \mathrm{SU}(2)$, and we identify $\mathfrak{g}^*$ with

$$\mathfrak{g} \simeq \mathfrak{su}(2) = \{X = \begin{pmatrix} ix_1 & ix_2 - x_3 \\ ix_2 + x_3 & -ix_1 \end{pmatrix} : x_1, x_2, x_3 \in \mathbb{R}\}.$$

Then the adjoint representation $\mathrm{Ad}(g) : X \mapsto gXg^{-1}$ preserves the determinant of X, that is, $x_1^2+x_2^2+x_3^2$. In fact, by an easy computation, each adjoint orbit $\mathcal{O}^G_X$ is identified with a sphere $\{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2+x_2^2+x_3^2 = n^2\}$ for some $n \in \mathbb{R}_{\geq 0}$. For an integer n, the orbit method 'attaches' an irreducible $(n + 1)$-dimensional representation of G to this sphere. As it is well-known, the restriction of π_n to a subgroup $K = \mathrm{SO}(2)$ decomposes into a multiplicity free direct sum of irreducible representations:

$$\pi_n|_{\mathrm{SO}(2)} \simeq \bigoplus_{\substack{m=-n \\ m\equiv n \bmod 2}}^{n} \chi_m. \tag{4.1}$$

Here, each one dimensional representation χ_m of SO(2) occurs with multiplicity free.

In the orbit picture, $\mathrm{pr}: \mathfrak{g}^* \to \mathfrak{h}^*$ is identified with the projection:

$$\mathbb{R}^3 \to \mathbb{R}, \quad (x_1, x_2, x_3) \mapsto x_3.$$

We also note that each coadjoint orbit in $\mathfrak{h}^*$ is a singlton, say $\{m\}$, because H is abelian. Then the intersection of $\mathcal{O}_X^G$ with $\mathrm{pr}^{-1}(\{m\})$ is given by

$$\{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = n^2\} \cap \{(x_1, x_2, m) : x_1, x_2 \in \mathbb{R}\},$$

which is a circle as in the Figure 4.1 if $|m| \le n$. This is obviously a single orbit of K. This geometry of coadjoint orbits reflects the multiplicity one property of the branching law (4.1).

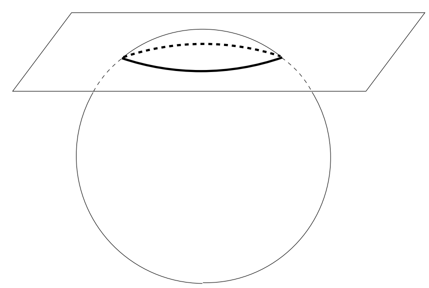

Figure 4.1

In the figure below (which does not come from any representation of SU(2)), the intersection consists of two disconnected parts. Such a figure does not arise in the setting of our theorems.

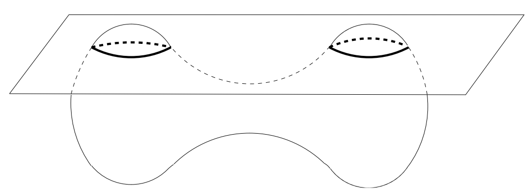

Figure 4.2

Next, let us consider infinite dimensional representations, with which our main concern is. Suppose $G = SL(2, \mathbb{R})$ and $K = SO(2)$. We identify $\mathfrak{g}^*$ with $\mathfrak{g} \simeq \mathfrak{sl}(2, \mathbb{R}) = \{\begin{pmatrix} x_1 & x_2 - x_3 \\ x_2 + x_3 & -x_1 \end{pmatrix} : x_1, x_2, x_3 \in \mathbb{R}\}$. A holomorphic discrete series representation π_n^+ $(n = 2, 3, 4, \cdots)$ is an irreducible representation of G realized in the space of square integrable and holomorphic sections of a G-equivariant holomorphic line bundle $G \times_K \chi_n \to G/K$. We put $z := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, which lies in the center of $\mathfrak{k} = \mathfrak{so}(2)$. The representation π_n^+ is a unitary highest weight module of scalar type, and is supposed to be attached to the coadjoint orbit $\mathcal{O}_{nz}^G = \mathrm{Ad}^*(G)(nz)$. We have

$$\begin{aligned} \mathcal{O}_{nz}^G =& \{X = \begin{pmatrix} x_1 & x_2 - x_3 \\ x_2 + x_3 & -x_1 \end{pmatrix} : \det X = \det \begin{pmatrix} 0 & -n \\ n & 0 \end{pmatrix}, \ x_3 > 0\} \\ \simeq& \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 - x_3^2 = -n^2, x_3 > 0\}, \end{aligned} \tag{4.2}$$

a connected component of a hyperboloid of two sheets.

We put $y := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, and $A := \exp \mathbb{R} y$. Let us use the identifications $\widehat{K} \simeq \mathbb{Z}$, $\chi_n \leftrightarrow n$; and $\widehat{A} \simeq \mathbb{R}$, $\chi_\xi \leftrightarrow \xi$. Then, the branching laws of $\pi_n^+ \in \widehat{G}$ with respect to the one-dimensional subgroups $K = \mathrm{SO}(2)$ and A are given respectively by

$$\pi_n^+|_K \simeq \sum_{k=0}^{\infty}{}^{\oplus} \chi_{n+2k}, \tag{4.3}$$

$$\pi_n^+|_A \simeq \int_{\mathbb{R}}^{\oplus} \chi_\xi \, d\xi. \tag{4.4}$$

The first formula (4.3) is discretely decomposable, while the second one (4.4) consists only of continuous spectrum. But in both branching laws, the multiplicity is free (this is a special case of Theorem 2.1). In the orbit pictures, the intersection of the hyperboloid (4.2) with a hyperplane, $x_3 =$ constant, is a circle, which is a single orbit of $K = \mathrm{SO}(2)$ (see Figure 4.3); while that with another hyperplane, $x_1 =$ constant, is a hyperbolic curve, which is a single orbit of $A \simeq \mathbb{R}$ (see Figure 4.4).

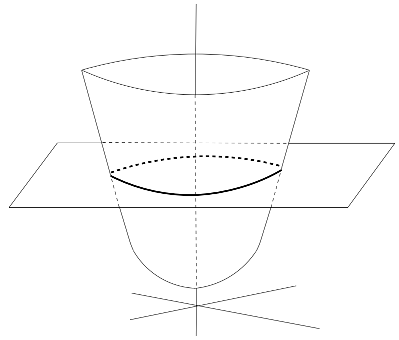

Figure 4.3

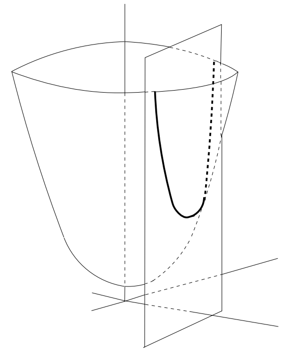

Figure 4.4

What Theorem A asserts is a higher dimensional generalization of Figures 4.3 and 4.4.

Finally, let us mention other representations which are **not** treated in our main theorems. For instance, let us consider a spherical principal series representation, denoted by π_λ, of $G = \mathrm{SL}(2,\mathbb{R})$, which is 'attached' to a coadjoint orbit

$$\mathcal{O}^G_{\lambda y} = \mathrm{Ad}^*(G)(\lambda y) = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 - x_3^2 = \lambda^2\},$$

by a real polarization. We note that $\mathcal{O}^G_{\lambda y}$ $(\lambda \neq 0)$ is a hyperboloid of one sheet.

The branching laws of π_λ when restricted to K and A are given respectively by

$$\pi_\lambda\big|_K \simeq \sum_{n\in\mathbb{Z}}{}^{\oplus} \chi_{2n}, \tag{4.5}$$

$$\pi_\lambda\big|_A \simeq \int_{\mathbb{R}}^{\oplus} 2\chi_\xi \, d\xi. \tag{4.6}$$

It happens that the first formula (4.5) is multiplicity free, and this property is reflected by the orbit picture (Figure 4.5), namely, the intersection is a circle which is a single orbit of $K = \mathrm{SO}(2)$. On the other hand, the multiplicity in (4.6) is two, and this property is reflected by the orbit picture (Figure 4.6), namely, the intersection consists of two hyperbolic curves on which A acts with two orbits.

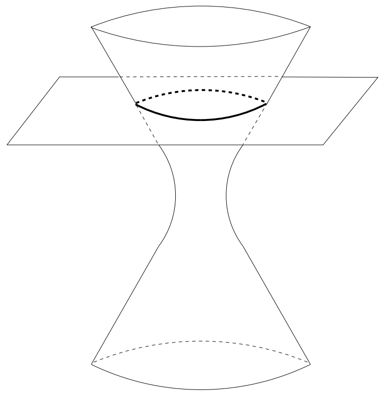

Figure 4.5

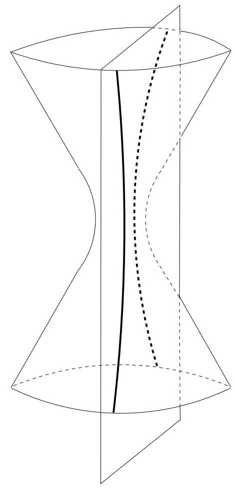

Figure 4.6

For higher dimensional generalizations of the last two examples, we should remark that the multiplicity is not finite in general. Correspondingly, the intersection $\mathcal{O}_\lambda^G \cap \mathrm{pr}^{-1}(\mathcal{O}^H)$ may consist of infinitely many H-orbits, that is, $n(\mathcal{O}_\lambda^G, \mathcal{O}^H)$ can be infinite for some coadjoint orbit $\mathcal{O}^H \subset \mathfrak{h}^*$. This is the case if $G = \mathrm{SL}(n,\mathbb{R})$ and $K = \mathrm{SO}(n)$ and if $\lambda = \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix}$ $(\sum \lambda_i = 0)$, with $\lambda_i \neq \lambda_j$ $(i \neq j)$ and $n \geq 3$. (The orbit method attaches $\mathcal{O}_\lambda^G$ to a spherical principal series representation of G by a real polarization.) This counterexample indicates an important role of our assumption on the coadjoint orbit $\mathcal{O}_z^G$, that is, the condition that z lies in the center of $\mathfrak{k}$.

To end this paper, we pin down some questions for further research:

1) Generalize Theorems A and B, of which a counterpart in unitary representation theory has not been known.

2) Find a feedback of (1) to unitary representation theory (namely, prove new multiplicity free results of branching laws of unitary representations which are predicted by the orbit method).

3) Find a refinement of Theorems A and B in the orbit method corresponding to the explicit Plancherel measure (description of its support and the Plancherel density).

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