Deformation of Properly Discontinuous Actions of $\mathbb{Z}^k$ on $\mathbb{R}^{k+1}$

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Abstract

We consider the deformation of a discontinuous group acting on the Euclidean space by affine transformations. A distinguished feature here is that even a 'small' deformation of a discrete subgroup may destroy proper discontinuity of its action. In order to understand the local structure of the deformation space of discontinuous groups, we introduce the concepts from a group theoretic perspective, and focus on 'stability' and 'local rigidity' of discontinuous groups. As a test case, we give an explicit description of the deformation space of $\mathbb{Z}^k$ acting properly discontinuously on $\mathbb{R}^{k+1}$ by affine nilpotent transformations. Our method uses an idea of 'continuous analogue' and relies on the criterion of proper actions on nilmanifolds.

1 Local rigidity and stability

Our concern in this article is with the deformation of discontinuous groups for non-Riemannian homogeneous spaces.

1.1 Deformation of discontinuous groups — in the non-Riemannian case

In contrast to the traditional case of discontinuous groups acting on Riemannian manifolds as isometries, our problem in the non-Riemannian case includes the following subproblem: if a discrete subgroup can be deformed, determine the range of the deformation parameters that does not destroy the proper discontinuity of its action.

As a clue to understanding the local structure of the 'deformation space', we consider a manifold X acted on by a Lie group G. Suppose Γ is a discontinuous group for X, that is, Γ is a discrete subgroup of G acting properly discontinuously and freely on X. Let Γ' be another discrete subgroup of G which is 'sufficiently close' to Γ. Now, our basic question is if the following statements hold or not:

(R)′ : (*Local Rigidity*) Γ' is conjugate to Γ by an inner automorphism of G.

(S)′ : (*Stability*) The Γ'-action on X is properly discontinuous and free.

For a homogeneous space $X = G/H$ (H being a closed subgroup of G), obvious remarks are:

1) If H is compact, then (S)′ automatically holds.

2) If (R)′ holds, so does (S)′.

This article studies the deformation of discontinuous groups for G/H in the case that (S)′ does not hold. This implies particularly that H is non-compact and that (R)′ does not hold.

Let us now formalize the above two statements (R)′ and (S)′ more rigorously. We begin with an (abstract) finitely generated group Γ, and denote by $\mathrm{Hom}(\Gamma, G)$ the set of all group homomorphisms of Γ into a Lie group G. Taking generators $\gamma_1, \ldots, \gamma_k$ of Γ, we use the injective map

$$\mathrm{Hom}(\Gamma, G) \hookrightarrow G \times \cdots \times G, \quad \varphi \mapsto (\varphi(\gamma_1), \ldots, \varphi(\gamma_k))$$

to endow $\mathrm{Hom}(\Gamma, G)$ with the induced topology from the direct product $G \times \cdots \times G$. This topology is independent of the choice of generators. Now, suppose G acts continuously on a manifold X. We recall from [6] the following definition:

$$R(\Gamma, G; X) := \{\varphi \in \mathrm{Hom}(\Gamma, G) \ : \quad \varphi \text{ is injective, and } \\ \varphi(\Gamma) \text{ acts properly discontinuously and freely on } X\}. \tag{1.1}$$

We remark that our notation here is slightly different from that in [6, 9]: for a homogeneous space $X = G/H$ our notation $R(\Gamma, G; G/H)$ here coincides with $R(\Gamma, G, H)$ loc. cit.

For each $\varphi \in R(\Gamma, G; X)$, the quotient space $\varphi(\Gamma)\backslash X \simeq \varphi(\Gamma)\backslash G/H$ (a *Clifford–Klein form* of X) becomes a Hausdorff topological space, and can be given a unique manifold structure for which the natural quotient map $X \to \varphi(\Gamma)\backslash X$ is a local diffeomorphism. Therefore, the Clifford–Klein form $\varphi(\Gamma)\backslash X$ enjoys all G-invariant local geometric structures on X. Thus, $R(\Gamma, G; X)$ may be regarded as a parameter space of Clifford–Klein forms $\varphi(\Gamma)\backslash X$ with parameter φ.

To be more precise on 'parameter', we should take into account 'unessential' deformation arising from inner automorphisms. If two homomorphisms φ_1 and φ_2 belonging to $R(\Gamma, G; X)$ satisfy $\varphi_2 = g \circ \varphi_1 \circ g^{-1}$ for some $g \in G$, then the corresponding Clifford–Klein forms are isomorphic to each other by the natural diffeomorphism $\varphi_1(\Gamma)\backslash X \xrightarrow{\sim} \varphi_2(\Gamma)\backslash X$, $\varphi_1(\Gamma)xH \mapsto \varphi_2(\Gamma)gxH$. In light of this observation, we define the *deformation space* as the quotient set

$$\mathcal{T}(\Gamma, G; X) := R(\Gamma, G; X)/G. \tag{1.2}$$

For example, if $G = PSL(2, \mathbb{R})$ and X is the upper half plane, and if Γ is the fundamental group of a closed Riemann surface M_g of genus $g \geq 2$, then $\mathcal{T}(\Gamma, G; X)$ is nothing other than the Teichmüller space of M_g. We refer the reader to an expository paper [9] for some elementary examples of the deformation space for non-Riemannian X.

Suppose now that $\varphi_0 : \Gamma \to G$ belongs to $R(\Gamma, G; X)$, and we reformalize (R)$'$ and (S)$'$ as follows:

(R): (*Local rigidity*) $G \cdot \varphi_0$ is open in $\mathrm{Hom}(\Gamma, G)$.

(S): (*Stability*) There is an open subset V of $\mathrm{Hom}(\Gamma, G)$ such that $\varphi_0 \in V \subset R(\Gamma, G; X)$.

We say $\varphi_0 \in R(\Gamma, G; X)$ is *locally rigid as a discontinuous group* for X if (R) holds. For a Riemannian symmetric space X, our terminology here is consistent with Weil's terminology used in [16].

A celebrated Selberg–Weil rigidity [16] for an irreducible **Riemannian** symmetric space X asserts that (R) holds for any torsion free uniform lattice $\varphi_0(\Gamma)$ of G unless G is locally isomorphic to $SL(2, \mathbb{R})$, whereas $R(\Gamma, G; X)$ is always open in $\mathrm{Hom}(\Gamma, G)$ and thus (S) holds. In contrast, there is an example that (R) fails for an irreducible **non-Riemannian** symmetric space X of an arbitrarily high dimension (see [6]).

The failure of (R) arouses our interest in the deformation space $\mathcal{T}(\Gamma, G; X)$ like the classical Teichmüller theory of Riemann surfaces. Besides, the concept of the stability (S) may be regarded as a first step to understand the local structure of the deformation space $\mathcal{T}(\Gamma, G; X)$ in the setting where (R) fails, in particular, where H is non-compact.

Such a viewpoint traces back to the paper [4] on three dimensional Lorentz space forms, where Goldman discovered a discontinuous group Γ for which (R) fails, and raised a question if (S) still holds (not exactly in the way formulated here). His case concerns with a semisimple Lie group G which is locally isomorphic to $SO(2, 1) \times SO(2, 1)$. This question was solved affirmatively, namely, there exists a cocompact discontinuous group Γ for which (R) fails but (S) holds in the generality that G is a semisimple Lie group which is locally isomorphic to the direct product of two copies of $SO(n, 1)$ or $SU(n, 1)$ (see [8, 15]). Its proof relies on the criterion for properly discontinuous actions [3, 7] on homogeneous spaces of reductive groups.

For a more general (Γ, G, X) such as the affine transformation group G, both (R) and (S) can fail, as is seen by the following one dimensional example:

Example 1.1. Let $\Gamma := \mathbb{Z}$, and G be the $ax + b$ group, that is, $G = \{(a, b) : a > 0, b \in \mathbb{R}\}$ with the multiplication given by $(a, b) \cdot (a', b') = (aa', ab' + b)$. Consider the affine transformation of G on $X := \mathbb{R}$. Then, $\mathrm{Hom}(\Gamma, G) \simeq G$, whereas $R(\Gamma, G; X) \simeq \{(1, b) : b \neq 0\}$. Hence, neither (R) nor (S) holds.

The above example deals with a homogeneous space X of a solvable group G and with a cocompact discontinuous group Γ.

1.2 Summary of this article

This article analyses the failure of (R) and (S) for homogeneous spaces of **nilpotent** Lie groups G. For a simply-connected nilpotent Lie group G,

$R(\Gamma, G; X)$ becomes open in $\mathrm{Hom}(\Gamma, G)$ and thus (S) always holds if $\Gamma\backslash X$ is compact ([19]). Thus, our interest here is in the case when $\Gamma\backslash X$ is non-compact. As a test case, we initiate a detailed analysis on the deformation space in the following setting:

$$\begin{aligned}
&\Gamma := \mathbb{Z}^k && \text{(free abelian group of rank } k\text{)},\\
&X := \mathbb{R}^{k+1} && \text{(nilmanifold)},\\
&G \subset \mathrm{Aff}(\mathbb{R}^{k+1}) && \text{(a two-step nilpotent subgroup)},
\end{aligned}$$

where Γ acts on X as nilpotent affine transformations via G. Then, we propose a method of giving a concrete description of $R(\Gamma, G; X)$ and the deformation space $\mathcal{T}(\Gamma, G; X)$ for a specific choice of G. Our main results are Theorems 2.3 and 5.1.

Besides, we shall see in Corollary 5.1.1 that the deformation space contains a smooth manifold $\mathcal{T}'(\Gamma, G; X)$ as its open dense subset such that

$$\dim \mathcal{T}'(\Gamma, G; X) = \begin{cases} 2k^2 - 1 & (k : \text{even}),\\ 2k^2 - 2 & (k : \text{odd}, \ \geq 3),\\ \quad 2 & (k = 1). \end{cases}$$

Thus, local rigidity (R) fails for any k because the dimension of the deformation space is positive. In the above formula, one sees that the dimension of the deformation space $\mathcal{T}(\Gamma, G; X)$ has a different feature according to whether k is even or odd. This will be explained by the criterion of properly discontinuous actions which involves the existence of a non-zero real eigenvalue of a certain $k \times k$ matrix, whence the parity of k counts. Moreover, it follows from the complete description of $R(\Gamma, G; X)$ that we can determine for which $\varphi_0 \in R(\Gamma, G; X)$ the stability (S) fails.

Our specific choice of G was motivated by Lipsman's classification [11] of maximal nilpotent affine transformation groups on $\mathbb{R}^3$, in which the two-step nilpotent group G for $k = 2$ played a crucial role. It is noteworthy that for any subgroup $\widetilde{G}$ of the affine transformation group $\mathrm{Aff}(\mathbb{R}^{k+1})$ containing our specific G, $R(\Gamma, \widetilde{G}; X)$ is not open in $\mathrm{Hom}(\Gamma, \widetilde{G})$ by Theorem 2.3, and consequently both (R) and (S) fail.

The key idea of our proof is to take a connected subgroup L that contains Γ as a cocompact discrete subgroup, and then to show that every injective homomorphism from Γ into G extends uniquely to a continuous homomorphism from L into G (an idea of *syndetic hull*). A second step is to find explicitly $\mathrm{Hom}(L,G)$ in place of $\mathrm{Hom}(\Gamma,G)$, and to determine which homomorphism yields a properly discontinuous action. Unlike the reductive case [3, 5, 7], properly discontinuous actions for affine transformation groups on $\mathbb{R}^{k+1}$ are far from being understood in general, as one sees the long-standing Auslander conjecture (see [1] and references therein). However, fortunately in our special setting, we can use the criterion [13] of proper actions for two-step nilpotent Lie groups, which was obtained as an affirmative solution to Lipsman's conjecture [11]. Then, the final step of the proof of Theorem 2.3 (the description of $R(\Gamma,G;X)$) is reduced to a certain problem of Lie algebras, which we can solve explicitly.

2 Description of deformation parameter

This section gives a complete description of the parameter space $R(\mathbb{Z}^k,G;\mathbb{R}^{k+1})$ of properly discontinuous $\mathbb{Z}^k$-actions on $\mathbb{R}^{k+1}$ through a certain nilpotent affine transformation group G. This is the first of the main results of this paper, and is stated in Theorem 2.3. Building on it, we shall determine the deformation space $\mathcal{T}(\Gamma,G;\mathbb{R}^{k+1}) \simeq R(\Gamma,G;\mathbb{R}^{k+1})/G$ in Section 5 (see Theorem 5.1).

2.1 Nilpotent affine transformation group

We fix a positive integer k. Our basic setting in this paper is:

$$\begin{aligned}
&\Gamma := \mathbb{Z}^k,\\
&G := \left\{ \begin{pmatrix} I_k & \vec{x} & \vec{y} \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} : \vec{x},\vec{y} \in \mathbb{R}^k, z \in \mathbb{R} \right\},\\
&H := \left\{ \begin{pmatrix} I_k & \vec{x} & \vec{0} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : \vec{x} \in \mathbb{R}^k \right\}.
\end{aligned} \tag{2.1}$$

Then G is a simply connected two-step nilpotent Lie group, and the homogeneous space G/H is diffeomorphic to $\mathbb{R}^{k+1}$. We shall first give a description of $\mathrm{Hom}(\Gamma,G)$ in Proposition 2.2.1, and then determine explicitly $R(\Gamma,G;\mathbb{R}^{k+1})$ as its subset in Theorem 2.3.

Geometrically, this means that we determine all possible properly discontinuous affine actions of $\mathbb{Z}^k$ on $\mathbb{R}^{k+1}$ preserving differential forms $d\xi_{k+1}$ and $d\xi_i \wedge d\xi_{k+1}$ $(1 \leq i \leq k)$, where $(\xi_1, \ldots, \xi_{k+1})$ is the coordinate of $\mathbb{R}^{k+1}$.

2.2 Description of $\mathrm{Hom}(\Gamma, G)$

Any group homomorphism $\varphi : \Gamma \to G$ is determined by its evaluation at generators of Γ. Taking a standard basis $\{e_1, \ldots, e_k\}$ of the abelian group $\Gamma = \mathbb{Z}^k$, we regard $\mathrm{Hom}(\Gamma, G)$ as a subset of the direct product $G \times \cdots \times G$ by the evaluation map:

$$\mathrm{Hom}(\Gamma, G) \hookrightarrow G \times \cdots \times G, \quad \varphi \mapsto (\varphi(e_1), \ldots, \varphi(e_k)). \tag{2.2}$$

Let us describe the image of the injective map (2.2). For this, first we set

$$M_1 := \{(\vec{x}, Y, \vec{z}) \in \mathbb{R}^k \oplus M(k, \mathbb{R}) \oplus \mathbb{R}^k : \vec{z} \neq \vec{0}\,\} \subset M(k, k+2; \mathbb{R}), \tag{2.3}$$

$$M_2 := M(k, 2k; \mathbb{R}). \tag{2.4}$$

Then, $\dim M_1 = k(k+2)$ and $\dim M_2 = 2k^2$. Second, for $\vec{x}, \vec{y} \in \mathbb{R}^k$ and $z \in \mathbb{R}$, we define a $(k+2) \times (k+2)$ matrix by

$$g(\vec{x}, \vec{y}, z) := \exp\begin{pmatrix} 0_k & \vec{x} & \vec{y} \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} I_k & \vec{x} & \vec{y} + \frac{1}{2}z\vec{x} \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix}. \tag{2.5}$$

With this expression, our groups G and H (see Section 2.1) are expressed as

$$G = \{g(\vec{x}, \vec{y}, z) : \vec{x}, \vec{y} \in \mathbb{R}^k, z \in \mathbb{R}\},$$
$$H = \{g(\vec{x}, \vec{0}, 0) : \vec{x} \in \mathbb{R}^k\}.$$

Now we define maps

$$\Psi_i \,:\, M_i \longrightarrow G \times \cdots \times G \quad (i = 1,\, 2)$$

such that their jth $(1 \leq j \leq k)$ components are respectively given by

$$\Psi_1(\vec{x}, Y, \vec{z})_j := g(z_j \vec{x}, \vec{y_j}, z_j) \quad \text{for } Y = (\vec{y_1}, \cdots, \vec{y_k}),\ {}^t\vec{z} = (z_1, \cdots, z_k), \tag{2.6}$$

$$\Psi_2(X, Y)_j := g(\vec{x_j}, \vec{y_j}, 0) \qquad \text{for } X = (\vec{x_1}, \cdots, \vec{x_k}),\ Y = (\vec{y_1}, \cdots, \vec{y_k}). \tag{2.7}$$

Then, the topological space $\mathrm{Hom}(\Gamma, G)$ is described via (2.2) as follows:

Proposition 2.2.1 (Description of $\mathrm{Hom}(\Gamma, G)$**).**
1) *The maps* Ψ_1 *and* Ψ_2 *induce a bijection*

$$\Psi_1 \cup \Psi_2 : M_1 \cup M_2 \xrightarrow{\sim} \mathrm{Hom}(\Gamma, G).$$

In particular, Ψ_1 *and* Ψ_2 *are injective, and their images* $\Psi_1(M_1)$ *and* $\Psi_2(M_2)$ *are disjoint subsets contained in* $\mathrm{Hom}(\Gamma, G)$.
2) (closure relation) $\Psi_2(M_2)$ *is closed in* $\mathrm{Hom}(\Gamma, G)$, *whereas the closure of* $\Psi_1(M_1)$ *is given as*

$$\overline{\Psi_1(M_1)} = \Psi_1(M_1) \cup \Psi_2(M_2^d). \tag{2.8}$$

Here, M_2^d *is a subset of* M_2 *defined by*

$$M_2^d := \{(X, Y) : X,\ Y \in M(k, \mathbb{R}),\ \mathrm{rank}\, X \leq 1\}. \tag{2.9}$$

We shall give a proof of this proposition in Section 3.

2.3 Description of $R(\Gamma, G; \mathbb{R}^{k+1})$

Let of introduce the following subsets of M_1 and M_2:

$$M_1^r := \{(\vec{x}, Y, \vec{z}) \in M(k, k+2; \mathbb{R}) : \vec{z} \neq \vec{0},\ \mathrm{rank}({}^tY, \vec{z}) = k\} \tag{2.10}$$
$$M_2^r := \{(X, Y) \in M(k, 2k; \mathbb{R}) : \det(Y - \lambda X) \neq 0 \text{ for any } \lambda \in \mathbb{R}\}. \tag{2.11}$$

We are now ready to characterize $R(\Gamma, G; \mathbb{R}^{k+1})$ as a subset of $\mathrm{Hom}(\Gamma, G)$. Here is the first of the main results in this paper:

Theorem 2.3 (Description of $R(\Gamma, G; \mathbb{R}^{k+1})$**).** *Let* G *be a nilpotent Lie group defined as* (2.1) *and* $\Gamma = \mathbb{Z}^k$. *Then, the maps* Ψ_1 *and* Ψ_2 *(see* (2.6) *and* (2.7)*) induce the bijection*

$$\Psi_1 \cup \Psi_2 : M_1^r \cup M_2^r \xrightarrow{\sim} R(\Gamma, G; \mathbb{R}^{k+1}).$$

We shall give a proof of this theorem in Section 4.

2.4 Generic points of $R(\Gamma, G; \mathbb{R}^{k+1})$

This subsection studies a generic part of $R(\Gamma, G; \mathbb{R}^{k+1})$ by analysing the sets M_1^r and M_2^r in detail.

For X, $Y \in M(k, \mathbb{R})$, we define a polynomial of λ by

$$f(X, Y; \lambda) := \det(Y - \lambda X) = \sum_{l=0}^{k} a_l(X, Y)\lambda^l. \tag{2.12}$$

Here, we note

$$\begin{aligned} a_0(X, Y) &= \det Y, \\ a_{k-1}(X, Y) &= (-1)^{k-1} \sum_{i,j=1}^{k} (-1)^{i+j} Y_{ij} \det \widehat{X}_{ij}, \\ a_k(X, Y) &= (-1)^k \det X, \end{aligned}$$

where $\widehat{X}_{ij}$ denotes the submatrix obtained by deleting row i and column j from X.

Let us recall from (2.11) and (2.12) that

$$M_2^r = \{(X, Y) \in M(k, 2k; \mathbb{R}) : f(X, Y; \lambda) \neq 0 \text{ for any } \lambda \in \mathbb{R}\}.$$

In order to give a 'generic' part of $R(\Gamma, G; \mathbb{R}^{k+1})$ by means of Theorem 2.3, we set

$$M_2^{\mathrm{ro}} := \begin{cases} M_2^r \cap \{(X, Y) : \det X \neq 0\} & (k\text{: even}), \\ M_2^r \cap \{(X, Y) : \operatorname{rank} X = k - 1, a_{k-1}(X, Y) \neq 0\} & (k\text{: odd}). \end{cases}$$

Proposition 2.4.1.
1) *M_1^r is open dense in M_1. In particular, it has dimension $k(k + 2)$.*
2) *M_2^{ro} is open in M_2^r, and the complement of M_2^{ro} in M_2^r has a smaller dimension than that of M_2^{ro}. The dimension of M_2^{ro} is given by*

$$\dim M_2^{\mathrm{ro}} = \begin{cases} 2k^2 & (k\text{: even}), \\ 2k^2 - 1 & (k\text{: odd}). \end{cases}$$

Proof. 1) Clear.

2) For an even integer l, we consider a monic polynomial of the real variable x:

$$g(x) = x^l + b_{l-1}x^{l-1} + \cdots + b_1 x + b_0.$$

Then, it attains its minimum, denoted by $m(b_0, b_1, \cdots, b_{l-1})$, which is a continuous function of the real coefficients $b_0, b_1, \cdots, b_{l-1}$.

Case 1 $(k$ **: even)** Suppose $(X,Y) \in M_2^{\mathrm{ro}}$. Then, the monic polynomial $\frac{f(X,Y;\lambda)}{a_k(X,Y)}$ must be positive for all $\lambda \in \mathbb{R}$. Thus we have

$$M_2^{\mathrm{ro}} = \left\{ (X,Y) \in M_2 : \det X \neq 0,\, m\left(\frac{a_0(X,Y)}{a_k(X,Y)}, \cdots, \frac{a_{k-1}(X,Y)}{a_k(X,Y)} \right) > 0 \right\}.$$

Hence, M_2^{ro} is open in $M_2 = M(k, 2k; \mathbb{R})$.

To see $M_2^{\mathrm{ro}} \neq \emptyset$, we set $J_k := \begin{pmatrix} 0 & -I_{\frac{k}{2}} \\ I_{\frac{k}{2}} & 0 \end{pmatrix} \in M(k,\mathbb{R})$. Then $(J_k,\, I_k) \in M_2^{\mathrm{ro}}$ because

$$f(J_k, I_k; \lambda) = \left(\det \begin{pmatrix} 1 & \lambda \\ -\lambda & 1 \end{pmatrix} \right)^{\frac{k}{2}} = (1+\lambda^2)^{\frac{k}{2}} > 0.$$

Hence, $M_2^{\mathrm{ro}} \neq \emptyset$ and $\dim M_2^{\mathrm{ro}} = \dim M_2 = 2k^2$.

Next, suppose $(X,Y) \in M_2^r \setminus M_2^{\mathrm{ro}}$. Then $\det X = 0$ by definition. Furthermore, it follows from $f(X,Y;\lambda) \neq 0$ for any $\lambda \in \mathbb{R}$ that the coefficient $a_{k-1}(X,Y)$ of λ^{k-1} must vanish because $f(X,Y;\lambda) = a_{k-1}(X,Y)\lambda^{k-1} + \cdots + a_0(X,Y)$ and $k-1$ is odd. Thus we have seen that

$$M_2^r \setminus M_2^{\mathrm{ro}} \subset \{ (X,Y) \in M_2 : \det X = a_{k-1}(X,Y) = 0 \}.$$

Hence the complement $M_2^r \setminus M_2^{\mathrm{ro}}$ has at least codimension two in M_2^r.

Case 2 $(k$**: odd)** First, we claim

$$M_2^r \subset \{ (X,Y) \in M(k,2k;\mathbb{R}) : \det X = 0 \}.$$

In fact, since k is odd, the polynomial $f(X,Y;\lambda)$ of the real variable λ has zeros unless the top term $a_k(X,Y)\lambda^k$ vanishes. Therefore, $a_k(X,Y)$ $(= (-1)^k \det X) = 0$ if $(X,Y) \in M_2^r$.

Next, let us prove that M_2^{ro} is open in the set

$$\begin{aligned} S &:= \{ (X,Y) \in M(k,2k;\mathbb{R}) : \det X = 0,\, \operatorname{grad} \det X \neq 0 \} \\ &= \{ (X,Y) \in M(k,2k;\mathbb{R}) : \operatorname{rank} X = k-1 \}. \end{aligned}$$

Suppose $(X,Y) \in M(k,2k;\mathbb{R})$ satisfies

$$\det X = 0 \text{ and } a_{k-1}(X,Y) \neq 0.$$

Then $f(X,Y;\lambda) \neq 0$ for any $\lambda \in \mathbb{R}$ if and only if the monic polynomial

$$\frac{f(X,Y;\lambda)}{a_{k-1}(X,Y)} = \lambda^{k-1} + \sum_{i=0}^{k-2} \frac{a_i(X,Y)}{a_{k-1}(X,Y)} \lambda^i$$

is positive for all λ. Thus, we have seen

$$M_2^{\mathrm{ro}} = \left\{ (X,Y) \in M(k,2k;\mathbb{R}) : \begin{array}{l} \operatorname{rank} X = k-1,\ a_{k-1}(X,Y) \neq 0, \\ m\left(\frac{a_0(X,Y)}{a_{k-1}(X,Y)}, \cdots, \frac{a_{k-2}(X,Y)}{a_{k-1}(X,Y)}\right) > 0. \end{array} \right\}.$$

It is now clear that M_2^{ro} is open in S. To see $M_2^{\mathrm{ro}} \neq \emptyset$, we set $J_k' := \begin{pmatrix} J_{k-1} & 0 \\ 0 & 0 \end{pmatrix} \in M(k;\mathbb{R})$. Then, $(J_k', I_k) \in M_2^{\mathrm{ro}}$ because

$$f(J_k', I_k; \lambda) = (1+\lambda^2)^{\frac{k-1}{2}} > 0.$$

Hence, M_2^{ro} is a non-empty open subset of S. Since S is a non-singular manifold of dimension $2k^2-1$, so is M_2^{ro}.

Finally, it follows from the definition of M_2^{ro} that $M_2^r \setminus M_2^{\mathrm{ro}}$ is contained in the algebraic variety:

$$\{(X,Y) : \det X = 0,\ a_{k-1}(X,Y) \| \operatorname{grad} \det X \|^2 = 0\},$$

which is of dimension $2k^2-2$. Thus, Proposition 2.4.1 has been proved. ☐

Remark 2.4.2. Proposition 2.4.1 implies that M_2^r contains an open subset of M_2 if and only if k is even. However, M_2^r itself is not open in M_2 even if k is even because $(O, I_k) \in M_2^r$ is not an inner point, as we shall see in the following example:

Example 2.4.3. Take a half dimensional affine subspace

$$V = \{(X, I_k) : X \in M(k,\mathbb{R})\}$$

of $M(k,2k;\mathbb{R})$. Then we have

$$V \cap M_2^r = \{(X, I_k) : \text{any eigenvalue of } X \text{ is in } \mathbb{C} \setminus \mathbb{R}^\times\}.$$

This gives a partial information on $R(\Gamma, G; \mathbb{R}^{k+1})$, and was proved in Lipsman [11, Theorem 4.4] for $k=2$ as a crucial step to the classification of maximal nilpotent affine subgroups that act properly on $\mathbb{R}^3$, and was generalized in [12] for $k \geq 3$. Our proof here is different and simpler.

3 Description of $\mathrm{Hom}(\Gamma, G)$

This section determines $\mathrm{Hom}(\Gamma, G)$ explicitly, and gives a proof of Proposition 2.2.1.

3.1 Parametrization of $\mathrm{Hom}(\Gamma, G)$

Recall from (2.2) that any $\varphi \in \mathrm{Hom}(\Gamma, G)$ is determined by $\varphi(e_j) \quad (1 \leq j \leq k)$, which we write as $\varphi(e_j) = g(\vec{x_j}, \vec{y_j}, z_j)$ for some $\vec{x_j}$, $\vec{y_j} \in \mathbb{R}^k$ and $z_j \in \mathbb{R}$ according to (2.5). Collecting these data $\vec{x_j}, \vec{y_j}$ $(1 \leq j \leq k)$ and $\vec{z} := {}^t(z_1, \ldots, z_k)$, we obtain an injective map defined by

$$\text{(3.1)} \qquad \mathrm{Hom}(\Gamma, G) \to M(k, 2k+1; \mathbb{R}),\ \varphi \mapsto \left(\vec{x_1}, \ldots, \vec{x_k}; \vec{y_1}, \ldots, \vec{y_k}; \vec{z}\right).$$

Let us determine the image of (3.1). Since $\varphi \in \mathrm{Hom}(\Gamma, G)$ satisfies

$$\varphi(e_i)\varphi(e_j) = \varphi(e_i + e_j) = \varphi(e_j)\varphi(e_i)$$

for any $i, j \quad (1 \leq i, j \leq k)$, we have

$$\text{(3.2)} \qquad z_i \vec{x_j} = z_j \vec{x_i} \quad \text{for any } i, j\ (1 \leq i, j \leq k).$$

Conversely, given k elements $g_1, \cdots, g_k$ in G that mutually commute, we can define a group homomorphism $\varphi : \Gamma \to G$ by $\varphi(\sum_{j=1}^k m_j e_j) := g_1^{m_1} \cdots g_k^{m_k}$. Therefore, the image of (3.1) is characterized by the condition (3.2), that is, we have a bijection:

$$\mathrm{Hom}(\Gamma, G) \simeq \{\left(\vec{x_1}, \ldots, \vec{x_k}; Y; \vec{z}\right) : Y \in M(k, \mathbb{R}); \vec{x_1}, \ldots, \vec{x_k}, \vec{z} \text{ satisfies (3.2)}\}.$$

We shall find all solutions of (3.2), according to the following two cases: (a) $\vec{z} \neq \vec{0}$ and (b) $\vec{z} = \vec{0}$.

In the case (a), there exists uniquely an element $\vec{r} \in \mathbb{R}^k$ such that $\vec{x_j} = z_j \vec{r}$ for all j $(1 \leq j \leq k)$. This amounts to $\Psi_1(M_1)$ (see (2.6) for the definition of Ψ_1).

In the case (b), any $\vec{x_1}, \ldots, \vec{x_k}$ solves (3.2). This amounts to $\Psi_2(M_2)$ (see (2.7) for the definition of Ψ_2).

Hence, $\mathrm{Hom}(\Gamma, G)$ is the disjoint union of $\Psi_1(M_1)$ and $\Psi_2(M_2)$. Thus we have completed the proof of Proposition 2.2.1 (1).

$\square$

3.2 Closure relation in $\mathrm{Hom}(\Gamma, G)$

This subsection gives a proof of Proposition 2.2.1 (2). It is clear that $\Psi_2(M_2)$ is a closed set. Let us consider the closure of $\Psi_1(M_1)$, and find its boundary. What we need is to prove:

$$\overline{\Psi_1(M_1)} \cap \Psi_2(M_2) = \Psi_2(M_2^d).$$

Proof of the inclusion $\supset$: Suppose $(X, Y) \in \Psi_2(M_2^d)$. Since $\mathrm{rank}\, X \leq 1$, we find $\vec{x} \in \mathbb{R}^k$ and $\vec{a} \in \mathbb{R}^k \setminus \{0\}$ such that $X = \vec{x}\,{}^t\vec{a}$. In light of the obvious formula

$$g(a_j\vec{x}, \vec{y_j}, 0) = \lim_{l\to\infty} g\big(\frac{a_j}{l} l\vec{x}, \vec{y_j}, \frac{a_j}{l}\big) \quad (1 \leq j \leq k),$$

we conclude from the definitions (2.6) and (2.7) of Ψ_1 and Ψ_2 that

$$\Psi_2(X, Y) = \lim_{l\to\infty} \Psi_1\big(l\vec{x}, Y, \frac{\vec{a}}{l}\big).$$

As $(l\vec{x}, Y, \frac{\vec{a}}{l})$ $(l = 1, 2, \cdots)$ is a sequence of M_1, we have proved the inclusion $\supset$.

Proof of the inclusion $\subset$: Take any sequence $(\overrightarrow{x^{(l)}}, Y^{(l)}, \overrightarrow{z^{(l)}})$ in M_1 such that $\Psi_1(\overrightarrow{x^{(l)}}, Y^{(l)}, \overrightarrow{z^{(l)}})$ converges to an element of $\Psi_2(M_2)$, say, $\Psi_2(X, Y)$ for some X, $Y \in M(k, \mathbb{R})$. Then the formula

$$\lim_{l\to\infty} \Psi_1(\overrightarrow{x^{(l)}}, Y^{(l)}, \overrightarrow{z^{(l)}}) = \Psi_2(X, Y)$$

implies that X is the limit of $X^{(l)} := (z_1^{(l)}\overrightarrow{x^{(l)}}, \cdots, z_k^{(l)}\overrightarrow{x^{(l)}})$ as l tends to infinity. Since $\mathrm{rank}\, X^{(l)} \leq 1$, its limit also satisfies $\mathrm{rank}\, X \leq 1$. Thus we have proved the inclusion $\subset$.

Thus, Proposition 2.2.1 (2) is proved. $\square$

4 Proof of Theorem 2.3

This section gives a proof of Theorem 2.3. Our strategy here is to rewrite the condition of $R(\Gamma, G; \mathbb{R}^{k+1})$, in particular, the condition for properly discontinuous actions in the following scheme:

	Γ	discrete subgroup	(see (1.1))
$\Rightarrow$	$L = \overline{\Gamma}$	its syndetic hull	(see Proposition 4.3.2)
$\Rightarrow$	$\mathfrak{l}$	its Lie algebra.	(see Section 4.4)

4.1 Proper actions and properly discontinuous actions

In dealing with properly discontinuous actions of a discrete group, a more general notion "proper action" is sometimes useful. We recall:

Definition 4.1.1 (Palais [14]). Suppose that a locally compact topological group L acts continuously on a Hausdorff, locally compact space X. For a subset S of X, we define a subset of L by $L_S = \{\gamma \in L : \gamma S \cap S \neq \emptyset\}$. The L-action on X is said to be *proper* if L_S is compact for every compact subset S of X.

We note that the L-action is *properly discontinuous* if L is a discrete group and if the L-action is proper.

The following elementary observation is a bridge between the action of a discrete group and that of a connected group.

Observation 4.1.2 ([5, Lemma 2.3]). *Suppose a locally compact group L acts on a Hausdorff, locally compact space X. Let Γ be a cocompact discrete subgroup of L. Then*

1) *The L-action on X is proper if and only if the Γ-action is properly discontinuous.*

2) *$L\backslash X$ is compact if and only if $\Gamma\backslash X$ is compact.*

4.2 Extension from a discrete subgroup

Suppose we are in the setting of Section 2.1. We set

$$L := \mathbb{R}^k$$

and regard $\Gamma = \mathbb{Z}^k$ as a cocompact discrete subgroup of L. We write $\mathrm{Hom}(L, G)$ for the set of continuous group homomorphisms from L into G. In our setting (2.1), every homomorphism from Γ into G extends uniquely to a continuous homomorphism from L to G. That is, we have:

Lemma 4.2.1. *The restriction map* $\mathrm{Hom}(L, G) \to \mathrm{Hom}(\Gamma, G), \psi \mapsto \psi|_\Gamma$ *is bijective.*

Proof. As G is a simply connected nilpotent group, the exponential map, $\exp : \mathfrak{g} \to G$ is bijective. We write log for its inverse. Then, $\psi \in \operatorname{Hom}(L, G)$ satisfies

$$\psi(\sum_{j=1}^{k} a_j e_j) = \exp\bigl(\sum_{j=1}^{k} a_j \log \psi(e_j)\bigr) \text{ for any } a_1, \cdots, a_k \in \mathbb{R}. \tag{4.1}$$

This shows that the homomorphism ψ is determined by its restriction $\psi|_\Gamma$. Conversely, the formula (4.1) also indicates how to extend a homomorphism from Γ to L. Thus we have proved Lemma 4.2.1. $\square$

In light of Lemma 4.2.1, any property of ψ should be expressed in terms of the restriction $\psi|_\Gamma$ in principle. We show:

Lemma 4.2.2. *The following two conditions on $\psi \in \operatorname{Hom}(L, G)$ are equivalent:*

(i) *ψ is injective.*

(ii) *$\psi|_\Gamma$ is injective and $\psi(\Gamma)$ is discrete in G.*

Proof. Since G is a simply connected nilpotent Lie group, any connected subgroup of G is closed. Therefore, $\psi : L/\operatorname{Ker}\psi \to G$ is a homeomorphism onto a closed subgroup of G. In particular, $\psi(\Gamma)$ is discrete in G if and only if $\Gamma/\Gamma \cap \operatorname{Ker}\psi$ is discrete in $L/\operatorname{Ker}\psi$. Now, it is clear that (i) implies (ii).

Conversely, if ψ is not injective and if $\psi|_\Gamma$ is injective, then the composition map $\Gamma \subset L \to L/\operatorname{Ker}\psi$ is injective with non-discrete image because $\operatorname{rank}\Gamma < \dim(L/\operatorname{Ker}\psi)$. Hence, $\psi(\Gamma)$ is not discrete in G, too. Thus, (ii) also implies (i). $\square$

4.3 A continuous analogue of properly discontinuous actions

Following Observation 4.1.2, we amplify Lemma 4.2.2 with the condition of proper actions on the homogeneous space G/H:

Lemma 4.3.1. *Let $\psi \in \operatorname{Hom}(L, G)$ and $\varphi = \psi|_\Gamma$ (see Lemma 4.2.1). Then the following two conditions are equivalent:*

(i) *$\psi : L \to G$ is injective and $\psi(L)$ acts properly on G/H.*

(ii) $\varphi : \Gamma \to G$ *is injective and* $\varphi(\Gamma)$ *acts properly discontinuously and freely on* G/H.

Proof. (i) $\Rightarrow$ (ii): Since ψ is injective, it follows from Lemma 4.2.2 that $\varphi(\Gamma)$ is discrete in a closed subgroup $\psi(L)$. Therefore, $\varphi(\Gamma)$ acts properly discontinuously on G/H because $\psi(L)$ acts properly on G/H. Furthermore, any properly discontinuous action of $\varphi(\Gamma)$ is automatically free because $\varphi(\Gamma) \simeq \Gamma$ is torsion-free. Hence (ii) is proved.

(ii) $\Rightarrow$ (i): If $\varphi(\Gamma)$ acts properly discontinuously on G/H then $\varphi(\Gamma)$ is discrete in G. Hence, $\psi : L \to G$ is injective by Lemma 4.2.2. Furthermore, $\psi(L)$ with its relative topology contains $\varphi(\Gamma)$ as a cocompact discrete subgroup. Therefore, $\psi(L)$ acts properly on G/H by Observation 4.1.2 (1). Thus, we have proved the implication (ii) $\Rightarrow$ (i). $\square$

We are ready to characterize $R(\Gamma, G; \mathbb{R}^{k+1})$ by means of the connected subgroup L:

Proposition 4.3.2. *Under the isomorphism* $\mathrm{Hom}(L, G) \xrightarrow{\sim} \mathrm{Hom}(\Gamma, G)$ *in Lemma 4.2.1, we have*

$$R(\Gamma, G; \mathbb{R}^{k+1}) \simeq \{\psi \in \mathrm{Hom}(L, G) : \text{i) } \psi \textit{ is injective}, \\ \text{ii) } \psi(L) \textit{ acts properly on } G/H\}.$$

4.4 Reformulation of $R(\Gamma, G; \mathbb{R}^{k+1})$

So far, we have transferred proper discontinuity and freeness of discrete group actions into a certain property of connected group actions. Now, let us rewrite the latter condition in terms of Lie algebras. We use the German lower case letters $\mathfrak{g}$, $\mathfrak{h}$ and $\mathfrak{l}$ to denote the Lie algebras of G, H and L respectively. We write $d\psi$ for the differential of $\psi \in \mathrm{Hom}(L, G)$. Consider the following conditions on $d\psi$:

$$d\psi : \mathfrak{l} \to \mathfrak{g} \text{ is injective.} \tag{4.2}$$

$$d\psi(\mathfrak{l}) \cap \bigcup_{g \in G} \mathrm{Ad}(g)\mathfrak{h} = \{0\}. \tag{4.3}$$

Now we can restate Proposition 4.3.2 as

Proposition 4.4.1. *Under the isomorphism* $\mathrm{Hom}(L,G) \xrightarrow{\sim} \mathrm{Hom}(\Gamma,G)$ *(see Lemma 4.2.1), we have*

$$R(\Gamma, G; \mathbb{R}^{k+1}) \simeq \{\psi \in \mathrm{Hom}(L,G) \ : d\psi \textit{ satisfies } (4.2) \textit{ and } (4.3)\}.$$

Proof. Any connected subgroup of a simply connected nilpotent Lie group is simply connected. Hence, $d\psi$ is injective if and only if ψ is injective. Now use the criterion of proper actions for a homogeneous space of a two-step nilpotent Lie group G as follows. ☐

Lemma 4.4.2 ([13, Theorem 2.11]). *Let G be a simply connected Lie group, and H, L its closed subgroups. Suppose G is a two-step nilpotent Lie group, which means that the commutator subgroup of G is contained in the centre of G. Then, the following three conditions on ψ are equivalent.*

(i) *$\psi(L)$ acts on G/H properly.*

(ii) *$\psi(L) \cap gHg^{-1} = \{e\}$ for all $g \in G$.*

(iii) *$d\psi(\mathfrak{l}) \cap \bigcup_{g\in G} \mathrm{Ad}(g)\mathfrak{h} = \{0\}$.*

Remark 4.4.3. Lemma 4.4.2 gives an affirmative solution to Lipsman's conjecture [11] for two-step nilpotent Lie groups. Recently, Baklouti–Khlif [2] and Yoshino [18] proved independently that Lipsman's conjecture is still true for three-step nilpotent Lie groups.

4.5 Completion of the proof of Theorem 2.3

Now let us complete the proof of Theorem 2.3. We have already reduced it to a problem of Lie algebras. Now, we use the following:

Lemma 4.5.1. *We define the variety in $\mathfrak{g}$ by*

$$\mathcal{V} = \bigcup_{g\in G} \mathrm{Ad}(g)\mathfrak{h}.$$

Then we have

$$\begin{aligned} \mathcal{V} &= \{W - [W,V] : W \in \mathfrak{h}, V \in \mathfrak{g}\} \\ &= \{ \begin{pmatrix} 0 & \vec{x} & b\vec{x} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} : \vec{x} \in \mathbb{R}^k, b \in \mathbb{R}\}. \end{aligned} \tag{4.4}$$

Proof. Elementary computation. $\square$

According to the parametrization

$$\Psi_1 \cup \Psi_2 : M_1 \cup M_2 \xrightarrow{\sim} \mathrm{Hom}(L, G) \xrightarrow{\sim} \mathrm{Hom}(\Gamma, G)$$

given in Proposition 2.2.1 and Lemma 4.2.1, we examine if $d\psi$ satisfies (4.2) and (4.3) for $\psi \in \Psi_1(M_1)$ and $\psi \in \Psi_2(M_2)$, respectively. The following proposition carries it out.

Proposition 4.5.2.
1) *Let* $\psi := \Psi_1(\vec{x}, Y, \vec{z})$ *for* $(\vec{x}, Y, \vec{z}) \in M_1$. *Then, we have the following equivalence:*

$$\psi \textit{ satisfies } (4.2) \iff \mathrm{rank}({}^tY, \vec{z}) = k.$$

In this case, ψ *satisfies* (4.3)*, too.*
2) *Let* $\psi := \Psi_2(X, Y)$ *for* $(X, Y) \in M_2$. *Then*

$$\psi \textit{ satisfies } (4.2) \iff \mathrm{rank} \begin{pmatrix} X \\ Y \end{pmatrix} = k.$$

In this case, we have the following equivalence:

$$\psi \textit{ satisfies } (4.3) \iff \det(Y - bX) \neq 0 \textit{ for any } b \in \mathbb{R}.$$

Proof. 1) It follows from the definition of Ψ_1 (see (2.6)) that

$$d\psi(\vec{a}) = \begin{pmatrix} 0 & \langle \vec{a}, \vec{z} \rangle \vec{x} & Y\vec{a} \\ 0 & 0 & \langle \vec{a}, \vec{z} \rangle \\ 0 & 0 & 0 \end{pmatrix}. \tag{4.5}$$

Here, $\langle\ ,\ \rangle$ denotes the standard inner product on $\mathbb{R}^k$. Then

$$\begin{aligned} \psi \text{ satisfies } (4.2) &\iff \{\vec{a} \in \mathfrak{l} : \langle \vec{z}, \vec{a} \rangle = 0,\, Y\vec{a} = \vec{0}\} = \{0\} \\ &\iff \mathrm{rank}({}^tY, \vec{z}) = k. \end{aligned}$$

Furthermore, by (4.4) and (4.5)

$$d\psi(\vec{a}) \in \mathcal{V} \iff d\psi(\vec{a}) = \vec{0}.$$

Therefore, (4.2) implies (4.3).

2) It follows from the definition of Ψ_2 (see (2.7)) that

$$d\psi(\vec{a}) = \begin{pmatrix} 0 & X\vec{a} & Y\vec{a} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{4.6}$$

Then,

$$\begin{aligned} \psi \text{ satisfies (4.2)} &\iff \{\vec{a} \in \mathfrak{l} : X\vec{a} = \vec{0},\, Y\vec{a} = \vec{0}\} = \{\vec{0}\} \\ &\iff \operatorname{rank}\begin{pmatrix} X \\ Y \end{pmatrix} = k. \end{aligned}$$

Suppose (4.2) is satisfied. It follows from (4.4) and (4.6) that

$$\begin{aligned} &\psi \text{ satisfies (4.3)} \\ \iff &\text{there is no } b \in \mathbb{R} \text{ and } \vec{a} \in \mathbb{R}^k \text{ such that } Y\vec{a} = bX\vec{a} \neq \vec{0} \\ \iff &\det(Y - bX) = 0 \text{ has no real solution for any } b \in \mathbb{R}. \end{aligned}$$

Hence, Proposition is proved. □

We note that if $\det(Y - bX) \neq 0$ for any $b \in \mathbb{R}$ then $\operatorname{rank}\begin{pmatrix} X \\ Y \end{pmatrix} = k$ because $\det Y \neq 0$. Then, Theorem 2.3 follows from Propositions 4.4.1 and 4.5.2. Hence, we have completed the proof of Theorem 2.3. □

5 Deformation space

Building on the description of the parameter space $R(\Gamma, G; \mathbb{R}^{k+1})$ of discontinuous groups given in Theorem 2.3, we determine explicitly the deformation space $\mathcal{T}(\Gamma, G; \mathbb{R}^{k+1})$. This is stated in Theorem 5.1 and Corollary 5.1.1, and is the second of the main results of this paper.

5.1 Description of the deformation space $\mathcal{T}(\Gamma, G; \mathbb{R}^{k+1})$

We define subsets of M_1^r and M_2^r, respectively, by

(5.1)

$$D_1 := \left\{ \left(\vec{x}, {}^t(\vec{\eta_1} \dots \vec{\eta_k}), \vec{z}\right) \in M(k, k+2; \mathbb{R}) : \begin{array}{ll} 1) & \vec{z} \neq \vec{0}, \\ 2) & \vec{\eta_j} \perp \vec{z} \ \ (1 \leq j \leq k), \\ 3) & \operatorname{rank}(\vec{\eta_1}, \cdots, \vec{\eta_k}) = k-1. \end{array} \right\},$$

(5.2)

$$D_2 := \left\{ (X, Y) \in M(k, \mathbb{R}) \oplus M(k, \mathbb{R}) : \begin{array}{ll} 1) & \operatorname{Trace}(X {}^t Y) = 0, \\ 2) & \det(Y - \lambda X) \neq 0 \text{ for any } \lambda \in \mathbb{R}. \end{array} \right\}.$$

We note that the third condition in (5.1) asserts that $\operatorname{rank}(\vec{\eta_1}, \cdots, \vec{\eta_k})$ attains its maximum because all the vectors $\vec{\eta_j}$ $(1 \leq j \leq k)$ are orthogonal to $\vec{z}$.

We retain the setting as in Section 2.1. In particular, $\Gamma = \mathbb{Z}^k$ and G is a nilpotent affine transformation group defined in (2.1). For $i = 1, 2$, we denote by

$$\overline{\Psi}_i : M_i^r \to \mathcal{T}(\Gamma, G; \mathbb{R}^{k+1})$$

the composition of $\Psi_i : M_i^r \to R(\Gamma, G; \mathbb{R}^{k+1})$ (see (2.6) and (2.7)) and the natural quotient map $R(\Gamma, G; \mathbb{R}^{k+1}) \to \mathcal{T}(\Gamma, G; \mathbb{R}^{k+1})$.

Here is an explicit description of the deformation space:

Theorem 5.1. *The maps $\overline{\Psi}_1$ and $\overline{\Psi}_2$ induce the following bijection:*

$$\overline{\Psi}_1 \cup \overline{\Psi}_2 : D_1 \cup D_2 \xrightarrow{\sim} \mathcal{T}(\Gamma, G; \mathbb{R}^{k+1}).$$

In particular, we find the dimension of the deformation space:

Corollary 5.1.1. *The deformation space $\mathcal{T}(\Gamma, G; \mathbb{R}^{k+1})$ contains a smooth manifold $\mathcal{T}'$ as its open dense subset, where the dimension of $\mathcal{T}'$ is given by*

$$\dim \mathcal{T}' = \begin{cases} 2k^2 - 1 & (k : \textit{even}), \\ 2k^2 - 2 & (k : \textit{odd}, \ \geq 3), \\ 2 & (k = 1). \end{cases}$$

5.2 Proof of Theorem 5.1 and Corollary 5.1.1

We let G act on $M(k, k+2; \mathbb{R})$ and $M(k, 2k; \mathbb{R})$, respectively, as follows: for $h = \begin{pmatrix} I_k & \vec{a} & \vec{b} \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \in G$, the actions of h are given by

$$(5.3) \qquad (\vec{x}, Y, \vec{z}) \mapsto (\vec{x}; Y + (\vec{a} - c\vec{x})^t\vec{z}, \vec{z}) \qquad \text{on} \quad M(k, k+2; \mathbb{R}),$$

$$(5.4) \qquad (X, Y) \mapsto (X, Y - cX) \qquad \text{on} \quad M(k, 2k; \mathbb{R}).$$

Proposition 5.2.1.
1) *Both M_1 and M_1^r are G-stable subsets of $M(k, k+2; \mathbb{R})$.*
2) *M_2^r is a G-stable subset of $M_2 = M(k, 2k; \mathbb{R})$.*
3) *For $i = 1$, 2, the maps $\Psi_i : M_i^r \to \mathrm{Hom}(\Gamma, G)$ respect G-actions.*
4) *For $i = 1$, 2, D_i are complete representatives of the G-orbit on M_i^r.*

Proof. 1) Clear from the definitions (2.3) and (2.10) of M_1 and M_1^r.
2) Clear from the definition (2.11) of M_2^r.
3) We first note that via (2.2) the G-action on $\mathrm{Hom}(\Gamma, G)$ is compatible with the diagonal G-action on $G \times \cdots \times G$:

$$(g_1, \cdots, g_k) \ \mapsto \ (hg_1h^{-1}, \cdots, hg_kh^{-1}).$$

Now, we compute the jth component of the image of Ψ_1 (see (2.6) for the definition) and Ψ_2 (see (2.7) for the definition), respectively, as follows: For $h = \begin{pmatrix} I_k & \vec{a} & \vec{b} \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \in G$, we have

$$h \exp \begin{pmatrix} 0 & z_j\vec{x} & \vec{y_j} \\ 0 & 0 & z_j \\ 0 & 0 & 0 \end{pmatrix} h^{-1} = \exp \begin{pmatrix} 0 & z_j\vec{x} & \vec{y_j} + z_j(\vec{a} - c\vec{x}) \\ 0 & 0 & z_j \\ 0 & 0 & 0 \end{pmatrix}$$

and

$$h \exp \begin{pmatrix} 0 & \vec{x_j} & \vec{y_j} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} h^{-1} = \exp \begin{pmatrix} 0 & \vec{x_j} & \vec{y_j} - c\vec{x_j} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

This is what we wanted to prove.
4) This is an elementary linear algebra. $\square$

Then, Theorem 5.1 is an immediate consequence of Theorem 2.3 and Proposition 5.2.1 (4). Corollary 5.1.1 now follows from Proposition 2.4.1, and from the G-action on $R(\Gamma, G; \mathbb{R}^{k+1})$ described in the above proof.

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