Lectures on restrictions of unitary representations of real reductive groups

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Contents

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Lecture 1 Reductive Lie groups

Classical groups such as the general linear group $GL(n,\mathbb{R})$ and the Lorentz group $O(p,q)$ are reductive Lie groups. This section tries to give an elementary introduction to the structures of reductive Lie groups based on examples, which will be used throughout these lectures. All the materials of this section and further details may be found in standard textbooks or lecture notes such as [23, 36, 99, 104].

1.1 Smallest objects

The "smallest objects" of representations are **irreducible representations**. All unitary representations are built up of irreducible unitary representations by means of direct integrals (see §3.1.2).

The "smallest objects" for Lie groups are those without non-trivial connected normal subgroups; they consist of **simple Lie groups** such as $SL(n,\mathbb{R})$, and one dimensional abelian Lie groups such as $\mathbb{R}$ and S^1. **Reductive Lie groups** are locally isomorphic to these Lie groups or their direct products.

Loosely, a theorem of Duflo [10] asserts that all irreducible unitary representations of a real algebraic group are built up from those of reductive Lie groups.

Throughout these lecture notes, our main concern will be with irreducible decompositions of unitary representations of reductive Lie groups.

In Lecture 1, we summarize necessary notation and basic facts on reductive Lie groups in an elementary way.

1.2 General linear group $GL(N,\mathbb{R})$

The general linear group $GL(N,\mathbb{R})$ is a typical example of reductive Lie groups. First of all, we set up notation for $G := GL(N,\mathbb{R})$.

We consider the following map

$$\theta : G \to G, \quad g \mapsto {}^t g^{-1}.$$

Clearly, θ is an **involutive automorphism** in the sense that θ satisfies:

$$\begin{cases} \theta \circ \theta = \mathrm{id}, \\ \theta \text{ is an automorphism of the Lie group } G. \end{cases}$$

The set of the fixed points of θ

$$\begin{aligned} K := G^\theta &= \{g \in G : \theta g = g\} \\ &= \{g \in GL(N,\mathbb{R}) : {}^t g^{-1} = g\} \end{aligned}$$

is nothing but the orthogonal group $O(N)$, which is compact because $O(N)$ is a bounded closed set in $M(N,\mathbb{R}) \simeq \mathbb{R}^{N^2}$ in light of the following inclusion:

$$O(N) \subset \{g = (g_{ij}) \in M(N,\mathbb{R}) : \sum_{i=1}^N \sum_{j=1}^N g_{ij}^2 = N\}.$$

Furthermore, there is no larger compact subgroup of $GL(N,\mathbb{R})$ which contains $O(N)$. Thus, $K = O(N)$ is a **maximal compact subgroup** of $G = GL(N,\mathbb{R})$. Conversely, any maximal compact subgroup of G is conjugate to K by an element of G. The involution θ (or its conjugation) is called a **Cartan involution** of $GL(N,\mathbb{R})$.

1.3 Cartan decomposition

The Lie algebras of G and K will be denoted by $\mathfrak{g}$ and $\mathfrak{k}$ respectively. Then, for $G = GL(N,\mathbb{R})$ and $K = O(N)$, we have the following sum decomposition:

$$\begin{aligned}
&\mathfrak{g} \ = \mathfrak{gl}(N,\mathbb{R}) && := \text{the set of } N \times N \text{ real matrices} \\
&\| && \\
&\mathfrak{k} \ = \mathfrak{o}(N) && := \{X \in \mathfrak{gl}(N,\mathbb{R}) : X = -{}^tX\} \\
&\oplus && \\
&\mathfrak{p} \ = \mathrm{Symm}(N,\mathbb{R}) && := \{X \in \mathfrak{gl}(N,\mathbb{R}) : X = {}^tX\}.
\end{aligned}$$

The decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ is called the **Cartan decomposition** of the Lie algebra $\mathfrak{g}$ corresponding to the Cartan involution θ. This decomposition lifts to the Lie group G in the sense that we have the following diffeomorphism

$$\mathfrak{p} \times K \xrightarrow{\sim} G, \quad (X,k) \mapsto e^X k. \tag{1.3.1}$$

The map (1.3.1) is bijective, as (X,k) is recovered from $g \in GL(N,\mathbb{R})$ by the following formula:

$$X = \frac{1}{2}\log(g{}^tg), \quad k = e^{-X}g.$$

Here, log is the inverse of the bijection

$$\exp : \mathrm{Symm}(N,\mathbb{R}) \to \mathrm{Symm}_+(N,\mathbb{R}) := \{X \in \mathrm{Symm}(N,\mathbb{R}) : X \gg 0\}.$$

It requires a small computation of the Jacobian to see the map (1.3.1) is a C^ω-diffeomorphism (see [23, Chapter II, Theorem 1.7]). The decomposition (1.3.1) is known as the polar decomposition of $GL(N,\mathbb{R})$ in linear algebra, and is a special case of a **Cartan decomposition** of a reductive Lie group in Lie theory (see (1.4.2) below).

1.4 Reductive Lie groups

A connected Lie group G is called **reductive** if its Lie algebra $\mathfrak{g}$ is reductive, namely, is isomorphic to the direct sum of simple Lie algebras and an abelian Lie algebra. In the literature on representation theory, however, different authors have introduced and/or adopted several variations of the category of reductive Lie groups, in particular, with respect to discreteness, linearity and covering. In this article we adopt the following definition of "reductive Lie group". Some advantages here are:

- Structural theory of reductive Lie group can be explained elementarily in parallel to that of $GL(n, \mathbb{R})$.
- It is easy to verify that (typical) classical groups are indeed real reductive Lie groups (see §1.5).

Definition 1.4. We say a Lie group G is **linear reductive** if G can be realized as a closed subgroup of $GL(N, \mathbb{R})$ satisfying the following two conditions

i) $\theta G = G$.

ii) G has at most finitely many connected components.

We say G is a **reductive Lie group** if it is a finite covering of a linear reductive Lie group.

Then, its Lie algebra $\mathfrak{g}$ is reductive, that is, a direct sum of simple Lie algebras and an abelian Lie algebra.

If G is linear reductive, then by using its realization in $GL(N, \mathbb{R})$ we define

$$K := G \cap O(N).$$

Then K is a maximal compact subgroup of G. The Lie algebra $\mathfrak{k}$ of K is given by $\mathfrak{g} \cap \mathfrak{o}(N)$. We define

$$\mathfrak{p} := \mathfrak{g} \cap \mathrm{Symm}(N, \mathbb{R}).$$

Similarly to the case of $GL(N, \mathbb{R})$, we have the following Cartan decompositions:

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{p} \qquad \text{(direct sum decomposition)} \tag{1.4.1}$$

$$\mathfrak{p} \times K \xrightarrow{\sim} G \qquad \text{(diffeomorphism)} \tag{1.4.2}$$

by taking the restriction of the corresponding decompositions for $GL(N, \mathbb{R})$ (see (1.3.1)) to $\mathfrak{g}$ and G, respectively.

1.5 Examples of reductive Lie groups

Reductive Lie groups in the above definition include all compact Lie groups (especially, finite groups), and classical groups such as the general linear group $GL(N, \mathbb{R}), GL(N, \mathbb{C})$, the special linear group $SL(N, \mathbb{R}), SL(N, \mathbb{C})$, the generalized Lorentz group (the indefinite orthogonal group) $O(p,q)$, the symplectic group $Sp(N,\mathbb{R})$, $Sp(N,\mathbb{C})$, and some others such as $U(p,q)$, $Sp(p,q)$, $U^*(2n)$, $O^*(2n)$ etc.

In this section, we review some of these classical groups, and explain how to prove that they are reductive Lie groups in the sense of Definition 1.4. For this purpose, the following Theorem is useful.

Theorem 1.5.1. *Let G be a linear algebraic subgroup of $GL(N,\mathbb{R})$. If $\theta G = G$, then G is a reductive Lie group.*

Here, we recall that a **linear algebraic group** (over $\mathbb{R}$) is a subgroup $G \subset GL(N,\mathbb{R})$ which is an algebraic subset in $M(N,\mathbb{R})$, that is , the set of zeros of an ideal of polynomial functions with coefficients in $\mathbb{R}$.

Sketch of proof. In order to prove that G has at most finitely many connected components, it is enough to verify the following two assertions:

- $K = G \cap O(N)$ is compact.
- The map $\mathfrak{p} \times K \xrightarrow{\sim} G, \quad (X,k) \mapsto e^X k$ is a homeomorphism (Cartan decomposition).

The first assertion is obvious because G is closed. The second assertion is deduced from the bijection (1.3.1) for $GL(N,\mathbb{R})$. For this, the non-trivial part is a proof of the implication "$e^X k \in G \Rightarrow X \in \mathfrak{p}$". Let us prove this. We note that if $e^X k \in G$ then $e^{2X} = (e^X k)(\theta(e^X k))^{-1} \in G$, and therefore $e^{2nX} \in G$ for all $n \in \mathbb{Z}$. To see $X \in \mathfrak{p}$, we want to show $e^{2tX} \in G$ for all $t \in \mathbb{R}$. This follows from Chevalley's lemma: let $X \in \mathrm{Symm}(N,\mathbb{R})$, if e^{sX} satisfies a polynomial equation of entries for any $s \in \mathbb{Z}$, then so does e^{sX} for any $s \in \mathbb{R}$. $\square$

As special cases of Theorem 1.5.1, we pin down Propositions 1.5.2, 1.5.3 and 1.5.6.

Proposition 1.5.2. $SL(N,\mathbb{R}) := \{g \in GL(N,\mathbb{R}) : \det g = 1\}$ *is a reductive Lie group.*

Proof. Obvious from Theorem 1.5.1. $\square$

Proposition 1.5.3. *Let A be an $N \times N$ matrix such that $A^2 = cI$ $(c \neq 0)$. Then*

$$G(A) := \{g \in GL(N, \mathbb{R}) : {}^t g A g = A\}$$

is a reductive Lie group.

Proof. First, it follows from the definition that $G(A)$ is a linear algebraic subgroup of $GL(N, \mathbb{R})$.

Second, let us prove $\theta(G(A)) = G(A)$. If ${}^t g A g = A$ then $g^{-1} = A^{-1}\,{}^t g A$. Since $A^{-1} = \frac{1}{c} A$, we have

$$I = gg^{-1} = g(\frac{1}{c}A)\,{}^t g(cA^{-1}) = gA\,{}^t gA^{-1}.$$

Hence $gA\,{}^t g = A$, that is $\theta g \in G(A)$. Then Proposition follows from Theorem 1.5.1. $\square$

Example 1.5.4. If $A = \begin{pmatrix} \overbrace{\begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix}}^{p} & O \\ O & \overbrace{\begin{matrix} -1 & & \\ & \ddots & \\ & & -1 \end{matrix}}^{q} \end{pmatrix}$ $(N = p + q)$, then

$$G(A) = O(p, q) \qquad \text{(indefinite orthogonal group).}$$

In this case

$$K = O(p, q) \cap O(p + q) \simeq O(p) \times O(q).$$
$$\mathfrak{p} = \{\begin{pmatrix} O & B \\ {}^t B & O \end{pmatrix} : B \in M(p, q; \mathbb{R})\}.$$

Example 1.5.5. If $A = \begin{pmatrix} O & I_n \\ -I_n & O \end{pmatrix}$ $(N = 2n)$, then

$$G(A) = Sp(n, \mathbb{R}) \qquad \text{(real symplectic group).}$$

In this case,

$$\begin{aligned}
K &= Sp(n,\mathbb{R}) \cap O(2n) && \simeq U(n). \\
\mathfrak{k} &= \{ \begin{pmatrix} A & B \\ -B & A \end{pmatrix} : A = -{}^tA, B = {}^tB \} && \xrightarrow{\sim} \mathfrak{u}(n), \\
& && \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \mapsto A + iB. \\
\mathfrak{p} &= \{ \begin{pmatrix} A & B \\ B & -A \end{pmatrix} : A = {}^tA, B = {}^tB \} && \xrightarrow{\sim} \mathrm{Symm}(n,\mathbb{C}), \\
& && \begin{pmatrix} A & B \\ B & -A \end{pmatrix} \mapsto A + iB.
\end{aligned}$$

Next, let $\mathbb{F} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$ (quaternionic number field). We regard $\mathbb{F}^n$ as a **right** $\mathbb{F}^\times$-module. Let

$M(n,\mathbb{F})$:= the ring of endomorphisms of $\mathbb{F}^n$, commuting with $\mathbb{F}^\times$-actions

$\cup$

$GL(n,\mathbb{F})$:= the group of all invertibles in $\mathrm{M}(n,\mathbb{F})$.

Proposition 1.5.6. *$GL(n,\mathbb{F})$ ($\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$) is a reductive Lie group.*

Sketch of proof. Use $\mathbb{C} \simeq \mathbb{R}^2$ as $\mathbb{R}$-modules and $\mathbb{H} \simeq \mathbb{C}^2$ as right $\mathbb{C}$-modules, and then realize $GL(n,\mathbb{C})$ in $GL(2n,\mathbb{R})$ and $GL(n,\mathbb{H})$ in $GL(2n,\mathbb{C})$. For example, $GL(n,\mathbb{H})$ can be realized in $GL(2n,\mathbb{C})$ as an algebraic subgroup:

$$U^*(2n) = \{g \in GL(2n,\mathbb{C}) : \bar{g}J = Jg\},$$

where $J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$. It can be also realized in $GL(4n,\mathbb{R})$ as an algebraic subgroup which is stable under the Cartan involution $\theta : g \mapsto {}^tg$. $\square$

Classical reductive Lie groups are obtained by Propositions 1.5.2, 1.5.3 and 1.5.6, and by taking their intersections and their finite coverings. For examples, in appropriate realizations, the following intersections

$$\begin{aligned}
U(p,q) &= SO(2p,2q) \cap GL(p+q,\mathbb{C}) \\
Sp(p,q) &= SO(4p,4q) \cap GL(p+q,\mathbb{H}) \\
SU^*(2n) &= U^*(2n) \cap SL(2n,\mathbb{C}) \\
SO^*(2n) &= GL(n,\mathbb{H}) \cap SO(2n,\mathbb{C})
\end{aligned}$$

are all reductive (linear) Lie groups.

1.6 Inclusions of groups and restrictions of representations

As the constructions in the previous section indicate, these classical Lie groups enjoy natural inclusive relations such as

$$
\begin{array}{ccccccccccccc}
\cdots \subset & GL(n,\mathbb{R}) & \subset & Sp(n,\mathbb{R}) & \subset & Sp(n,\mathbb{C}) & \subset & Sp(2n,\mathbb{R}) & \subset & GL(4n,\mathbb{R}) & \subset \cdots \\
& \cup & & \cup & & \cup & & \cup & & \cup & \\
\cdots \subset & O(p,q) & \subset & U(p,q) & \subset & Sp(p,q) & \subset & U(2p,2q) & \subset & O(4p,4q) & \subset \cdots \\
& \cap & & \cap & & \cap & & \cap & & \cap & \\
\cdots \subset & GL(n,\mathbb{R}) & \subset & GL(n,\mathbb{C}) & \subset & GL(n,\mathbb{H}) & \subset & GL(2n,\mathbb{C}) & \subset & GL(4n,\mathbb{R}) & \subset \cdots \\
& \cap & & \cap & & \cap & & \cap & & \cap & \\
& \vdots & & \vdots & & \vdots & & \vdots & & \vdots &
\end{array}
$$

where $p + q = n$. Our object of the lectures will be the restrictions of unitary representations of a group to its subgroups. This may be regarded as a representation theoretic counterpart of inclusive relations between Lie groups as above.

Lecture 2 Unitary representations and admissible representations

To deal with infinite dimensional representations, we need a good category to work with. This section introduces some standard notation such as continuous representations followed by more specialized category such as discrete decomposable representations and admissible representations.

2.1 Continuous representations

Let $\mathcal{H}$ be a topological vector space over $\mathbb{C}$. We shall write:

$$\begin{aligned} \mathrm{End}(\mathcal{H}) &:= \text{the ring of continuous endomorphisms of } \mathcal{H}, \\ GL(\mathcal{H}) &:= \text{the group of all invertibles in } \mathrm{End}(\mathcal{H}). \end{aligned}$$

Definition 2.1.1. Let G be a Lie group and $\pi : G \to GL(\mathcal{H})$ a group homomorphism. We say $(\pi, \mathcal{H})$ is a **continuous representation**, if the following map

$$G \times \mathcal{H} \to \mathcal{H}, \quad (g, v) \mapsto \pi(g)v \tag{2.1.1}$$

is continuous.

If $\mathcal{H}$ is a Fréchet space (a Banach space, a Hilbert space, etc.), the continuity condition in this definition is equivalent to strong continuity — that $g \mapsto \pi(g)v$ is continuous from G to $\mathcal{H}$ for each $v \in \mathcal{H}$.

A continuous representation $(\pi, \mathcal{H})$ is **irreducible** if there is no invariant closed subspace of $\mathcal{H}$ but for obvious ones $\{0\}$ and $\mathcal{H}$.

2.2 Examples

Example 2.2.1. Let $G = \mathbb{R}$ and $\mathcal{H} = L^2(\mathbb{R})$. For $a \in G$ and $f \in \mathcal{H}$, we define

$$T(a) : L^2(\mathbb{R}) \to L^2(\mathbb{R}), \quad f(x) \mapsto f(x - a).$$

Then, it is easy to see that the map (2.1.1) is continuous. The resulting continuous representation $(T, L^2(\mathbb{R}))$ is called the **regular representation** of $\mathbb{R}$. We note that $T : G \to GL(\mathcal{H})$ is not continuous if $GL(\mathcal{H})$ is equipped with operator norm $\| \ \|$ because

$$\lim_{a \to a_0} \|T(a) - T(a_0)\| = \sqrt{2} \neq 0. \tag{2.2.1}$$

This observation explains that the continuity of (2.1.1) is a proper one to define the notion of continuous representations.

2.3 Unitary representations

Definition 2.3.1. A unitary representation is a continuous representation π defined on a Hilbert space $\mathcal{H}$ such that $\pi(g)$ is a unitary operator for all $g \in G$.

The set of equivalent classes of irreducible unitary representaitons of G is called the **unitary dual** of G, and will be denoted by $\widehat{G}$.

2.4 Admissible restrictions

2.4.1 (Analytically) discretely decomposable representations

Let G' be a Lie group, and π a unitary representation of G' on a (separable) Hilbert space.

Definition 2.4.1. We say the unitary representation π is **(analytically) discretely decomposable** if π is unitarily equivalent to a discrete sum of irreducible unitary representations of G':

$$\pi|_{G'} \simeq \sum_{\sigma \in \widehat{G'}}^{\oplus} n_\pi(\sigma)\sigma. \tag{2.4.1}$$

Here, $n_\pi : \widehat{G'} \to \{0, 1, 2, \ldots, \infty\}$, and $\sum^{\oplus} n_\pi(\sigma)\sigma$ denotes the Hilbert completion of an algebraic direct sum of irreducible representations

$$\bigoplus_{\sigma \in \widehat{G'}} (\underbrace{\sigma \oplus \sigma \oplus \cdots \oplus \sigma}_{n_\pi(\sigma)}).$$

If we have an unitary equivalence (2.4.1), then $n_\pi(\sigma)$ is given by

$$n_\pi(\sigma) = \dim \operatorname{Hom}_{G'}(\sigma, \pi|_{G'}),$$

the dimension of the space of continuous G'-homomorphisms.

The point of Definition 2.4.1 is that there is no continuous spectrum in the decomposition (see Theorem 3.1.2 for a general nature of irreducible decompositions).

2.4.2 Admissible representations

Let π be a unitary representation of G'.

Definition 2.4.2. We say π is **G'-admissible** if it is (analytically) discretely decomposable and if $n_\pi(\sigma) < \infty$ for any $\sigma \in \widehat{G'}$.

Example. 1) Any finite dimensional unitary representation is admissible.

2) The regular representation of a compact group K on $L^2(K)$ is also admissible by the Peter-Weyl theorem.

We shall give some more important examples where admissible representations arise in various contexts.

2.4.3 Gelfand-Piateski-Shapiro's theorem

Let Γ be a discrete subgroup of a unimodular Lie group (for example, any reductive Lie group is unimodular). Then we can induce a G-invariant measure on the coset space G/Γ from the Haar measure on G, and define a unitary representation of G on the Hilbert space $L^2(G/\Gamma)$, consisting of square integrable functions on G/Γ.

Theorem 2.4.3 (Gelfand-Piateski-Shapiro). *If G/Γ is compact, then the representation on $L^2(G/\Gamma)$ is G-admissible.*

Proof. See [105, Proposition 4.3.1.8] for a proof due to Langlands. □

2.4.4 Admissible restrictions

Our main object of these lectures is to study restrictions of a unitary representation π of G with respect to its subgroup G'.

Definition 2.4.4. We say the restriction $\pi|_{G'}$ is **(analytically) discretely decomposable**, **G'-admissible**, if it is analytically discretely decomposable (Definition 2.4.1), G'-admissible (Definition 2.4.2), respectively.

Our main concern in later sections will be with the case where G' is noncompact. We end up with some basic results on admissible restrictions for later purposes.

2.4.5 Chain rule of admissible restrictions

Theorem 2.4.5. *Suppose $G \supset G_1 \supset G_2$ are (reductive) Lie groups, and π is a unitary representation of G. If the restriction $\pi|_{G_2}$ is G_2-admissible, then the restriction $\pi|_{G_1}$ is G_1-admissible.*

Sketch of proof. Use Zorn's lemma. The proof parallels to that of the Gelfand-Piateski-Shapiro. See [41, Theorem 1.2] for details. $\square$

Here is an immediate consequence of Theorem 2.4.5

Corollary. *Let π be a unitary representation of G, and K' a compact subgroup of G. Assume that* $\dim \operatorname{Hom}_{K'}(\sigma, \pi|_{K'}) < \infty$ *for any $\sigma \in \widehat{K'}$. Then the restriction $\pi|_{G'}$ is G'-admissible for any G' containing K'.*

This corollary is a key to a criterion of admissible restrictions, which we shall return in later sections.

2.4.6 Harish-Chandra's admissibility theorem

Here is a special, but very important example of admissible restrictions:

Theorem 2.4.6 (Harish-Chandra, [21]).
Let G be a reductive Lie group with a maximal compact subgroup K. For any $\pi \in \widehat{G}$, the restriction $\pi|_K$ is K-admissible.

Since K is compact, the point of Theorem 2.4.6 is the finiteness of K-multiplicities:

$$\dim \operatorname{Hom}_K(\sigma, \pi|_K) < \infty \quad \text{for any } \sigma \in \widehat{K}. \tag{2.4.6}$$

Remark. In a traditional terminology, a continuous (not necessarily unitary) representation π of G is called **admissible** if (2.4.6) holds.

For advanced readers, we give a flavor of the proof of Harish-Chandra's admissibility theorem without going into details.

Sketch of proof of Theorem 2.4.6. See [104], Theorem 3.4.10; [105], Theorem 4.5.2.11.

Step 1. Any principal series representation is K-admissible. This is an easy consequence of the Frobenius reciprocity because the K-structure of a principal series is given as the induced representation of K from an irreducible representation of a subgroup of K.

Step 2. Let π be any irreducible unitary representation of G. The center $Z(\mathfrak{g})$ of the enveloping algebra $U(\mathfrak{g})$ acts on π^∞ as scalars. Here, π^∞ denotes the representation on smooth vectors. Step 2 is due to Segal and Mautner. It may be regarded as a generalization of Schur's lemma.

Step 3. Any π can be realized in a subquotient of some principal series representation (Harish-Chandra, Lepowsky, Rader), or more strongly, in a subrepresentation of some principal series representation (Casselman's embedding theorem). Together with Step 1, Theorem 2.4.6 follows. Casselman's proof is to use the theory of the Jacquet module and the $\mathfrak{n}$-homologies of representations.

There is also a more analytic proof for the Casselman's embedding theorem without using $\mathfrak{n}$-homologies: This proof consists of two steps:. (i) to compactify G ([79]), (ii) to realize π^∞ in $C^\infty(G)$ via matrix coefficients, (iii) to take the boundary values into principal series representations (see [80] and references therein).

2.4.7 Further readings

For further details, see [105] for general facts on continuous representations in 2.1; [41] (and also an exposition [48, 47, 50]) on some aspect of G'-admissible restrictions where G' is not necessarily compact; [104, Chapter 3] (and also an exposition [102]) for Harish-Chandra's admissibility theorem 2.4.6 and for the idea of $\mathfrak{n}$-homologies.

Lecture 3 $SL(2,\mathbb{R})$ and Branching Laws

Branching laws of unitary representations are related with many different areas of mathematics —spectral theory of partial differential operators, harmonic analysis, combinatorics, differential geometry, complex analysis, $\cdots$. The aim of this section is to give a flavor of various aspects on branching laws through a number of examples arising from $SL(2,\mathbb{R})$.

3.1 Branching Problems

3.1.1 Direct integral of Hilbert spaces

We recall briefly how to generalize the concept of the discrete direct sum of Hilbert spaces into the direct integral of Hilbert spaces, and explain the general theory of irreducible decomposition of unitary representations that may contain continuous spectrum. Since the aim for this is just to provide a wider perspective on discrete decomposable restrictions that will be developed in later chapters, we shall try to minimize the exposition. See [34] for further details on §3.1.

Let $\mathcal{H}$ be a (separable) Hilbert space, and (Λ, μ) a measure space. We construct a Hilbert space, denoted by

$$\int_{\Lambda}^{\oplus} \mathcal{H} d\mu(\lambda),$$

consisting of those $\mathcal{H}$-valued functions $s : \Lambda \to \mathcal{H}$ with the following two properties:

i) For any $v \in \mathcal{H}$, $(s(\lambda), v)$ is measurable with respect to μ.
ii) $\|s(\lambda)\|_{\mathcal{H}}^2$ is square integrable with respect to μ.

The inner product on $\int_{\Lambda}^{\oplus} \mathcal{H} d\mu(\lambda)$ is given by

$$(s, s') := \int_{\Lambda} (s(\lambda), s'(\lambda))_{\mathcal{H}} d\mu(\lambda).$$

Example. If $\mathcal{H} = \mathbb{C}$ then $\int_{\Lambda}^{\oplus} \mathcal{H} d\mu(\lambda) \simeq L^2(\Lambda)$.

More generally, if a "measurable" family of Hilbert spaces $\mathcal{H}_\lambda$ parameterized by $\lambda \in \Lambda$ is given, then one can also define a direct integral of Hilbert spaces $\int_\Lambda^\oplus \mathcal{H}_\lambda d\mu(\lambda)$ in a similar manner. Furthermore, if $(\pi_\lambda, \mathcal{H}_\lambda)$ is a "measurable" family of unitary representations of a Lie group G' for each λ, then the map

$$(s(\lambda))_\lambda \mapsto (\pi_\lambda(g)s(\lambda))_\lambda$$

defines a unitary operator on $\int_\Lambda^\oplus \mathcal{H}_\lambda d\mu(\lambda)$. The resulting unitary representation is called the **direct integral of unitary representaions** $(\pi_\lambda, \mathcal{H}_\lambda)$ and will be denoted by

$$(\int_\Lambda^\oplus \pi_\lambda d\mu(\lambda), \int_\Lambda^\oplus \mathcal{H}_\lambda d\mu(\lambda)).$$

3.1.2 Irreducible decomposition

The following theorem holds more generally for a group of type I in the sense of von Neumann algebras.

Theorem 3.1.2. *Every unitary representation π of a reductive Lie group G' on a (separable) Hilbert space is unitarily equivalent to a direct integral of irreducible unitary representations of G'.*

$$\pi \simeq \int_{\widehat{G'}}^\oplus n_\pi(\sigma)\sigma \, d\mu(\sigma). \tag{3.1.2}$$

Here, $d\mu$ is a Borel measure on the unitary dual $\widehat{G'}$, $n_\pi : \widehat{G'} \to \mathbb{N} \cup \{\infty\}$ is a measurable function, and $n_\pi(\sigma)\sigma$ is a multiple of the irreducible unitary representation σ.

3.1.3 Examples

Example 3.1.3. 1) (Decomposition of $L^2(\mathbb{R})$)

Let $G' = \mathbb{R}$. For a parameter $\xi \in \mathbb{R}$, we define a one-dimensional unitary representation of the abelian Lie group $\mathbb{R}$ by

$$\chi_\xi : \mathbb{R} \to \mathbb{C}^\times, \quad x \mapsto e^{ix\xi}.$$

Then we have a bijection $\widehat{\mathbb{R}} \simeq \mathbb{R}, \chi_\xi \leftrightarrow \xi$. The regular representation T of $\mathbb{R}$ on $L^2(\mathbb{R})$ is decomposed into irreducibles of $\mathbb{R}$ by the Fourier transform:

$$T \simeq \int_{\mathbb{R}}^\oplus \chi_\xi \, d\xi.$$

2) (Decomposition of $L^2(S^1)$)

Let $G' = S^1 \simeq \mathbb{R}/2\pi\mathbb{Z}$. By the Fourier series expansion, we have a discrete sum of Hilbert spaces:

$$L^2(S^1) \simeq \sum_{n\in\mathbb{Z}}^{\oplus} \mathbb{C}\, e^{in\theta}.$$

This is regarded as the irreducible decomposition of the regular representation of the compact abelian Lie group S^1 on $L^2(S^1)$. Here, we have identified $\widehat{S^1}$ with $\mathbb{Z}$ by $e^{in\theta} \leftrightarrow n$.

We note that there is no discrete spectrum in (1), while there is no continuous spectrum in (2). In particular, $L^2(S^1)$ is S^1-admissible, as already mentioned in §2.4.2 in connection with the Peter-Weyl theorem.

3.1.4 Branching problems

Let π be an irreducible unitary representation of a group G, and G' be its subgroup. By a **branching law**, we mean the irreducible decomposition of π when restricted to the subgroup G'. **Branching problems** ask to find branching laws as explicitly as possible.

3.2 Unitary dual of $SL(2,\mathbb{R})$

Irreducible unitary representations of $G := SL(2,\mathbb{R})$ were classified by Bargmann in 1947. In this section, we recall some of important family of them.

3.2.1 $SL(2,\mathbb{R})$-action on $\mathbb{P}^1\mathbb{C}$

The Riemann sphere $\mathbb{P}^1\mathbb{C} = \mathbb{C} \cup \{\infty\}$ splits into three orbits $\mathcal{H}_+, \mathcal{H}_-$ and S^1 under the linear fractional transformation of $SL(2,\mathbb{R})$, $z \mapsto \dfrac{az+b}{cz+d}$.

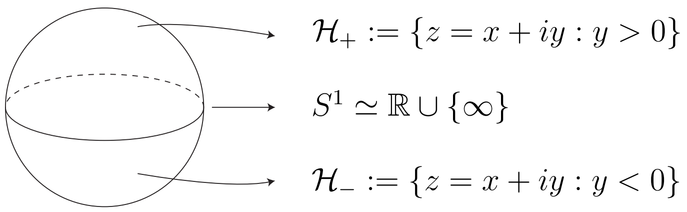


Figure 3.2.1

We shall associate a family of irreducible unitary representations of $SL(2,\mathbb{R})$ to these orbits in §3.2.2 and §3.2.3.

3.2.2 Unitary principal series representations

First, we consider the closed orbit

$$S^1 \simeq \mathbb{R} \cup \{\infty\}.$$

For $\lambda \in \mathbb{C}$ such that $\operatorname{Re}\lambda = 1$, and for $g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$, we define

$$\pi_\lambda(g) : L^2(\mathbb{R}) \to L^2(\mathbb{R}), \quad f \mapsto (\pi_\lambda(g)f)(x) := |cx+d|^{-\lambda} f\left(\frac{ax+b}{cx+d}\right).$$

Then the following holds:

Proposition 3.2.2. *For any $\lambda \in \mathbb{C}$ such that $\operatorname{Re}\lambda = 1$, $(\pi_\lambda, L^2(\mathbb{R}))$ is an irreducible unitary representation of $G = SL(2,\mathbb{R})$.*

Exercise. Prove Proposition 3.2.2.

Hint 1) It is straightforward to see

$$\begin{aligned} &\pi_\lambda(g_1 g_2) = \pi_\lambda(g_1)\pi_\lambda(g_2) \quad (g_1, g_2 \in G), \\ &\|\pi_\lambda(g)f\|_{L^2(\mathbb{R})} = \|f\|_{L^2(\mathbb{R})} \quad (g \in G). \end{aligned}$$

2) Let $K = SO(2)$. For the irreducibility, it is enough to verify the following two claims:

a) Any closed invariant subspace W of $L^2(\mathbb{R})$ contains a non-zero K-invariant vector.

b) Any K-invariant vector in $L^2(\mathbb{R})$ is a scalar multiple of $|1+x^2|^{-\frac{\lambda}{2}}$.

3.2.3 Holomorphic discrete series representations

Next, we construct a family of representations attached to the open orbit $\mathcal{H}_+$ (see Figure 3.2.1). Let $\mathcal{O}(\mathcal{H}_+)$ be the space of holomorphic functions on the upper half plane $\mathcal{H}_+$. For an integer $n \geq 2$, we define

$$V_n^+ := \mathcal{O}(\mathcal{H}_+) \cap L^2(\mathcal{H}_+, y^{n-2}\,dx\,dy).$$

Then, it turns out that V_n^+ is a non-zero closed subspace of the Hilbert space $L^2(\mathcal{H}_+, y^{n-2}\,dx\,dy)$, from which the Hilbert structure is induced.

For $g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$, we define

$$\pi_n^+(g) : V_n^+ \to V_n^+, \quad f(z) \mapsto (cz+d)^{-n} f\left(\frac{az+b}{cz+d}\right).$$

Then the following proposition holds:

Proposition 3.2.3 (holomorphic discrete series).
Each (π_n^+, V_n^+) $(n \geq 2)$ is an irreducible unitary representation of G.

Let us discuss this example in the following exercises:

Exercise. Verify that $\pi_n^+(g)$ $(g \in G)$ is a unitary operator on V_n^+ by a direct computation.

We shall return the irreducibility of (π_n^+, V_n^+) in §3.3.4

Exercise. Let $k_\theta := \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$, and $f_n(z) := (z+i)^{-n}$. Prove the following formula:

$$\pi_n^+(k_\theta) f_n = e^{in\theta} f_n.$$

As we shall see in Proposition 3.3.3 (1), the K-types occurring in π_n^+ are $\chi_n, \chi_{n+2}, \chi_{n+4}, \ldots$. Hence, the vector $f_n \in V_n^+$ is called a **minimal K-type** vector of the representation (π_n^+, V_n^+).

In §7.1, we shall construct a family of unitary representations attached to elliptic coadjoint orbits. The above construction of (π_n^+, V_n^+) is a simplest example, where the Dolbeault cohomology group turns up in the degree 0 because the elliptic coadjoint orbit is biholomorphic to a Stein manifold $\mathcal{H}_+$ in this case.

Similarly to (π_n^+, V_n^+), we can construct another family of irreducible unitary representations π_n^- $(n = 2, 3, 4, \ldots)$ of G on the Hilbert space

$$V_n^- := \mathcal{O}(\mathcal{H}_-) \cap L^2(\mathcal{H}_-, |y|^{n-2}\,dx\,dy).$$

The representation (π_n^-, V_n^-) is called the **anti-holomorphic discrete series representation**.

3.2.4 Restriction and proof for irreducibility

Given a representation π of G, how can one find finer properties of π such as irreducibility, Jordan-Hölder series, etc? A naive (and sometimes powerful) approach is to take the restriction of π to a suitable subgroup H. For instance, the method of $(\mathfrak{g}, K)$-modules (see Lecture 4) is based on "discretely decomposable" restrictions to a maximal compact subgroup K. Together with "transition coefficients" that describe the actions of $\mathfrak{p}$ on $\mathfrak{g}$-modules, the method of $(\mathfrak{g}, K)$-modules provides an elementary and alternative proof of irreducibility of both π_λ and $\pi_n^{\pm}$, simultaneously (e.g. [95, Chapter 2]; see also Howe and Tan [26] for some generalization to the Lorentz group $O(p, q)$).

On the other hand, the restriction to non-compact subgroups is sometimes effective in studying representations of G. We shall return this point in Lecture 8.

For a better understanding, let us observe a number of branching laws of unitary representations with respect to both compact and non-compact subgroups in the case $SL(2, \mathbb{R})$.

3.3 Branching laws of $SL(2, \mathbb{R})$

3.3.1 Subgroups of $SL(2, \mathbb{R})$

Let us consider three subgroups of $G = SL(2, \mathbb{R})$:

$$G \supset H := \begin{cases} K := \{k_\theta : \theta \in \mathbb{R}/2\pi\mathbb{Z}\} \simeq S^1, \\ N := \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} : b \in \mathbb{R} \right\} \simeq \mathbb{R}, \\ A := \left\{ \begin{pmatrix} e^s & 0 \\ 0 & e^{-s} \end{pmatrix} : s \in \mathbb{R} \right\} \simeq \mathbb{R}. \end{cases}$$

For $\xi \in \mathbb{R}$, we define a one-dimensional unitary representation of $\mathbb{R}$ by

$$\chi_\xi : \mathbb{R} \to \mathbb{C}^\times, \quad x \mapsto e^{ix\xi}.$$

For $n \in \mathbb{Z}$, χ_n is well-defined as a unitary representation of $S^1 \simeq \mathbb{R}/2\pi\mathbb{Z}$.

3.3.2 Branching laws $G \downarrow K, A, N$

Proposition 3.3.2. *Fix $\lambda \in \mathbb{C}$ such that $\operatorname{Re}\lambda = 1$. Then the branching laws of a principal series representation π_λ of $G = SL(2,\mathbb{R})$ to the subgroups K, N and A are given by:*

1) $\pi_\lambda\big|_K \simeq \sum_{n\in\mathbb{Z}}^{\oplus} \chi_{2n}.$

2) $\pi_\lambda\big|_N \simeq \int_{\mathbb{R}}^{\oplus} \chi_\xi \, d\xi.$

3) $\pi_\lambda\big|_A \simeq \int_{\mathbb{R}}^{\oplus} 2\chi_\xi \, d\xi.$

Here, we have used the identifications $\widehat{K} \simeq \mathbb{Z}$, and $\widehat{N} \simeq \widehat{A} \simeq \mathbb{R}$.

Sketch of proof. In all three cases, the proof reduces to the (Euclidean) harmonic analysis. Here are some more details.

1) We define a unitary map (up to scalar) T_λ by

$$T_\lambda : L^2(\mathbb{R}) \to L^2(S^1), \quad f \mapsto \left|\cos\frac{\psi}{2}\right|^{-\lambda} f\left(\tan\frac{\psi}{2}\right). \tag{3.3.2}$$

Then T_λ respects the K-actions as follows:

$$T_\lambda(\pi_\lambda(k_\varphi)f)(\theta) = (T_\lambda f)(\theta + 2\varphi).$$

Thus, the branching law (1) follows from the (discrete) decomposition of $L^2(S^1)$ by the Fourier series expansion:

$$L^2(S^1) \simeq \sum_{n\in\mathbb{Z}}^{\oplus} \mathbb{C}\, e^{in\theta},$$

as was given in Example 3.1.3 (2).

2) Since $(\pi_\lambda \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} f)(x) = f(x-b)$, the restriction $\pi_\lambda\big|_N$ is nothing but the regular representation of $\mathbb{R}$ on $L^2(\mathbb{R})$. Hence the branching law (2) is given by the Fourier transform as we saw in Example 3.1.3

3) We claim that the multiplicity in the continuous spectrum is uniformly two. To see this, we consider the decomposition

$$L^2(\mathbb{R}) \simeq L^2(\mathbb{R}_+) \oplus L^2(\mathbb{R}_-) \xrightarrow[T_+ + T_-]{\sim} L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$$

where T_+ (likewise, T_-) is defined by

$$T_+ : L^2(\mathbb{R}_+) \to L^2(\mathbb{R}), \quad f(x) \mapsto e^{\frac{\lambda}{2}t} f(e^t).$$

Then, the unitary representation $(\pi_\lambda\big|_A, L^2(\mathbb{R}_+))$ of A is unitarily equivalent to the regular representation of $A \simeq \mathbb{R}$ because

$$T_+(\pi_\lambda \begin{pmatrix} e^s & 0 \\ 0 & e^{-s} \end{pmatrix} f)(t) = (T_+ f)(t - 2s).$$

Hence, the branching law $\pi_\lambda\big|_A$ follows from the Plancherel formula for $L^2(\mathbb{R})$ as was given in Example 3.1.3 (1). □

3.3.3 Restriction of holomorphic discrete series

Without a proof, we present branching laws of holomorphic discrete series representations π_n^+ $(n = 2, 3, 4, \ldots)$ of $G = SL(2, \mathbb{R})$ with respect to its subgroups K, N and A:

Proposition 3.3.3. *Fix* $n = 2, 3, 4, \ldots$.

1) $\pi_n^+\big|_K \simeq \sum_{k=0}^{\infty}{}^{\oplus} \chi_{n+2k}$.

2) $\pi_n^+\big|_N \simeq \int_{\mathbb{R}_+}^{\oplus} \chi_\xi \, d\xi$.

3) $\pi_n^+\big|_A \simeq \int_{\mathbb{R}}^{\oplus} \chi_\xi \, d\xi$.

It is remarkable that the multiplicity is free in all of the above three cases in Proposition 3.3.3 (compare with the multiplicity 2 results in Proposition 3.3.2 (3)). This multiplicity-free result holds in more general branching laws of unitary highest weight representations (see [42, 54]).

3.3.4 Irreducibility of π_n^+ $(n = 2, 3, 4, \ldots)$

One of traditional approaches to show the irreducibility of π_n^+ is to use $(\mathfrak{g}, K)$-modules together with explicit transition coefficients of Lie algebra actions.

Aside from this, we shall explain another approach based on the restriction to non-compact subgroups N and A. This is a small example that restrictions to non-compact subgroups are also useful in studying representations of the whole group.

Suppose that W is a G-invariant closed subspace in V_n^+. In view of the branching law of the restriction to the subgroup N (Proposition 3.3.3 (3)), the representation (π_n^+, W) of N is unitarily equivalent to the direct integral

$$\int_E^{\oplus} \chi_\xi \, d\xi$$

for some measurable set E in $\mathbb{R}_+$, as a unitary representation of $N \simeq \mathbb{R}$. Furthermore, since W is invariant also by the subgroup A, E must be stable under the dilation, namely, E is either ϕ or $\mathbb{R}_+$ (up to a measure zero set). Hence, the invariant subspace W is either $\{0\}$ or V_n^+. This shows that (π_n^+, V_n^+) is already irreducible as a representation of the subgroup AN. $\square$

3.4 $\otimes$-product representations of $SL(2,\mathbb{R})$

3.4.1 Tensor product representations

Tensor product representations are a special case of restrictions, and their decompositions are a special case of branching laws. In fact, suppose π and π' are unitary representations of G. Then the outer tensor product $\pi \boxtimes \pi'$ is a unitary representation of the direct product group $G \times G$. Its restriction to the diagonally embedded subgroup G:

$$G \hookrightarrow G \times G, \quad g \mapsto (g, g)$$

gives rise to the tensor product representation $\pi \otimes \pi'$ of the group G.

Let us consider the irreducible decomposition of the tensor product representations, a special case of branching laws.

3.4.2 $\pi_\lambda \otimes \pi_{\lambda'}$ (principal series)

Proposition 3.4.2. *Let* $\operatorname{Re}\lambda = \operatorname{Re}\lambda' = 1$. *Then, the tensor product representation of two (unitary) principal series representations* π_λ *and* $\pi_{\lambda'}$ *decomposes into irreducibles of* $G = SL(2,\mathbb{R})$ *as follows:*

$$\pi_\lambda \otimes \pi_{\lambda'} \simeq \int^{\oplus}_{\substack{\operatorname{Re}\nu = 1 \\ \operatorname{Im}\nu \geq 0}} 2\pi_\nu \, d\nu + \sum_{n=1}^{\infty}{}^{\oplus} (\pi_{2n}^+ + \pi_{2n}^-).$$

A distinguishing feature here is that both continuous and discrete spectrum occur. Discrete spectrum occurs with multiplicity free, while continuous spectrum with multiplicity two.

Sketch of proof. Unlike the branching laws in §3.3, we shall use **non-commutative** harmonic analysis. That is, the decomposition of the tensor product representation reduces to the Plancherel formula for the hyperboloid. To be more precise, we divide the proof into three steps:

Step 1. The outer tensor product $\pi_\lambda \boxtimes \pi_{\lambda'}$ is realized on

$$L^2(\mathbb{R})\widehat{\otimes}L^2(\mathbb{R}) \simeq L^2(\mathbb{R}^2),$$

or equivalently, on $L^2(\mathbb{T}^2)$ via the intertwining operator $T_\lambda \otimes T_{\lambda'}$ (see (3.3.2)).

Step 2. Consider the hyperboloid of one sheet:

$$X := \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 - z^2 = 1\}$$

We identify X with the set of matrices:

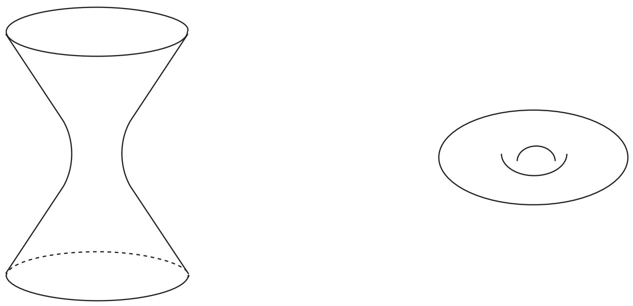
$$\{B = \begin{pmatrix} z & x+y \\ x-y & -z \end{pmatrix} : \operatorname{Trace} B = 0, \det B = 1\}.$$

Then, $G = SL(2, \mathbb{R})$ acts on X by

$$X \to X, \ B \mapsto gBg^{-1}.$$

Step 3. Embed the G-space X into an open dense subset of the $(G \times G)$-space $\mathbb{T}^2$ so that it is equivariant with respect to the diagonal homomorphism $G \hookrightarrow G \times G$. (This is a conformal embedding with respect to natural pseudo-Riemannian metrics).

$$\iota : X \qquad \hookrightarrow \qquad \mathbb{T}^2$$



Step 4. The pull-back ι^* followed by a certain twisting (depending on λ and λ') sends $L^2(\mathbb{T}^2)$ onto $L^2(X)$. The Laplace-Beltrami operator on X (a wave operator) commutes with the action of G (not by $G \times G$), and its spectral decomposition gives rise to the Plancherel formula for $L^2(X)$, or equivalently the branching law $\pi_\lambda \otimes \pi_{\lambda'}$.

For advanced readers who are already familiar with basic theory of representations of semisimple Lie groups, it would be easier to understand some of the above steps by the (abstract) Mackey theory. For example, the above embedding $\iota : X \to \mathbb{T}^2$ may be written as

$$X \simeq G/MA \simeq G/(P \cap \overline{P}) \hookrightarrow (G \times G)/(P \times \overline{P}),$$

where P and $\overline{P}$ are opposite parabolic subgroups of G such that $P \cap \overline{P} = MA := \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} : a \in \mathbb{R}^\times \right\}$.

A remaining part is to prove that the irreducible decomposition is essentially independent of λ and λ' (see [45]). See, for example, [14, 81] for the Plancherel formula for the hyperboloid $O(p,q)/O(p-1,q)$ (the above case is essentially the same with $(p,q) = (2,1)$), and articles of van den Ban, Delorme and Schlichtkrull in this volume for more general case (reductive symmetric spaces). $\square$

3.4.3 $\pi_m^+ \otimes \pi_n^+$ (holomorphic discrete series)

Proposition 3.4.3. *Let $m, n \geq 2$. Then*

$$\pi_m^+ \otimes \pi_n^+ \simeq \sum_{j=0}^{\infty}{}^{\oplus} \pi_{m+n+2j}^+$$

A distinguishing feature here is that there is no continuous spectrum (i.e. "discretely decomposable restriction"), even though it is a branching law with respect to a non-compact subgroup.

Sketch of proof. Realize $\pi_m^+ \otimes \pi_n^+$ as holomorphic functions of two variables on $\mathcal{H}_+ \times \mathcal{H}_+$. Take their restrictions to the diagonally embedded submanifold

$$\iota : \mathcal{H}_+ \hookrightarrow \mathcal{H}_+ \times \mathcal{H}_+.$$

Then the "bottom" representation π_{m+n}^+ (see irreducible summands in Proposition 3.4.3) arises. Other representations π_{m+n+2j}^+ $(j = 1, 2, \ldots)$ are obtained by taking normal derivatives with respect to the embedding ι. $\square$

Lecture 4 $(\mathfrak{g}, K)$-modules and infinitesimal discrete decomposition

4.1 Category of $(\mathfrak{g}, K)$-modules

Lecture 4 starts with a brief summary of basic results on $(\mathfrak{g}, K)$-modules. Advanced readers can skip §4.1, and go directly to §4.2 where the concept of **infinitesimal discrete decomposition** is introduced. This concept is an algebraic analog of the property "having no continuous spectrum", and is powerful in the algebraic study of admissible restrictions of unitary representations.

4.1.1 K-finite vectors

Let G be a reductive Lie group with a maximal compact subgroup K.

Suppose $(\pi, \mathcal{H})$ is a (K-)admissible representation of G on a Fréchet space. (See §2.4.6 and Remark there for the definition.) We define a subset of $\mathcal{H}$ by

$$\mathcal{H}_K := \{v \in \mathcal{H} : \dim_{\mathbb{C}} \langle K \cdot v \rangle < \infty\}.$$

Here, $\langle K \cdot v \rangle$ denotes the complex vector space spanned by $\{\pi(k)v : k \in K\}$. Elements in $\mathcal{H}_K$ are called **K-finite vectors**. Then, it turns out that $\mathcal{H}_K$ is a dense subspace of $\mathcal{H}$, and decomposes into an algebraic direct sum of irreducible K-modules:

$$\mathcal{H}_K \simeq \bigoplus_{(\sigma, V_\sigma) \in \widehat{K}} \operatorname{Hom}_K(\sigma, \pi|_K) \otimes V_\sigma \quad \text{(algebraic direct sum)}.$$

4.1.2 Underlying $(\mathfrak{g}, K)$-modules

Retain the setting as before. Suppose $(\pi, \mathcal{H})$ is a continuous representation of G on a Fréchet space $\mathcal{H}$. If $\pi|_K$ is K-admissible, then the limit

$$d\pi(X)v := \lim_{t \to 0} \frac{\pi(e^{tX})v - v}{t}$$

exists for $v \in \mathcal{H}_K$ and $X \in \mathfrak{g}$, and $d\pi(X)v$ is again an element of $\mathcal{H}_K$. Thus, $\mathfrak{g} \cup K$ acts on $\mathcal{H}_K$.

Definition 4.1.2. With this action, $\mathcal{H}_K$ is called **the underlying $(\mathfrak{g}, K)$-module** of $(\pi, \mathcal{H})$.

To axiomize “abstract $(\mathfrak{g}, K)$-modules”, we pin down the following three properties of $\mathcal{H}_K$ (here, we omit writing π or $d\pi$):

$$k \cdot X \cdot k^{-1} \cdot v = \mathrm{Ad}(k) X \cdot v \qquad (X \in \mathfrak{g}, k \in K), \tag{4.1.2)(a}$$

$$\dim_{\mathbb{C}} \langle K \cdot v \rangle < \infty, \tag{4.1.2)(b}$$

$$Xv = \frac{d}{dt}\Big|_{t=0} \frac{e^{tX} \cdot v - v}{t} \qquad (X \in \mathfrak{k}). \tag{4.1.2)(c}$$

(Of course, (4.1.2)(c) holds for $X \in \mathfrak{g}$ in the above setting.)

4.1.3 $(\mathfrak{g}, K)$-modules

Now, we forget (continuous) representations of a group G and consider only the action of $\mathfrak{g} \cup K$.

Definition (Lepowsky). We say W is a $(\mathfrak{g}, K)$-module if W is $\mathfrak{g} \cup K$-module satisfying the axioms (4.1.2)(a), (b) and (c).

The point here is that no topology is specified. Nevertheless, many of the fundamental properties of a continuous representation are preserved when passing to the underlying $(\mathfrak{g}, K)$-module. For example, irreducibility (or more generally, Jordan-Hölder series) of a continuous representation is reduced to that of $(\mathfrak{g}, K)$-modules, as the following theorem indicates:

Theorem 4.1.3. *Let $(\pi, \mathcal{H})$ be a (K-)admissible continuous representation of G on a Fréchet space $\mathcal{H}$. Then there is a lattice isomorphism between*

$$\{\textit{closed } G\textit{-invariant subspaces of } \mathcal{H}\}$$

and

$$\{\mathfrak{g}\textit{-invariant subspaces of } \mathcal{H}_K\}.$$

The correspondence is given by

$$\mathcal{V} \longmapsto \mathcal{V}_K := \mathcal{V} \cap \mathcal{H}_K,$$
$$\mathcal{V}_K \longmapsto \overline{\mathcal{V}_K} \ (\textit{closure in } \mathcal{H}).$$

In particular, $(\pi, \mathcal{H})$ is irreducible if and only if its underlying $(\mathfrak{g}, K)$-module is irreducible.

4.1.4 Infinitesimally unitary representations

Unitary representations form an important class of continuous representations. Unitarity can be also studied algebraically by using $(\mathfrak{g}, K)$-modules. Let us recall some known basic results in this direction.

A unitary representation of G gives rise to a representation of $\mathfrak{g}$ by essentially skew-adjoint operators (Segal and Mautner). Conversely, let us start with a representation of $\mathfrak{g}$ having this property:

Definition. A $(\mathfrak{g}, K)$-module W is **infinitesimally unitary** if it admits a positive definite invariant Hermitian form.

Here, by "invariant", we mean the following two conditions: for any $u, v \in W$,

$$\begin{aligned} &(Xu, v) + (u, Xv) = 0 && (X \in \mathfrak{g}), \\ &(k \cdot u, k \cdot v) = (u, v) && (k \in K). \end{aligned}$$

The point of the following theorem is that analytic objects (unitary representations of G) can be studied by algebraic objects (their underlying $(\mathfrak{g}, K)$-modules).

Theorem 4.1.4. 1) *Any irreducible infinitesimally unitary $(\mathfrak{g}, K)$-module is the underlying $(\mathfrak{g}, K)$-modules of some irreducible unitary representation of G on a Hilbert space.*

2) *Two irreducible unitary representations of G on Hilbert spaces are unitarily equivalent if and only if their underlying $(\mathfrak{g}, K)$-modules are isomorphic as $(\mathfrak{g}, K)$-modules.*

4.1.5 Scope

Our interest throughout this article focuses on the restriction of unitary representations. The above mentioned theorems suggest us to deal with restrictions of unitary representations by algebraic methods of $(\mathfrak{g}, K)$-modules. We shall see that this idea works quite successfully for admissible restrictions.

Along this line, we shall introduce the concept of infinitesimally discretely decomposable restrictions in §4.2.

4.1.6 For further reading

See [105, Chapter 4], [104, Chapter 3] for §4.1.

4.2 Infinitesimal discrete decomposition

The aim of Lecture 4 is to establish an algebraic formulation of the condition "having no continuous spectrum". We are ready to explain this notion by means of $(\mathfrak{g}, K)$-modules with some background of motivations.

4.2.1 Wiener subspace

We begin with an observation in the opposite extremal case — "having no **discrete** spectrum".

Observation 4.2.1. There is no discrete spectrum in the Plancherel formula for $L^2(\mathbb{R})$. Equivalently, there is no closed $\mathbb{R}$-invariant subspace in $L^2(\mathbb{R})$.

An easy proof for this is given simply by observing

$$e^{ix\xi} \notin L^2(\mathbb{R}) \quad \text{for any } \xi \in \mathbb{R}.$$

A less easy proof is to deduce from the following claim:

Claim. For any non-zero closed $\mathbb{R}$-invariant subspace W in $L^2(\mathbb{R})$, there exists an infinite **decreasing** sequence $\{W_j\}$ of closed $\mathbb{R}$-invariant subspaces:

$$W \supsetneqq W_1 \supsetneqq W_2 \supsetneqq \cdots .$$

Exercise. Prove that there is no discrete spectrum in the Plancherel formula for $L^2(\mathbb{R})$ by using the above claim.

Sketch of the proof of Claim. Let W be a closed $\mathbb{R}$-invariant subspace in $L^2(\mathbb{R})$ (a **Wiener subspace**). Then, one can find a measurable subset E of $\mathbb{R}$ such that

$$W = \mathcal{F}(L^2(E)),$$

that is, the image of the Fourier transform $\mathcal{F}$ of square integrable functions supported on E. Then take a decreasing sequence of measurable sets

$$E \supsetneqq E_1 \supsetneqq E_2 \supsetneqq \cdots$$

and put $W_j := \mathcal{F}(L^2(E_j))$. $\square$

4.2.2 Discretely decomposable modules

Let us consider an opposite extremal case, namely, the property "having no continuous spectrum". We shall introduce a notion of this nature for representations of Lie algebras as follows.

Definition 4.2.2. Let $\mathfrak{g}'$ be a Lie algebra, and X a $\mathfrak{g}'$-module. We say X is **discretely decomposable as a $\mathfrak{g}'$-module** if there is an **increasing** sequence $\{X_j\}$ of $\mathfrak{g}'$-modules satisfying both (1) and (2):

(1) $X = \displaystyle\bigcup_{j=0}^{\infty} X_j$.

(2) X_j is of finite length as a $\mathfrak{g}'$-module for any j.

Here, we note that X_j is not necessarily infinite dimensional.

4.2.3 Infinitesimally discretely decomposable representations

Let us turn to representations of Lie groups.

Suppose $G \supset G'$ is a pair of reductive Lie groups with maximal compact subgroups $K \supset K'$, respectively. Let $(\pi, \mathcal{H})$ be a (K-)admissible representation of G.

Definition 4.2.3. The restriction $\pi|_{G'}$ is **infinitesimally discretely decomposable** if the underlying $(\mathfrak{g}, K)$-module $(\pi_K, \mathcal{H}_K)$ is discretely decomposable as a $\mathfrak{g}'$-module (Definition 4.2.2).

4.2.4 Examples

1) It is always the case if G' is compact, especially if $G' = \{e\}$.

2) Let π_λ ($\lambda \in 1 + \sqrt{-1}\mathbb{R}$) be a principal series representation of $G = SL(2, \mathbb{R})$ (see §3.3.2). Then, the restriction $\pi_\lambda\big|_K$ is infinitesimally discretely decomposable, while the restriction $\pi_\lambda\big|_A$ is not infinitesimally discretely decomposable. See also explicit branching laws given in Proposition 3.3.2.

4.2.5 Unitary case

So far, we have not assumed the unitarity of π. The terminology "discretely decomposable" fits well if π is unitary, as is seen in the following theorem.

Theorem 4.2.5. *Let $G \supset G'$ be a pair of reductive Lie groups with maximal compact subgroups $K \supset K'$, respectively. Suppose $(\pi, \mathcal{H}) \in \widehat{G}$. Then the following three conditions on the triple (G, G', π) are equivalent:*

i) *The restriction $\pi|_{G'}$ is infinitesimally discretely decomposable.*

ii) *The underlying $(\mathfrak{g}, K)$-module $(\pi_K, \mathcal{H}_K)$ decomposes into an algebraic direct sum of irreducible $(\mathfrak{g}', K')$-modules:*

$$\pi_K \simeq \bigoplus_Y n_\pi(Y) Y \qquad \textit{(algebraic direct sum)},$$

where Y runs over all irreducible $(\mathfrak{g}', K')$-modules and we have defined

$$n_\pi(Y) := \dim \operatorname{Hom}_{\mathfrak{g}', K'}(Y, \mathcal{H}_K) \in \mathbb{N} \cup \{\infty\}. \tag{4.2.5}$$

iii) *There exists an irreducible $(\mathfrak{g}', K')$-module Y such that $n_\pi(Y) \neq 0$.*

Sketch of proof.

(ii) $\Rightarrow$ (i) $\Rightarrow$ (iii) : Obvious.

(i) $\Rightarrow$ (ii) : Use the assumption that π is unitary.

(iii) $\Rightarrow$ (i) : Use the assumption that π is irreducible.

(ii) $\Rightarrow$ (i): If X is decomposed into the algebraic direct sum of irreducible $(\mathfrak{g}', K')$-modules, say $\bigoplus_{i=0}^{\infty} Y_i$, then we put $X_j := \bigoplus_{i=0}^{j} Y_i$ $(j = 0, 1, 2, \ldots)$.

□

We end this section with two important theorems without proof (their proof is not very difficult).

4.2.6 Infinitesimal $\Rightarrow$ analytic discrete decomposability

For $\pi \in \widehat{G}$ and $\sigma \in \widehat{G'}$, we define

$$m_\pi(\sigma) := \dim \operatorname{Hom}_{G'}(\sigma, \pi|_{G'}),$$

the dimension of continuous G'-intertwining operators. Then, in general, the following inequality holds;

$$n_\pi(\sigma_{K'}) \leq m_\pi(\sigma).$$

This inequality becomes an equality if the restriction $\pi|_{G'}$ is infinitesimally discretely decomposable. More precisely, we have:

Theorem 4.2.6. *In the setting of Theorem 4.2.5, if one of (therefore, any of) the three equivalent conditions is satisfied, then the restriction $\pi|_{G'}$ is (analytically) discretely decomposable (Definition 2.2.1), that is, we have an equivalence of unitary representations of G':*

$$\pi|_{G'} \simeq \sum_{\sigma \in \widehat{G'}}^{\oplus} m_\pi(\sigma)\sigma \ \textit{(discrete Hilbert sum).}$$

Furthermore, the above multiplicity $m_\pi(\sigma)$ coincides with the algebraic multiplicity $n_\pi(\sigma_{K'})$ given in (4.2.5).

It is an open problem (see [48, Conjecture D]), whether (analytically) discretely decomposability implies infinitesimally discrete decomposability.

4.2.7 K'-admissibility $\Rightarrow$ infinitesimally discrete decomposability

Theorem 4.2.7. *Retain the setting of Theorem 4.2.5. If π is K'-admissible, then the restriction $\pi|_{G'}$ is infinitesimally discretely decomposable.*

4.2.8 For further reading

Materials of §4.2 are taken from the author's papers [44] and [48]. See also [47] and [49] for related topics.

Lecture 5 Algebraic theory of discretely decomposable restrictions

The goal of this section is to introduce associated varieties to the study of infinitesimally discretely decomposable restrictions.

5.1 Associated varieties

Associated varieties give a coarse approximation of modules of Lie algebras. This subsection summarizes quickly some known results on associated varieties (see Vogan's treatise [98] for more details).

5.1.1 Graded modules

Let V be a finite dimensional vector space over $\mathbb{C}$. We use the following notation:

$$\begin{aligned}
&V^* : \text{ the dual vector space of } V.\\
&S(V) = \bigoplus_{k=0}^{\infty} S^k(V) : \text{ the symmetric algebra of } V.\\
&S_k(V) := \bigoplus_{j=0}^{k} S^j(V).
\end{aligned}$$

Let $M = \bigoplus_{k=0}^{\infty} M_k$ be a finitely generated $S(V)$-module. We say M is a **graded $S(V)$-module** if

$$S^i(V) M_j \subset M_{i+j} \quad \text{for any } i, j.$$

The annihilator $\mathrm{Ann}_{S(V)}(M)$ is an ideal of $S(V)$ defined by

$$\mathrm{Ann}_{S(V)}(M) := \{f \in S(V) : f \cdot u = 0 \quad \text{for any } u \in M\}.$$

Then, $\mathrm{Ann}_{S(V)}(M)$ is a homogeneous ideal, namely,

$$\mathrm{Ann}_{S(V)}(M) = \bigoplus_{k=0}^{\infty} (\mathrm{Ann}_{S(V)}(M) \cap S^k(V)),$$

and thus,

$$\mathrm{Supp}_{S(V)}(M) := \{\lambda \in V^* : f(\lambda) = 0 \quad \text{for any } f \in \mathrm{Ann}_{S(V)}(M)\}$$

is a closed cone in V^*. Hence, we have defined a functor

$$\{\text{graded } S(V)\text{-modules}\} \rightsquigarrow \{\text{closed cones in } V^*\}$$
$$M \mapsto \operatorname{Supp}_{S(V)}(M).$$

5.1.2 Associated varieties of $\mathfrak{g}$-modules

Let $\mathfrak{g}_{\mathbb{C}}$ be a Lie algebra over $\mathbb{C}$. For each integer $n \geq 0$, we define a subspace $U_n(\mathfrak{g}_{\mathbb{C}})$ of the enveloping algebra $U(\mathfrak{g}_{\mathbb{C}})$ of the Lie algebra $\mathfrak{g}_{\mathbb{C}}$ by

$$U_n(\mathfrak{g}_{\mathbb{C}}) := \mathbb{C}\text{-span}\{Y_1 \cdots Y_k \in U(\mathfrak{g}_{\mathbb{C}}) : Y_1, \cdots, Y_k \in \mathfrak{g}_{\mathbb{C}}, k \leq n\}.$$

It follows from definition that

$$U(\mathfrak{g}_{\mathbb{C}}) = \bigcup_{n=0}^{\infty} U_n(\mathfrak{g}_{\mathbb{C}}),$$
$$\mathbb{C} = U_0(\mathfrak{g}_{\mathbb{C}}) \subset U_1(\mathfrak{g}_{\mathbb{C}}) \subset U_2(\mathfrak{g}_{\mathbb{C}}) \subset \cdots .$$

Then, the graded ring

$$\operatorname{gr} U(\mathfrak{g}_{\mathbb{C}}) := \bigoplus_{k=0}^{\infty} U_k(\mathfrak{g}_{\mathbb{C}})/U_{k-1}(\mathfrak{g}_{\mathbb{C}})$$

is isomorphic to the symmetric algebra

$$S(\mathfrak{g}_{\mathbb{C}}) = \bigoplus_{k=0}^{\infty} S^k(\mathfrak{g}_{\mathbb{C}})$$

by the Poincaré-Birkhoff-Witt theorem. The point here is that the Lie algebra structure on $\mathfrak{g}_{\mathbb{C}}$ is forgotten in the graded ring $\operatorname{gr} U(\mathfrak{g}_{\mathbb{C}}) \simeq S(\mathfrak{g}_{\mathbb{C}})$.

Suppose X is a finitely generated $\mathfrak{g}_{\mathbb{C}}$-module. We fix its generators $v_1, \cdots, v_m \in X$, and define a filtration $\{X_j\}$ of X by

$$X_j := \sum_{i=1}^{m} U_j(\mathfrak{g}_{\mathbb{C}}) v_i.$$

Then the graded module $\operatorname{gr} X := \bigoplus_{j=0}^{\infty} X_j/X_{j-1}$ becomes naturally a finitely generated graded module of $\operatorname{gr} U(\mathfrak{g}_{\mathbb{C}}) \simeq S(\mathfrak{g}_{\mathbb{C}})$. Thus, as in §5.1.1, we can define its support by

$$\mathcal{V}_{\mathfrak{g}_{\mathbb{C}}}(X) := \operatorname{Supp}_{S(\mathfrak{g}_{\mathbb{C}})}(\operatorname{gr} X).$$

It is known that the variety $\mathcal{V}_{\mathfrak{g}_{\mathbb{C}}}(X)$ is independent of the choice of generators $v_1, \cdots, v_m$ of X.

Definition 5.1.2. We say $\mathcal{V}_{\mathfrak{g}_\mathbb{C}}(X)$ is the **associated variety** of a $\mathfrak{g}_\mathbb{C}$-module X. Its complex dimension is called the **Gelfand-Kirillov dimension**, denote by $\mathrm{Dim}(X)$.

By definition, the associated variety $\mathcal{V}_{\mathfrak{g}_\mathbb{C}}(X)$ is a closed cone in $\mathfrak{g}_\mathbb{C}^*$. For a reductive Lie algebra, we shall identify $\mathfrak{g}_\mathbb{C}^*$ with $\mathfrak{g}_\mathbb{C}$, and thus regard $\mathcal{V}_{\mathfrak{g}_\mathbb{C}}(X)$ as a subset of $\mathfrak{g}_\mathbb{C}$.

5.1.3 Associated varieties of G-representations

So far, we have considered a representation of a Lie algebra. Now, we consider the case where X comes from a continuous representation of a reductive Lie group G. The scheme is :

$$\begin{array}{cl}
(\pi, \mathcal{H}) & : \text{an admissible representation of } G \text{ of finite length,} \\
\Downarrow & \\
(\pi_K, \mathcal{H}_K) & : \text{its underlying } (\mathfrak{g}, K)\text{-module,} \\
\Downarrow & \\
\mathcal{H}_K & : \text{regarded as a } U(\mathfrak{g}_\mathbb{C})\text{-module,} \\
\Downarrow & \\
\operatorname{gr}\mathcal{H}_K & : \text{its graded module (an } S(\mathfrak{g}_\mathbb{C})\text{-module),} \\
\Downarrow & \\
\mathcal{V}_{\mathfrak{g}_\mathbb{C}}(\mathcal{H}_K) & : \text{the associated variety (a subset of } \mathfrak{g}_\mathbb{C}^*).
\end{array}$$

5.1.4 Nilpotent cone

Let $\mathfrak{g}$ be the Lie algebra of a reductive Lie group G. We define the **nilpotent cone** by

$$\mathcal{N}_{\mathfrak{g}_\mathbb{C}} := \{H \in \mathfrak{g}_\mathbb{C} : \mathrm{ad}(H) \text{ is a nilpotent endomorphism}\}.$$

Then, $\mathcal{N}_{\mathfrak{g}_\mathbb{C}}$ is an $\mathrm{Ad}(G_\mathbb{C})$-invariant closed cone in $\mathfrak{g}_\mathbb{C}$.

Let $\mathfrak{g}_\mathbb{C} = \mathfrak{k}_\mathbb{C} + \mathfrak{p}_\mathbb{C}$ be the complexification of a Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$. We take a connected complex Lie group $K_\mathbb{C}$ with Lie algebra $\mathfrak{k}_\mathbb{C}$.

Theorem 5.1.4 ([98]). *If $(\pi, \mathcal{H}) \in \widehat{G}$, then its associated variety $\mathcal{V}_{\mathfrak{g}_\mathbb{C}}(\mathcal{H}_K)$ is a $K_\mathbb{C}$-invariant closed subset of*

$$\mathcal{N}_{\mathfrak{p}_\mathbb{C}} := \mathcal{N}_{\mathfrak{g}_\mathbb{C}} \cap \mathfrak{p}_\mathbb{C}.$$

Theorem 5.1.4 holds in a more general setting where π is a (K-)admissible (non-unitary) representation of finite length.

Sketch of proof. Here are key ingredients:

1) The center $Z(\mathfrak{g}_{\mathbb{C}})$ of $U(\mathfrak{g}_{\mathbb{C}})$ acts on $\mathcal{H}_K$ as scalars $\Rightarrow \mathcal{V}_{\mathfrak{g}_{\mathbb{C}}}(X) \subset \mathcal{N}_{\mathfrak{g}_{\mathbb{C}}}$.

2) K acts on $\mathcal{H}_K \Rightarrow \mathcal{V}_{\mathfrak{g}_{\mathbb{C}}}(X)$ is $K_{\mathbb{C}}$-invariant.

3) $\mathcal{H}_K$ is locally K-finite $\Rightarrow \mathcal{V}_{\mathfrak{g}_{\mathbb{C}}}(X) \subset \mathfrak{p}_{\mathbb{C}}$.

□

In [60], Kostant and Rallis studied the $K_{\mathbb{C}}$-action on the nilpotent cone $\mathcal{N}_{\mathfrak{p}_{\mathbb{C}}}$ and proved that the number of $K_{\mathbb{C}}$-orbits on $\mathcal{N}_{\mathfrak{p}_{\mathbb{C}}}$ is finite ([60, Theorem 2]; see also [8, 89]). Therefore, there are only a finitely many possibilities of the associated varieties $\mathcal{V}_{\mathfrak{g}_{\mathbb{C}}}(\mathcal{H}_K)$ for admissible representations $(\pi, \mathcal{H})$ of G. As an illustrative example, we shall give an explicit combinatorial description of all $K_{\mathbb{C}}$-orbits on $\mathcal{N}_{\mathfrak{p}_{\mathbb{C}}}$ in §5.2 for $G = U(2,2)$ case.

5.2 Restrictions and associated varieties

This subsection presents the behavior of associated varieties with respect to infinitesimally discretely decomposable restrictions.

5.2.1 Associated varieties of irreducible summands

Let $\mathfrak{g}$ be a reductive Lie algebra, and $\mathfrak{g}'$ its subalgebra which is reductive in $\mathfrak{g}$. This is the case if $\mathfrak{g}' \subset \mathfrak{g} \subset \mathfrak{gl}(n, \mathbb{R})$ are both stable under the Cartan involution $X \mapsto -{}^tX$.

We write

$$\mathrm{pr}_{\mathfrak{g}\to\mathfrak{g}'} : \mathfrak{g}_{\mathbb{C}}^* \to (\mathfrak{g}'_{\mathbb{C}})^*$$

for the natural projection dual to $\mathfrak{g}'_{\mathbb{C}} \hookrightarrow \mathfrak{g}_{\mathbb{C}}$. Suppose X is an irreducible $\mathfrak{g}$-module, and Y is an irreducible $\mathfrak{g}'$-module. Then, we can define their associated varieties in $\mathfrak{g}_{\mathbb{C}}^*$ and $(\mathfrak{g}'_{\mathbb{C}})^*$, respectively. Let us compare them in the following diagram.

$$\begin{array}{ccc} \mathcal{V}_{\mathfrak{g}_{\mathbb{C}}}(X) & \subset & \mathfrak{g}_{\mathbb{C}}^* \\ & & \big\downarrow {\scriptstyle \mathrm{pr}_{\mathfrak{g}\to\mathfrak{g}'}} \\ \mathcal{V}_{\mathfrak{g}'_{\mathbb{C}}}(Y) & \subset & (\mathfrak{g}'_{\mathbb{C}})^* \end{array}$$

Theorem 5.2.1. *If* $\mathrm{Hom}_{\mathfrak{g}'}(Y, X) \neq \{0\}$, *then*

$$\mathrm{pr}_{\mathfrak{g}\to\mathfrak{g}'}(\mathcal{V}_{\mathfrak{g}_{\mathbb{C}}}(X)) \subset \mathcal{V}_{\mathfrak{g}'_{\mathbb{C}}}(Y). \tag{5.2.1}$$

Sketch of proof. In order to compare two associated varieties $\mathcal{V}_{\mathfrak{g}_{\mathbb{C}}}(X)$ and $\mathcal{V}_{\mathfrak{g}'_{\mathbb{C}}}(Y)$, we shall take a double filtration as follows: First take an $\mathrm{ad}(\mathfrak{g}')$-invariant complementary subspace $\mathfrak{q}$ of $\mathfrak{g}'$ in $\mathfrak{g}$:

$$\mathfrak{g} = \mathfrak{g}' \oplus \mathfrak{q}.$$

Choose a finite dimensional vector space $F \subset Y$, and we define a subspace of X by

$$X_{ij} := \mathbb{C}\text{-span}\left\{ A_1 \cdots A_p B_1 \cdots B_q v : \begin{array}{l} A_1, \cdots, A_p \in \mathfrak{g}\ (p \leq i), \\ B_1, \cdots, B_q \in \mathfrak{g}'\ (q \leq j), \\ v \in F \end{array} \right\}.$$

Then

$$\begin{array}{ll} \left\{ \bigoplus_{i+j\leq N} X_{i,j} \right\}_N & \text{is a filtration of } X, \\ \{X_{0,j}\}_j & \text{is a filtration of } Y, \end{array}$$

through which we can define and compare the associated varieties $\mathcal{V}_{\mathfrak{g}_{\mathbb{C}}}(X)$ and $\mathcal{V}_{\mathfrak{g}'_{\mathbb{C}}}(Y)$. This gives rise to (5.2.1). $\square$

Remark 5.1. The above approach based on double filtration is taken from [44, Theorem 3.1]; Jantzen ([30, page 119]) gave an alternative proof to Theorem 5.2.1.

5.2.2 G': compact case

Let us apply Theorem 5.2.1 to a very special case, namely, $G' = K' = K$ (a maximal compact subgroup).

Obviously, there exists a (finite dimensional) irreducible representation of K such that $\mathrm{Hom}_{\mathfrak{k}'}(Y, X) \neq \{0\}$. Since Y is finite dimensional, we have $\dim Y < \infty$, and therefore $\mathcal{V}_{\mathfrak{k}'_{\mathbb{C}}}(Y) = \{0\}$. Then Theorem 5.2.1 means that

$$\mathrm{pr}_{\mathfrak{g}\to\mathfrak{k}'}(\mathcal{V}_{\mathfrak{g}_{\mathbb{C}}}(X)) = \{0\},$$

or equivalently,

$$\mathcal{V}_{\mathfrak{g}_{\mathbb{C}}}(X) \subset \mathfrak{p}_{\mathbb{C}} \tag{5.2.2}$$

if we identify $\mathfrak{g}_{\mathbb{C}}^*$ with $\mathfrak{g}_{\mathbb{C}}$. This inclusion (5.2.2) is the one given in Theorem 5.1.4.

5.2.3 Criterion for infinitesimally discretely decomposable restrictions

For a non-compact G', Theorem 5.2.1 leads us to a useful criterion for infinitesimal discrete decomposability by means of associated varieties:

Corollary 5.2.3. *Let $(\pi, \mathcal{H}) \in \widehat{G}$, and $G \supset G'$ be a pair of reductive Lie groups. If the restriction $\pi|_{G'}$ is infinitesimally discretely decomposable, then*

$$\mathrm{pr}_{\mathfrak{g}\to\mathfrak{g}'}(\mathcal{V}_{\mathfrak{g}_{\mathbb{C}}}(\mathcal{H}_K)) \subset \mathcal{N}_{\mathfrak{g}'_{\mathbb{C}}}.$$

Here, $\mathcal{N}_{\mathfrak{g}'_{\mathbb{C}}}$ is the nilpotent cone of $\mathfrak{g}'_{\mathbb{C}}$.

Proof. Corollary readily follows from Theorem 5.2.1 and Theorem 5.1.4. $\square$

The converse statement also holds if (G, G') is a reductive symmetric pair. That is, if $\mathrm{pr}_{\mathfrak{g}\to\mathfrak{g}'}(\mathcal{V}_{\mathfrak{g}_{\mathbb{C}}}(\mathcal{H}_K))$ is contained in the nilpotent cone $\mathcal{N}_{\mathfrak{g}'_{\mathbb{C}}}$, then the restriction $\pi|_{G'}$ is infinitesimally discretely decomposable. This result was formulated and proved in [44] in the case where π_K is of the form $A_{\mathfrak{q}}(\lambda)$ (e.g. discrete series representations). A general case was conjectured in [48, Conjecture B], and Huang and Vogan announced recently an affirmative solution to it.

See Remark 8.1 after Theorem 8.2.2 for further illuminating examples of associated varieties regarding the theta correspondence for reductive dual pairs.

5.3 Examples

In this section, we examine Corollary 5.2.3 for the pair of reductive groups (G, G_i) $(i = 1, 2, 3)$:

$$\begin{array}{ccl} & & G_1 = Sp(1,1) \\ & \rotatebox{45}{$\supset$} & \\ G = U(2,2) & \supset & G_2 = U(2,1) \times U(1) \\ & \rotatebox{-45}{$\supset$} & \\ & & G_3 = U(2) \times U(2) \quad (= K). \end{array}$$

The goal of this subsection is to classify all $K_{\mathbb{C}}$-orbits in the nilpotent variety $\mathcal{N}_{\mathfrak{p}_{\mathbb{C}}}$ corresponding to the conditions in Corollary 5.2.3 for infinitesimally discretely decomposable restrictions with respect to the subgroups G_i $(i = 1, 2, 3)$. (The case $i = 3$ is trivial.)

5.3.1 Strategy

We divide into two steps:

Step 1. Classify $K_{\mathbb{C}}$-orbits on $\mathcal{N}_{\mathfrak{p}_{\mathbb{C}}}$ (§5.3.2).

Step 2. List all $K_{\mathbb{C}}$-orbits $\mathcal{O}$ on $\mathcal{N}_{\mathfrak{p}_{\mathbb{C}}}$ such that $\mathrm{pr}_{\mathfrak{g}\to\mathfrak{g}_i}(\mathcal{O}) \subset \mathcal{N}_{\mathfrak{p}_{i\mathbb{C}}}$ (§5.3.4, §5.3.5, §5.3.6).

5.3.2 $K_{\mathbb{C}}$-orbits on $\mathcal{N}_{\mathfrak{p}_{\mathbb{C}}}$

For $G = U(2,2)$, let us write down explicitly all $K_{\mathbb{C}}$-orbits on $\mathcal{N}_{\mathfrak{g}_{\mathbb{C}}}$ by using elementary linear algebra. Relation to representation theory will be discussed in §5.3.3.

We realize in matrices the group $G = U(2,2)$ as

$$\{g \in GL(4,\mathbb{C}) : \overline{{}^t g}\ I_{2,2} g = I_{2,2}\},$$

where $I_{2,2} := \mathrm{diag}(1,1,-1,-1)$. We shall identify

$$\begin{aligned} K_{\mathbb{C}} &\simeq GL(2,\mathbb{C}) \times GL(2,\mathbb{C}), \quad \begin{pmatrix} g_1 & O \\ O & g_2 \end{pmatrix} \leftrightarrow (g_1, g_2), \\ \mathfrak{p}_{\mathbb{C}} &\simeq M(2,\mathbb{C}) \oplus M(2,\mathbb{C}), \quad \begin{pmatrix} O & A \\ B & O \end{pmatrix} \leftrightarrow (A, B). \end{aligned}$$

Then the adjoint action of $K_{\mathbb{C}}$ on $\mathfrak{p}_{\mathbb{C}}$ is given by

$$(A, B) \mapsto (g_1 A g_2^{-1}, g_2 B g_1^{-1}) \tag{5.3.2}$$

for $(g_1, g_2) \in GL(2,\mathbb{C}) \times GL(2,\mathbb{C})$.

Furthermore, the nilpotent cone $\mathcal{N}_{\mathfrak{p}_{\mathbb{C}}} = \mathcal{N}_{\mathfrak{g}_{\mathbb{C}}} \cap \mathfrak{p}_{\mathbb{C}}$ is given by

$$\{(A,B) \in M(2,\mathbb{C}) \oplus M(2,\mathbb{C}) : \text{Both } AB \text{ and } BA \text{ are nilpotent matrices}\}.$$

We define a subset (possibly an empty set) of $\mathcal{N}_{\mathfrak{p}_{\mathbb{C}}}$ for $0 \le i,j,k,l \le 2$ by

$$\begin{aligned} \mathcal{O}_{\substack{i\,j\\k\,l}} := \{(A,B) \in \mathcal{N}_{\mathfrak{p}_{\mathbb{C}}} : \ & \operatorname{rank} A = i, \operatorname{rank} B = j, \\ & \operatorname{rank} AB = k, \operatorname{rank} BA = l\}. \end{aligned}$$

Since $\operatorname{rank} A$, $\operatorname{rank} B$, $\operatorname{rank} AB$, and $\operatorname{rank} BA$ are invariants of the $K_{\mathbb{C}}$-action (5.3.2), all $\mathcal{O}_{\substack{i\,j\\k\,l}}$ are $K_{\mathbb{C}}$-invariant sets. Furthermore, the following proposition holds:

Proposition 5.3.2. 1) *Each* $\mathcal{O}_{\substack{i\,j\\k\,l}}$ *is a single* $K_{\mathbb{C}}$*-orbit if it is not empty.*

2) *The nilpotent cone* $\mathcal{N}_{\mathfrak{p}_{\mathbb{C}}}$ *splits into* 10 $K_{\mathbb{C}}$*-orbits. These orbits together with their dimensions are listed in the following diagram:*

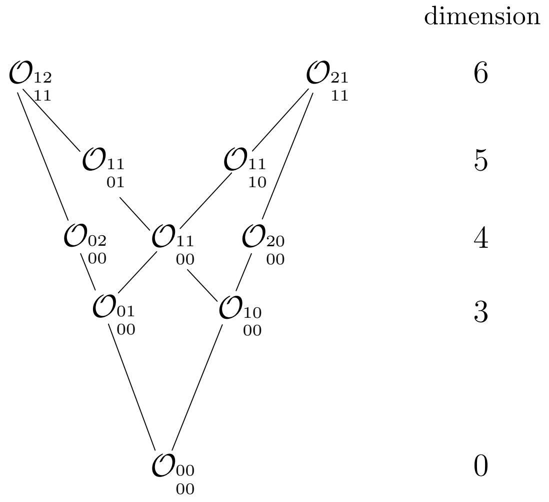


Figure 5.3.2

Here, $\begin{array}{c}\mathcal{O}'\\ |\\ \mathcal{O}''\end{array}$ *stands for the closure relation* $\overline{\mathcal{O}'} \supset \mathcal{O}''$.

There is also known a combinatorial description of nilpotent orbits of classical reductive Lie groups. For example, nilpotent orbits in $\mathfrak{u}(p,q)$ are parameterized by **signed Young diagrams** ([8, Theorem 9.3.3]). For the convenience of readers, let us illustrate briefly it by the above example.

A signed Young diagram of signature (p,q) is a Young diagram in which every box is labeled with a $+$ or $-$ sign with the following two properties:

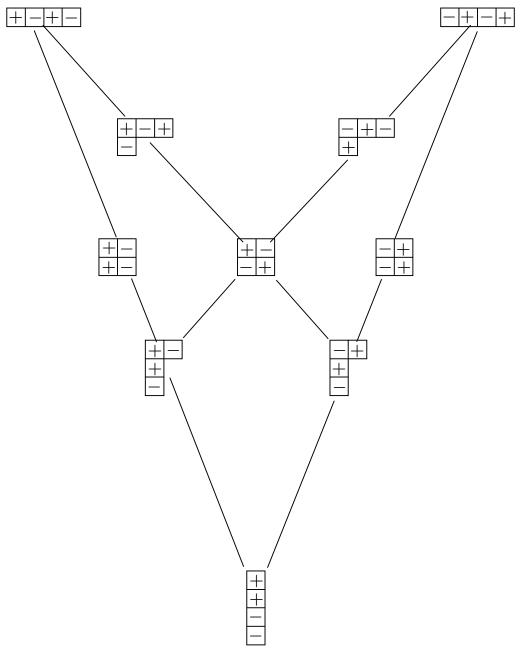
1) The number of boxes labeled $+$ is p, while that of boxes labeled $-$ is q.

2) Signs alternate across rows (they do not need to alternate down columns).

Two such signed diagrams are regarded as equivalent if and only if one can be obtained by interchanging rows of equal length. For a real reductive Lie group G, the number of $K_{\mathbb{C}}$-orbits on $\mathcal{N}_{\mathfrak{p}_{\mathbb{C}}}$ is finite. Furthermore, there is a bijection (the Kostant-Sekiguchi correspondence [89])

$$\{K_{\mathbb{C}}\text{-orbits on } \mathcal{N}_{\mathfrak{p}_{\mathbb{C}}}\} \leftrightarrow \{G\text{-orbits on } \mathcal{N}_{\mathfrak{g}}\}.$$

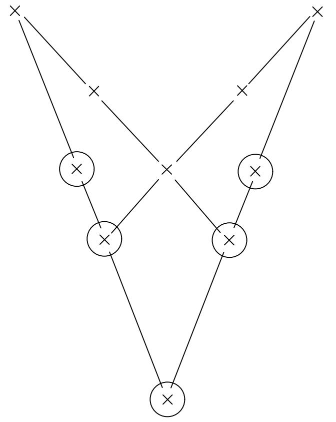
Therefore, $K_{\mathbb{C}}$-orbits on $\mathcal{N}_{\mathfrak{p}_{\mathbb{C}}}$ are also parameterized by signed Young diagrams. For example, Figure 5.3.2 may be written as



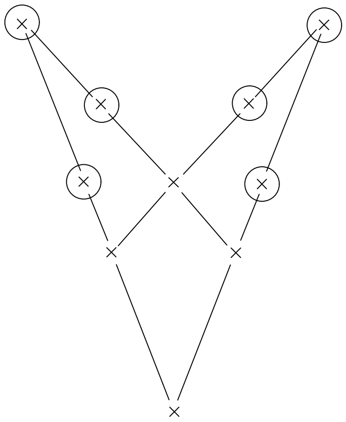
5.3.3 Meanings from representation theory

We have given a combinatorial description of all $K_{\mathbb{C}}$-orbits on $\mathcal{N}_{\mathfrak{p}_{\mathbb{C}}}$. The goal of this subsection is to provide (without proof) a list of the orbits which are realized as the associated varieties of specific representations of $G = U(2,2)$.

1) $\overline{\mathcal{O}_{\substack{i\,j\\k\,l}}}$ is the associated variety of some highest (or lowest) weight module of G if and only if $i = 0$ or $j = 0$, namely, $\mathcal{O}_{\substack{i\,j\\k\,l}}$ corresponds to one of the circled points below:



2) $\overline{\mathcal{O}_{\substack{i\,j\\k\,l}}}$ is the associated variety of some (Harish-Chandra's) discrete series representations of G (i.e. irreducible unitary representations that can be realized in $L^2(G)$) if and only if $\mathcal{O}_{\substack{i\,j\\k\,l}}$ corresponds to one of the circled points below:



To see this, we recall the work of Beĭlinson and Bernstein that realizes irreducible $(\mathfrak{g}, K)$-modules with regular infinitesimal characters by using $\mathcal{D}$-modules on the flag variety of $G_{\mathbb{C}} \simeq GL(4, \mathbb{C})$. The $\mathcal{D}$-module that corresponds to a discrete series representation is supported on a closed $K_{\mathbb{C}}$-orbit $\mathcal{W}$, and its associated variety is given by the image of the moment map of the conormal bundle of $\mathcal{W}$ ([4]). A combinatorial description of this map for some classical groups may be found in [109]. For the case of $G = U(2, 2)$, compare [41, Example 3.7] for the list of all $K_{\mathbb{C}}$-orbits on the flag variety of

$G_{\mathbb{C}}$ with [44, §7] (a more detailed explanation of Figure 5.3.2) for those of the image of the moment map of the conormal bundles.

5.3.4 Case $(G, G_1) = (U(2,2), Sp(1,1))$ (essentially, $(SO(4,2), SO(4,1))$)

The goal of this subsection is to classify the $K_{\mathbb{C}}$-orbits $\mathcal{O}_{\substack{i\,j\\k\,l}}$ on $\mathcal{N}_{\mathfrak{p}_{\mathbb{C}}}$ such that $\mathrm{pr}_{\mathfrak{g}\to\mathfrak{g}_1}(\mathcal{O}_{\substack{i\,j\\k\,l}})$ is contained in the nilpotent cone of $\mathfrak{g}_{1\mathbb{C}}$, in the case $G_1 = Sp(1,1) \simeq Spin(4,1)$.

Let $\mathfrak{g}_1 = \mathfrak{k}_1 + \mathfrak{p}_1$ be a Cartan decomposition of $\mathfrak{g}_1$, and we shall identify $\mathfrak{p}_{1\mathbb{C}}$ with $M(2,\mathbb{C})$.

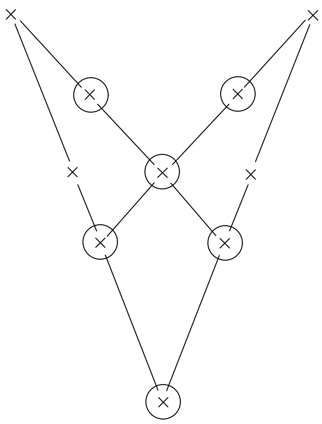
The projection $\mathrm{pr}_{\mathfrak{g}\to\mathfrak{g}_1} : \mathfrak{g}_{\mathbb{C}} \to \mathfrak{g}_{1\mathbb{C}}$ when restricted to $\mathfrak{p}_{\mathbb{C}} \simeq M(2,\mathbb{C}) \oplus M(2,\mathbb{C})$ is given by

$$\begin{array}{rccc} \mathrm{pr}_{\mathfrak{g}\to\mathfrak{g}_1} : & M(2,\mathbb{C}) \oplus M(2,\mathbb{C}) & \to & M(2,\mathbb{C}), \\ & (A, B) & \mapsto & A + \tau B, \end{array}$$

where $\tau \begin{pmatrix} a & b \\ c & d \end{pmatrix} := \begin{pmatrix} -d & b \\ c & -a \end{pmatrix}$.

Then we have

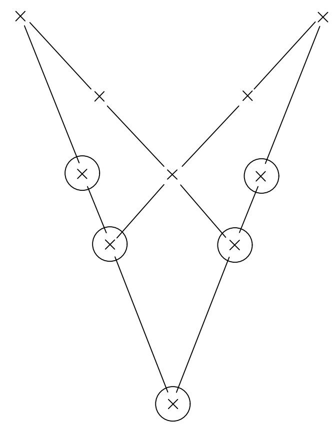
Proposition 5.3.4. $\mathrm{pr}_{\mathfrak{g}\to\mathfrak{g}_1}(\mathcal{O}_{\substack{i\,j\\k\,l}}) \subset \mathcal{N}_{\mathfrak{p}_{1\mathbb{C}}}$ *if and only if* $0 \le i, j, k, l \le 1$, *namely,* $\mathcal{O}_{\substack{i\,j\\k\,l}}$ *corresponds to one of the circled points below:*



5.3.5 Case $(G, G_2) = (U(2,2), U(2,1) \times U(1))$

We write down the result without proof:

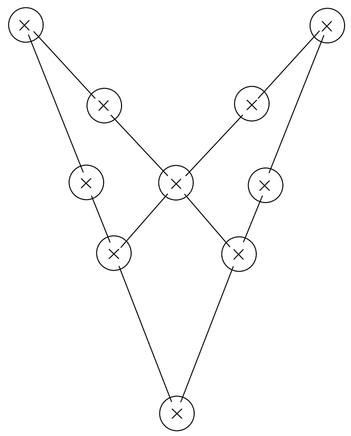
Proposition 5.3.5. *Let $\mathcal{O}_{\substack{i\,j\\k\,l}}$ be a $K_{\mathbb{C}}$-orbit on $\mathcal{N}_{\mathfrak{p}_{\mathbb{C}}}$. Then, $\mathrm{pr}_{\mathfrak{g}\to\mathfrak{g}_2}(\mathcal{O}_{\substack{i\,j\\k\,l}}) \subset \mathcal{N}_{\mathfrak{p}_{2\mathbb{C}}}$ if and only if $l = 0$, namely, $\mathcal{O}_{\substack{i\,j\\k\,l}}$ corresponds to one of the circled points below:*



5.3.6 Case $(G, G_3) = (U(2,2), U(2) \times U(2))$

Since G_3 is compact, we have obviously

Proposition 5.3.6. $\mathrm{pr}_{\mathfrak{g}\to\mathfrak{g}_3}(\mathcal{O}_{\substack{i\,j\\k\,l}}) \subset \mathcal{N}_{\mathfrak{p}_{3\mathbb{C}}}$ *for any $K_{\mathbb{C}}$-orbit $\mathcal{O}_{\substack{i\,j\\k\,l}}$ in $\mathcal{N}_{\mathfrak{p}_{\mathbb{C}}}$, namely $\mathcal{O}_{\substack{i\,j\\k\,l}}$ is one of the circled points below:*



It would be an interesting exercise to compare the above circled points with those in §5.3.3 (meaning from representation theory).

Lecture 6 Admissible restriction and microlocal analysis

The sixth lecture tries to explain how the idea coming from microlocal analysis leads to a criterion for admissibility of restrictions of unitary representations. Main results are given in §6.3, for which the key ideas of the proof are explained in §6.1. Basic concepts such as asymptotic K-support are introduced in §6.2. They will play a crucial role in the main results of §6.3. In §6.4, we give a sketch of the idea of an alternative (new) proof of Theorem 6.3.3 by using symplectic geometry.

6.1 Hyperfunction characters

This subsection explains very briefly reasons why we need distributions or hyperfunctions in defining characters of **infinite dimensional** representations and how they work. The exposition for Schwartz's distribution characters (§6.1.1- 6.1.3) is influenced by Atiyah's article [1], and the exposition for Sato's hyperfunction characters (§6.1.4 - 6.1.6) is intended to explain the idea of our main applications to restricting unitary representations in spirit of the papers [33, 43] of Kashiwara-Vergne and the author.

6.1.1 Finite group

Let G be a finite group. Consider the (left) regular representation π of G on

$$L^2(G) \simeq \bigoplus_{g \in G} \mathbb{C}g.$$

In light of a matrix expression of $\pi(g)$ with respect to the basis on the right-hand side, the character χ_π of π is given by the following form:

$$\operatorname{Trace} \pi(g) = \begin{cases} \#G & (g = e), \\ 0 & (g \neq e). \end{cases}$$

6.1.2 Dirac's delta function

How does the character formula look like as the order $\#G$ of G goes to infinity? The character will not be a usual function if $\#G = \infty$.

We shall take $G = S^1$ as an example of an infinite group below, and consider the character of the regular representation π. Then we shall interpret $\operatorname{Trace}\pi$ as a Dirac delta function both from Schwartz's distributions and Sato's hyperfunctions.

6.1.3 Schwartz's distributions

Let $G = S^1 \simeq \mathbb{R}/2\pi\mathbb{Z}$. Let π be the regular representation π of S^1 on $L^2(S^1)$. From the viewpoint of Schwartz's distributions, the character $\chi_\pi = \operatorname{Trace}\pi$ is essentially Dirac's delta function:

Lemma 6.1.3. $\operatorname{Trace}\pi(\theta)d\theta = 2\pi\delta(\theta)$

Sketch of proof. Take a test function $f \in C^\infty(S^1)$, and we develop f into the Fourier series:

$$f(\theta) = \frac{1}{2\pi}\sum_{n\in\mathbb{Z}} a_n(f)e^{in\theta},$$

where $a_n(f) := \int_{S^1} f(\theta)e^{-in\theta}d\theta$.

We do not go into technical details here (such as the proof of the fact that $\int_{S^1} f(\theta)\pi(\theta)d\theta$ is a trace class operator on $L^2(S^1)$), but present only a formal computation

$$\begin{aligned}
\langle f, \operatorname{Trace}\pi(\theta)d\theta\rangle &= \int_{S^1} f(\theta)\operatorname{Trace}\pi(\theta)d\theta \\
&= \sum_{m\in\mathbb{Z}}\int_{S^1} f(\theta)e^{-im\theta}d\theta \\
&= \sum_{m\in\mathbb{Z}} a_m(f) \\
&= 2\pi f(0) \\
&= \langle f, 2\delta(\theta)\rangle.
\end{aligned}$$

This is what we wanted to verify. □

6.1.4 Sato's hyperfunctions

Let us give another interpretation of the character of the regular representation of $G = S^1$. We regard S^1 as the unit circle in $\mathbb{C}$. From the viewpoint of Sato's hyperfunctions [32, 83], the character χ_π is given as boundary values of holomorphic functions:

Lemma 6.1.4.

$$\operatorname{Trace}\pi(\theta) = \lim_{\substack{z\to e^{i\theta}\\ |z|\uparrow 1}} \frac{1}{1-z} + \lim_{\substack{z\to e^{i\theta}\\ |z|\downarrow 1}} \frac{-1}{1-z}.$$

Sketch of proof (heuristic argument). Formally, we may write

$$\operatorname{Trace}\pi(\theta) = \sum_{n\in\mathbb{Z}} e^{in\theta} = \sum_{n\in\mathbb{Z}} z^n,$$

where we put $z = e^{i\theta}$. Let us "compute" this infinite sum as follows:

$$\begin{aligned}\sum_{n\in\mathbb{Z}} z^n &= \sum_{n\geq 0} z^n + \sum_{n<0} z^n \\ &= \frac{1}{1-z} + \frac{-1}{1-z}.\end{aligned} \tag{6.1.4}$$

Since the first term converges in $|z| < 1$ and the second one in $|z| > 1$, there is no intersection of domains where the above formula makes sense in a usual way. However, it can be justified as boundary values of holomorphic functions, in the theory of hyperfunctions. This is Lemma 6.1.4. ☐

6.1.5 Distributions or hyperfunctions

The distribution character in Lemma 6.1.3 is essentially the same with the hyperfunction character in Lemma 6.1.4. Cauchy's integral formula bridges them.

To see this, we take an arbitrary test function $F(z) \in \mathcal{A}(S^1)$. Then $F(z)$ extends holomorphically in the complex neighborhood of S^1, say, $1 - 2\varepsilon < |z| < 1 + 2\varepsilon$ for some $\varepsilon > 0$. Then,

$$\begin{aligned}&\frac{1}{2\pi i}\oint_{|z|=1-\varepsilon} \frac{1}{1-z}F(z)dz + \frac{1}{2\pi i}\oint_{|z|=1+\varepsilon} \frac{-1}{1-z}F(z)dz \\ &= \frac{1}{2\pi i}\oint_{\gamma} \frac{F(z)}{z-1}dz = F(1)\end{aligned} \tag{6.1.4}$$

where γ is a contour surrounding $z = 1$. We define a function f on $\mathbb{R}/2\pi\mathbb{Z}$ by

$$f(\theta) = e^{i\theta}F(e^{i\theta}).$$

We note $F(1) = f(0)$. If $z = e^{i\theta}$, then $F(z)dz = if(\theta)d\theta$. Hence, the formula (6.1.4) amounts to

$$\frac{1}{2\pi}\int_0^{2\pi}(\lim_{\substack{z\to e^{i\theta}\\ |z|\uparrow 1}}\frac{1}{1-z}+\lim_{\substack{z\to e^{i\theta}\\ |z|\downarrow 1}}\frac{-1}{1-z})f(\theta)d\theta = f(0). \tag{6.1.5}$$

Since any smooth test function on S^1 can be approximated by real analytic functions, the formula (6.1.5) holds for any $f \in C^\infty(S^1)$. Hence, we have shown the relation between Lemma 6.1.3 and Lemma 6.1.4.

6.1.6 Strategy

We recall that our main object of these lectures is the restriction $\pi|_{G'}$ in the setting where

- π is an irreducible unitary representation of G,
- G' is a reductive subgroup of G.

We are particularly interested in the (non-)existence of continuous spectrum in the irreducible decomposition of the restriction $\pi|_{G'}$. For this, our strategy is summarized as follows:

1) Restriction of a **representation** π to a subgroup G'.

$$\Uparrow$$

2) Restriction of its **character** $\operatorname{Trace}\pi$ to a subgroup.

$$\Uparrow$$

3) Restriction of a **holomorphic function** to a complex submanifold.

For (2), we shall find that the interpretation of a character as a hyperfunction fits well. Then (3) makes sense if the domain of holomorphy is large enough. In turn, the domain of holomorphy will be estimated by the invariants of the representation π, namely the asymptotic K-support $\mathrm{AS}_K(\pi)$, which we are going to explain in the next section.

6.2 Asymptotic K-support

Associated to a representation π of a compact Lie group K, the asymptotic K-support $\mathrm{AS}_K(\pi)$ is defined as a closed cone in a positive Weyl chamber. Here, we are mostly interested in the case where π is infinite dimensional (therefore, not irreducible as a K-module). The goal of this subsection is to introduce the definition of the asymptotic K-support and to explain how it works in the admissibility of the restriction of a unitary representation of a (non-compact) reductive Lie group G based on the ideas explained in §6.1.

6.2.1 Asymptotic cone

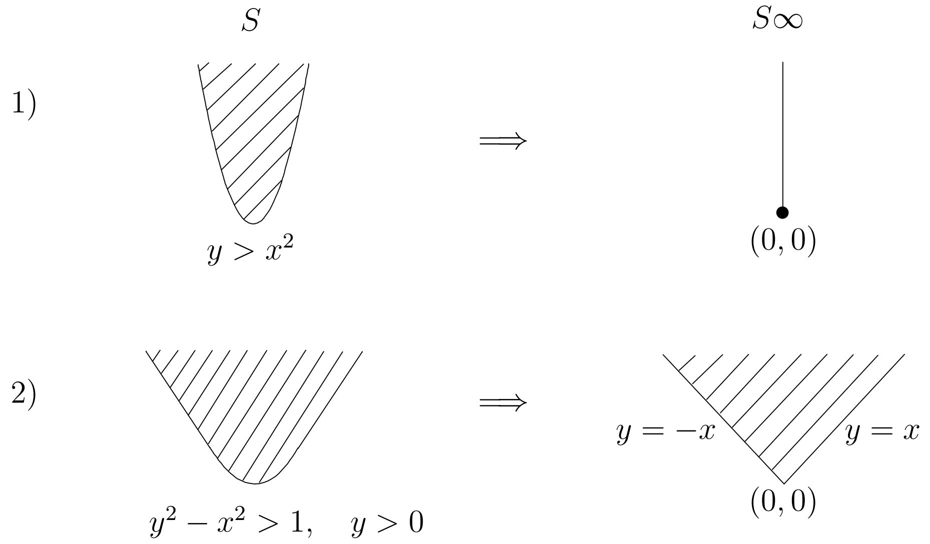
Let S be a subset of a real vector space $\mathbb{R}^N$. The asymptotic cone $S\infty$ is an important notion in microlocal analysis (e.g. [[32], Definition 2.4.3]), which is a closed cone defined by

$$S\infty := \{y \in \mathbb{R}^N : \text{ there exists a sequence } (y_n, \varepsilon_n) \in S \times \mathbb{R}_{>0} \\ \text{such that } \lim_{n\to\infty} \varepsilon_n y_n = y \text{ and } \lim_{n\to\infty} \varepsilon_n = 0\}.$$

Example 6.2.1. We illustrate the correspondence

$$S \;\Rightarrow\; S\infty$$

by two dimensional examples:



6.2.2 The Cartan-Weyl highest weight theory

We review quickly a well-known fact on finite dimensional representations of compact Lie groups and fix notation as follows.

Let K be a connected compact Lie group, and take a maximal torus T. We write $\mathfrak{k}$ and $\mathfrak{t}$ for their Lie algebras, respectively. Fix a positive root system $\Delta^+(\mathfrak{k}, \mathfrak{t})$, and write C_+ ($\subset \sqrt{-1}\mathfrak{t}^*$) for the corresponding closed Weyl chamber. We regard $\widehat{T}$ as a lattice of $\sqrt{-1}\mathfrak{t}^*$ and put

$$\Lambda_+ := \widehat{T} \cap C_+.$$

We note that the asymptotic cone $\Lambda_+\infty$ is equal to C_+.

For $\lambda \in \Lambda_+$, we shall denote by τ_λ an irreducible (finite dimensional) representation of K whose highest weight is λ. Then, the Cartan-Weyl highest weight theory (for a connected compact Lie group) establishes a bijection between $\widehat{K}$ and Λ_+:

$$\begin{array}{ccccccc} \widehat{K} & \simeq & \Lambda_+ & \subset & C_+ & \subset & \sqrt{-1}\mathfrak{t}^* \\ \cup\!\!\!\shortmid & & \cup\!\!\!\shortmid & & & & \\ \tau_\lambda & \leftrightarrow & \lambda & & & & \end{array}$$

6.2.3 Asymptotic K-support

For a representation π of K, we define the **K-support of π** by

$$\mathrm{Supp}_K(\pi) := \{\lambda \in \Lambda_+ : \mathrm{Hom}_K(\tau_\lambda, \pi) \neq 0\}.$$

Later, π will be a representation of a non-compact reductive Lie group G when restricted to its maximal compact subgroup K. Thus, we have in mind the case where π is infinite dimensional (and therefore, not irreducible). Its asymptotic cone

$$\mathrm{AS}_K(\pi) := \mathrm{Supp}_K(\pi)\infty$$

is called the **asymptotic K-support** of π. We note that it is a closed cone contained in the closed Weyl chamber C_+, because $\mathrm{Supp}_K(\pi) \subset \Lambda_+$ and $C_+ = \Lambda_+\infty$. The asymptotic K-support $\mathrm{AS}_K(\pi)$ was introduced by Kashiwara and Vergne [33] and has the following property (see [43, Proposition 2.7] for a rigorous statement):

The smaller the asymptotic K-support $\mathrm{AS}_K(\pi)$ is, the larger the domain of $K_{\mathbb{C}}$ in which the character $\mathrm{Trace}(\pi)$ extends holomorphically.

For further properties of the asymptotic K-support $\mathrm{AS}_K(\pi)$, see[33, 43, 53]. In particular, we mention:

Theorem 6.2.3 ([53]). *Suppose $\pi \in \widehat{G}$, or more generally, π is a (K-)admissible representation of G of finite length. (π is not necessarily unitary.) Then the asymptotic K-support $\mathrm{AS}_K(\pi)$ is a finite union of polytopes, namely, there exists a finite set $\{\alpha_{ij}\} \subset C_+$ such that*

$$\mathrm{AS}_K(\pi) = \bigcup_{k=1}^{l} \mathbb{R}_{\geq 0}\text{-}\operatorname{span}\{\alpha_{k1}, \alpha_{k2}, \ldots, \alpha_{km_k}\},$$

6.2.4 Examples from $SL(2, \mathbb{R})$

Let $G = SL(2, \mathbb{R})$. We recall the notation in Lecture 3, in particular, we identify $\widehat{K}$ with $\mathbb{Z}$. Here is a list of $\mathrm{Supp}_K(\pi)$ and $\mathrm{AS}_K(\pi)$ for some typical representations π of G.

	π	$\mathrm{Supp}_K(\pi)$	$\mathrm{AS}_K(\pi)$
(1)	$\mathbf{1}$	$\{0\}$	$\{0\}$
(2)	π_λ	$2\mathbb{Z}$	$\mathbb{R}$
(3)	π_n^+	$\{n, n+2, n+4, \ldots\}$	$\mathbb{R}_{\geq 0}$
(4)	π_n^-	$\{-n, -n-2, -n-4, \ldots\}$	$\mathbb{R}_{\leq 0}$

Here, $\mathbf{1}$ denotes the trivial one dimensional representation, π_λ a principal series representation, and $\pi_n^\pm$ ($n \geq 2$) a holomorphic or anti-holomorphic representation of G.

Now, let us recall §6.1.4 for a hyperfunction character. In the case (2), the character $\mathrm{Trace}(\pi) = \sum_{k \in 2\mathbb{Z}} e^{ik\theta}$ has a similar nature to the character of $L^2(S^1)$ computed in §6.1. In the cases (3) and (4), the asymptotic K-support is smaller, and the characters $\mathrm{Trace}(\pi) = \sum_{k \in n+2\mathbb{N}} e^{\pm ik\theta}$ converges in a larger complex domain. This resembles to each of the summands in (6.1.4).

6.3 Criterion for the admissible restriction

6.3.1 The closed cone $C_K(K')$

Let K be a connected compact Lie group, and K' its closed subgroup. Dual to the inclusion $\mathfrak{k}' \subset \mathfrak{k}$ of Lie algebras, we write

$$\mathrm{pr}_{\mathfrak{k} \to \mathfrak{k}'} : \mathfrak{k}^* \to (\mathfrak{k}')^*$$

for the projection, and $(\mathfrak{k}')^\perp$ for the kernel of $\mathrm{pr}_{\mathfrak{k}\to\mathfrak{k}'}$. We fix a positive definite and $\mathrm{Ad}(K)$-invariant bilinear form on $\mathfrak{k}$, and regard $\mathfrak{t}^*$ as a subspace of $\mathfrak{k}^*$.

Associated to the pair (K, K') of compact Lie groups, we define a closed cone in $\sqrt{-1}\mathfrak{t}^*$ by

$$C(K') \equiv C_K(K') := C_+ \cap \sqrt{-1}\,\mathrm{Ad}^*(K)(\mathfrak{k}')^\perp. \tag{6.3.1}$$

6.3.2 Symmetric pair

For a compact symmetric pair (K, K'), the closed cone $C_K(K')$ takes a very simple form. Let us describe it explicitly.

Suppose that (K, K') is a symmetric pair defined by an involutive automorphism σ of K. As usual, the differential of σ will be denoted by the same letter. By taking a conjugation by K if necessary, we may and do assume that $\mathfrak{t}$ and $\Delta^+(\mathfrak{k}, \mathfrak{t})$ are chosen so that

1) $\mathfrak{t}^{-\sigma} := \mathfrak{t} \cap \mathfrak{k}^{-\sigma}$ is a maximal abelian subspace of $\mathfrak{k}^{-\sigma}$,

2) $\sum^+(\mathfrak{k}, \mathfrak{t}^{-\sigma}) := \{\lambda|_{\mathfrak{t}^{-\sigma}} : \lambda \in \Delta^+(\mathfrak{k}, \mathfrak{t})\} \setminus \{0\}$ is a positive system of the restricted root system $\sum(\mathfrak{k}, \mathfrak{t}^{-\sigma})$.

We write $(\mathfrak{t}^{-\sigma})^*_+$ $(\subset \sqrt{-1}(\mathfrak{t}^{-\sigma})^*)$ for the corresponding dominant Weyl chamber.

Then we have (see [23])

Proposition 6.3.2. $C_K(K') = (\mathfrak{t}^{-\sigma})^*_+$.

One can also give an alternative proof of Proposition 6.3.2 by using Theorem 6.4.3 and a Cartan-Helgason theorem [105].

6.3.3 Criterion for the K'-admissibility

We are ready to explain the main results of Lecture 6, namely, Theorem 6.3.3 and Theorem 6.3.4 on a criterion for admissible restrictions.

Theorem 6.3.3 ([43, 53]). *Let $K \supset K'$ be a pair of compact Lie groups, and X a K-module. Then the following two conditions are equivalent:*

1) *X is K'-admissible.*

2) $C_K(K') \cap \mathrm{AS}_K(\pi) = \{0\}$.

Sketch of proof. Let us explain an idea of the proof (2) $\Rightarrow$ (1). The K-character $\mathrm{Trace}(\pi|_K)$ of X is a distribution (or a hyperfunction) on K. The condition (2) implies that its wave front set (or its singularity spectrum) is transversal to the submanifold K'. Then the restriction of this distribution (or hyperfunction) to K' is well-defined, and coincides with the K'-character of X (see [43] for details). $\square$

6.3.4 Sufficient condition for the G'-admissibility

Now, let us return our original problem, namely, the restriction to a non-compact reductive subgroup.

Theorem 6.3.4. *Let $\pi \in \widehat{G}$, and $G \supset G'$ be a pair of reductive Lie groups. Take maximal compact subgroups $K \supset K'$, respectively. If*

$$C_K(K') \cap \mathrm{AS}_K(\pi) = \{0\},$$

then the restriction $\pi|_{G'}$ is G'-admissible, that is, $\pi|_{G'}$ splits discretely:

$$\pi|_{G'} \simeq \sum_{\tau \in G'}^{\oplus} n_\pi(\tau)\tau$$

into irreducible representations of G' with $n_\pi(\tau) < \infty$ for any $\tau \in \widehat{G}'$.

6.3.5 Remark

The assumption of Theorem 6.3.4 is obviously fulfilled if $C_K(K') = \{0\}$ or if $\mathrm{AS}_K(\pi) = \{0\}$. What are the meanings of these extremal cases? Here is the answer:

1) $C_K(K') = \{0\} \Leftrightarrow K' = K$.

In this case, Theorem 6.3.4 is nothing but Harish-Chandra's admissibility theorem, as we explained in Lecture 2 (see Theorem 2.4.6).

2) $\mathrm{AS}_K(\pi) = \{0\} \Leftrightarrow \dim \pi < \infty$.

In this case, the conclusion of Theorem 6.3.4 is nothing but the complete reducibility of a finite dimensional unitary representation. The second case is more or less trivial.

In this connection, one might be tempted to ask when $\mathrm{AS}_K(\pi)$ becomes the second smallest, namely, of the form $\mathbb{R}_+ v$ generated by a single element

v? It follows from the following result of Vogan [94] that this is always the case if π is a **minimal representation** of G (in the sense that its annihilator is the Joseph ideal).

Theorem 6.3.5 (Vogan). *Let π be a minimal representation of G. Then there exists a weight ν such that*

$$\pi_K \simeq \bigoplus_{m \in \mathbb{N}} \tau_{mv+\nu},$$

as K-modules, where v is the highest weight of $\mathfrak{p}$, and $\tau_{mv+\nu}$ is the irreducible representation of K with highest weight $mv + \nu$. In particular,

$$AS_K(\pi) = \mathbb{R}_+ v.$$

This reflects the fact that there are fairly rich examples of discretely decomposable restriction of the minimal representation π of a reductive group G with respect to noncompact reductive subgroup G' (e.g. Howe [24] for $G = Mp(n, \mathbb{R})$, Kobayashi-Ørsted [57] for $G = O(p, q)$, and Gross-Wallach [19] for some exceptional Lie groups G).

6.4 Application of symplectic geometry

6.4.1 Hamiltonian action

Let (M, ω) be a symplectic manifold on which a compact Lie group K acts as symplectic automorphisms by $\tau : K \times M \to M$. We write $\mathfrak{k} \to \mathfrak{X}(M), X \mapsto X_M$ for the vector field induced from the action. The action τ is called **Hamiltonian** if there exists a map

$$\Psi : M \to \mathfrak{k}^*$$

such that $d\Psi^X = \iota(X_M)\omega$ for all $X \in \mathfrak{k}$, where we put $\Psi^X(m) = \langle X, \Psi(m) \rangle$. We say Ψ is the **momentum map**. For a subset Y of M, The **momentum set** $\Delta(Y)$ is defined by

$$\Delta(Y) = \sqrt{-1}\Psi(Y) \cap C_+.$$

6.4.2 Affine varieties

Let V be a complex Hermitian vector space. Assume that a compact Lie group K acts on V as a unitary representation. Then the action of K is also symplectic if we equip V with the symplectic form $\omega_V(u,v) = -\operatorname{Im}(u,v)$, as we saw the inclusive relation $U(n) \subset Sp(n,\mathbb{R})$ in Lecture 1 (see Example 1.5.5). Then, V is Hamiltonian with the momentum map

$$\Psi : V \to \mathfrak{k}^*, \Psi(v)(X) = \frac{\sqrt{-1}}{2}(Xv, v), \ (X \in \mathfrak{k}).$$

We extend the K-action to a complex linear representation of $K_{\mathbb{C}}$. Suppose $\mathcal{V}$ is a $K_{\mathbb{C}}$-stable closed irreducible affine variety of V. We have naturally a representation of K on the space of regular functions $\mathbb{C}[\mathcal{V}]$. Then $\operatorname{Supp}_K(\mathbb{C}[\mathcal{V}])$ is a monoid, namely, there exists a finite number of elements in Λ_+, say $\lambda_1, \dots, \lambda_k$, such that

$$\operatorname{Supp}_K(\mathbb{C}[\mathcal{V}]) = \mathbb{Z}_{\geq 0}\text{-span}\{\lambda_1, \dots, \lambda_k\}. \tag{6.4.2}$$

The momentum set $\Delta(\mathcal{V})$ is a "classical" analog of the set of highest weights $\operatorname{Supp}_K(\mathbb{C})$. That is, the following theorem holds:

Proposition 6.4.2 ([88]). $\Delta(\mathcal{V}) = \mathbb{R}_{\geq 0} \operatorname{Supp}_K(\mathbb{C}[\mathcal{V}])$.

Together with (6.4.2), we have

$$\Delta(\mathcal{V}) = \mathbb{R}_{\geq 0}\text{-span}\{\lambda_1, \dots, \lambda_k\}. = \operatorname{AS}_K(\operatorname{Supp}_K(\mathbb{C}[\mathcal{V}])).$$

6.4.3 Cotangent bundle

Let $M = T^*(K/K')$, the cotangent bundle of the homogeneous space K/K'. We define an equivalence relation on the direct product $K \times \mathfrak{k}^{\perp}$ by $(k, \lambda) \sim (kh, \operatorname{Ad}^*(h)^{-1}\lambda)$ for some $h \in H$, and write $[k, \lambda]$ for its equivalence class. Then the set of equivalent classes, denoted by $K \times_{K'} \mathfrak{k}^{\perp}$, becomes a K-equivariant homogeneous bundle over K/K', and is identified with $T^*(K/K')$.

Then the symplectic manifold $T^*(K/K')$ is naturally a Hamiltonian with the momentum map

$$\Psi : T^*(K/K') \to \mathfrak{k}^*, [k, \lambda] \mapsto \operatorname{Ad}^*(k)\lambda.$$

The momentum set $\Delta(T^*(K/K'))$ equals to $C_K(K')$, as follows from the definition (6.3.1).

The closed cone $C_K(K')$ can be realized as the asymptotic K-support of a certain induced representation of K. That is, we can prove:

Theorem 6.4.3 ([53]). *Let τ be an arbitrary finite dimensional representation of K'. Then,*

$$C_K(K') = \mathrm{AS}_K(\mathrm{Ind}_{K'}^{K}(\tau)).$$

In particular,

$$C_K(K') = \mathrm{AS}_K(L^2(K/K')). \tag{6.4.3}$$

Here, we note that the asymptotic K-support $\mathrm{AS}_K(\mathrm{Ind}_{K'}^{K}(\tau))$ does not change whatever we take the class of functions (algebraic, square integrable, hyperfunctions, ...) in the definition of the induced representation $\mathrm{Ind}_{K'}^{K}(\tau)$.

Lecture 7 Discretely decomposable restriction of $A_{\mathfrak{q}}(\lambda)$

In the philosophy of the Kostant-Kirillov orbit method, there is an important family of irreducible unitary representations of a reductive Lie group G, "attached to" elliptic coadjoint orbits.

Geometrically, they are realized in dense subspaces of Dolbeault cohomology groups of certain equivariant holomorphic line bundles. This is a vast generalization due to Langland, Schmid, and others of the Borel-Weil-Bott theorem: from compact to non-compact; from finite dimensional to infinite dimensional.

Algebraically, they are also expressed as Zuckerman's derived functor modules $A_{\mathfrak{q}}(\lambda)$ by using so called cohomological parabolic induction. Among all, important is the unitarizability theorem of these modules under certain positivity condition on parameter proved by Vogan and Wallach.

Analytically, some of them can be realized in L^2-spaces on homogeneous spaces. For example, Harish-Chandra's discrete series representations for group manifolds and Flensted-Jensen's ones for symmetric spaces are the cases.

§7.1 collects some of basic results on these representations from these three – geometric, algebraic, and analytic – aspects.

§7.2 speculates their restrictions to non-compact subgroups, and examines the results of Lectures 5 and 6 for these modules.

7.1 Elliptic orbits and geometric quantization

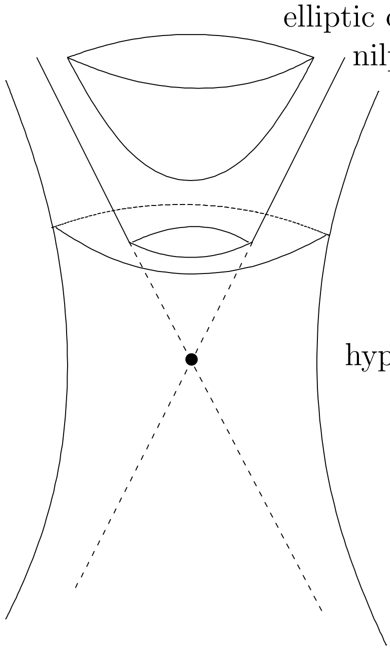


Figure 7.1 : (co)adjoint orbits of $G = SL(2, \mathbb{R})$

7.1.1 Elliptic orbits

Consider the adjoint action of a Lie group G on its Lie algebra $\mathfrak{g}$. For $X \in \mathfrak{g}$, we define the adjoint orbit $\mathcal{O}_X$ by

$$\mathcal{O}_X := \mathrm{Ad}(G)X \simeq G/G_X.$$

Here, G_X is the isotropy subgroup at X, given by $\{g \in G : \mathrm{Ad}(g)X = X\}$.

Definition 7.1.1. An element $X \in \mathfrak{g}$ is **elliptic** if $\mathrm{ad}(X) \in \mathrm{End}_{\mathbb{C}}(\mathfrak{g}_{\mathbb{C}})$ is diagonalizable and if all eigenvalues are purely imaginary. Then, $\mathcal{O}_X$ is called an **elliptic orbit**.

Example. If G is a compact Lie group, then any adjoint orbit is elliptic.

Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be a Cartan decomposition of the Lie algebra $\mathfrak{g}$ of G. We fix a maximal abelian subspace $\mathfrak{t}$ of $\mathfrak{k}$.

Any elliptic element is conjugate to an element in $\mathfrak{k}$ under the adjoint action of G. Furthermore, any element of $\mathfrak{k}$ is conjugate to an element in $\mathfrak{t}$ under the adjoint action of K. Hence, for $X \in \mathfrak{g}$, we have the following equivalence:

$$\begin{aligned} \mathcal{O}_X \text{ is an elliptic orbit } &\Leftrightarrow \mathcal{O}_X \cap \mathfrak{k} \neq \phi \\ &\Leftrightarrow \mathcal{O}_X \cap \mathfrak{t} \neq \phi. \end{aligned}$$

From now on, without loss of generality, we can and do take $X \in \mathfrak{t}$ when we deal with an elliptic orbit.

7.1.2 Complex structure on an elliptic orbit

Every elliptic orbit $\mathcal{O}_X$ carries a G-invariant complex structure. This subsection provides a sketch of this fact. (See, for example, [56] for further details.)

First, we note that $\sqrt{-1}\,\mathrm{ad}(X) \in \mathrm{End}(\mathfrak{g}_{\mathbb{C}})$ is a semisimple transformation with all eigenvalues real. Then, the eigenspace decomposition of $\sqrt{-1}\,\mathrm{ad}(X)$ leads to the Gelfand-Naimark decomposition:

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{u}^- + (\mathfrak{g}_X)_{\mathbb{C}} + \mathfrak{u}^+. \tag{7.1.2}$$

Here, $\mathfrak{u}^+$ (respectively, $\mathfrak{u}^-$) is the direct sum of eigenspaces of $\sqrt{-1}\,\mathrm{ad}(X)$ with positive (respectively, negative) eigenvalues. We define a parabolic subalgebra of $\mathfrak{g}_{\mathbb{C}}$ by

$$\mathfrak{q} := (\mathfrak{g}_X)_{\mathbb{C}} + \mathfrak{u}^+.$$

For simplicity, suppose G is a connected reductive Lie group contained in a (connected) complex Lie group $G_{\mathbb{C}}$ with Lie algebra $\mathfrak{g}_{\mathbb{C}}$. Let Q be the parabolic subgroup of $G_{\mathbb{C}}$ with Lie algebra $\mathfrak{q}$. We note that Q is a connected complex subgroup of $G_{\mathbb{C}}$. Then the key ingredients here are

$$\begin{aligned} G \cap Q &= G_X, \\ \mathfrak{g} + \mathfrak{q} &= \mathfrak{g}_{\mathbb{C}}. \end{aligned}$$

Hence, the natural inclusion $G \subset G_{\mathbb{C}}$ induces an open embedding of $\mathcal{O}_X$ into the generalized flag variety $G_{\mathbb{C}}/Q$:

$$\mathcal{O}_X \simeq G/G_X \underset{\text{open}}{\hookrightarrow} G_{\mathbb{C}}/Q.$$

Hence, we can define a complex structure on the adjoint orbit $\mathcal{O}_X$ from that on $G_{\mathbb{C}}/Q$. Obviously, the action of G on $\mathcal{O}_X$ is biholomorphic. For a further

structure on $\mathcal{O}_X$, we note that $\mathcal{O}_X$ contains another (smaller) generalized flag variety
$$\mathcal{O}_X^K := \mathrm{Ad}(K)X \simeq K/K_X,$$
of which the complex dimension will be denoted by S.

In summary, we have

$$\mathcal{O}_X^K \underset{\text{closed}}{\subset} \mathcal{O}_X \underset{\text{open}}{\hookrightarrow} G_{\mathbb{C}}/Q,$$

and in particular,

Proposition 7.1.2. *Any elliptic orbit $\mathcal{O}_X$ carries a G-invariant complex structure through an open embedding into the generalized flag variety $G_{\mathbb{C}}/Q$. Furthermore, $\mathcal{O}_X$ contains a compact complex submanifold $\mathcal{O}_X^K$.*

7.1.3 Elliptic coadjoint orbit

For a reductive Lie group G, adjoint orbits on $\mathfrak{g}$ and coadjoint orbits on $\mathfrak{g}^*$ can be identified via a non-degenerate G-invariant bilinear form. For example, such a bilinear form is given by

$$\mathfrak{g} \times \mathfrak{g} \to \mathbb{R}, \ (X, Y) \mapsto \mathrm{Trace}(XY)$$

if $\mathfrak{g}$ is realized in $\mathfrak{gl}(N, \mathbb{R})$ such that ${}^t\mathfrak{g} = \mathfrak{g}$.

Let $\lambda \in \sqrt{-1}\mathfrak{g}^*$. We write $X_\lambda \in \sqrt{-1}\mathfrak{g}$ for the corresponding element via the isomorphism
$$\sqrt{-1}\mathfrak{g}^* \simeq \sqrt{-1}\mathfrak{g}.$$
We write $X := -\sqrt{-1}X_\lambda \in \mathfrak{g}$. Then isotropy subgroups of the adjoint action and the coadjoint action coincides: $G_X = G_\lambda$.

Assume that X is an elliptic element. With analogous notation as in §7.1.2, Proposition 7.1.2 tells that the coadjoint orbit

$$\mathcal{O}_\lambda := \mathrm{Ad}^*(G) \cdot \lambda \simeq G/G_\lambda = G/G_X \simeq \mathrm{Ad}(G) \cdot X =: \mathcal{O}_X$$

carries a G-invariant complex structure.

The Lie algebra of G_λ is given by $\mathfrak{g}_\lambda := \{X \in \mathfrak{g} : \lambda(X) = 0\}$. We define $\rho_\lambda \in \sqrt{-1}\mathfrak{g}_\lambda^*$ by
$$\rho_\lambda(Y) := \mathrm{Trace}(\mathrm{ad}(Y) : \mathfrak{u}^+ \to \mathfrak{u}^+),$$

for $Y \in \mathfrak{g}_\lambda$. We say λ is **integral** if the Lie algebra homomorphism

$$\lambda + \rho_\lambda : \mathfrak{g}_\lambda \to \mathbb{C}$$

lifts to a character of G_λ. For simplicity, we shall write $\mathbb{C}_{\lambda+\rho_\lambda}$ for the lifted character. Then

$$\mathcal{L}_\lambda := G \times_{G_\lambda} \mathbb{C}_{\lambda+\rho_\lambda} \to \mathcal{O}_\lambda$$

is a G-equivariant holomorphic line bundle over the coadjoint orbit $\mathcal{O}_\lambda$.

7.1.4 Geometric quantization a la Schmid-Wong

This subsection completes the following scheme of the "geometric quantization" of an elliptic coadjoint orbit $\mathcal{O}_\lambda$.

$$\begin{array}{cl}
\lambda \in \sqrt{-1}\mathfrak{g}^* & \text{an elliptic and integral element} \\
\downarrow & \\
\mathcal{L}_\lambda \to \mathcal{O}_\lambda & \text{a } G\text{-equivariant holomorphic line bundle} \\
\downarrow & \\
H^*_{\bar\partial}(\mathcal{O}_\lambda, \mathcal{L}_\lambda) & \text{a representation } \Pi(\lambda) \text{ of } G
\end{array}$$

We have already explained the first step. Here is a summary on the second step:

Theorem 7.1.4. *Let $\lambda \in \sqrt{-1}\mathfrak{g}^*$ be elliptic and integral.*

1) *The Dolbeault cohomology group $H^j_{\bar\partial}(\mathcal{O}_\lambda, \mathcal{L}_\lambda)$ carries a Fréchet topology, on which G acts continuously.*

2) *(vanishing theorem) $H^j_{\bar\partial}(\mathcal{O}_\lambda, \mathcal{L}_\lambda) = 0$ if $j \neq S$.*

3) *(unitarizability) There is a dense subspace $\mathcal{H}$ in $H^S_{\bar\partial}(\mathcal{O}_\lambda, \mathcal{L}_\lambda)$ with which a G-invariant Hilbert structure can be equipped.*

4) *If λ is "sufficiently regular", then the unitary representation of G on $\mathcal{H}$ is irreducible and non-zero.*

We shall denote by $\Pi(\lambda)$ the unitary representation constructed in Theorem 7.1.4 (3).

Here are some comments on and further introductions to Theorem 7.1.4.

1) The non-trivial part of the statement (1) is that the range of the $\bar{\partial}$-operator is closed with respect to the Fréchet topology on the space of $(0, q)$-forms. The difficulty arises from the fact that $\mathcal{O}_\lambda \simeq G/G_\lambda$ is non-compact. This **closed range problem** was solved affirmatively by Schmid in the case G_λ compact, and by H. Wong for general G_λ early in 1990s [107].

2) This vanishing result is an analogue of Cartan's Theorem for Stein manifolds.
We note that $\mathcal{O}_\lambda$ is Stein if and only if $S = 0$. In this case, $\mathcal{O}_\lambda$ is biholomorphic to a Hermitian symmetric space of non-compact type, and the statement (2) asserts the vanishing of all cohomologies in higher degrees. The resulting representations in the 0th degree in this special case are highest weight representations.
An opposite extremal case is when G is compact (see the next subsection §7.1.5). In this case, our choice of the complex structure on $\mathcal{O}_X$ implies that the dual of $\mathcal{L}_\lambda$ is ample, and the statement (2) asserts that all the cohomologies vanish except for the top degree.

3) The unitarizability was conjectured by Zuckerman, and proved under certain positivity condition on the parameter λ by Vogan [96] and Wallach independently[103] in 1980s. See also a proof in treatises by Knapp-Vogan [37] or by Wallach[104]. In our formulation that is suitable for the orbit method, this positivity condition is automatically satisfied.

4) Vogan introduced the condition "good range" and a slightly weaker one "fair range". The statement (4) of Theorem 7.1.4 holds if λ is in the good range ([96]).

Special cases of Theorem 7.1.4 contain many interesting representations as we shall see in §7.1.5 ∼§7.1.7.

7.1.5 Borel-Weil-Bott theorem

If G is a compact Lie group, then any coadjoint orbit $\mathcal{O}_\lambda$ is elliptic and becomes a compact complex manifold (a generalized flag variety). Then by a theorem of Kodaira-Serre, the Dolbeault cohomology groups $H^j_{\bar{\partial}}(\mathcal{O}_\lambda, \mathcal{L}_\lambda)$ are finite dimensional.

The representations constructed in Theorem 7.1.4 are always irreducible, and exhaust all irreducible (finite dimensional, unitary) representations of G. This is known as the Borel-Weil-Bott construction.

7.1.6 Discrete series representations

Suppose (X, μ) is a G-space with G-invariant measure μ. Then, on the Hilbert space $L^2(X, d\mu)$ of square integrable functions, there is a natural unitary representation of G by translations.

Definition 7.1.6. An irreducible unitary representation π of G is called a **discrete series representation** for $L^2(X, d\mu)$ (or simply, for X) if π can be realized in a G-invariant closed subspace of $L^2(X, d\mu)$, or equivalently, if

$$\dim \operatorname{Hom}_G(\pi, L^2(X)) \neq 0,$$

where Hom_G denotes the space of continuous G-intertwining operators.

We shall write $\operatorname{Disc}(X)$ for the subset of $\widehat{G}$ consisting of all discrete series representations for $L^2(X, d\mu)$. It may happen that $\operatorname{Disc}(X) = \phi$.

7.1.7 Harish-Chandra's discrete series representations

Let G be a real reductive linear Lie group. If $(X, \mu) = (G, \text{Haar measure})$ with left G-action, then $\operatorname{Disc}(G)$ was classified by Harish-Chandra. In the context of Theorem 7.1.4, $\operatorname{Disc}(G)$ is described as follows:

Theorem 7.1.7. *Let G be a real reductive linear Lie group.*

$$\operatorname{Disc}(G) = \{\Pi(\lambda) : \lambda \textit{ is integral and elliptic, } G_\lambda \textit{ is a compact torus.}\}$$

This theorem presents a geometric construction of discrete series representations. Such a construction was conjectured by Langlands, and proved by Schmid [85].

We can see easily that there exists λ such that G_λ is a compact torus if and only if $\operatorname{rank} G = \operatorname{rank} K$. In this case, there are countably many integral and elliptic λ such that G_λ is a compact torus. In particular, the above formulation of Theorem 7.1.7 includes a Harish-Chandra's celebrated criterion:

$$\operatorname{Disc}(G) \neq \phi \Leftrightarrow \operatorname{rank} G = \operatorname{rank} K. \tag{7.1.7}$$

7.1.8 Discrete series representations for symmetric spaces

Suppose G/H is a reductive symmetric space. Here are some typical examples.

$$\begin{aligned}&SL(p+q,\mathbb{R})/SO(p,q),\\&GL(p+q,\mathbb{R})/(GL(p,\mathbb{R})\times GL(q,\mathbb{R})),\\&GL(n,\mathbb{C})/GL(n,\mathbb{R}).\end{aligned}$$

Without loss of generality, we may assume that H is stable under a fixed Cartan involution θ of G. Then, we may formulate the results of Flensted-Jensen, Matsuki-Oshima and Vogan on discrete series representations for reductive symmetric spaces as follows:

Theorem 7.1.8. *Let G/H be a reductive symmetric space. Then,*

$$\mathrm{Disc}(G/H)=\left\{\Pi(\lambda):\begin{array}{l}\lambda \textit{ is elliptic, and satisfies a certain}\\ \textit{integral condition, } \lambda|_{\mathfrak{h}}\equiv 0,\\ G_\lambda/(G_\lambda\cap H) \textit{ is a compact torus}\end{array}\right\}.$$

We note that the original construction of discrete series representations for G/H did not use Dolbeault cohomology groups but used the Poisson transform of hyperfunctions (or distributions) on real flag varieties. It follows from the duality theorem due to Hecht-Miličić-Schmid-Wolf [22] that these discrete series representations are isomorphic to some $\Pi(\lambda)$. The above formulation on the description of discrete series representations is taken from the author's exposition ([45, Example 2.9]).

Like the case of Harish-Chandra's discrete series representations, we can see easily that such λ exists if and only if $\operatorname{rank} G/H = \operatorname{rank} K/H\cap K$. Hence, Theorem 7.1.8 contains a criterion for the existence of discrete series representations for reductive symmetric spaces:

$$\mathrm{Disc}(G/H)\neq\phi \Leftrightarrow \operatorname{rank} G/H = \operatorname{rank} K/H\cap K.$$

This generalizes (7.1.7) (since the group case can be regarded as a symmetric space $(G\times G)/\operatorname{diag}(G)$), and was proved by Flensted-Jensen, Matsuki and Oshima.

7.2 Restriction of $\Pi(\lambda)$ attached to elliptic orbits

7.2.1 Asymptotic K-support, associated variety

Throughout this section, $\Pi(\lambda)$ will be a unitary representation of G attached to an integral elliptic coadjoint orbit $\mathcal{O}_\lambda$. We assume that $\Pi(\lambda)$ is non-zero. This is the case if λ is in the good range ([96, 97]).

We may and do assume $X_\lambda \in \sqrt{-1}\mathfrak{t}$ (see (7.1.1)). Let $\mathfrak{g}_{\mathbb{C}} = \mathfrak{k}_{\mathbb{C}} + \mathfrak{p}_{\mathbb{C}}$ be the complexification of a Cartan decomposition, and

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{u}_\lambda^- + (\mathfrak{g}_\lambda)_{\mathbb{C}} + \mathfrak{u}_\lambda^+$$

the Gelfand-Naimark decomposition (7.1.2) corresponding to the action of $\sqrt{-1}\,\mathrm{ad}(-\sqrt{-1}X_\lambda) = \mathrm{ad}(X_\lambda)$. We define the set of $\mathfrak{t}$-weights (positive non-compact roots) by

$$\Delta_\lambda^+(\mathfrak{p}) := \Delta(\mathfrak{u}_\lambda^+ \cap \mathfrak{p}_{\mathbb{C}}, \mathfrak{t}) \subset \sqrt{-1}\mathfrak{t}^*. \tag{7.2.1}$$

Here is an explicit estimate on the asymptotic K-support, and the associated variety of $\Pi(\lambda)$.

Theorem 7.2.1. *Suppose that* $\Pi(\lambda) \neq 0$ *in the above setting.*
1) $\mathrm{AS}_K(\Pi(\lambda)) \subset \mathbb{R}_{\geq 0}\text{-span}\,\Delta_\lambda^+(\mathfrak{p})$.
2) $\mathcal{V}_{\mathfrak{g}_{\mathbb{C}}}(\Pi(\lambda)) = \mathrm{Ad}^*(K_{\mathbb{C}})(\mathfrak{u}_\lambda^- \cap \mathfrak{p}_{\mathbb{C}})$.

Proof. See [43, §3] for the Statement (1). See [4] or [97] for the Statement (2) for a regular λ; and [44] for a general case. $\square$

7.2.2 Restriction to a symmetric subgroup

We recall the notation and convention in §6.3.2, where (G, G') is a reductive symmetric pair defined by an involutive automorphism σ of G. In particular, we have

$$C_K(K') = (\mathfrak{t}^{-\sigma})^*_+.$$

Then the following theorem tells us explicitly when the restriction $\Pi(\lambda)|_{G'}$ is infinitesimally discretely decomposable.

Theorem 7.2.2 ([44, Theorem 4.2]). *Let* $\Pi(\lambda)$ *be a non-zero unitary representation of* G *attached to an integral elliptic coadjoint orbit, and* (G, G') *a reductive symmetric pair defined by an involutive automorphism* σ *of* G*. Then the following four conditions on* (G, σ, λ) *are equivalent:*

i) $\Pi(\lambda)|_{K'}$ *is* K'*-admissible.*

ii) $\Pi(\lambda)|_{G'}$ *is infinitesimally discretely decomposable.*

iii) $\mathbb{R}_{\geq 0}\text{-span}\,\Delta^+_\lambda(\mathfrak{p}) \cap (\mathfrak{t}^{-\sigma})^*_+ = \{0\}$.

iv) $\mathrm{pr}_{\mathfrak{g}\to\mathfrak{g}'}(\mathrm{Ad}(K_{\mathbb{C}})(\mathfrak{u}_- \cap \mathfrak{p}_{\mathbb{C}})) \subset \mathcal{N}_{\mathfrak{p}'_{\mathbb{C}}}$.

Sketch of proof.

(iii)$\Rightarrow$(i) This follows from the criterion of K'-admissibility given in Theorem 6.3.3 together with Theorem 7.2.1

(i)$\Rightarrow$(ii) See Theorem 4.2.7.

(ii)$\Rightarrow$(iv) This follows from the criterion for infinitesimally discretely decomposable restrictions by means of associated varieties. Use Corollary 5.2.3 and Theorem 7.2.1.

(iv)$\Rightarrow$(iii) This part can be proved only by techniques of Lie algebras (without representation theory).

□

7.3 $U(2,2) \downarrow Sp(1,1)$

This subsection examines Theorem 7.2.2 by an example

$$(G, G') = (U(2,2), Sp(1,1)).$$

More precisely, we shall take a discrete series representation of $U(2,2)$ (with Gelfand-Kirillov dimension 5), and explain how to verify the criteria (iii) (root data) and (iv) (associated varieties) in Theorem 7.2.2.

7.3.1 Non-holomorphic discrete series representations for $U(2,2)$

Let $\mathfrak{t}$ be a maximal abelian subspace of $\mathfrak{k} \simeq \mathfrak{u}(2) + \mathfrak{u}(2)$, and we take a standard basis $\{e_1, e_2, e_3, e_4\}$ of $\sqrt{-1}\mathfrak{t}^*$ such that

$$\Delta(\mathfrak{p}_{\mathbb{C}}, \mathfrak{t}) = \{\pm(e_i - e_j) : 1 \leq i \leq 2, 3 \leq j \leq 4\}.$$

We shall fix once and for all

$$\lambda = \lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3 + \lambda_4 e_4 \in \sqrt{-1}\mathfrak{t}^*$$

such that

$$\lambda_1 > \lambda_3 > \lambda_4 > \lambda_2, \ \lambda_j \in \mathbb{Z} \quad (1 \le j \le 4). \tag{7.3.1}$$

Such λ is integral and elliptic, and the resulting unitary representation $\Pi(\lambda)$ is non-zero irreducible. We note that G_λ is isomorphic to a compact torus $\mathbb{T}^4$. Then, $\Pi(\lambda)$ is a Harish-Chandra's discrete series representation by Theorem 7.1.7. Its Gelfand-Kirillov dimension is 5, as we shall see in (7.3.3) that its associated variety is five dimensional. Furthermore, $\Pi(\lambda)$ is not a holomorphic discrete series representation which has Gelfand-Kirillov dimension 4 for $G = U(2,2)$.

7.3.2 Criterion for the K'-admissibility for $U(2,2) \downarrow \mathrm{Sp}(1,1)$

Retain the setting as in §7.3.1. In light of (7.3.1), the set of non-compact positive roots $\Delta_\lambda^+(\mathfrak{p})$ (see (7.2.1) for definition) is given by

$$\Delta_\lambda^+(\mathfrak{p}) = \{e_1 - e_3, e_1 - e_4, e_3 - e_2, e_4 - e_2\}.$$

Hence, via the identification $\sqrt{-1}\mathfrak{t}^* \simeq \mathbb{R}^4$, we have

$$\mathbb{R}_{\geq 0}\text{-span}\,\Delta_\lambda^+(\mathfrak{p}) = \left\{ \begin{pmatrix} a+b \\ -c-d \\ -a+c \\ -b+d \end{pmatrix} ; a,b,c,d \geq 0 \right\}. \tag{7.3.2}$$

On the other hand, for $(K, K') \equiv (K, K^\sigma) = (U(2) \times U(2), \mathrm{Sp}(1) \times \mathrm{Sp}(1))$, we have

$$(\mathfrak{t}^{-\sigma})^*_+ = \mathbb{R} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \mathbb{R} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}.$$

Thus, $\mathbb{R}_{\geq 0}\text{-span}\,\Delta_\lambda^+(\mathfrak{p}) \cap (\mathfrak{t}^{-\sigma})^*_+ = \{0\}$ because the condition

$$\begin{pmatrix} a+b \\ -c-d \\ -a+c \\ -b+d \end{pmatrix} \in \mathbb{R} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \mathbb{R} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}$$

leads to $a + b = -c - d$, which occurs only if $a = b = c = d$ under the assumption $a, b, c, d \geq 0$ (see (7.3.2)). Thus, the condition (iii) of Theorem 7.2.2 holds. Therefore, the restriction $\Pi(\lambda)|_{K'}$ is K'-admissible.

7.3.3 Associated variety of $\Pi(\lambda)$

Retain the setting of the example as above. Although we have already known that all of the equivalent conditions in Theorem 7.2.2 hold, it is illustrative to verify them directly. So, let us compute the associated variety $\mathcal{V}_{\mathfrak{g}_{\mathbb{C}}}(\Pi(\lambda))$ and verify the condition (iv) of Theorem 7.2.2 by using the computation given in Proposition 5.3.4.

In light of

$$\begin{aligned}\Delta(\mathfrak{u}_\lambda^- \cap \mathfrak{p}_{\mathbb{C}}, \mathfrak{t}) &= \{-e_1 + e_3, -e_1 + e_4, e_2 - e_3, e_2 - e_4\}, \\ &(= -\Delta_\lambda^+(\mathfrak{p}))\end{aligned}$$

we have

$$\mathfrak{u}_\lambda^- \cap \mathfrak{p}_{\mathbb{C}} \simeq \left\{ \left(\begin{pmatrix} 0 & 0 \\ x & y \end{pmatrix}, \begin{pmatrix} z & 0 \\ w & 0 \end{pmatrix} \right) : x, y, z, w \in \mathbb{C} \right\}$$

via the identification $\mathfrak{p}_{\mathbb{C}} \simeq M(2, \mathbb{C}) \oplus M(2, \mathbb{C})$ (see (5.3.2) in §5). Hence, we have

$$\mathcal{V}_{\mathfrak{g}_{\mathbb{C}}}(\Pi(\lambda)) = K_{\mathbb{C}}(\mathfrak{u}^- \cap \mathfrak{p}_{\mathbb{C}}) = \overline{\mathcal{O}_{\substack{1\,1 \\ 1\,0}}} \tag{7.3.3}$$

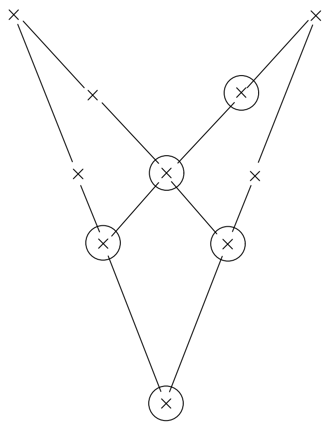
by Theorem 7.2.1, where we recall that $\mathcal{O}_{\substack{1\,1 \\ 1\,0}}$ is a five dimensional manifold defined by

$$\mathcal{O}_{\substack{1\,1 \\ 1\,0}} = \{(A, B) : \operatorname{rank} A = \operatorname{rank} B = \operatorname{rank} AB = 1, \operatorname{rank} BA = 0\}.$$

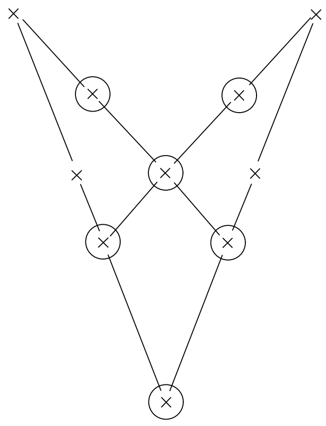
In the Figure 5.3.2, the closure of $\mathcal{O}_{\substack{1\,1 \\ 1\,0}}$ consists of 5 $K_{\mathbb{C}}$-orbits, which are described by circled points below in the left. In view of the classification of $K_{\mathbb{C}}$-orbits $\mathcal{O}$ on $\mathcal{N}_{\mathfrak{p}_{\mathbb{C}}}$ such that $\mathrm{pr}_{\mathfrak{g} \to \mathfrak{g}'}(\mathcal{O}) \subset \mathcal{N}_{\mathfrak{p}'_{\mathbb{C}}}$ by Proposition 5.3.4, one can observe that all the circled points below in the left are also those in the right. Hence, we conclude that

$$\mathrm{pr}_{\mathfrak{g} \to \mathfrak{g}'}(\overline{\mathcal{O}_{\substack{1\,1 \\ 1\,0}}}) \subset \mathcal{N}_{\mathfrak{p}'_{\mathbb{C}}}.$$

Therefore, $\mathrm{pr}_{\mathfrak{g} \to \mathfrak{g}'}(\mathrm{Ad}(K_{\mathbb{C}})(\mathfrak{u}_- \cap \mathfrak{p}_{\mathbb{C}})) \subset \mathcal{N}_{\mathfrak{p}'_{\mathbb{C}}}$, namely, the condition (iv) of Theorem 7.2.2 holds.



Closure $\overline{\mathcal{O}_{\substack{1\,1\\1\,0}}}$ in $\mathcal{N}_{\mathfrak{p}_{\mathbb{C}}}/K_{\mathbb{C}}$.



$\mathrm{pr}^{-1}_{\mathfrak{g}\to\mathfrak{g}'}(\mathcal{N}_{\mathfrak{p}'_{\mathbb{C}}})$
(see Proposition 5.3.4)

Lecture 8 Applications of branching problems

So far, we have explained a basic theory of restrictions of unitary representations of reductive Lie groups, with emphasis on discrete spectrum for non-compact subgroups. In Lecture 8, let us discuss what restrictions can do for representation theory. Furthermore, we shall discuss briefly some new interactions of unitary representation theory with other branches of mathematics through **restrictions** of representations to subgroups.

For further details on applications, we refer to [45, Sections 3 and 4], and [50, Sections 4, 5 and 6] and references therein.

8.1 Understanding representations via restrictions

8.1.1 Analysis and Synthesis

An idea of "analysis through decomposition" is to try to decompose the object into the smallest units, and to try to understand how it can be built up from the smallest units. This method may be pursued until one reaches the smallest units. Then, what can we do with the analysis on the smallest units? Only a complete change of viewpoint allows us to go further. For example, molecules may be regarded as the "smallest units"; but they could be decomposed into atoms which may be regarded as the "smallest" in another sense, and then they consist of electrons, protons and neutrons, and then... Each step for these decompositions requires a different viewpoint.

An irreducible representations π of a group is the "smallest unit" as representations of G. Now, a subgroup G' could provide a "different viewpoint". In fact, π is no more the "smallest unit" as representations of G'. This leads us a **method** to study the irreducible representations of G by taking the restriction to G', still in the spirit of "analysis through decomposition".

8.1.2 The Cartan-Weyl highest weight theory, revisited

Let G be a connected compact Lie group, and $G' = T$ its maximal toral subgroup.

Let π be an irreducible representation of G. Obviously π is T-admissible. We write the branching law of the restriction $\pi|_T$ as

$$\pi|_T \simeq \bigoplus_{\lambda} n_\pi(\lambda)\mathbb{C}_\lambda, \tag{8.1.2}$$

where $\mathbb{C}_\lambda$ is a one dimensional representation of T, and $n_\pi(\lambda)$ is its multiplicity. Of course, the whole branching law (or **weight decomposition**) (8.1.2) determines the representation π in this case because finite dimensional representations are determined by their characters and because characters are determined on their restrictions to the maximal torus T. Much more strongly, the Cartan-Weyl highest weight theory asserts that a single element λ (the "largest" piece in the decomposition (8.1.2)) is sufficient to determine π.

This may be regarded as an example of the spirit: understanding of representations through their restrictions to subgroups.

8.1.3 Vogan's minimal K-type theory

Let G be a reductive linear Lie group, and $G' = K$ a maximal compact subgroup.

Let π be an irreducible unitary representation of G. Then π is K-admissible (Harish-Chandra's admissibility theorem, see Theorem 2.4.6). As we have already mentioned in Lecture 4, Harish-Chandra's admissibility laid the foundation of the theory of $(\mathfrak{g}, K)$-modules, an algebraic approach to infinite dimensional representations of reductive Lie groups.

Let us write the branching law as

$$\pi|_K \simeq \sum_{\tau \in \widehat{K}}^{\oplus} n_\pi(\tau)\tau. \tag{8.1.3}$$

Then, Vogan's minimal K-type theory shows that a single element (or a few number of elements) τ (the "smallest ones" in the branching law (8.1.3)) gives a crucial information for the classification of irreducible $(\mathfrak{g}, K)$-modules ([93, 95]) and and also for the understanding of the unitary dual $\widehat{G}$ ([82, 100]).

This may be regarded as another example of the spirit: understanding of representations through their restrictions to subgroups.

8.1.4 Restrictions to non-compact groups

In contrast to the restriction to compact subgroups as we discussed in §8.1.2, §8.1.3, not much is known about the restriction of irreducible unitary representations to non-compact subgroups.

I try to indicate some few examples where the restriction to non-compact subgroups G' gives a successful clue to understand representations π of G.

This idea works better, particularly, in the case where π belongs to "singular" (or "small") representations, which are the most mysterious part of the unitary dual $\widehat{G}$ in the current status.

1) (Parabolic restriction) Some of "small" representations of G remain irreducible when restricted to parabolic subgroups. (See, for example, [59, 106].) Conversely, Torasso [90] constructed minimal representations by making use of their restriction to maximal parabolic subgroups.

2) (Non-vanishing condition for $A_{\mathfrak{q}}(\lambda)$) In the philosophy of the orbit method, it is perhaps a natural problem to classify all integral orbits $\mathrm{Ad}(G)\lambda$ such that the corresponding unitary representation $\Pi(\lambda) \neq 0$) (or equivalently, the underlying $(\mathfrak{g}, K)$-module $A_{\mathfrak{q}}(\lambda) \neq 0$, see §7.1.4). This is always the case if λ is in the good range [96], but the general case remains open.

There are some partial results on this problem by using the restriction of certain non-compact reductive subgroup G' such that the restriction $\Pi(\lambda)|_{G'}$ is G'-admissible (see Theorem 7.2.2 for a criterion). See [39, Chapter 4] for combinatorial computations on singular parameters λ such that $A_{\mathfrak{q}}(\lambda) \neq 0$ for some classical groups. There are also different approaches to this problem (see [39, Chapter 5]; [91]).

3) (Jordan-Hölder series) Even in the case where π is neither unitary nor irreducible, the restriction to subgroups may give a good tool to study the representation. For example, Lee and Loke [62] determined explicitly the Jordan-Hölder series of certain degenerate principal series representations and classified which irreducible subquotients are unitarizable. Previously known results of this feature were mostly in the case where degenerate principal series representations have a multiplicity free K-type decomposition as was in the Howe and Tan's paper [26]. Lee and Loke were able to treat a more general case by replacing K by certain non-compact subgroup G' such that the restriction to G' splits discretely with multiplicity free.

These three examples may be also regarded as concrete cases where one can have a better understanding of unitary representations through their restrictions to subgroups.

8.2 Construction of representations of subgroups

Suppose we are given a representation π of G. Then the knowledge of (a part of) the branching law of the restriction $\pi|_{G'}$ may be regarded as a **construction** of irreducible representations of G'. One of advantages of admissible restrictions is that there is no continuous spectrum in the branching law so that each irreducible summand could be explicitly captured. In this way, admissible restrictions may also serve as a **method** to study representations of **subgroups**.

8.2.1 Finite dimensional representations

Consider the natural representation of $G' = GL(n, \mathbb{C})$ on $V = \mathbb{C}^n$. Then the m-th tensor power $T^m(V) = V \otimes \cdots \otimes V$ becomes an irreducible representation of the direct product group $G = G' \times \cdots \times G'$. The restriction from G to the diagonally embedded subgroup of G' is no more irreducible, and its branching law gives rise to irreducible representations of G' as irreducible summands. When restricted to the diagonally embedded subgroup $G' \simeq GL(n, \mathbb{C})$, it decomposes into irreducible representations of G'. (A precise description is given by the Schur-Weyl duality that treats $T^m(V)$ as a representation of the direct product group $GL(n, \mathbb{C}) \times \mathfrak{S}_m$.) Conversely, any polynomial representation of G' can be obtained in this way for some $m \in \mathbb{N}$.

This may be regarded as a construction of irreducible representations of the subgroup $GL(n, \mathbb{C})$ via branching laws.

8.2.2 highest weight modules

The above idea can be extended to construct some irreducible infinite dimensional representations, called **highest weight representations**. For example, let us consider the Weil representation π of the metaplectic representation $G'_1 = Mp(n, \mathbb{R})$. Then, the m-th tensor power $\pi \otimes \cdots \otimes \pi$ becomes a representation of $G'_1 \times O(m)$, which decomposes discretely into irreducible representations of $G'_1 \times O(m)$ with multiplicity free, and in particular, giving rise to irreducible highest weight representations of G'_1. More generally, we have the following theorem which is a part of Howe's theory on reductive dual pairs:

Theorem 8.2.2 (Theta correspondence, [24]). *Let π be the Weil representation of the metaplectic group $G = Mp(n, \mathbb{R})$, and $G' = G'_1 G'_2$ forms a dual pair with G'_2 compact. Then the restriction $\pi|_{G'}$ decomposes discretely into irreducible representations of $G' = G'_1 G'_2$ with multiplicity free. In particular, the restriction $\pi|_{G'}$ is G'-admissible. Furthermore, each irreducible summand is a highest weight representation of G'.*

A typical example of the setting of Theorem 8.2.2 is given locally as

$$(\mathfrak{g}, \mathfrak{g}'_1, \mathfrak{g}''_2) = (\mathfrak{sp}(nm, \mathbb{R}), \mathfrak{sp}(n, \mathbb{R}), \mathfrak{o}(m)), (\mathfrak{sp}((p+q)k, \mathbb{R}), \mathfrak{u}(p,q), \mathfrak{u}(k)).$$

The classification of irreducible highest weight representations was accomplished in early 1980s by Enright-Howe-Wallach and Jakobsen, independently (see [12, 28]). Quite a large part of irreducible highest weight representations were constructed as irreducible summands of the restriction of the Weil representation by Howe, Kashiwara, Vergne and others in 1970s.

Remark 8.1. Theorem 8.2.2 fits into our framework of Sections 4 – 7. Very recently, Nishiyama-Ochiai-Taniguchi [75] and Enright-Willenbring [13] have made a detailed study on irreducible summands occurring in the restriction $\pi|_{G'}$ in the setting of Theorem 8.2.2 under the assumption that (G, G') is in the stable range with G'_2 smaller, that is, $m \leq \mathbb{R}$-rank G'_1. Then, as was pointed out in Enright-Willenbring [13, Theorem 6], the Gelfand-Kirillov dimension of each irreducible summand Y is dependent only on the dual pair and is independent of Y. They proved this result based on case-by-case argument. We note that this result follows directly from a general theory ([44, Theorem 3.7]) of discretely decomposable restrictions. Furthermore, one can show by using the results in [13, 44] that the associated variety $\mathcal{V}_{\mathfrak{g}'_{\mathbb{C}}}(Y)$ coincides with the projection $\mathrm{pr}_{\mathfrak{g}\to\mathfrak{g}'}(\mathcal{V}_{\mathfrak{g}_{\mathbb{C}}}(\pi))$. (See Theorem 5.2.1 for the inclusive relation in the general case).

8.2.3 Small representations

The idea of constructing representations of subgroups as irreducible summands works also for non-highest weight representations.

As one can observe from the criterion for the admissibility of the restrictions (see Theorem 6.3.4), the restriction $\pi|_{G'}$ tends to be discretely decomposable if its asymptotic K-support $AS_K(\pi)$ is small. In particular, if π is a minimal representation, then $AS_K(\pi)$ is one dimensional (Remark 6.3.5). Thus, there is a good possibility that $AS_K(\pi)$ is discretely decomposable if π is "small".

For example, if $G = O(p,q)$ $(p, q \geq 2; p+q \geq 8$ even) and π is the minimal representation of G, then the restriction to its natural subgroup $G' = O(p,q') \times O(q'')$ is G'-admissible for any q', q'' such that $q' + q'' = q$ (see [57, Theorem 4.2]). In this case, branching laws give rise to unitary representations "attached to" minimal elliptic orbits (recall the terminology from §7.1). It is also G''-admissible if $G'' = U(p', q')$, $p = 2p'$ and $q = 2q'$.

The idea of constructing representations via branching laws was also used in a paper by Gross and Wallach [19], where they constructed interesting "small" unitary representations of exceptional Lie groups G' by taking the restrictions of another small representation of G.

Discretely decomposable branching laws for non-compact G' are used also in the theory of automorphic forms for exceptional groups by J.-S. Li [64].

Note also that Neretin [74] recently constructed some of irreducible unitary representations of $O(p,q)$ with K-fixed vector (namely, so called **spherical unitary representations**) as discrete spectrum of the restriction of irreducible representations of $U(p,q)$. In his case, the restriction contains both continuous and discrete spectrum.

8.3 Branching problems

In general, it is a hard problem to find explicitly branching laws of unitary representations. Except for highest weight modules (e.g. the Weil representation, holomorphic discrete series representations) or principal series representations, not much has been studied systematically on the branching laws with respect to non-compact subgroups until recently.

From the viewpoint of finding explicit branching law, an advantage of admissible restrictions is that we may employ algebraic techniques because there is no continuous spectrum. Recently, a number of explicit branching laws have been found (e.g. [19, 20, 38, 40, 41, 57, 64, 65, 66, 108]) in the context of admissible restrictions to non-compact reductive subgroups.

8.4 Global analysis

Let G/H be the homogeneous space of a Lie group G by a closed subgroup H. The idea of non-commutative harmonic analysis is to try to understand functions on G/H by means of representations of the group G. For example, we refer to the articles in the same series by van den Ban, Delorme, Schlichtkrull on this subject where G/H is a reductive symmetric space.

Our main concern here is with non-symmetric spaces for which very little has been known.

8.4.1 Global analysis and restriction of representations

Our approach on this problem is different from traditional approaches: The main machinery here is the restriction of representations.

1) Embed G/H into a larger homogeneous space $\widetilde{G}/\widetilde{H}$.

2) Realize a representation $\widetilde{\pi}$ of $\widetilde{G}$ on a subspace $\widetilde{V} \hookrightarrow C^\infty(\widetilde{G}/\widetilde{H})$.

3) Restrict the space $\widetilde{V}$ of functions to the submanifold G/H, and understand it by the restriction $\widetilde{\pi}|_G$ of representations.

One may also consider variants of this idea by replacing C^∞ by L^2, holomorphic functions, sections of vector bundles, or cohomologies. Also, one may consider the restriction to submanifolds after taking normal derivatives (e.g. [72, 29]).

Our optimistic idea here (which is used in [38, 41], for example) is that the knowledge of any two of the three would be useful in understanding the remaining one.

$$\begin{tikzcd} & \text{Restriction from } \widetilde{G} \text{ to } G \arrow[dl, Leftrightarrow] \arrow[dr, Leftrightarrow] & \\ \text{Analysis on } G/H \arrow[rr, Leftrightarrow] & & \text{Analysis on } \widetilde{G}/\widetilde{H} \end{tikzcd}$$

In particular, we shall consider the setting where G/H is the space on which we wish to develop harmonic analysis, and where $\widetilde{G}/\widetilde{H}$ is the space on which we have already a good understanding of harmonic analysis (e.g. a group manifold or a symmetric space).

8.4.2 Discrete series and admissible representations

As we explained in §7.1.7 and §7.1.8, a necessary and sufficient condition for the existence of discrete series representations is known for a reductive group and also for a reductive symmetric space. However, in the generality that $H \subset G$ are a pair of reductive subgroups, it is still an open problem to determine which homogeneous space G/H admits discrete series representations.

Let us apply the strategy in §8.4.1 to a non-symmetric homogeneous space G/H. We start with a discrete series $\widetilde{\pi}$ for $\widetilde{G}/\widetilde{H}$, and consider the branching law of the restriction $\widetilde{\pi}|_G$.

We divide the status of an embedding $G/H \hookrightarrow \widetilde{G}/\widetilde{H}$ into three cases, from the best to more general settings.

1) The case $G/H \simeq \widetilde{G}/\widetilde{H}$.

If subgroups $G, \widetilde{H} \subset \widetilde{G}$ satisfy $G\widetilde{H} = \widetilde{G}$ then we have a natural diffeomorphism $G/H \simeq \widetilde{G}/\widetilde{H}$. For example,

$$\begin{aligned}
U(p,q)/U(p-1,q) &\simeq SO_0(2p,2q)/SO_0(2p-1,2q),\\
G_2(\mathbb{R})/SL(3,\mathbb{R}) &\simeq SO_0(3,4)/SO_0(3,3),\\
G_2(\mathbb{R})/SU(2,1) &\simeq SO_0(3,4)/SO_0(4,2).
\end{aligned}$$

(see [41, Example 5.2] for more examples). In this case, any discrete spectrum of the branching law contribute to discrete series representations for G/H, and conversely, all discrete series representations for G/H are obtained in this way.

2) (Generic orbit) Suppose G/H is a principal orbit in $\widetilde{G}/\widetilde{H}$ in the sense of Richardson, namely, there is a G-open subset U in $\widetilde{G}/\widetilde{H}$ such that any G-orbit in U is isomorphic to G/H. Then, any discrete spectrum of the branching law $\widetilde{\pi}|_G$ contributes to discrete series representations for G/H [46, §8]. See [38, 41, 63] for concrete examples.

3) (General case; e.g. sigular orbits) For a general embedding $G/H \hookrightarrow \widetilde{G}/\widetilde{H}$ the above strategy may not work; the restriction of L^2-functions does not always yield L^2-functions on submanifolds. A remedy for this is to impose the G-**admissibility of the restriction** of $\widetilde{\pi}$, which justifies again the above strategy ([46]).

For instance, let us consider the action of a group G on G itself. If we consider the left action, then the action is transitive and we cannot get new results from the above strategy. However, the action is non-trivial if we consider it from both the left and the right such as

$$G \to G, x \mapsto gx\sigma(g)^{-1}$$

for some group automorphism σ of G (e.g. σ is the identity, a Cartan involution, etc.). For example, if $G = Sp(2n,\mathbb{R})$ and take an involutive automorphism σ such that $G^\sigma \simeq Sp(n,\mathbb{C})$, then the following homogeneous manifolds

$$G/H = Sp(2n,\mathbb{R})/(Sp(n_0,\mathbb{C}) \times GL(n_1,\mathbb{C}) \times \cdots \times GL(n_k,\mathbb{C}))$$

occur as G-orbits on G for any partition $(n_0, \ldots, n_k)$ of n. (These orbits arise as "general case", namely, in the case (3).) The above method tells that there always exist discrete series for the above homogeneous spaces for any partition $(n_0, \dots, n_k)$ of n. We note that G/H is a symmetric space if and only if $n_1 = \cdots = n_k = 0$.

There are also further results, for example, by Neretin, Olshanski-Neretin, and Ørsted-Vargas that interact the restriction of representations and harmonic analysis on homogeneous manifolds [73, 74, 78].

8.5 Discrete groups and restriction of unitary representations

8.5.1 Matsushima-Murakami's formula

Let G be a reductive linear Lie group, and Γ a cocompact discrete subgroup without torsion. By Gelfand-Piateski-Shapiro's theorem (see Theorem 2.4.3), the right regular representation on $L^2(\Gamma\backslash G)$ is G-admissible, so that it decomposes discretely:

$$L^2(\Gamma\backslash G) \simeq \sum_{\pi \in \widehat{G}} n_\Gamma(\pi)\pi$$

with $n_\Gamma(\pi) < \infty$.

Since Γ acts on the Riemannian symmetric space G/K properly discontinuously and freely, the quotient space $X = \Gamma\backslash G/K$ becomes a compact smooth manifold. Its cohomology group can be described by means of the multiplicities $n_\Gamma(\pi)$ by a theorem of Matsushima-Murakami (see a treatise of Borel-Wallach [6]):

Theorem 8.5.1. *Let $X = \Gamma\backslash G/K$ be as above. Then*

$$H^*(X; \mathbb{C}) \simeq \bigoplus_{\pi \in \widehat{G}} \mathbb{C}^{n_\Gamma(\pi)} \otimes H^*(\mathfrak{g}, K; \pi_K).$$

Here $H^*(\mathfrak{g}, K; \pi_K)$ denotes the $(\mathfrak{g}, K)$ cohomology of the $(\mathfrak{g}, K)$-module π_K ([6, 95]). We say $\mathbb{C}^{n_\Gamma(\pi)} \otimes H^*(\mathfrak{g}, K; \pi_K)$ is the π-component of $H^*(X; \mathbb{C})$, and denote by $H^*(X)_\pi$.

Furthermore, $H^*(\mathfrak{g}, K; \pi_K)$ is non-zero except for a finite number of π, for which the $(\mathfrak{g}, K)$-cohomologies are explicitly computed by Vogan and Zuckerman ([101]). (More precisely, such π is exactly the representations that we explained in §7.1.3, namely, π is isomorphic to certain $\Pi(\lambda)$ with $\lambda = \rho_\lambda$.)

8.5.2 Vanishing theorem for modular varieties

Matsushima-Murakami's formula interacts the topology of a compact manifold $X = \Gamma\backslash G/K$ with unitary representations of G. Its object is a single manifold X. Let us consider the topology of morphisms, that is our next object is a pair of manifolds Y, X. For this, we consider the following setting:

$$\begin{array}{ccccc} \Gamma' & \subset & G' & \supset & K', \\ \cap & & \cap & & \cap \\ \Gamma & \subset & G & \supset & K, \end{array}$$

such that $G' \subset G$ are a pair of reductive linear Lie groups $K' := K \cap G'$ is a maximal compact subgroup, and $\Gamma' := \Gamma \cap G$ is a cocompact in G'. Then $Y := \Gamma'\backslash G'/K'$ is also a compact manifold. We have a natural map

$$\iota : Y \to X,$$

and the **modular variety** $\iota(Y)$ defines a totally geodesic manifold in X. We write $[Y] \in H_m(Y;\mathbb{Z})$ for the fundamental class defined by Y, where we put $m = \dim Y$. Then, the cycle $\iota_*[Y]$ in the homology group $H_m(X;\mathbb{Z})$ is called the **modular symbol**.

Theorem 8.5.2 (a vanishing theorem for modular symbols). *If* $\mathrm{AS}_K(\pi) \cap C_K(K') = \{0\}$ *(see Theorem 6.3.4) and if* $\pi \neq \mathbf{1}$ *(the trivial one dimensional representation), then the modular symbol* $\iota_*[Y]$ *is annihilated by the* π*-component* $H^m(X)_\pi$ *in the perfect pairing* $H_m(X;\mathbb{C}) \times H^m(X;\mathbb{C}) \to \mathbb{C}$.

The discreteness of irreducible decomposition plays a crucial role both in Matsushima-Murakami's formula and in a vanishing theorem for modular varieties. In the former, $L^2(\Gamma\backslash G)$ is G-admissible (Gelfand-Piateski-Shapiro), while the restriction $\pi|_{G'}$ is G'-admissible (see Theorem 6.3.4) in the latter.

8.5.3 Clifford-Klein problem

A Clifford-Klein form of a homogeneous space G/H is the quotient manifold $\Gamma\backslash G/H$ where Γ is a discrete subgroup of G acting properly discontinuously and freely on G/H. Any Riemannian symmetric space G/K admits a compact Clifford-Klein form (Borel [5]). On the other hand, there is no compact Clifford-Klein form of $O(n,1)/O(n-1)$, namely, any complete Lorentz manifold with constant sectional curvature is non-compact (Calabi-Markus phenomenon [7]).

It is an unsolved problem to classify homogeneous spaces G/H which admit compact Clifford-Klein forms even for the special case where G/H is a symmetric space such as $SL(n, \mathbb{R})/SO(p, n-p)$.

Recently, Margulis revealed a new connection of this problem with restrictions of unitary representations. He found an obstruction for the existence of compact Clifford-Klein forms for G/H. His approach is to consider the unitary representation of G on the Hilbert space $L^2(\Gamma\backslash G)$ from the right (Γ is a discrete subgroup of G), and to take the restriction to the subgroup H. The key technique is to study the asymptotic behavior of matrix coefficients of these unitary representations (see a paper of Margulis[70] and also of Oh [76]).

We refer to [49, 71] and references therein for an overall exposition and open questions related to this problem.

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