Integral formulas for the minimal representation of $O(p, 2)$

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Abstract

The minimal representation π of $O(p, q)$ ($p + q$: even) is realized on the Hilbert space of square integrable functions on the conical subvariety of $\mathbb{R}^{p+q-2}$. This model presents a close resemblance of the Schrödinger model of the Segal-Shale-Weil representation of the metaplectic group. We shall give explicit integral formulas for the 'inversion' together with the analytic continuation to a certain semigroup of $O(p+2, \mathbb{C})$ of the minimal representation of $O(p, 2)$ by using Bessel functions.

1 Introduction

Our concern in this paper is with the L^2-model of the **minimal representation** π of the indefinite orthogonal group $O(p, q)$ with $p + q$ even, in particular, integral formulas of unitary or contraction operators for $q = 2$.

In order to explain our motivation, we recall the Segal-Shale-Weil representation ϖ of the metaplectic group $\widetilde{Sp}(n, \mathbb{R})$, the twofold cover of the real

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symplectic group $Sp(n,\mathbb{R})$. Among many beautiful aspects of this representation, we take up the Schrödinger model which realizes the representation ϖ on the Hilbert space $L^2(\mathbb{R}^n)$ with the following well-known features:

1) The representation space is not complicated; it is just $L^2(\mathbb{R}^n)$.

2) The whole group $\widetilde{Sp}(n,\mathbb{R})$ acts on the function space $L^2(\mathbb{R}^n)$, while only the Siegel parabolic subgroup P can act on the manifold $\mathbb{R}^n$.

3) The restriction $\varpi|_P$ is still irreducible. The action of P on $L^2(\mathbb{R}^n)$ is given simply by translations and multiplications by unitary characters of an abelian group.

4) The infinitesimal action $d\varpi$ of the Lie algebra $\mathfrak{sp}(n,\mathbb{R})$ is given by differential operators of at most second order.

5) There is a distinguished element w_0 (the "inversion" for P). Then, the unitary operator $\varpi(w_0)$ on $L^2(\mathbb{R}^n)$ is essentially the Fourier transform.

The Segal-Shale-Weil representation splits into two irreducible unitary representations of $\widetilde{Sp}(n,\mathbb{R})$, which are "minimal representations" in the sense of Joseph. In the last decade, minimal representations of reductive groups have been extensively studied by many authors, especially by algebraic approaches (e.g. [12]).

As for the indefinite orthogonal group $O(p,q)$, Vogan pointed out that there is no minimal representation if $p+q>8$ is odd [13]. For $p+q$ even, Kostant [10] first constructed a minimal representation in the case $p=q=4$, and Binegar-Zierau [1] generalized his construction to the general p,q (≥ 2). Many other different models of the same representation have been also found: for example, as the θ-lifting of the trivial representation of $SL(2,\mathbb{R})$ [5], and also as solution space of the Yamabe operator [8] in the context of conformal geometry (see also [6] for an exposition). Among other models, it is proved in Kobayashi-Ørsted [9] that the same representation can be realized on the Hilbert space of L^2-functions on the conical subvariety C in $\mathbb{R}^{p+q-2}$ associated to the quadratic form of signature $(p-1,q-1)$.

Sections 2 and 3 present a close resemblance of this realization for the group $O(p,q)$ to the Schödinger model of the Segal-Shale-Weil representation for the group $\widetilde{Sp}(n,\mathbb{R})$ with regard to the above features (1) $\sim$ (4). For example, the pseudo-Euclidean motion group $O(p-1,q-1)\ltimes\mathbb{R}^{p+q-2}$ acts naturally on $L^2(C)$. Then, a maximal parabolic subgroup $\overline{P^{\max}}$ containing $O(p-1,q-1) \ltimes \mathbb{R}^{p+q-2}$ plays the role of the Siegel parabolic.

Sections 4 and 5 are devoted to the case $q=2$, where C splits in two connected components C_+ and C_-, a forward and a backward light cone, and functions supported on the forward cone yield a unitary lowest weight representation of the connected group $SO_0(p,2)$.

We shall consider the (holomorphic) semigroup of Hilbert-Schmidt operators $\pi(e^{tZ}) = \exp(t d\pi(Z))$ $(\operatorname{Re} t > 0)$ on $L^2(C_+)$ generated by the following self-adjoint operator

$$d\pi(Z) = \frac{r}{4}\frac{\partial^2}{\partial r^2} + \frac{p-2}{4}\frac{\partial}{\partial r} + \frac{\Delta_{S^{p-2}}}{4r} - r.$$

Here, we have identified C_+ with $\mathbb{R}_+ \times S^{p-2}$ by the polar coordinate. It turns out that the operator norm of $\pi(e^{tZ})$ on $L^2(C_+)$ equals $e^{-\frac{p-2}{2}\operatorname{Re} t}$, and thus it is a contraction. We shall find in Theorem B that the operator $\pi(e^{tz})$ on $L^2(C_+)$ is given as the integral transform on C_+ against the kernel

$$K^+(\zeta,\zeta';t) := \frac{2e^{-\sqrt{2}(|\zeta|+|\zeta'|)\coth\frac{t}{2}}}{\pi^{\frac{p-2}{2}}\sinh^{\frac{p}{2}}\frac{t}{2}} \sqrt{2\langle\zeta,\zeta'\rangle}^{-\frac{p-4}{2}} I_{\frac{p-4}{2}}\Big(\frac{2\sqrt{2\langle\zeta,\zeta'\rangle}}{\sinh\frac{t}{2}}\Big),$$

where $I_\nu(z) = \sqrt{-1}^{-\nu} J_\nu(\sqrt{-1}z)$ is the modified Bessel function.

The semigroup $\{\pi(e^{tZ}) : \operatorname{Re} t > 0\}$ on $L^2(C_+)$ may be regarded as an analogue of the **Hermite semigroup** on $L^2(\mathbb{R}^n)$ given by the Gaussian kernel (see Howe [4] for the connection with the Segal-Shale-Weil representation; see also [3]).

Furthermore, in light that the inversion element w_0 is given by $e^{\pi\sqrt{-1}Z}$, we can obtain the integral formula for the unitary operator $\pi(w_0)$ as the "boundary value" of the contraction operator $\pi(e^{tZ})$ as t tends to $\pi\sqrt{-1}$. In the case of the metaplectic group $\widetilde{Sp}(n,\mathbb{R})$, such an operator $\varpi(w_0)$ is nothing but the Fourier transform on $L^2(\mathbb{R}^n)$ (see the above feature (5)), while in the case of the indefinite orthogonal group $O(p,2)$ it turns out that the Fourier-Bessel transform arises in describing $\pi(w_0)$ (see Theorem D).

This article is an outgrowth of the lecture delivered by the first author at the 2002 Twente Conference on Lie Groups. Detailed proof of Theorems B, C and D will be given in [7]. He expresses his sincere gratitude to the organizers for the warm hospitality and the opportunity to participate in the conference. In particular, he admires with all his heart and soul the dedicated work of Professor Gerard Helminck who organized the conference successfully in spite of the difficulties of the fire of the mathematics building of Twente University.

2 Square integrable functions on the cone

In this section, we describe an irreducible unitary representation π of the semidirect product group $O(p-1,q-1)\ltimes\mathbb{R}^{p+q-2}$ on the Hilbert space $L^2(C)$ obtained by translation together with multiplications by unitary characters of $\mathbb{R}^{p+q-2}$. All the materials here are standard. In Section 3, the representation π will be extended to the minimal representation of $O(p,q)$ for $p+q\in 2\mathbb{N}$ (see Theorem A).

Let $\mathbb{R}^{p-1,q-1}$ be the pseudo-Riemannian Euclidean space $\mathbb{R}^{p+q-2}$ equipped with the standard indefinite metric

$$ds^2 = d\zeta_1^2+\cdots+d\zeta_{p-1}^2-d\zeta_p^2-\cdots-d\zeta_{p+q-2}^2. \tag{2.1}$$

Then, the group $\mathrm{Isom}(\mathbb{R}^{p-1,q-1})$ of isometries on $\mathbb{R}^{p-1,q-1}$ is isomorphic to the semidirect product group $O(p-1,q-1)\ltimes\mathbb{R}^{p+q-2}$.

Let C be the cone in $\mathbb{R}^{p+q-2}$ given by

$$C:=\{(\zeta_1,\cdots,\zeta_{p+q-2})\in\mathbb{R}^{p+q-2}:\zeta_1^2+\cdots+\zeta_{p-1}^2-\zeta_p^2-\cdots-\zeta_{p+q-2}^2=0\}\backslash\{0\}.$$

Then C is of dimension $p+q-3$ and is acted transitively by the indefinite orthogonal group $O(p-1,q-1)$. With respect to the polar coordinate

$$\mathbb{R}_+\times S^{p-2}\times S^{q-2}\to C,\quad (r,\omega,\eta)\mapsto(r\omega,r\eta), \tag{2.2}$$

we define a measure $d\mu$ on C by

$$d\mu=\frac{1}{2}r^{p+q-5}drd\omega d\eta.$$

Then $d\mu$ is $O(p-1,q-1)$ invariant because we have

$$\theta|_C=d\mu$$

for any $(p+q-3)$-form θ on $\mathbb{R}^{p+q-2}$ satisfying

$$d(\zeta_1^2+\cdots+\zeta_{p-1}^2-\zeta_p^2-\cdots-\zeta_{p+q-2}^2)\wedge\theta=d\zeta_1\wedge\cdots\wedge d\zeta_{p+q-2}.$$

Hence, we have naturally a unitary representation π of $O(p-1,q-1)$ on the Hilbert space $L^2(C,d\mu)\equiv L^2(C)$ by translations.

Next, let the abelian group $\mathbb{R}^{p+q-2}$ act on $L^2(C)$ by the formula:

$$\pi(b):L^2(C)\to L^2(C),\quad \psi(\zeta)\mapsto e^{2\sqrt{-1}(b_1\zeta_1+\cdots+b_{p+q-2}\zeta_{p+q-2})}\psi(\zeta),$$

where $b = (b_1, \cdots, b_{p+q-2}) \in \mathbb{R}^{p+q-2}$.

The above actions of $O(p-1, q-1)$ and $\mathbb{R}^{p+q-2}$ on $L^2(C)$ respectively give rise to a representation (we shall use the same notation π) of the semidirect group $O(p-1, q-1) \ltimes \mathbb{R}^{p+q-2}$. Then, we have readily the following proposition (see [9], Proposition 3.3):

Proposition 2.1. $(\pi, L^2(C))$ *is an irreducible unitary representation of the semidirect product group* $O(p-1, q-1) \ltimes \mathbb{R}^{p+q-2}$.

3 Schrödinger model of the minimal representation of $O(p, q)$

In general, an irreducible representation of a group G is no more irreducible when restricted to a subgroup G'. In other words, it is quite rare that an irreducible representation of a subgroup G' extends to that of the whole group G (on the same representation space). Hence, it should be noted that the irreducible unitary representation in Proposition 2.1 can be extended with respect to the following embedding:

$$O(p-1, q-1) \ltimes \mathbb{R}^{p+q-2} \subset O(p, q). \tag{3.1}$$

Theorem A ([9], Theorem 4.9). *Suppose* $p+q$ *is even,* ≥ 6, *and* $p, q \geq 2$. *Then, the representation* $(\pi, L^2(C))$ *of the semidirect product group* $O(p-1, q-1) \ltimes \mathbb{R}^{p+q-2}$ *extends to an irreducible unitary representation of* $O(p, q)$.

The resulting representation, denoted by the same π, of $G := O(p, q)$ is the minimal representation in the sense of Joseph if $p + q \geq 8$. The Gelfand-Kirillov dimension of π is $p + q - 3$, which attains its minimum among all infinite dimensional irreducible unitary representations of G.

Another point of Theorem A is that it gives a model of the minimal representation of G on the Hilbert space $L^2(C)$, resembling the Schrödinger model for the Segal-Shale-Weil representation of the metaplectic group $\widetilde{Sp}(n, \mathbb{R})$. In the papers [2, 11], one finds a similar construction of Hilbert spaces (i.e., $L^2(C)$ for some conical variety C) of minimal unitary representations of other groups (e.g., Koecher-Tits groups associated with semisimple Jordan algebras) under the assumption that π is a highest weight representation or a spherical representation. We note that our representation is neither a highest weight representation nor a spherical representation if $p, q \geq 3$ and $p \neq q$.

Let us explain more about Theorem A, especially about how the group G or the Lie algebra $\mathfrak{g}$ acts on $L^2(C)$.

First, we fix some notation and explain the inclusion (3.1). Let $e_0, \cdots, e_{p+q-1}$ be the standard basis of $\mathbb{R}^{p+q}$, E_{ij} the matrix unit, and

$$\begin{aligned}
\varepsilon_j &:= \begin{cases} 1 & (1 \le j \le p-1) \\ -1 & (p \le j \le p+q-2), \end{cases} \\
\overline{N_j} &:= E_{j,0} + E_{j,p+q-1} - \varepsilon_j E_{0,j} + \varepsilon_j E_{p+q-1,j} \quad (1 \le j \le p+q-2), \\
N_j &:= E_{j,0} - E_{j,p+q-1} - \varepsilon_j E_{0,j} - \varepsilon_j E_{p+q-1,j} \quad (1 \le j \le p+q-2), \\
E &:= E_{0,p+q-1} + E_{p+q-1,0}.
\end{aligned}$$

We define some subalgebras of the Lie algebra $\mathfrak{g}$ by

$$\overline{\mathfrak{n}^{\max}} := \sum_{j=1}^{p+q-2} \mathbb{R}\overline{N_j}, \quad \mathfrak{n}^{\max} := \sum_{j=1}^{p+q-2} \mathbb{R}N_j, \quad \mathfrak{a} := \mathbb{R}E,$$

and define some subgroups of G as follows:

$$\begin{aligned}
M_+^{\max} &:= \{g \in G : g \cdot e_0 = e_0,\ g \cdot e_{p+q-1} = e_{p+q-1}\} \ \simeq \ O(p-1, q-1), \\
M^{\max} &:= M_+^{\max} \cup \{-I_{p+q}\} \cdot M_+^{\max} \ \simeq \ O(p-1, q-1) \times \mathbb{Z}_2, \\
A &:= \exp(\mathfrak{a}), \\
N^{\max} &:= \exp(\mathfrak{n}^{\max}), \\
\overline{N^{\max}} &:= \exp(\overline{\mathfrak{n}^{\max}}).
\end{aligned}$$

Then the subgroup $M_+^{\max}\overline{N^{\max}}$ is isomorphic to the semidirect product group $O(p-1, q-1) \ltimes \mathbb{R}^{p+q-2}$ via the bijection:

$$\overline{N^{\max}} \xrightarrow{\sim} \mathbb{R}^{p+q-2}, \quad \exp(\sum_{j=1}^{p+q-2} b_j \overline{N_j}) \mapsto (b_1, \cdots, b_{p+q-2}).$$

Hence, the natural inclusion $M_+^{\max}\overline{N^{\max}} \subset G$ amounts to (3.1). Another meaning of (3.1) is that $O(p-1, q-1) \ltimes \mathbb{R}^{p+q-2}$ is the group of isometries of the pseudo-Riemannian Euclidean space $\mathbb{R}^{p-1,q-1}$, while $O(p, q)$ is the group of Möbius transformations on $\mathbb{R}^{p-1,q-1}$ preserving the conformal structure.

Next, we define a maximal parabolic subgroup

$$\overline{P^{\max}} := M^{\max} A \overline{N^{\max}},$$

which plays an analogous role to the Siegel parabolic subgroup of the metaplectic group $\widetilde{Sp}(n,\mathbb{R})$.

With regard to the inclusive relation

$$M_+^{\max}\overline{N^{\max}} \subset \overline{P^{\max}} \subset G,$$

the extension of the unitary representation π from $M_+^{\max}\overline{N^{\max}}$ to $\overline{P^{\max}}$ is easily achieved by defining

$$\begin{aligned} \pi(-I_{p+q})\psi &:= (-1)^{\frac{p-q}{2}}\psi \\ \pi(e^{tE})\psi(\zeta) &:= e^{-\frac{p+q-4}{2}t}\psi(e^{-t}\zeta), \quad t \in \mathbb{R}. \end{aligned}$$

Here we recall that $\overline{P^{\max}}$ is generated by $M_+^{\max}, \overline{N^{\max}}, -I_{p+q}$ and $e^{tE}(t \in \mathbb{R})$.

In order to describe the extension of the unitary representation π from $\overline{P^{\max}}$ to G, we use the Gelfand-Naimark decomposition

$$\mathfrak{g} = \overline{\mathfrak{n}^{\max}} \oplus \mathfrak{a} \oplus \mathfrak{m}^{\max} \oplus \mathfrak{n}^{\max} = \overline{\mathfrak{p}^{\max}} \oplus \mathfrak{n}^{\max}.$$

Then, the representation π of G will be determined if we give the differential representation $d\pi(X)$ for $X \in \mathfrak{n}^{\max}$. For this, we denote by E_ζ and $\Box_\zeta$ the Euler and Laplace operators, respectively, namely,

$$\begin{aligned} E_\zeta &:= \zeta_1\frac{\partial}{\partial\zeta_1} + \cdots + \zeta_{p+q-2}\frac{\partial}{\partial\zeta_{p+q-2}}, \\ \Box_\zeta &:= \frac{\partial^2}{\partial\zeta_1^2} + \cdots + \frac{\partial^2}{\partial\zeta_{p-1}^2} - \frac{\partial^2}{\partial\zeta_p^2} - \cdots - \frac{\partial^2}{\partial\zeta_{p+q-2}^2}. \end{aligned}$$

Then, the differential representation $d\pi(X)$ is given in [9], Lemma 3.2 as follows:

$$d\pi(\sum_{j=1}^{p+q-2} b_jN_j) = \sqrt{-1}((-\frac{p+q}{2} - E_\zeta)\sum_{j=1}^{p+q-2} b_j\frac{\partial}{\partial\zeta_j} + \frac{1}{2}(\sum_{j=1}^{p+q-2} b_j\varepsilon_j\zeta_j)\Box_\zeta)), \tag{3.2}$$

where we regard $d\pi(X)$ as a differential operator acting on the space of Schwartz's distributions $\mathcal{S}'(\mathbb{R}^{p+q-2})$ via the inclusion

$$L^2(C) \hookrightarrow \mathcal{S}'(\mathbb{R}^{p+q-2}), \quad \psi \mapsto \psi d\mu.$$

The differential operator (3.2) is of second order. This reflects the fact that the subgroup $N^{\max}$ does not act on the cone C itself, but only on the function space $L^2(C)$.

Instead of differential actions of the Lie algebra $\mathfrak{n}^{\max}$, we will in Section 5 deal with integral formulas for the action of the group G on $L^2(C)$.

4 Integral formulas for the minimal representation of $O(p,2)$

For the rest of this article, we shall assume $q = 2$. Then, the cone C naturally splits into two connected components

$$C = C_+ \cup C_-,$$

where $C_\pm := \{(\zeta_1, \cdots, \zeta_p) \in C : \pm\zeta_p > 0\}$. The polar coordinate (2.2) then reduces to

$$\begin{aligned} \mathbb{R}_+ \times S^{p-2} \to C_+, \quad (r,\omega) \mapsto (r\omega, r), \\ \mathbb{R}_+ \times S^{p-2} \to C_-, \quad (r,\omega) \mapsto (r\omega, -r). \end{aligned}$$

Accordingly, we have a direct sum decomposition:

$$L^2(C) = L^2(C_+) \oplus L^2(C_-),$$

where integrations are defined against the measure $d\mu = \frac{1}{2}r^{p-3}drd\omega$. This gives the branching law $\pi = \pi_+ \oplus \pi_-$ with respect to the restriction $G \downarrow G_0$ where $G_0 := SO_0(p,2)$, the identity component of $O(p,2)$.

Let K be the standard maximal compact subgroup of G. Then $K \simeq O(p) \times O(2)$, and $K_0 := K \cap G_0 (\simeq SO(p) \times SO(2))$ is a maximal compact subgroup of G_0. We write $L^2(C_+)_{K_0}$ for the space of K_0-finite vectors in $L^2(C_+)$. Then $L^2(C_+)_{K_0}$ is a dense subspace, on which the Lie algebra $\mathfrak{g}$ naturally acts as differentials.

Let $\mathfrak{z}(\mathfrak{k})$ be the center of the Lie algebra $\mathfrak{k}$ of K. Then $\mathfrak{z}(\mathfrak{k})$ is one dimensional if $p > 2$. We take a generator Z of $\mathfrak{z}(\mathfrak{k}) \otimes_{\mathbb{R}} \mathbb{C}$ as

$$Z := \sqrt{-1}(E_{p,p+1} - E_{p+1,p}) \in \sqrt{-1}\mathfrak{z}(\mathfrak{k}),$$

then the set of eigenvalues of $d\pi_+(Z)$ on $L^2(C_+)_{K_0}$ is given by

$$\{-(j + \frac{p-2}{2}) : j = 0, 1, 2 \cdots\},$$

and thus is upper bounded. Hence, $(\pi_+, L^2(C_+))$ is a lowest weight module of G_0. Similarly, $(\pi_-, L^2(C_-))$ is a highest weight module of G_0.

For $t \in \mathbb{C}$ we define a linear map $\pi_+(e^{tZ}) : L^2(C_+)_{K_0} \to L^2(C_+)_{K_0}$ by

$$\pi_+(e^{tZ}) := \sum_{n=0}^{\infty} \frac{1}{n!}(d\pi_+(tZ))^n.$$

In light of our observation on the eigenvalues of $d\pi_+(tZ)$, $\pi_+(e^{tZ})$ extends to a continuous operator on $L^2(C_+)$ if $\mathrm{Re}\, t \geq 0$. Then the set of continuous operators $\{\pi_+(e^{tZ}) : \mathrm{Re}\, t \geq 0\}$ forms a semigroup, whose generator is given by the self-adjoint operator on $L^2(C_+)$:

$$d\pi_+(Z) = \frac{r}{4}\frac{\partial^2}{\partial r^2} + \frac{p-2}{4}\frac{\partial}{\partial r} + \frac{\Delta_{S^{p-2}}}{4r} - r, \tag{4.1}$$

where $\Delta_{S^{p-2}}$ denotes the Laplace-Beltrami operator on the standard sphere S^{p-2}.

We shall give an explicit integral formula for the operator $\exp(td\pi_+(Z)) = \pi_+(e^{tZ})$ for $t \in D$, where set

$$D := \{t \in \mathbb{C} : \mathrm{Re}\, t \geq 0\} \setminus 2\pi\sqrt{-1}\mathbb{Z}.$$

We write $\langle \cdot, \cdot \rangle$ for the standard inner product of $\mathbb{R}^p$, and define the norm $|\zeta|$ by $|\zeta| := \sqrt{\langle \zeta, \zeta \rangle}$. Let us define a kernel function $K^+(\zeta, \zeta'; t)$ on $C_+ \times C_+ \times D$ by the following formula:

$$K^+(\zeta, \zeta'; t) := \frac{2e^{-\sqrt{2}(|\zeta|+|\zeta'|)\coth\frac{t}{2}}}{\pi^{\frac{p-2}{2}}\sinh^{\frac{p}{2}}\frac{t}{2}}\sqrt{2\langle \zeta, \zeta' \rangle}^{-\frac{p-4}{2}} I_{\frac{p-4}{2}}\Big(\frac{2\sqrt{2\langle \zeta, \zeta' \rangle}}{\sinh\frac{t}{2}}\Big), \tag{4.2}$$

where $I_\nu(z)$ is the modified Bessel function of the first kind, i.e., $I_\nu(z) = \sqrt{-1}^{-\nu} J_\nu(\sqrt{-1}z)$ [14]. We note that $\sinh\frac{t}{2}$ in the denominator is non-zero because $t \notin 2\pi\sqrt{-1}\mathbb{Z}$, and that $\langle \zeta, \zeta' \rangle > 0$ if $\zeta, \zeta' \in C_+$.

Here is an integration formula of the (holomorphic) semigroup $\pi_+(e^{tZ})$:

Theorem B (integral formula for a semigroup). *For $t \in D$, the operator $\pi_+(e^{tZ}) : L^2(C_+) \to L^2(C_+)$ is given by the integral transform:*

$$(\pi_+(e^{tZ})u)(\zeta) = \int_{C_+} K^+(\zeta, \zeta'; t) u(\zeta') d\mu(\zeta'), \quad u \in L^2(C_+). \tag{4.3}$$

Let us comment on the convergence of the integral (4.3); If $\mathrm{Re}\, t > 0$, then for each fixed t, $K^+(\zeta, \zeta'; t) \in L^2(C_+ \times C_+)$ and consequently $\pi_+(e^{tZ})$ becomes a Hilbert-Schmidt operator. If $t \in \sqrt{-1}\mathbb{R} \setminus 2\pi\sqrt{-1}\mathbb{Z}$, then $K^+(\zeta, \zeta'; t) \notin L^2(C_+ \times C_+)$ but the integral (4.3) converges absolutely if $u \in L^2(C_+)_{K_0}$ and yields an L^2-function on C_+.

Next, let us rewrite the formula (4.3) of Theorem B in the case where u is of the form

$$u(r\omega, r) = f(r)\phi(\omega) \tag{4.4}$$

for some $f \in L^2((0,\infty), r^{p-3}dr)$ and $\phi \in \mathcal{H}^l(\mathbb{R}^{p-1})$, where $\mathcal{H}^l(\mathbb{R}^{p-1})$ denotes the space of spherical harmonics on S^{p-2} of degree l ($l = 0, 1, 2, \cdots$), that is,

$$\mathcal{H}^l(\mathbb{R}^{p-1}) = \{\phi \in C^\infty(S^{p-2}) : \Delta_{S^{p-2}}\phi = -l(l+p-3)\phi\}.$$

For each l, we introduce the kernel function $K_l^+(r, r'; t)$ on $\mathbb{R}_+ \times \mathbb{R}_+ \times D$ by the formula:

$$K_l^+(r, r'; t) := \frac{2e^{-2(r+r')\coth\frac{t}{2}}}{\sinh\frac{t}{2}}(rr')^{-\frac{p-3}{2}} I_{p-3+2l}\Big(\frac{4\sqrt{rr'}}{\sinh\frac{t}{2}}\Big). \tag{4.5}$$

Then the point of the following theorem is that the operator $\pi_+(e^{tZ})$ essentially reduces to the integration of a function $f(r)$ of one variable if u is of the form (4.4), that is, we have

Theorem C. *If u is of the form $u(r\omega, r) = f(r)\phi(\omega)$, $\phi \in \mathcal{H}^l(\mathbb{R}^{p-1})$, then*

$$(\pi_+(e^{tZ})u)(r\omega, r) = \phi(\omega)\int_0^\infty K_l^+(r, r'; t) f(r') r'^{p-3} dr'. \tag{4.6}$$

Owing to Theorem C, the semigroup law

$$\pi_+(e^{t_1 Z})\pi_+(e^{t_2 Z}) = \pi_+(e^{(t_1+t_2)Z}) \quad (t_1, t_2 \in D)$$

is equivalent to the integral equation of the kernel

$$\int_0^\infty K_l^+(r, s; t_1) K_l^+(s, r'; t_2) s^{p-3} ds = K_l^+(r, r'; t_1 + t_2), \tag{4.7}$$

which is closely related to the classical formula, called **Weber's second exponential integral** (see [14], §13.31 (1)):

$$\int_0^\infty e^{-\rho x^2} J_\nu(\alpha x) J_\nu(\beta x) x dx = \frac{1}{2\rho^2}\exp\Big(-\frac{\alpha^2+\beta^2}{4\rho^2}\Big) I_\nu\Big(\frac{\alpha\beta}{2\rho^2}\Big).$$

5 Integral formula for the inversion operator

We define the "inversion element" w_0 of order two in G_0 by

$$w_0 := \begin{pmatrix} I_p & 0 \\ 0 & -I_2 \end{pmatrix}.$$

Then, w_0 normalizes $M^{\max}A$ and

$$\mathrm{Ad}(w_0)\mathfrak{n}^{\max} = \overline{\mathfrak{n}^{\max}}. \tag{5.1}$$

Hence, the group G is generated by $\overline{P^{\max}}$ and w_0.

The goal of this section is to give an explicit integral formula of the unitary operator $\pi_+(w_0)$ on $L^2(C_+)$.

In light of $w_0 = e^{\pi\sqrt{-1}Z}$, we define the following kernel functions by substituting $t = \pi\sqrt{-1}$ into (4.2) and (4.5), respectively:

$$K^+(\zeta,\zeta') := K^+(\zeta,\zeta';\pi\sqrt{-1}) = \frac{2}{(-1)^{\frac{p-2}{2}}\pi^{\frac{p-2}{2}}}\sqrt{2\langle\zeta,\zeta'\rangle}^{-\frac{p-4}{2}} J_{\frac{p-4}{2}}(2\sqrt{2\langle\zeta,\zeta'\rangle}),$$

$$K_l^+(r,r') := K_l^+(r,r';\pi\sqrt{-1}) = 2(-1)^{-\frac{p-2}{2}+l}(rr')^{-\frac{p-3}{2}} J_{p-3+2l}(4\sqrt{rr'}).$$

Then, the following result is obtained as a special case of Theorems B and C.

Theorem D. 1) *The unitary operator $\pi_+(w_0) : L^2(C_+) \to L^2(C_+)$ coincides with the integral transform defined by*

$$T : L^2(C_+) \to L^2(C_+), \quad u \mapsto \int_{C_+} K^+(\zeta,\zeta')u(\zeta')d\mu(\zeta'). \tag{5.2}$$

2) *If u is of the form $u(r\omega, r) = f(r)\phi(\omega)$ with $\phi \in \mathcal{H}^l(\mathbb{R}^{p-1})$ ($l = 0, 1, \cdots$), then the integral (5.2) is reduced to that of one variable:*

$$T : L^2(C_+) \to L^2(C_+), \quad u(r\omega, r) \mapsto \phi(\omega)(T_l f)(r). \tag{5.3}$$

Here, the operator $T_l : L^2((0,\infty), r^{p-3}dr) \to L^2((0,\infty), r^{p-3}dr)$ is defined by

$$(T_l f)(r) := \int_0^\infty K_l^+(r,r')f(r')r'^{p-3}dr'. \tag{5.4}$$

We note that T_l is essentially the Fourier-Bessel transform.

We can prove similar integral formulas for the unitary operator $\pi_-(w_0)$ on $L^2(C_-)$ for the backward cone C_-, and also for $\pi(w_0)$ on $L^2(C)$ for $C = C_+ \cup C_-$.

Finally, we end up this exposition with some immediate consequences of Theorem D. Since the relation $w_0^2 = I_{p+2}$ implies $\pi_+(w_0)^2 = \mathrm{Id}$, we have:

Corollary E (Inversion and Plancherel formula). *The integral operator T (see (5.2)) on $L^2(C_+)$ is of order two, that is, the inversion formula is given simply as*

$$T^{-1} = T.$$

Furthermore, T is unitary:

$$\|Tu\|_{L^2(C_+)} = \|u\|_{L^2(C_+)}, \quad u \in L^2(C_+).$$

Corollary F (Inversion and Plancherel formula for the Fourier-Bessel transform). *Fix $l = 0, 1, 2, \cdots$. Then the integral operator T_l (see (5.4)) on $L^2((0,\infty), r^{p-3}dr)$ is an unitary operator of order two. Hence,*

$$T_l^{-1} = T_l,$$
$$\|T_l f\|_{L^2((0,\infty),r^{p-3}dr)} = \|f\|_{L^2((0,\infty),r^{p-3}dr)}, \quad f \in L^2((0,\infty), r^{p-3}dr).$$

The statement $T_l^{-1} = T_l$ in Corollary F is equivalent to the integral formula:

$$f(r)r^{\frac{p-3}{2}} = 4\int_0^\infty \Big(\int_0^\infty f(r')r'^{\frac{p-3}{2}} J_{p-3+2l}(4\sqrt{r'r''})dr'\Big) J_{p-3+2l}(4\sqrt{rr''})dr''.$$

In turn, this is closely related to the **reciprocal formula of the Fourier-Bessel transform** (see [14], §14.3 (3)):

$$F(x) = \int_0^\infty \Big(\int_0^\infty F(y)J_\nu(y\xi)ydy\Big) J_\nu(\xi x)\xi d\xi.$$

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