On discontinuous group actions on non-Riemannian homogeneous spaces [*]

Toshiyuki Kobayashi

Abstract

This article gives an up-to-date account of the theory of discrete group actions on non-Riemannian homogeneous spaces.

As an introduction of the motifs of this article, we begin by reviewing the current knowledge of possible global forms of pseudo-Riemannian manifolds with constant curvatures, and discuss what kind of problems we propose to pursue.

For pseudo-Riemannian manifolds, isometric actions of discrete groups are not always properly discontinuous. The fundamental problem is to understand when discrete subgroups of Lie groups G act properly discontinuously on homogeneous spaces G/H for non-compact H. For this, we introduce the concepts from a group-theoretic perspective, including the 'discontinuous dual' of G/H that recovers H in a sense.

We then summarize recent results giving criteria for the existence of properly discontinuous subgroups, and the known results and conjectures on the existence of cocompact ones. The final section discusses the deformation theory and in particular rigidity results for cocompact properly discontinuous groups for pseudo-Riemannian symmetric spaces.

1   Introduction: The problem of space forms

1.1   Pseudo-Riemannian manifolds of constant curvature

The local to global study of geometries was a major trend of 20th century geometry, with remarkable developments achieved particularly in Riemannian geometry. In contrast, in areas such as Lorentz geometry, familiar to us as the space-time of relativity theory, and more generally in pseudo-Riemannian[1] geometries, as well as in various other kinds of geometry (symplectic, complex geometry,...), surprising little is known about global properties of the geometry even if we impose a locally homogeneous structure.

On the other hand, in the representation theory of Lie groups and in the area of global analysis that applies it (noncommutative harmonic analysis), the great trends of development throughout the 20th century include the generalizations

$$
\begin{array}{rcl}
\text{compact} & \mapsto & \text{noncompact} \\
\text{Riemannian} & \mapsto & \text{pseudo-Riemannian manifolds} \\
\text{finite} & \mapsto & \text{infinite dimensional representations}
\end{array}
$$

within their terms of study, and together with these, the appearance of ground-breaking new research methods; these moreover deepened relations with various areas of mathematics, such as PDEs, functional analysis, differential and algebraic geometry.

Against this background, from around the mid 1980s, I began to envisage the possibility of developing the theory of discontinuous groups also in the world of pseudo-Riemannian manifolds. Soon afterwards, I succeeded in proving a necessary and sufficient condition for the Calabi–Markus phenomenon to occur, and, stimulated by this, launched a systematic study of the general theory of discontinuous groups of homogeneous spaces that have good geometric structures, but are not necessarily Riemannian manifolds; for example, semisimple symmetric spaces or adjoint orbit spaces.

---

[1]That is, having a pseudometric 2-form that is not necessarily positive definite: whereas a Riemannian metric is given by a positive definite 2-form at every point of a manifold, a *pseudo-Riemannian metric* is the generalization obtained by replacing the positive definite condition on the form by nondegenerate. The case of *Lorentz manifolds* corresponds to the nondegenerate 2-form having signature $(n-1, 1)$.

Whereas the theory of discontinuous groups of Riemannian symmetric spaces had been the center of wide and deep developments for more than a hundred years, at the time in the 1980s, practically no other researchers were interested in the theory of discontinuous groups for non-Riemannian homogeneous spaces; although the starting point was solitary, whatever I did was a new development. After the publication of the series of papers containing foundational results [17]–[19], [28], from the early 1990s, specialists in other areas from France and the United States such as Benoist, Labourie, Zimmer, Lipsman, Witte and so on eventually started to join in the study of this problem. After this, research methods developed rapidly over the following 10 years or so, and the ideas concerning discontinuous groups of non-Riemannian homogeneous spaces have come to relate to many areas of mathematics, including not only Lie group theory and discrete group theory, but also differential geometry, algebra, ergodic theory, mechanical systems, unitary representation theory and so on ([4], [5], [9], [15], [17], [28], [34], [35], [40], [43], [46], [56], ...), so much so that already no single mathematician can hope to cover them all.

For example, the recent researches of Margulis, Oh, Shalom and myself ([23], [35], [40], [46]) can be viewed as another instance of the same trend, where the fundamental question of understanding the distinction between a discrete group action and a discontinuous group for a non-Riemannian homogeneous space begins to tie in to the at first sight unrelated subject of the restriction of a unitary representation to a noncompact subgroup.

As an introduction to this article, we review and put in order the simplest examples (in some sense) of spaces "of constant curvature", and discuss what kind of problems we propose to pursue, and what is the current state of knowledge concerning the "possible global forms" of these spaces. To set these things up precisely we recall the following definitions.

Definition A pseudo-Riemannian manifold of constant sectional curvature is a *space form*.

For example, for signature $(n, 0)$ (Riemannian manifolds), the sphere S^n is a space form of positive curvature, and hyperbolic space a space form of negative curvature. For signature $(n - 1, 1)$ (Lorentz manifolds), *de Sitter space* is a space form of positive curvature,[2] *Minkowski space* a space form of

---

[2]In Calabi and Markus [8], in connection with the use of 4-dimensional Lorentz manifold as the space-time continuum of relativity theory, the Lorentz space form of positive curvature (that is, de Sitter space) is called the *relativistic spherical space form.*

zero curvature, and *anti-de Sitter space* a space form of negative curvature.

Here since we are interested in global properties, when we say space form, we assume that the geometry is geodesically complete. The main topic we consider in this section is the following question:

Local assumption: among pseudo-Riemannian space forms[3] of signature (p, q) and curvature κ,

Global conclusion: do there exist any compact examples?

And if so, what types of group can appear as their fundamental groups?

1.2 The two dimensional case

The sphere S^2, the torus T^2, and the closed Riemann surface M_g of genus $g \geq 2$ can be given Riemannian metrics to make them respectively into space forms of positive, zero and negative curvature. In other words, in two dimensional Riemannian geometry, there exists a space form of any curvature κ. The same holds in general dimensions.

However, in the case of Lorentz signature $(1, 1)$, there do not exist any compact space form with $\kappa \neq 0$. In fact, the sphere S^2 and the Riemann surface M_g with $g \geq 2$ do not even admit a Lorentz metric.[4] And if T^2 can be given a Lorentz metric of constant curvature κ then $\kappa = 0$ by the Gauss–Bonnet theorem.

1.3 The case of positive curvature

Among Riemannian manifolds, the sphere S^n is the typical model for a space form of positive curvature. Conversely, the only complete space forms with this property are S^n, or at most S^n divided by a suitable finite group.[5] We recall two classical theorems generalizing the fact that "a space form of positive curvature has finite fundamental group".

In one direction, we leave unchanged the positive definite property of the metric form (that is, a Riemannian manifold), and perturb the curvature (or the metric itself).

---

[3] Multiplying a pseudo-Riemannian metric by -1 changes its signature from (p, q) to (q, p), and its curvature from κ to $-\kappa$.

[4] Any paracompact manifold admits a Riemannian structure, but the analogous result does not hold for pseudo-Riemannian structure.

[5] See Wolf [50] for details on what kind of finite groups one can divide by.

Theorem 1 (Myers 1941 [38]) *Suppose that the Ricci curvature of a complete Riemannian manifold has a positive lower bound. Then the fundamental group is finite, and the manifold is compact.*

In the other direction, we now leave unchanged the positive definite property of the curvature, but vary the positive definite assumption on the metric form (that is, the condition for a Riemannian manifold).

Theorem 2 (Calabi and Markus 1962 [8]) *In Lorentz geometry of dimension ≥ 3, a space form of positive curvature has finite fundamental group, and is noncompact.*

Theorem 2 can be formulated more generally, with *pseudo-Riemannian manifold of general signature* in place of Lorentz manifold, and *locally homogeneous space* in place of constant sectional curvature. We can formalize this as the problem of discontinuous groups for homogeneous spaces. Here we say that a discrete subgroup Γ of G is a discontinuous group for the homogeneous space G/H to mean that the left action of Γ on G/H is properly discontinuous and free (for more details, see Section 2). The following result is formulated so as to contain Theorem 2 as a special case.

Theorem 3 (Criterion for the Calabi–Markus phenomenon 1989 [17]) *Let $G \supset H$ be a pair of reductive Lie groups; then the homogeneous space G/H admits a discontinuous group of infinite order if and only if $\mathrm{rank}_\mathbb{R} G > \mathrm{rank}_\mathbb{R} H$.*

In the opposite directions, testing this theorem on various examples of homogenous spaces leads one to believe that the following conjecture may hold:

Conjecture 4 (see [27]) *Assume that $p \geq q > 0$ and $p + q \geq 3$. Suppose that the sectional curvature of a complete pseudo-Riemannian manifold of signature (p, q) has a positive lower bound.[6] Then the fundamental group is finite and the manifold is noncompact.*

---

[6]Here we are using sectional curvature. With Ricci curvature only, the assumption is too weak.

5

1.4   The case of curvature zero

Among Riemannian manifolds, the n-dimensional torus T^n is the typical example of a space form of curvature 0. Its fundamental group $\mathbb{Z}^n$ is an Abelian group. More generally, the following theorem says that the fundamental group of a space form of curvature 0 is also close to an Abelian group:

Theorem 5 (Bieberbach 1911) *The fundamental group of a complete Riemannian manifold of constant sectional curvature* 0 *contains an Abelian subgroup of finite index.*

Whether the analogous result holds for a pseudo-Riemannian manifold is still unknown:

Conjecture 6 (Auslander Conjecture – Special case) *The fundamental group of a compact pseudo-Riemannian manifold of constant sectional curvature* 0 *contains a solvable subgroup of finite index.*

Conjecture 6 is true for a Lorentz manifold (Goldman and Kamishima 1984 [12], Tomanov). More generally, one can extend the Bieberbach theorem under the assumption of an affine manifold. This is the original Auslander Conjecture. One can also envisage a stronger form:

Problem 7 (Milnor 1977 [37]) Is it true that any discontinuous group (see Section 2) for affine space $\mathbb{R}^n = (\mathrm{GL}(n, \mathbb{R}) \ltimes \mathbb{R}^n)/\mathrm{GL}(n, \mathbb{R})$ contains a solvable subgroup of finite index?

In 1983 Margulis gave a counterexample to this conjecture of Milnor in dimension $n = 3$ (Theorem 11). On the other hand, the following proposition is a continuous analog of Milnor's problem:

> A connected subgroup of the affine group that acts properly on $\mathbb{R}^n$ (see Section 2) is amenable; that is, it is a compact extension of a solvable group.

This statement (also in a more general form) is known to hold (1993 [20], Lipsman 1995 [34]). Although the original Auslander Conjecture remains open, Abels, Margulis and Soifer (1997 [1], [2], [36]) have announced that it holds in dimension ≤ 6. Also, on a related topic, the Lipsman conjecture

(1995, [34]) on discontinuous groups of nilmanifolds and their proper actions is known. Definitive results on the Lipsman conjecture have been obtained in very recent work of Nasrin, Yoshino, Baklouti and Khlif. Namely, it is true if the nilmanifold has dimension ≤ 4 [51], and there is a counterexamples in dimension ≥ 5 [52]. Moreover, it is true for Lie groups that are at most 3-step nilpotent ([3], [39], [54]), and there is a counterexample for 4-step nilpotent or more [52].

1.5 The case of negative curvature

In the case of Riemannian manifolds, negatively curved compact space forms (hyperbolic manifolds) exist. This is equivalent to the fact that the Lorentz groups $O(n, 1)$ admit uniform lattices.[7] However, for general pseudo-Riemannian manifolds, the fundamental problem of knowing for which signature (p, q) there exist *compact space forms (of negative curvature)* is still not completely settled as things stand. The following conjecture addresses this question; it is a specialization to the case of the homogeneous spaces $O(p, q + 1)/ O(p, q)$ of Conjecture 17, the existence problem for uniform lattices for a reductive homogeneous space.

Conjecture 8 (Conjecture on Space Forms, 1996) *The necessary and sufficient condition for the existence of a compact pseudo-Riemannian manifold of signature (p, q) with constant negative sectional curvature is that (p, q) is in the following list:*

q	N	0	1	3	7
p	0	N	$2N$	$4N$	8

where $N = 1, 2, 3, \ldots$.

The sufficiency of the condition is proved. As already discussed, for $q = 0$, these are hyperbolic (Riemannian) manifolds; for $q = 1, 3$, examples were discovered in 1981 by Kulkarni [30]. In the case $q = 7$, examples were discovered from the 1990s by applying the existence theorem of uniform lattices for general homogeneous spaces (Theorem 15), see [21].

---

[7]For arithmetic uniform lattices, this is general theory, due to Borel and Harish-Chandra (1962), Mostow and Tamagawa (1962) and Borel (1963); for nonarithmetic uniform lattices (in hyperbolic spaces) examples were constructed by Makarov (1966), Vinberg, Gromov and Pyatetskiĭ-Shapiro (1981).

Whether the condition of Conjecture 8 is necessary is still not settled, although it is proved to hold in many cases, such as $q = 1$, or $p \leq q$, or pq odd. The final "odd condition" on p, q was extended to general reductive homogeneous spaces (the most typical pseudo-Riemannian homogeneous spaces) by Ono and myself [28], by a method generalizing Hirzebruch proportionality for characteristic classes. (For more details on these topics, we refer to the references given in [27] and [31]. See also [58].)

2 Discontinuous actions and Clifford–Klein forms

Even though the problem of space forms treats extremely special spaces, as discussed in the preceding section, many problems remain open. However, even restricting to these cases, there are instances when, rather than studying individual isolated examples, we obtain a clearer perspective from the general viewpoint of *discontinuous group actions on (non-Riemannian) homogeneous spaces*.[8] In this direction, this section explains basic notions and concrete examples, while emphasizing the distinction between "discontinuous groups acting on homogeneous spaces" and "discrete subgroups". In the remainder of the article, keeping at the back of our minds the point of view on the problem of space forms explained above, we want to discuss how far the world of discontinuous groups can be extended in the general framework, avoiding as far as possible the technical terms of the theory of Lie group.

First, in the case of Riemannian manifolds, subgroups consisting of isometries satisfy

$$\text{discrete group} \quad \Longleftrightarrow \quad \text{discontinuous group}.$$

However, for pseudo-Riemannian manifolds and for subgroups consisting of isometries, these conditions are not equivalent:

$$\text{discrete group} \quad \overset{\Longrightarrow}{\underset{\Longleftarrow}{\not\Leftarrow}} \quad \text{discontinuous group},$$

and the quotient space by the action of a discrete group is not necessarily Hausdorff. For example, the orbit of a discrete group is not necessarily a

---

[8]From this point of view, any pseudo-Riemannian space forms of signature (p, q) with $q \neq 1$ is a Clifford–Klein form of the rank 1 semisimple symmetric space $\mathrm{O}(p, q+1)/\mathrm{O}(p, q)$.

closed set; this corresponds to the fact that in the topology of the quotient space, a single point is not a closed point, and in particular the quotient topology is not Hausdorff. There are thus cases when the quotient is non-Hausdorff for local reasons. However, Hausdorff is a global property of a topological space, and there are also more curious counterexamples. The following example is one such. Here the quotient topology is non-Hausdorff for a global reason, because accumulation points do not exist.[9]

Example Make the discrete group $\mathbb{Z}$ act on $X = \mathbb{R}^2 \setminus \{(0,0)\}$ by the map $\mathbb{Z} \times X \to X$ given by $(n, (x, y)) \mapsto (2^n x, 2^{-n} y)$. Thus the $\mathbb{Z}$-orbits are contained in the hyperbolas $xy = \text{const.}$ or in the x- and y-axes; in
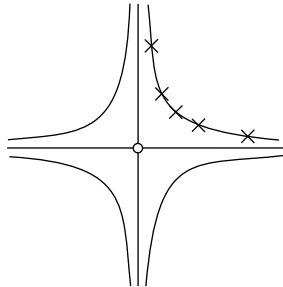

Figure 1: $\mathbb{Z}$-orbits

Figure 1 the crosses $\times$ represent a $\mathbb{Z}$-orbit contained in the first quadrant. This $\mathbb{Z}$-action does not have any accumulation points in X, but the quotient space $\mathbb{Z} \backslash X$ is non-Hausdorff. In fact, by considering the fiber bundle $\mathbb{Z} \backslash \mathbb{R} \to \mathbb{Z} \backslash X \to \mathbb{R} \backslash X$, one sees that the quotient space $\mathbb{Z} \backslash X$ is homeomorphic to an S^1 fiber bundle over the base space illustrated in Figure 2, consisting of four half-lines and four points, given a non-Hausdorff topology.

How to understand this kind of example in group theoretic terms is the main topic of Sections 2–3. As a preparation, we introduce some pieces of basic terminology. The set-up we consider is a topological group Γ with a continuous action on a space X; we write $\Gamma \curvearrowright X$ for this action. For $S \subset X$ a subset, we define the subset $\Gamma_S \subset \Gamma$ as follows:

$$\Gamma_S := \big\{ \gamma \in \Gamma \mid \gamma \cdot S \cap S \neq \emptyset \big\}.$$

---

[9]This example can be reformulated in group theoretic terms as a homogeneous space of $\mathrm{SL}(2, \mathbb{R})$.
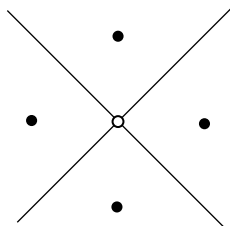
Figure 2: $\mathbb{R}\backslash X \simeq$ four half-lines and four points
(a complete system of representatives)

Definition   (1) The action $\Gamma \curvearrowright X$ is *properly discontinuous* if Γ_S is finite for any compact subset $S \subset X$.

    (2) The action $\Gamma \curvearrowright X$ is *proper* if Γ_S is compact for any compact subset $S \subset X$.

    (3) The action $\Gamma \curvearrowright X$ is *free* if the stabilizer subgroup $\Gamma_{\{p\}}$ of any point $p \in X$ consists of the identity element only.

The action of a noncompact group is not necessarily well behaved. The notion of a "proper action" (Palais 1961 [44]) abstracts out the "good property" of a compact group action. Putting together as above the notion of properly discontinuous gives rise to the following equation:

$$\Gamma \text{ acts properly discontinuously} \iff \begin{cases} \Gamma \text{ acts properly,} \\ and \ \Gamma \text{ is a discrete group.} \end{cases}$$

Thus, in order to determine whether the action of a discrete group is properly discontinuous, it is enough to determine whether the action is proper. This last point has wide applications.

Now, for a group Γ acting on a set X, we write $\Gamma\backslash X$ for the set of equivalence classes of the equivalence relation given by

$$x \sim x' \iff x' = \gamma \cdot x \text{ for some } \gamma \in \Gamma.$$

We can view $\Gamma\backslash X$ as the set of Γ-orbits in X, so we call it the *orbit space* (or Γ-orbit space). The reason for considering properly discontinuous actions of Γ is the following well-known result:

Lemma *Let X be a manifold (respectively C^∞ manifold, pseudo-Riemannian manifold, complex manifold, etc.), and suppose that a discrete group Γ acts on X continuously (respectively smoothly, isometrically, biholomorphically, etc.). If the action of Γ is properly discontinuous and free then the quotient topology on $\Gamma\backslash X$ is Hausdorff, and the quotient $\Gamma\backslash X$ can be given a unique manifold structure for which the quotient map $X \to \Gamma\backslash X$ is a local homeomorphism (respectively local diffeomorphism, local pseudo-isometry, locally biholomorphic).*

In what follows, $X = G/H$ is a homogeneous space for a Lie group G, and Γ is a discrete subgroup of G, so we have the following triple of groups:

$$\Gamma \subset G \supset H.$$

Definition We say that Γ is a *discontinuous group* for the homogeneous space G/H if Γ acts properly discontinuously and freely on G/H. Here "properly discontinuously" is the most important condition; papers in the literature sometimes omit the condition that the action is free in the definition of a discontinuous group. If Γ is a discontinuous group for G/H, the manifold obtained as the double coset space $\Gamma\backslash G/H$ is a *Clifford–Klein form* of G/H. If in addition $\Gamma\backslash G/H$ is compact, we say that Γ is a *uniform lattice* of the homogeneous space G/H.

Examples (1) Let $(G, \Gamma, H) = (\mathbb{R}^n, \mathbb{Z}^n, \{0\})$; then the Clifford–Klein form $\Gamma\backslash G/H$ is diffeomorphic to the n-dimensional torus T^n, and so is a compact manifold.

(2) Nilmanifold: set

$$G := \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \middle| a, b, c, \in \mathbb{R} \right\}, \quad \Gamma := G \cap \mathrm{GL}(3, \mathbb{Z}), \quad H = \{e\}.$$

The Clifford–Klein form $\Gamma\backslash G/H$ is a 3-dimensional compact manifold (the *Iwasawa manifold*).

(3) The modular group: set $(G, \Gamma, H) = (\mathrm{SL}(2, \mathbb{R}), \mathrm{SL}(2, \mathbb{Z}), \{e\})$. The Clifford–Klein form $\Gamma\backslash G/H$ is noncompact, but has finite volume (with respect to a naturally defined measure). Moreover $\Gamma\backslash G/H$ is homeomorphic to the knot complement $\mathbb{R}^3 \setminus$ (trefoil knot).

(4) Closed Riemann surface: the closed Riemann surface M_g of genus $g \geq 2$ can be realized as a Clifford–Klein form $\Gamma \backslash G/H$ of the Poincaré upper half-space $G/H = \mathrm{SL}(2, \mathbb{R})/\mathrm{SO}(2)$; here $\Gamma \simeq \pi_1(M_g)$.

(5) The Calabi–Markus phenomenon: let $G = \mathrm{SL}(2, \mathbb{R})$, and let $H \subset G$ be any noncompact closed subgroup. Then the only discontinuous groups of G/H are finite groups. In particular, if H and G/H are both noncompact then there does not exist any uniform lattice for G/H.

(6) Compact Lorentz space form: let $(G, H) = (\mathrm{SO}(2n, 2), \mathrm{SO}(2n, 1))$ and let $\Gamma \subset \mathrm{U}(n, 1)$ be a uniform lattice without torsion elements. If we view Γ as a discrete subgroup of G by the embedding $\Gamma \subset \mathrm{U}(n, 1) \subset G$ then Γ is also a uniform lattice of the homogeneous space G/H. However, Γ cannot be a uniform lattice of G.

The above examples (5) and (6) illustrate the following important warning for noncompact subgroups:

> for a noncompact subgroup H, a uniform lattice for G is *not* the same thing as a uniform lattice for the homogeneous space G/H.

3   Criterion for an action to be discontinuous

This section discusses the following problem:

Problem A Find effective methods of determining whether a discrete subgroup Γ acts properly discontinuously on a homogeneous space G/H.

The definition of a properly discontinuous action on a topological space was easy enough to formulate. However, in general, given a discrete subgroup Γ of a Lie group G, actually determining whether or not the action of Γ on a homogeneous space G/H is properly discontinuous is not at all easy. One aims for "criteria" in Problem A that are so concrete and powerful that we can, for example, obtain the following various theorems as sample applications.

Theorem 9 (Pseudo-Riemannian manifold space form of signature (p, q) **with negative curvature [8], [48], [30], [17])** *The homogeneous space* $\mathrm{O}(p, q+1)/\mathrm{O}(p, q)$ *has only finite groups as discontinuous groups* $\iff$ $p \leq q$.

Theorem 10 (Solvable manifolds 1993 [20], [34]) *Any homogeneous space for a solvable Lie group has a Clifford–Klein form with fundamental group of infinite order.*

Theorem 11 (Affinely flat manifold, Margulis 1983 [2]) *Affine 3-space* $(\mathrm{GL}(3,\mathbb{R}) \ltimes \mathbb{R}^3)/\mathrm{GL}(3,\mathbb{R}) \simeq \mathbb{R}^3$ *admits a free non-Abelian group as a discontinuous group.*

Theorem 12 (Pseudo-Riemannian homogeneous space, Benoist 1996 [4]) $\mathrm{SL}(3,\mathbb{R})/\mathrm{SL}(2,\mathbb{R})$ *does not admit a free non-Abelian group as a discontinuous group.*

Now for a non-Riemannian homogeneous space G/H, the usual approach to its discontinuous groups was to restrict the study to extremely special situations (for example, the rank 1 symmetric spaces of the type treated in Section 1) and to make clever use of the special properties enjoyed by the individual homogeneous spaces G/H; compare [8], [30], [48], [49],.... This method requires huge calculations, even if G/H is a rank 1 symmetric space (as in [30]). Instead of this, to deal with discontinuous groups for more general pseudo-Riemannian homogeneous spaces (for noncompact H), I introduced the following idea in [17], [22].

1. *Forget* that H is a group and that the homogeneous space G/H is a manifold.

2. *Forget* that Γ is discrete and that it is a group.

Having thus thrown away all of the (at first sight) most important information, we are left with the following possibilities:

3. View Γ and H *on an equal footing*, simply as subsets of G.

4. Control the discontinuous property of the action $\Gamma \curvearrowright G/H$ using the representation theory of G.

In order to implement this idea, we introduce the following two relations $\pitchfork$ and $\sim$ on subsets H and[10] L of a locally compact group G.

---

[10]In what follows, we often use L as an alternative notation to Γ.

Definition **(see [22])** (1) We say that the pair (L, H) is *proper* in G and write $L \pitchfork H$ (in G) if and only if for any compact subset S of G the intersection $L \cap SHS$ is relatively compact.[11]

(2) We write $L \sim H$ (in G) if and only if there exists a compact subset S of G such that $L \subset SHS$ and $H \subset SLS$.

For an Abelian group G, the relations $\pitchfork$ and $\sim$ are remarkably simple.

Example Let H and L be subspaces of the Abelian group $G := \mathbb{R}^n$.

(1) $H \pitchfork L$ in $G \iff H \cap L = \{0\}$.

(2) $H \sim L$ in $G \iff H = L$.

Now if L and H are closed subgroups of G then

$$L \pitchfork H \text{ in } G \iff L \text{ acts properly on } G/H.$$

Thus we can view $\pitchfork$ as a notion generalizing properness of a group action. In other words, to understand whether an action is proper, or is properly discontinuous, it is enough to understand the relation $\pitchfork$. Moreover

$$L \pitchfork H \iff H \pitchfork L$$

This reflects a kind of symmetry between the action of L on G/H and of H on G/L. We also have

$$\text{if } H \sim H' \text{ then } \quad L \pitchfork H \iff L \pitchfork H'.$$

Thus the use of $\sim$ provides economies in considering $\pitchfork$. We define the *discontinuous dual* $H^{\pitchfork}$ of a subset H of G as follows:

$$H^{\pitchfork} := \left\{ L \mid L \text{ is a subset of } G \text{ satisfying } L \pitchfork H \right\}.$$

First, we have the following theorem, which I proved in 1996 [22] for G a reductive Lie group; whether it holds in general was one of the unsolved problems discussed in [27], and was settled positively by Yoshino in 2004 [53].

---

[11]In differential geometry, $\pitchfork$ is often used to mean that two submanifolds intersect transversally; here we use the notation in a completely different meaning.

Theorem 13 (Duality theorem) *A subset H of a Lie group G can be reconstructed from its discontinuous dual $H^{\pitchfork}$ up to the equivalence $\sim$.*

Our original aim was to determine by explicit methods whether the action of a discrete group is properly discontinuous. Problem A can be formalized again in the following more general form.

Problem A′ Let H and L be subsets of a group G. Find criteria to determine whether $H \pitchfork L$.

Let G be a reductive linear Lie group (for example, $\mathrm{GL}(n, \mathbb{R})$ or $\mathrm{O}(p, q)$, etc.). Write $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ for a Cartan decomposition of the Lie algebra $\mathfrak{g}$ of G, and choose a maximal Abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$. Write

$$d(G) := \dim \mathfrak{p}, \quad \mathrm{rank}_{\mathbb{R}} G := \dim a.$$

Also, using a Cartan decomposition $G = K \exp(\mathfrak{a}) K$, we define the *Cartan projection* $\nu \colon G \to \mathfrak{a}$, which is determined up to the action of the Weyl group.

For example, for $G = \mathrm{GL}(n, \mathbb{R})$, we have $d(G) = \binom{n+1}{2}$ and $\mathrm{rank}_{\mathbb{R}} G = n$. For a square matrix $g \in G$, the product ${}^t g g$ is a positive definite symmetric matrix, and we write out its eigenvalues in order, from the largest down:

$$\lambda_1 \geq \cdots \geq \lambda_n \ (> 0).$$

Then the Cartan projection $\nu \colon G \to \mathbb{R}^n$ is given by the formula: $g \mapsto \frac{1}{2}(\log \lambda_1, \ldots, \log \lambda_n)$.

After these preparations, the answer to Problem A (or Problem A′) is as follows:

Theorem 14 (Criterion for a properly discontinuous action [17], [22])
Let H and L be subsets of a reductive linear Lie group G. Then

(1) $L \sim H$ *in* $G \iff \nu(L) \sim \nu(H)$ *in* $\mathfrak{a}$.

(2) $L \pitchfork H$ *in* $G \iff \nu(L) \pitchfork \nu(H)$ *in* $\mathfrak{a}$.

For the Abelian group $\mathfrak{a} \simeq \mathbb{R}^n$ the relations $\pitchfork$ and $\sim$ have a very simple meaning. Thus Theorem 14 is useful as a criterion.

I solved Problem A′ in 1989 [17] for a triple of reductive Lie groups (G, H, L), and subsequently generalized the result in [22] in the above form,

15

without even assuming that L and H have group structures; Benoist [4] proved a similar generalization independently of [22]. The above results Theorems 3, 9 and 12 can be obtained as corollaries of Theorem 14. Also, using a generalization in the style of Theorem 14 made it possible to study the extent to which the proper discontinuity of an action is preserved on deforming a discrete group (for example, Goldman's conjecture 1985 [11]); see [24], [29], [41], [45]; we return to this topic in Section 5. In addition to this, recent work (from 2001 onwards) due to Iozzi, Oh, Witte and others [14], [27], [42] contain results arising from applications of Theorem 14 to individual homogeneous spaces.

The implications $\Leftarrow$ of Theorem 14, (1) and $\Rightarrow$ of Theorem 14, (2) are obvious. And the implication $\Rightarrow$ in (1) is related to giving uniform bounds on the errors in the eigenvalues when a matrix is perturbed.[12]

4 The existence problem for compact Clifford–Klein forms

4.1 Existence and nonexistence theorems for uniform lattices

In what follows, we let $G \supset H$ be a pair of reductive linear Lie groups. G/H is the typical model of a pseudo-Riemannian homogeneous space (Riemannian if H is compact). This section discusses the following problem.

Problem B For which homogeneous spaces G/H does a uniform lattice exist? In other words, classify the homogeneous spaces that have compact Clifford–Klein forms.

Among the various currently known results, we discuss two that have the widest field of applications. The key points in the proof of these results are the criterion for a proper action (Theorem 14) and the cohomology of discrete groups.

Theorem 15 (1989 [17]) *If a reductive subgroup $L \subset G$ satisfies*

$$L \pitchfork H \quad and \quad d(L) + d(H) = d(G),$$

---

[12]Various inequalities concerning this are known, of which a theorem of Weyl is the prototype.

then a compact Clifford–Klein form of G/H exists.

Theorem 16 (1992 [19]) *If $L \subset G$ is a reductive subgroup and there exists H with*

$$L \sim H \quad and \quad d(L) > d(H)$$

then there does not exist a compact Clifford–Klein form of G/H.

We refer to [21] for the list of pairs (G, H) that satisfy the conditions of Theorem 15[13] and to [19], [21] for the list of (G, H) that satisfy those of Theorem 16.

If Γ is a torsion-free uniform lattice of a group L then, provided the assumptions of Theorem 15 are satisfied, $\Gamma\backslash G/H$ is a compact Clifford–Klein form. Conversely, even if we assume that $\Gamma\backslash G/H$ is a compact Clifford–Klein form, it does not necessarily follow that there exists a reductive Lie subgroup L containing Γ satisfying the assumptions of Theorem 15 (1998 [24], Salein 1999 [45]). However, in a slightly weaker form, the following conjecture is still unsolved. The special case $G/H = O(p, q + 1)/O(p, q)$ of Conjecture 17 is Conjecture 6 on space forms.

Conjecture 17 (1996 [21]) *The converse of Theorem 15 also holds.*

4.2 Uniform lattices of an adjoint orbit

We consider semisimple orbits as examples of homogeneous spaces. If we choose an element X of the Lie algebra $\mathfrak{g}$ of a Lie group G, the adjoint orbit $\mathcal{O}_X = \text{Ad}(G)X$ is a submanifold of $\mathfrak{g}$ that we can identify with the homogeneous space G/G_X, where $G_X = \{g \in G \mid \text{Ad}(g)X = X\}$ is the stabilizer of X.

When G is a reductive Lie group and $\text{ad}\, X \in \text{End}(\mathfrak{g})$ is semisimple, we say that $\mathcal{O}_X$ is a *semisimple orbit*. If in addition all of the eigenvalues of $\text{ad}\, X$ are purely imaginary numbers then we say that $\mathcal{O}_X$ is an *elliptic orbit*. For example, the adjoint orbits of a compact Lie group are always elliptic orbits.

A number of important classes of homogeneous spaces arise as semisimple orbits. For example, all the flag manifolds, Hermitian symmetric spaces,

---

[13]It follows of course from Borel's Theorem [5] that this list includes the case that H is compact, and the case of the group manifold itself (that is $G = G' \times G'$ and $H = \text{diag}\, G'$).

para-Hermitian symmetric spaces, and so on can be realized as semisimple orbits.[14]

One can define a natural G-invariant symplectic structure and pseudo-Riemannian structure on a semisimple orbit.[15] Also, various unitary representations appear as geometric quantizations of these orbits, including principal series representations (more generally, degenerate principal series representations), and discrete series representations (more generally, Zuckerman's derived functor modules $A_{\mathfrak{q}}(\lambda)$), and so on. The latter in particular corresponds to geometric quantization of elliptic orbits.

We have the following theorem concerning the existence problem[16] of compact Clifford–Klein forms of semisimple orbits.

Theorem 18 (see [19]) *The only semisimple orbits having a uniform lattice are elliptic orbits. In particular, these have an invariant complex structure.*

Hermitian symmetric spaces always have a uniform lattice (Borel [6]), and are elliptic orbits. We give one example of an elliptic orbit that is not a Hermitian symmetric space: consider the Hermitian form of signature $(2, n)$

$$z_1\overline{z_1} + z_2\overline{z_2} - z_3\overline{z_3} - \cdots - z_{n+2}\overline{z_{n+2}},$$

and write $\mathcal{O} \subset \mathbb{P}_{\mathbb{C}}^{n+1}$ for the set of all complex lines such that the restriction of the form is positive definite. Then $\mathcal{O}$ is an open subset of complex projective space $\mathbb{P}_{\mathbb{C}}^{n+1}$ (so is in particular a complex manifold), and can be identified with an elliptic orbit of $\mathrm{U}(2, n)$. Note that $\mathcal{O}$ does not have the structure of a Hermitian symmetric space. Moreover, if n is even then we can use Theorem 15 to see that there exists a uniform lattice for $\mathcal{O}$ (for this, one need only set $L = \mathrm{Sp}(1, \frac{n}{2})$). In particular, one can use this to construct a compact symplectic complex manifold for which the natural form has indefinite signature (see [17], [21]).

Theorem 18 was discovered by myself; its proof uses the cohomology of discrete groups, and is based on Theorem 16 ([18], [19]). Subsequently Benoist and Labourie [5] gave a different proof using symplectic geometry.

---

[14]The first two cases are even elliptic orbits.

[15]Plus, in the case of an elliptic orbit, a G-invariant complex structure. A Kähler or pseudo-Kähler structure can also be defined.

[16]Although extremely special, the case that a compact Clifford–Klein form exists has known analytic applications, such as Atiyah and Schmid's construction [57] of the discrete series representations via the L^2 index theorem.

4.3 Uniform lattices of $\mathrm{SL}(n)/\mathrm{SL}(m)$

This section discusses the question of whether there exist compact Clifford–Klein forms of the non-symmetric homogeneous space $\mathrm{SL}(n)/\mathrm{SL}(m)$. This space is special from our point of view; from the mid-1990s onwards, the question of the existence of its compact Clifford–Klein forms was attacked using a variety of different methods, with the same result being obtained using many different methods of proof, resulting in an attractive amalgam with other areas.

The original model is the following result, obtained by applying $L = \mathrm{SU}(2,1)$ to Theorem 16.

Theorem 19 (1990 [18]) *There do not exist any compact Clifford–Klein forms of* $\mathrm{SL}(3,\mathbb{C})/\mathrm{SL}(2,\mathbb{C})$.

The following theorem is deduced from Theorem 16 by the same principle (replacing $\mathbb{R}$ by $\mathbb{C}$ or by the quaternions $\mathbb{H}$ leads to similar results).

Theorem 20 (1992 [19]) *There do not exist any compact Clifford–Klein forms of the homogeneous spaces* $\mathrm{SL}(n,\mathbb{R})/\mathrm{SL}(m,\mathbb{R})$ *if* $\frac{n}{3} > \left[\frac{m+1}{2}\right]$.

Now for the Clifford–Klein forms of $\mathrm{SL}(n,\mathbb{R})/\mathrm{SL}(m,\mathbb{R})$, we can also consider the right action of $\mathrm{SL}(n-m,\mathbb{R})$. Taking note of this point, Zimmer and his collaborators used machinery such as the superrigidity theorem for cocycles and Ratner's theorem on the closure of orbits to prove the following theorems.

Theorem 21 (Zimmer 1994 [56]) *There do not exist any compact Clifford–Klein forms of* $\mathrm{SL}(n,\mathbb{R})/\mathrm{SL}(m,\mathbb{R})$ *when* $n > 2m$.

Theorem 22 (Labourie, Mozes and Zimmer 1995 [32]) *There do not exist any compact Clifford–Klein forms of* $\mathrm{SL}(n,\mathbb{R})/\mathrm{SL}(m,\mathbb{R})$ *if* $n \geq 2m$.

Theorem 23 (Corlette and Zimmer 1994 [9], [10]) *There do not exist any compact Clifford–Klein forms of* $\mathrm{Sp}(n,2)/\mathrm{Sp}(m,1)$ *if* $n > 2m$.

The following result was obtained as an application of the criterion for a proper action (Theorem 14).

Theorem 24 (Benoist 1996 [4]) *There do not exist any compact Clifford–Klein forms of* $\mathrm{SL}(n,\mathbb{R})/\mathrm{SL}(n-1,\mathbb{R})$ *for odd* n.

Moreover, for $\mathrm{SL}(m)$ embedded in $\mathrm{SL}(n)$ by an irreducible representation φ (not just the natural inclusion), Margulis considered the restriction of unitary representations of $\mathrm{SL}(n,\mathbb{R})$ to noncompact subgroups, and used methods involving delicate estimates for the asymptotic behavior of matrix coefficients to prove the following theorem.

Theorem 25 (Margulis 1997 [35]) *For $n \geq 5$, there does not exist any compact Clifford–Klein form of $\mathrm{SL}(n,\mathbb{R})/\varphi(\mathrm{SL}(2,\mathbb{R}))$.*

The systematic study of Margulis' method was taken further by Oh 1998 [40]. The method based on unitary representation theory was developed futher, and Shalom proved the following theorem.

Theorem 26 (Shalom 2000 [46]) *For $n \geq 4$, there does not exist any compact Clifford–Klein form of $\mathrm{SL}(n,\mathbb{R})/\mathrm{SL}(2,\mathbb{R})$.*

These results were obtained by taking up recent development in other areas of mathematics, and the methods of proof extend over many branches. However, as things stand at present, all the currently known results support Conjecture 17 (which, applied to this case, states that "there do not exist any compact Clifford–Klein forms of $\mathrm{SL}(n,\mathbb{R})/\mathrm{SL}(m,\mathbb{R})$ for $n > m$").

Note that, among these theorems, Theorems 21, 22, 23 and 26 are contained in an extremely special case of Theorem 16: although the references [9], [10], [31], [46], [56] mentioned above do not refer explicitly to this, Theorem 16 or its corollary Theorem 20 is actually a stronger result even when restricted to these special cases. On the other hand, Benoist's Theorem 24 and Margulis' Theorem 25 are not contained in Theorem 16. For various results concerning these explicit kinds of homogeneous spaces, many details are contained in my lecture notes [21] in the proceedings of a European School.

5 Rigidity and deformations of Clifford–Klein forms

This section discusses the following problem.

Problem C Is it possible to deform a uniform lattice Γ for a homogenous space G/H?

For an irreducible Riemannian symmetric space G/H of dimension ≥ 3 with a compact subgroup H, and Γ a uniform lattice of G/H, there do not exist any essential deformations of Γ (Theorem 27). This result is the original model for various kinds of rigidity theorems (in Riemannian geometry).

Now, does there exist a similar *rigidity result* in the case that H is noncompact (the pseudo-Riemannian case)? We can view a "rigidity theorem" as an assertion of the type that the fundamental group determines not just the topological structure, but also the geometric structure. Now, does the "rigidity theorem" also hold for an (irreducible) pseudo-Riemannian symmetric space?

In fact, for a noncompact subgroup H, the situation for the rigidity theorem is quite different from the case of Riemannian symmetric spaces. More precisely, there exist (irreducible) pseudo-Riemannian symmetric spaces of arbitrarily high dimension, that admit uniform lattices for which the rigidity theorem does not hold (see [20], [24]). We give such examples in Theorem 28, where we give a more precise formulation of Problem C. But first, we note that Problem C includes the following two subproblems.

Problem C-1 For a discrete subgroup $\Gamma \subset G$, describe the deformations of Γ as an abstract group inside G.

Problem C-2 If a discrete subgroup $\Gamma \subset G$ can be deformed inside G, determine the range of the deformation parameters that does not destroy the proper discontinuity of its action on G/H.

Bearing these questions in mind, we now try to describe abstractly the set of deformations of a discontinuous group.

Let G be a Lie group and Γ a finitely generated group; we give the set $\mathrm{Hom}(\Gamma, G)$ of group homomorphisms from Γ to G the pointwise convergence topology. The same topology is obtained by taking generators $\gamma_1, \ldots, \gamma_k$ of Γ, then using the injective map

$$\mathrm{Hom}(\Gamma, G) \hookrightarrow G \times \cdots \times G \quad \text{given by} \quad \varphi \mapsto (\varphi(\gamma_1), \ldots, \varphi(\gamma_k))$$

to give $\mathrm{Hom}(\Gamma, G)$ the relative topology induced from the direct product $G \times \cdots \times G$.

Let H be a closed subgroup of G. As already explained, if H is noncompact then a discrete subgroup of G does not necessarily act properly discontinuously on G/H. Here rather than $\mathrm{Hom}(\Gamma, G)$, it is the subset $R(\Gamma, G, H)$

defined below that plays the important role (see [20]):

$$R(\Gamma, G, H) := \left\{ u \in \mathrm{Hom}(\Gamma, G) \,\middle|\, \begin{array}{l} u \text{ is injective; and } u(\Gamma) \text{ acts properly} \\ \text{discontinuously and freely on } G/H \end{array} \right\}.$$

Then for each $u \in R(\Gamma, G, H)$ we obtain a Clifford–Klein form $u(\Gamma)\backslash G/H$. Now the direct product group $\mathrm{Aut}(\Gamma) \times G$ acts naturally on $\mathrm{Hom}(\Gamma, G)$, leaving $R(\Gamma, G, H)$ invariant. We now define the following two spaces:

the deformation space $\mathcal{T}(\Gamma, G, H) := R(\Gamma, G, H)/G$; and

the moduli space $\mathcal{M}(\Gamma, G, H) := \mathrm{Aut}(\Gamma)\backslash R(\Gamma, G, H)/G$.

For example, if $(G, H) = (\mathrm{PSL}(2, \mathbb{R}), \mathrm{PSO}(2))$ and $\Gamma = \pi_1(M_g)$ for $g \geq 2$ then $\mathcal{T}(\Gamma, G, H)$ is the Teichmüller space of M_g, and $\mathcal{M}(\Gamma, G, H)$ is nothing other than the moduli space of complex structures on M_g.

We formalize local rigidity of a discontinuous group as saying that it corresponds to an "isolated point" of the deformation space $\mathcal{T}(\Gamma, G, H)$:

Definition (Local rigidity in a non-Riemannian homogeneous space 1993 [20]) Let $u \in R(\Gamma, G, H)$. We say that the discontinuous group $u(\Gamma)$ for the homogeneous space G/H determined by u is *locally rigid* as a discontinuous group of G/H if the single point $[u]$ is an open set of the quotient space $\mathrm{Hom}(\Gamma, G)/G$; this means that any point sufficiently close to u is conjugate to u under an inner automorphism of G. If u is not locally rigid, we say that *u admits continuous deformations.*

When H is compact, this terminology coincides with the original notion (see Weil [47]).

In higher dimensions, let us compare whether the local rigidity theorem holds in the cases that H is compact or noncompact. Let G' be a noncompact simple linear Lie group, and K' its maximal compact subgroup. We can use the vanishing and nonvanishing theorems for cohomology of Lie algebras together with the criterion for a properly discontinuous action discussed above (Theorem 14) and so on, to prove the following theorem.

Theorem 27 (Local rigidity theorem – the Riemannian case: Selberg and Weil 1964 [47]) *Let $(G, H) := (G', K')$. Then the following two conditions on G' are equivalent:*

(i) there exists a uniform lattice $\iota\colon \Gamma \to G'$ such that $\iota \in R(\Gamma, G, H)$ admits continuous deformations.

(ii) G' is locally isomorphic to $\mathrm{SL}(2, \mathbb{R})$.

Theorem 28 (Local rigidity theorem – the non-Riemannian case 1998 [24])
Let $(G, H) := (G' \times G', \mathrm{diag}(G'))$. Then the following two conditions on G' are equivalent:

(i) there exists a uniform lattice $\iota\colon \Gamma \to G'$ such that $\iota \times 1 \in R(\Gamma, G, H)$ admits continuous deformations.

(ii) G' is locally isomorphic to $\mathrm{SO}(n, 1)$ or $\mathrm{SU}(n, 1)$.

From a different point of view, for the group manifold G' and its uniform lattice Γ, Theorem 27 only treats the rigidity of left actions, whereas Theorem 28 considers the rigidity of both left and right actions. (Note that $G/H \cong G'$ in the latter case.)

The deformation spaces have also been studied in the following cases:

(1) The Poincaré disk $G/H = \mathrm{SL}(2, \mathbb{R})/\mathrm{SO}(2)$.

(2) $G/H = G' \times G'/\mathrm{diag}\, G'$ for $G' = \mathrm{SL}(2, \mathbb{R})$ (Goldman 1985 [11], Salein 1999 [45]).

(3) $G/H = G' \times G'/\mathrm{diag}\, G'$ for $G' = \mathrm{SL}(2, \mathbb{C})$ (Ghys 1995 [13]).

In these cases the deformation space $\mathcal{T}(\Gamma, G, H)$ corresponds respectively to:

(1) The deformations of complex structures on a Riemann surface M_g of genus $g \geq 2$.

(2) The deformations of Lorentz structures on a 3-dimensional manifold.

(3) The deformation of complex structures on a 3-dimensional complex manifolds.

Moreover, (2) and (3) correspond to the cases $n = 1, 2, 3$ of Theorem 28. Indeed, this follows because we have the local isomorphisms of Lie groups

$$G' = \mathrm{SL}(2, \mathbb{R}) \approx \mathrm{SO}(2, 1) \approx \mathrm{SU}(1, 1) \quad \text{and} \quad G' = \mathrm{SL}(2, \mathbb{C}) \approx \mathrm{SO}(3, 1).$$

In Theorem 28, as n increases, one sees that one can construct irreducible pseudo-Riemannian symmetric spaces of arbitrarily high dimension, together with a uniform lattice Γ for which the local rigidity theorem does not hold. In [24], for general n, we obtained quantitative estimates for deformations of this type of uniform lattice Γ (that is, the range within which discontinuity is preserved) using the diameter of locally Riemannian symmetric spaces $\Gamma \backslash G'/K'$. Now the proposition

> a "small" deformation of a discrete subgroup preserves the discontinuity of the action

is false for general Lie groups (see [29]).However, in the case of semisimple Lie groups discussed above this proposition holds, and in particular this settles Goldman's conjecture [11] positively. The key to the proof is the criterion for a properly discontinuous action (Theorem 14).

References

[1]   H. Abels, G. A. Margulis and G. A. Soifer, Properly discontinuous groups of affine transformations with orthogonal linear part, C. R. Acad. Sci. Paris Sér. I Math., **324** (1997), 253–258.

[2]   H. Abels, G. A. Margulis and G. A. Soifer, On the Zariski closure of the linear part of a properly discontinuous group of affine transformations, J. Differential Geom., **60** (2002), 315–344.

[3]   A. Baklouti and F. Khlif, Proper actions on some exponential solvable homogeneous spaces, Internat. J. Math. **16** (2005), 941–955

[4]   Y. Benoist, Actions propres sur les espaces homogènes réductifs, Ann. of Math. (2), **144** (1996), 315–347.

[5]   Y. Benoist and F. Labourie, Sur les espaces homogènes modèles de variétés compactes, Inst. Hautes Études Sci. Publ. Math., **76** (1992), 99–109.

[6]   A. Borel, Compact Clifford–Klein forms of symmetric spaces, Topology, **2** (1963), 111–122.

[7]   A. Borel and Harish-Chandra, Arithmetic subgroups of algebraic groups, Ann. of Math. (2), **75** (1962), 485–535.

[8]   E. Calabi and L. Markus, Relativistic space forms, Ann. of Math. (2), **75** (1962), 63–76.

[9]   K. Corlette, Harmonic maps, rigidity, and Hodge theory, In: Proceedings of the International Congress of Mathematicians (Zürich, 1994), Birkhäuser, Basel, 1995. pp. 465–471.

[10] K. Corlette and R. Zimmer, Superrigidity for cocycles and hyperbolic geometry, Internat. J. Math. **5** (1994), 273–290.

[11] W. M. Goldman, Nonstandard Lorentz space forms, J. Differential Geom. **21** (1985), 301–308.

[12] W. M. Goldman and Y. Kamishima, The fundamental group of a compact flat Lorentz space form is virtually polycyclic. J. Differential Geom. **19** (1984), 233–240.

[13] É. Ghys, Déformations des structures complexes sur les espaces homogènes de SL(2, $\mathbb{C}$), J. reine angew. Math. **468** (1995), 113–138.

[14] A. Iozzi and D. Witte, Cartan-decomposition subgroups of SU(2, n), J. Lie Theory, **11** (2001), 505–543.

[15] A. Iozzi and D. Witte, Tessellations of homogeneous spaces of classical groups of real rank two, Geom. Dedicata, **103** (2004), 115–191.

[16] B. Klingler, Complétude des variétés lorentziennes à courbure constante, Math. Ann., **306** (1996), 353–370.

[17] T. Kobayashi, Proper action on a homogeneous space of reductive type, Math. Ann., **285** (1989), 249–263.

[18] T. Kobayashi, Discontinuous groups acting on homogeneous spaces of reductive type, In: Proceedings of ICM-90 satellite conference on Representation Theory of Lie Groups and Lie Algebras, Fuji-Kawaguchiko, 1990 (T. Kawazoe, T. Oshima and S. Sano eds.), World Scientific, 1992, pp. 59–75.

[19] T. Kobayashi, A necessary condition for the existence of compact Clifford–Klein forms of homogeneous spaces of reductive type, Duke Math. J., **67** (1992), 653–664.

[20] T. Kobayashi, On discontinuous groups acting on homogeneous spaces with noncompact isotropy subgroups, J. Geom. Phys., **12** (1993), 133–144.

[21] T. Kobayashi, Discontinuous groups and Clifford–Klein forms of pseudo-Riemannian homogeneous manifolds, In: Algebraic and analytic methods in representation theory (Sønderborg, 1994), Perspect. in Math., **17**, Academic Press, San Diego, CA, 1997, pp. 99–165.

[22] T. Kobayashi, Criterion for proper actions on homogeneous spaces of reductive groups, J. Lie Theory, **6** (1996), 147–163. http://www.emis.de/journals/JLT/vol.6_no.2

[23] T. Kobayashi, Discrete decomposability of the restrictions of $A_{\mathfrak{q}}(\lambda)$ with respect to reductive subgroups and its applications, I. Invent. Math., **117** (1994), 181–205; II. Ann. of Math. (2), **147** (1998), 709–729; III. Invent. Math., **131** (1998), 229–256.

[24] T. Kobayashi, Deformations of compact Clifford-Klein forms of indefinite-Riemannian homogeneous manifolds, Math. Ann. **310** (1998), 395–409.

[25] T. Kobayashi, Introduction to actions of discrete groups on pseudo–Riemannian homogeneous manifolds, Acta Appl. Math., **73** (2002), 115–131.

[26] T. Kobayashi, Branching problems of unitary representations. In: Proceedings of the International Congress of Mathematicians, Vol. II (Beijing, 2002), Higher Ed. Press, Beijing, 2002, pp. 615–627.

[27] T. Kobayashi, Discontinuous groups for non-Riemannian homogeneous spaces, In: Mathematics Unlimited—2001 and Beyond, B. Engquist and W. Schmid (eds.) Springer, Berlin, 2001, pp. 723–747. Japanese text: Discontinuous groups for non-Riemannian homogeneous spaces, In: Advanced Mathematics, a challenge to the 21st century, T. Sunada (ed.), vol. 1, pp. 18–73.

[28] T. Kobayashi, and K. Ono, Note on Hirzebruch's proportionality principle. J. Fac. Sci. Univ. Tokyo Sect. IA Math., **37** (1990), 71–87.

[29] T. Kobayashi and S. Nasrin, Deformation of properly discontinuous actions of $\mathbb{Z}^k$ on $\mathbb{R}^{k+1}$, to appear in Internat. J. Math.

[30] R. S. Kulkarni, Proper actions and pseudo-Riemannian space forms, Adv. Math., **40** (1981), 10–51.

[31] F. Labourie, Quelques résultats récents sur les espaces localement homogènes compacts, In: Manifolds and geometry (Pisa, 1993), Sympos. Math., XXXVI, Cambridge Univ. Press, 1996, pp. 267–283.

[32] F. Labourie, S. Mozes, and R. J. Zimmer, On manifolds locally modelled on non-Riemannian homogeneous spaces, Geom. Funct. Anal., **5** (1995), 955–965.

[33] F. Labourie and R. J. Zimmer, On the non-existence of cocompact lattices for $\mathrm{SL}(n)/\mathrm{SL}(m)$, Math. Res. Lett., **2** (1995), 75–77.

[34] R. Lipsman, Proper actions and a compactness condition, J. Lie Theory, **5** (1995), 25–39. http://www.emis.de/journals/JLT/vol.5_no.1

[35] G. A. Margulis, Existence of compact quotients of homogeneous spaces, measurably proper actions, and decay of matrix coefficients, Bull. Soc. Math. France, **125** (1997), 447–456.

[36] G. A. Margulis, Problems and conjectures in rigidity theory, in Mathematics: Frontiers and Perspectives, Amer. Math. Soc., Providence, RI, 2000, pp. 161–174.

[37] J. Milnor, On fundamental groups of complete affinely flat manifolds, Adv. Math., **25** (1977), 178–187.

[38] S. B. Myers, Riemannian manifolds with positive mean curvature, Duke Math. J., **8** (1941), 401–404.

[39] S. Nasrin, Criterion of proper actions for 2-step nilpotent Lie groups, Tokyo J. Math., **24** (2001), 535–543.

[40] H. Oh, Tempered subgroups and representations with minimal decay of matrix coefficients, Bull. Soc. Math. France, **126** (1998), 355–380.

[41] H. Oh and D. Witte, New examples of compact Clifford–Klein forms of homogeneous spaces of SO(2, n), Internat. Math. Res. Notices, **5** (2000), 235–251.

[42] H. Oh and D. Witte, Cartan-decomposition subgroups of SO(2, n), Trans. Amer. Math. Soc., **356** (2004), 1–38.

[43] H. Oh and D. Witte, Compact Clifford–Klein forms of homogeneous spaces of SO(2, n), Geom. Dedicata, **89** (2002), 25–57.

[44] R. S. Palais, On the existence of slices for actions of noncompact Lie groups, Ann. of Math. (2), **73** (1961), 295–323.

[45] F. Salein, Variétés anti–de Sitter de dimension 3 possédant un champ de Killing non trivial, ENS de Lyons Ph.D. thesis, Dec. 1999, see also C. R. Acad. Sci. Paris Sér. I Math., **324** (1997), 525–530, and François Salein, Variétés anti-de Sitter de dimension 3 exotiques, Ann. Inst. Fourier, **50** (2000) 257–284.

[46] Y. Shalom, Rigidity, unitary representations of semisimple groups, and fundamental groups of manifolds with rank one transformation group, Ann. of Math. (2), **152** (2000), 113–182.

[47] A. Weil, Remarks on the cohomology of groups, Ann. of Math. (2), **80** (1964), 149–157.

[48] J. A. Wolf, The Clifford–Klein space forms of indefinite metric, Ann. of Math. (2), **75** (1962), 77–80.

[49] J. A. Wolf, Isotropic manifolds of indefinite metric, Comment. Math. Helv., **39** (1964), 21–64.

[50] J. A. Wolf, Spaces of constant curvature (5th ed.), Publish or Perish, Inc., Houston Texas, 1984.

[51] T. Yoshino, A solution to Lipsman's conjecture for $\mathbb{R}^4$, Int. Math. Res. Notices., **80** (2004), 4293–4306.

[52] T. Yoshino, A counterexample to Lipsman's conjecture, Internat. J. Math., **16** (2005), 561–566.

[53] T. Yoshino, On a discontinuous duality theorem, In: Notes from Symposium on Representation Theory (Awajishima 2004), T. Kobayashi, H. Ochiai, and H. Tagawa (eds.), ISBN 4-9902328-0-1, pp. 23–27.

[54] T. Yoshino, Criterion of proper actions for three step nilpotent Lie groups, preprint, 2004.

[55] A. Zeghib, On closed anti-de Sitter spacetimes, Math. Ann., **310** (1998), 695–716.

[56] R. J. Zimmer, Discrete groups and non-Riemannian homogeneous spaces, J. Amer. Math. Soc., **7** (1994), 159–168.

[57] M. Atiyah and W. Schmid, A geometric construction of the discrete series for semisimple Lie groups, Invent. Math., **42** (1977), 1–62.

[58] T. Kobayashi and T. Yoshino, Compact Clifford–Klein forms of symmetric spaces – revisited, Pure and Applied Mathematics Quarterly, **1** (2005), Special Issue: In Memory of Armand Borel, 591–663.

[59] T. Kobayashi, Theory of discrete decomposable branching laws of unitary representations of semisimple Lie groups and some applications (in Japanese), Sugaku, Math. Soc. Japan, **51** (1999), 337–356; (English translation) Sugaku Exposition, **18**, Amer. Math. Soc., 2005, 1–37.

KOBAYASHI Toshiyuki, Kyoto University,
Research Institute for Mathematical Sciences,
e-mail `toshi@kurims.kyoto-u.ac.jp`

Translated by M. Reid
`miles@maths.warwick.ac.uk`