On Mikio Sato's works related to automorphic forms –history and later developments—

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To the memory of 佐藤幹夫先生

Professor Mikio Sato (1928/4/18∼2023/1/09).

Sato had strong interest also in number theory, including that of automorphic forms, mainly on (RP) the Ramanujan -Petersson conjecture and (ST) on (what is later called) the Sato-Tate conjecture. And his fundamental contributions to these subjects, succesively within two years in the early 1960s. Although unpublished, they strongly motivated later works by others. Today, I shall speak mainly on (ST).

On the Sato-Tate Conjecture

(1) Sato's initial interest in the distribution of Frobenius arguments for non-CM elliptic curves, and how he reached his conjecture on this distribution (with the aids by Namba et al.),

(2) an independently raised conjecture by Tate, and the connection between them,

(3) a final solution (around 2010) by Taylor et. al, and related open problems on the Langlands correspondence.

1 Around 1962, Mikio Sato asked himself the following question.

Fix any elliptic curve E over $\mathbb{Q}$ with conductor N. For each prime $p \nmid N$, the numerator of the congruence zeta function of E_p (the good reduction of E mod p) has the form

$$1-a_p u+pu^2 = (1-\sqrt{p}e^{i\theta_p}u)(1-\sqrt{p}e^{-i\theta_p}u) \quad (0 \leq \theta_p \leq \pi)$$

with $a_p = 1 + p - |E_p| = 2\sqrt{p}\cos(\theta_{\mathrm{p}})$, $| \quad |$ denoting the number of rational points. Now let p run over all primes $p \nmid N$. Disregarding the exceptional case where E has CM (complex multiplications), his question was:

"*How is* θ_p *distributed on* $[0, \pi]$ *as* p *runs over all primes* $\nmid N$?"

He was interested in this, and closely related questions associated to modular cusp forms.

He wished to use the newly installed (old) computer HIPAC 103 at Tokyo Univ.of Education,to obtain some convincing insight. The first example was $E : y^2 = x(x^2 + x - 1)$, where $N = 20$. Its L-function is

$$\sum_{n\geq 1} a_n n^{-s} = (1+5^{-s})^{-1} \prod_{p\neq 2,5} (1 - a_p p^{-s} + p^{1-2s})^{-1},$$

where a_n can be computed from the Dedekind η-function

$$\eta(\tau) = q^{1/24} \prod_{m\geq 1} (1-q^m) \qquad (q = e^{2\pi i\tau}), \text{ as}$$

$$\sum_{n\geq 1,\ odd} a_n q^{n/2} = (\eta(\tau)\eta(5\tau))^2 = q^{1/2} \prod_{m\geq 1} (1-q^m)^2(1-q^{5m})^2$$

(*a cusp form of weight 2, level 20*)

K.Namba managed to use the *infant computer for common use* at that time to obtain the values of $a_p = 2\sqrt{p}\cos(\theta_{\mathrm{p}})$ for all $p < 14000$ $(p \neq 2, 5)$. Then Sato asked T. Nagashima for the computations of "θ_p" using Namba's data for "a_p".

[Namba]

(a) Dedekind η-function and Sato's $\sin^2$-Conjecture. 難波完爾 津田塾大学数学，計算機科学研究所報27，第16回数学史シンポジウム(2005), pp.95–149.

(b) Hyperelliptic curves of genus 2 and the $\sin^2$-Conjecture. 難波完爾　津田塾大学数学，計算機科学研究所報28，第17回数学史シンポジウム(2006), pp.101–174.

(c) On Sato's $\sin^2$-Conjecture. 難波完爾　佐藤$\sin^2$–予想の話，「佐藤幹夫の数学[増補版]」木村達雄編，日本評論社(2007) pp.344–367.

With their data, Sato, after he moved to Osaka Univ, wrote to Namba (May, 1963)

"The hypothesis, that $\{\theta_p\}$ *has the distribution density (at* θ*) proportional to* $\sin^2\theta$, *seems highly probable"*.

Computations for some other non-CM $E/\mathbb{Q}$, some modular forms of weight > 2, and hyperelliptic curves $/\mathbb{Q}$, had also been carried out by Namba and others [Namba(b)], but no explicit conjectures seem to have been raised outside non-CM elliptic curves.

–A remote view—

Sato is a celebrated founder of algebraic analysis. A starting point was his pinpointed observation that a more natural theoretic basis for hyperfunctions should be obtained from analytic functions on local complement spaces "$\mathbf{C} \setminus \mathbf{R}$", by a relative cohomological approach. But when he talked about it during his 2 years stay at IAS, it was considered too abstract while lacking an impressive application. This and also an encounter with Michio Kuga at IAS, probably drove him into a more 'fashionable subject' in number theory, i.e., the connection between each Hecke Dirichlet series and a Hasse zeta function of *unknown* algebraic variety. In fact, Sato found a correct track for proving Kuga's conjecture *in embryo* on what the unknown variety would be. As a graduate student, I was impressed that even a non-specialist can see matters correctly if one only faces it directly with penetrating eyes. Then Sato returned to Japan, and was expected to complete this task too. So he had already too many things to think about deeply.

Yet, his spirit seemed to have been looking also for *unexpected* phenomena in arithmetic whose finding certainly required extensive numerical experiments.

2 J. Tate, "Algebraic cycles and poles of zeta functions", Proc.Purdue Univ.Conf.1963,New York (1965), cited [Tate].

$$\begin{aligned} \text{Let} \quad L^1(E,s) &= \prod_{p\nmid N}(1-a_p p^{-s}+p^{1-2s})^{-1} \\ &= \prod_{p\nmid N}((1-\alpha_p p^{-s})(1-\overline{\alpha}_p p^{-s}))^{-1} \end{aligned}$$

be the partial L-function of E (where $\alpha_p\overline{\alpha}_p = p$), and for each integer $m \geq 1$, consider the new L-function

$$L^1_m(E,s) = \prod_{p\nmid N}\prod_{j=0}^{m}(1-\alpha_p^{m-j}(\overline{\alpha}_p)^j p^{-s})^{-1}.$$

In terms of L-functions associated with (good) ℓ-adic Galois representations, this corresponds to the Galois representation on the symmetric $\otimes m$ of $H^1(E)$.

This converges absolutely on $\mathcal{R}e(s) > \frac{m}{2} + 1$; hence it is holomorphic and non-zero here.

Conjecture (a special case of Conjecture 2 of [Tate])
$L^1_m(E, s)$ is holomorphic and $\neq 0$ on $\mathcal{R}e(s) \geq 1 + \frac{m}{2}$ for any $m \geq 1$.

Sato's conjecture on the $\sin^2 \theta$ distribution of $\{\theta_p\}$ is (via H. Weyl) equivalent with that $L^1_m(E, s)$ is holomorphic and $\neq 0$ at $s = 1 + \frac{m}{2}$ *for any* $m \geq 1$; hence the validity of the above Conjecture will imply Sato's conjecture. In fact, Tate added:

"I understand that M. Sato has found this $\sin^2$ *distribution law experimentally with machine computations. Conjecture* 2 *seems to offer a partial explanation for it!"*

cf. also Y. Yamamoto's unpublished Master's thesis, Osaka Univ. (1966 February) on this relation.

3 The proof, by Taylor, Clozel, Harris, Shepherd-Barron et al.

The Sato–Tate conjecture was proved for non-CM elliptic curves E over any totally real field F, and also for some wide class of elliptic modular cusp forms of weight > 2 that are "non-CM".

(!1) L. Clozel, M. Harris and R. Taylor, "Automorphy for some ℓ-adic lifts of automorphic $\mathrm{mod}\ \ell$ Galois representations", Publ. Math. IHES 108 (2008).
(!2) R. Taylor, "Automorphy for some ℓ-adic lifts of automorphic $\mathrm{mod}\ \ell$ representations II", ibid.
(!3) M. Harris, N. Shepherd-Barron and R. Taylor, "A family of Calabi-Yau varieties and potential automorphy", Ann. of Math. 171 (2010).
(!4) T. Barnet-Lamb, D. Geraghty, M. Harris and R. Taylor, "A family of Calabi-Yau varieties and potential automorphy II", Publ. RIMS, Kyoto Univ. 47(2011)

The Sato–Tate conjecture is proved in (!4) in Tate's form.

(a) They proved that for any elliptic curve E/F over a totally real number field, $L^1_m(E,s)$ (for any $m>0$) has a meromorphic continuation to $\mathbb{C}$ and satisfies the expected functional equation relating the values at s and $m+1-s$. If E is not CM, it is holomorphic and non-zero on $\mathcal{R}e(s) \geq 1+m/2$; hence the Sato-Tate conjecture holds for all such E/F.

(b) Similar results on 'non-CM' modular cusp forms of weight not necessarily equal to 2, including the Δ-function

$$\sum_{n=1}^{\infty} \tau(n) q^n = q \prod_{n=1}^{\infty} (1-q^n)^{24},$$

where $\cos(\theta_p) = \tau(p)/(2p^{11/2})$.

Evolutions during the process (!1) to (!4) involve:

Effective use of Calabi-Yau (or Dwork) family, as a generalization of the family of elliptic curves (!3)(!4)

Effective uses of bad primes (multiplicative reduction in (!3)) evolved to even more impressive uses of *ordinary* primes (!4)

In the earlier papers (!1) and in a preprint version of (!3), the results including their proofs of the generalized Sato-Tate conjecture were *conditional*, i.e., depended on a hypothetical validity of a generalization of "Ihara's lemma" proposed in (!1).

I am very glad that such important achivements eventually became unconditional.

Now, coming back to the published papers (!1) $\sim$ (!4), they contain the proofs of new significant cases of the global Langlands duality over number fields, i.e., the duality between

(i)$_n$ automorphic representations of a certain type, RAESDC *on* GL_n over totally real number fields,

(ii)$_n$ (ℓ-adic) Galois representations *into* GL_n satisfying certain good properties.

Given an automorphic representation for (i)$_2$, with the Langlands dual (ii)$_2$, it is expected that the symmetric power $\otimes m$ of the latter, which belongs to (ii)$_n$ for $n = m + 1$, also has a dual automorphic representation in (i)$_n$. In (!4), the authors prove more generally that any element of (ii)$_n$ possessing some good properties at each place $v \mid (\infty\ell)$ has, potentially, its dual in (i)$_n$ (Theorem A), and in particular, prove (Theorem B) a “potential modularity” for such $\otimes m$ of (ii)$_2$ that arises from (i)$_2$ (over $\mathbb{Q}$).

This focuses attention of a wider mathematical public, to the following general question as to whether the tensor product *operation* on the Galois representation (ii)-side *itself* has a natural Langlands dual on the automorphic representation (i)-side.

In February 2021, I asked R. Taylor whether there appeared a more general solution to this question and obtained a very kind and detailed explanation as to what are known or unknown. Here, I add only that no sufficiently general solutions are publicized.

What Sato might have been thinking furthur

Also, I wonder whether Sato, upon raising his conjecture, thought about the possibility that the (first) proof may be obtained by an ingenious construction of a suitable ℓ-adic representation of the Galois group. Neither Langlands program nor etale cohomology was at hand. Still, Sato's seminal works on hyperfunctions that had started earlier were based on advanced sheaf cohomology theories, and he had an extraoridinary power of penetration; so I cannot help imagining that he might have had at least dreamed of such a possibility.

Monuments stand above past great works, and in this case, not to be forgotten include the following works of Y.Taniyama and of G.Shimura.

(YT) The duality was first recognized and established by Taniyama for $n = 1$.

(GS) The proofs in (!4) still needed Shimura varieties as parts of logical basis; i.e., for some construction from (i) to (ii), for some compatibility, as well as for the proof of the local Langlands conjecture.

Reading the manuscript version of (!4) and enjoying the beauty of their use of Calabi-Yau varieties, I became curious to know whether the use of Calabi-Yau varieties completely replaced that of Shimura varieties. So I asked R. Taylor whether there still remained some logical dependences on Shimura varieties, and on this occasion also, obtained a very prompt and helpful reply (March, 2010) upon which the above remark (GS) depends.

(Related written Commentaries by Y.I.)

(RP) Ramanujan-Petersson Conjecture
(ST) Sato-Tate Conjecture

both submitted October 2019, and after minor revisions accepted for publication in a planned "Collected Works of Mikio Sato".

The copyright for the present version in the *publisher's* format certainly belongs to the publisher, while it owes a responsibility to publish the contributions within a reasonable length of time. So we have been and are waiting for its publication (*).

(*) If some of you happen to be interested now in some details, please consult me for the above texts in *my original* format with almost the same mathematical contents.

In Sato's case, several important works were left unpublished and hence the corresponding commentaries were asked for this Volume and contributed. The number-theory related is just a small part. Until publication, they are not directly accessible. I strongly hope that this Volume containing such commentaries as a characteristic part,be published in time.

A generation-gap can be fatal for the Volume to remain vivid as a legend.

Supplement: The Ramanujan-Petersson conjecture.

This conjecture is on the absolute values of eigenvalues of Hecke operators acting on modular cusp forms, and M.Sato's contribution is on its reduction to the Weil conjecture; i.e., a conjecture (later solved affirmatively by P. Deligne) on the absolute values of eigenvalues of the Frobenius action on etale cohomology groups of complete smooth algebraic varieties over finite fields. Around that time, the asked-for comparison was between
(i') Dirichlet series associated with modular cusp forms
(ii') Hasse zeta functions of algebraic varieties over number fields.

Michio Kuga communicated his idea to Sato, that the variety corresponding to (i') with weight k would be a fiber variety whose base is the modular curve and whose fiber at each point j is the $(k-2)$-th power of the elliptic curve having j as its modulus. When $k = 2$ this was known. The Eichler-Selberg trace formula for Hecke operators makes the connection. Kuga missed to realize that each term of the trace formula for $T(p^m)$ in the case $k > 2$ looks like corresponding to the number of rational points over $\mathbf{F}_{p^m}$ of the reduction mod p. Sato observed it.

Though his arguments needed to be supplemented by using Deuring's finer arguments, he was on the right track and had motivated later works by others. The crucial thing still missing was that (ii') should have been replaced by a finer

(ii) ℓ-adic Galois representations (*a la* J-P. Serre, P.Deligne)