Danwa-kai, Wednesday June 29, Kyoto

Title:
Old and recent topics on the subject of resolution of singularities

Abstract:
The main theme of this talk is the following:

Problem of resolution of singularities Given an algebraic variety (say, over an algebraically closed field k), find a proper birational morphism $\pi : \widetilde{X} \to X$ from a nonsingular variety $\widetilde{X}$.

The original variety X may have singularities, but $\widetilde{X}$ does not by requirement. Thus π "resolves" the singularities of X. A typical example can be seen when one considers $X = \{x^2 - y^3 = 0\} \subset \mathbb{A}^2 = \text{Spec } k[x, y]$ and the map $\pi : \widetilde{X} = \mathbb{A}^1 = \text{Spec } k[t] \to X$ given by $x = t^3, y = t^2$. As in this example, resolution of singularities provides a "nice" parametrization of the solution of a set of equations (i.e., an algebraic variety) locally, and hence carries another name "local uniformization".

One hopes to solve the problem of resolution of singularities by "induction".

However, if we stick to the original formulation of the problem, the structure of "induction" may not be apparent. Through the notion of "idealistic exponent" (initiated by Hironaka and later given different names by various authors), it is reformulated into the following problem, where the inductive structure is more obvious:

Reformulation : Given a nonsingular variety W and an ideal $\mathcal{I}$ on W, reduce its multiplicity to less than a fixed number a, by a sequence of blowups whose center is transversal to a simple normal crossing divisor E. We symbolize this reformulation by $(W, (\mathcal{I}, a), E)$.

In characteristic zero (i.e., when $\text{char}(k) = 0$), the problem of $(W, (\mathcal{I}, a), E)$. can be reduced to $(H, (\mathcal{J}, b), F)$ with $\dim H = \dim W - 1$, hence achieving the "induction" on dimension, by taking **a hypersurface of maximal contact** H.

In contrast, it has been long known that in positive characteristic (i.e., when $\text{char}(k) = p > 0$) there is no hope of finding a hypersurface of maximal contact, as long as we insist H to be nonsingular. The lack of an "obvious" inductive structure has been the biggest obstacle in positive characteristic. Recently some approaches to find an alternative inductive structure to the problem of resolution of singularities have been proposed. One is the method of "generic" projection by Villamayor, and another is the method of a leading generator system by Kawanoue (roughly speaking, the latter corresponds to considering singular hypersurfaces of maximal contact), among others.

In this talk, we start with a brief introduction to the classical inductive solution in characteristic zero by the notion of a hypersurface of maximal contact, and then show an easy example where there is no such in positive characteristic. Toward the end, if time permits, we would like to explain how the proposed approaches can be successfully carried out in dimension 3 (even though the higher dimensional case is still out of reach).