# Natural boundaries revisited through differential equations, infinite order or non-linear

*Dedicated to Professor Y. Ihara on his sixtieth birthday*

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## § 0. Introduction.

Study of natural boundaries has been a central issue in the theory of analytic functions of several complex variables; we also know that most of them are related to important results and techniques in the theory of differential equations. (See, e.g. [H1], [E1], [H2] and references cited therein.) Actually intriguing phenomena observed in analytic function theory such as Hartogs' phenomenon and the Levi convexity seem to have been guideposts in developing general theory of overdetermined systems of linear differential equations. The object studied most extensively is, however, the geometry of a domain whose boundary is, or, to the contrary, can never be, a natural boundary for any holomorphic function defined on it; study of the natural boundary of an individual function has been seldom done in recent years. The aim of this

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report is to discuss how to analyze this classical problem with the aid of differential equations. My hope is that analysis of this sort not only helps us to acquire better understanding of the problem but also gives us new problems in the theory of differential equations. Section 1 explains the background ideas of the articles [Ka], [BKS] and [KS], which make full use of linear differential operators of infinite order. The content of Section 1 is quite "modern", as it essentially depends upon microlocal analysis (cf. e.g. [SKK]) and the recent progress of the theory of interpolating varieties (cf. e.g. [BT]). Section 2 is more "classical", particularly compared with Section 1; I discuss there the Jacobi (non-linear differential) equation that the theta-zerovalue satisfies by the method which Joshi and Kruskal used to analyze the Chazy equation in a very inspiring manner ([JK]). Although I learned the Jacobi equation from Ehrenpreis (cf. [E2, p.69]) only in 1987, the equation itself had been written down in 1847. ([J]) At the present moment the method explained in Section 1 is analytically more sharp and powerful than the asymptotic method used in Section 2. Still I hope the latter approach based upon the use of non-linear differential equations should eventually turn out to be more far-reaching; my dream is to construct a structure theory for non-linear differential equations by algebraic analysis of singular perturbations, as has been done for systems of linear differential equations by microlocal analysis in [SKK]. Such a dream might be too challenging, but I would count our success in the analysis of the Painlevé transcendents ([KT 1], [KT 2], [AKT 1], [KT 3], [KT 4],$\cdots$) as a supporting evidence for the existence of such a theory.

# § 1. Linear differential operators of infinite order and natural boundaries.

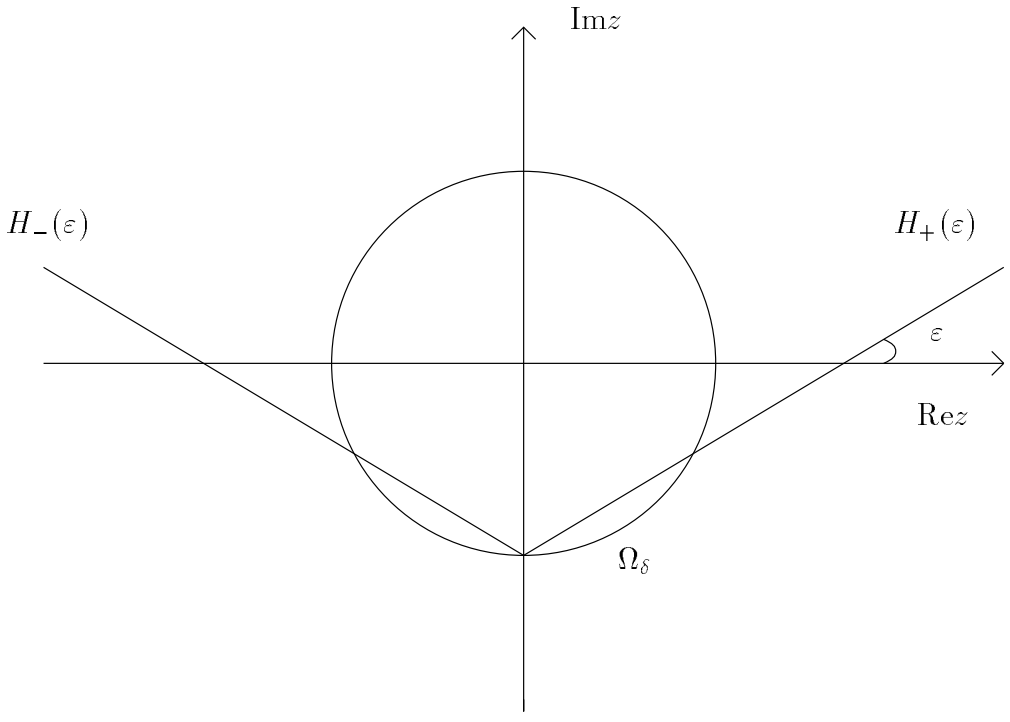
The theta-zerovalue $\vartheta_3(t) = \sum_\nu \exp(\pi i \nu^2 t)$ is a remarkable special function in that it has a natural boundary $\{t \in \mathbb{C}; \operatorname{Im} t = 0\}$. As the classical result of Fabry tells us, this transcendental property of the theta-zerovalue originates from the lacunary character of the series. Can we grasp such analytic properties of lacunary series in terms of differential equations? This is a question raised by Ehrenpreis [E1, p.397] in its spirit, although he proposed to use convolution equations there. Our answer ([Ka]) to this question is as follows:

Let us consider a Fourier series

$$f(z) = \sum_n a_n \exp(i\lambda_n z) \tag{1.1}$$

Suppose, for simplicity, $\lambda_n$ is real. Assume that $f(z)$ converges uniformly on each compact subset of $U = \{z \in \mathbb{C}; \operatorname{Im} z > 0\}$ and that it can be holomorphically extended onto $\Omega_\delta = \{z \in \mathbb{C}; |z| < \delta\}$. (Cf. Figure 1 below.) If the series $\{\lambda_n\}$ is sufficiently lacunary, $f(z)$ satisfies a differential equation of infinite order for which half-lines $H_\pm(\varepsilon)$ $(\varepsilon > 0)$ described in Figure 1 below are non-characteristic, and hence $f(z)$ can be eventually extended holomorphically onto $\{z \in \mathbb{C}; \operatorname{Im} z > -\delta\}$. Here the non-characteristic property of $H_\pm(\varepsilon)$ follows from the reality of $\lambda_n$. Stating this result in a different manner we conclude the following: if $f(z)$ has some singularities at all on the real axis $\{\operatorname{Im} z = 0\}$, $f(z)$ can never be extended beyond the real axis in a neighborhood of each point in the real axis, that is, the real axis is a natural boundary for $f(z)$.

**Figure 1.**



The infinite order differential operator is constructed in the form of infinite product, i.e., $\prod_n (1-(\lambda_n)^{-2}d^2/dz^2)$, and the lacunary condition required in the classical result of Fabry supplemented by a kind of regularity condition

$$|\lambda_n - \lambda_m| > c|n-m| \qquad (c>0) \tag{1.2}$$

guarantees that the infinite series is well-defined and that it determines a differential operator (of infinite order) in the sense of [SKK]; we note that one important property of such an operator is that it preserves the local character, that is, it determines a sheaf homomorphism of the sheaf $\mathcal{O}$ of holomorphic functions, and hence that of the sheaf $\mathcal{B}$ of hyperfunctions. Thus the lacunary character of a Fourier series and its relevance to the existence of the natural boundary of the series have been translated into the language of differential equations. Furthermore, to circumvent some technical complexity in finding a suitable family of non-characteristic hypersurfaces needed for the analytic continuation, Berenstein, Struppa and I have related the analytic continuation problem with the interpolating properties of the characteristic variety of the system of linear differential equations of infinite order. ([BKS]) Needless to say, in the higher dimensional case we encounter several simultaneous differential equations that the Fourier series $f(z_1,\cdots,z_n)$ in question satisfies. In order to imagine the geometrical subtlety of the problem, it might suffice to recall that the natural boundary of the Riemann theta-zerovalue $\vartheta(t)=\sum_{\nu\in\mathbb{Z}^n}\exp(\pi i\langle\nu,t\nu\rangle)$ ($t$: a complex symmetric $n\times n$ matrix) is given by a cone $\{t;\mathrm{Im}\,t \gg 0\}$. A main result of [BKS] (Theorem 4.1) asserts that, if the characteristic variety of the simultaneous differential equations that a Fourier series $f(z)$ with purely imaginary frequencies satisfies is discrete and interpolating (in the sense of e.g. [BT]), then for any real vector $a=(a_1,\cdots,a_n)$ we can find a linear differential operator (of infinite order) $G_a(\partial/\partial z_1,\cdots,\partial/\partial z_n)$ which satisfies

$$f(z+a)=G_a(\partial/\partial z_1,\cdots,\partial/\partial z_n)f(z). \tag{1.3}$$

Let us emphasize again that a differential operator of infinite order is of local character and hence, in particular, preserves the analyticity. Thus, for example, in a situation described in Figure 1, if $f(z)$ is extended onto $\Omega_\delta$, then $f(z)$ is extensible also onto the translate $\Omega_\delta+a$ of $\Omega_\delta$. [To make the argument run smoothly the best way is to consider the boundary value of

$f(z)$ onto $\{z \in \mathbb{C}; \operatorname{Im} z = 0\}$ in the situation described in Figure 1. Then it determines a hyperfunction on the real axis, and the relation (1.3) holds also for the hyperfunction. Hence the local character of the operator $G_a$ entails the analyticity of $f(z)$ near $z = a$ from its analyticity near $z = 0$, as we wanted to show. The argument of this sort equally holds for the higher dimensional case. See [KK] and [BKS] for details.] As one can readily imagine, the relation (1.3) is a very powerful relation (in a sense, it gives a fundamental solution of the Cauchy problem for an infinite order differential equation) and that is why we can bypass the somewhat clumsy step of constructing a family of non-characteristic hypersurfaces. To find such a powerful relation, we have assumed the interpolating property of the characteristic variety involved. [Note that the discreteness assumption on the characteristic variety reflects the lacunary character of the Fourier series in question. Hence some discreteness assumption should be needed in any formulation of the problem of this kind.] From the viewpoint of general theory of differential equations the assumption that the characteristic variety in question be interpolating is not too restrictive; probably this is the only case where we might discuss all the relevant cohomology groups (i.e., not just the 0-th cohomology groups of the solution complex). However, from the viewpoint of the application of the theory of differential equations to the better understanding of gap theorems, the assumption is a little too restrictive (cf. [Ko] and references cited therein for recent results in this field.); even if the variety is not interpolating, we often encounter the situation where a solution of the equation can be represented as a conditionally convergent exponential sum, i.e., a series suitable grouped like $\sum_{k=1}^{\infty}(\sum_{j\in J_k} c_j(z)\exp(i\lambda_j z))$ with $c_j(z)$ being a polynomial. (See e.g. [St] for the precise meaning of "grouping".) Actually grouping becomes needed in general if the regularity assumption (1.2) is omitted. But, as Ehrenpreis [E1, p.438] first observed, the situation is analytically interesting because it is related to the Ostrowsky-type overconvergence phenomena. As a matter of fact, I have long been puzzled how to make the observation of Ehrenpreis concrete. Fortunately Struppa and I have recently realized that not the Ostrowsky-type but rather the Bernstein-type overconvergence ([B]) can be equally handled with the aid of infinite order differential equations. ([KS]) I imagine that non-slowly-decreasing convolution equations should be relevant to discuss the Ostrowski-type overconvergence and that a more subtle and involved argument would be needed.

## § 2. Non-linear differential equations whose solutions may have natural boundaries.

Since I learned about the Jacobi equation

$$(2.1)\quad (y(t)^2\frac{\partial^3 y(t)}{\partial t^3} - 15y(t)\frac{\partial y(t)}{\partial t}\frac{\partial^2 y(t)}{\partial t^2} + 30(\frac{\partial y(t)}{\partial t})^3)^2$$
$$+32(y(t)\frac{\partial^2 y(t)}{\partial t^2} - 3(\frac{\partial y(t)}{\partial t})^2)^3 = \pi^2 y(t)^{10}(y(t)\frac{\partial^2 y(t)}{\partial t^2} - 3(\frac{\partial y(t)}{\partial t})^2)^2$$

from Ehrenpreis in 1987, I have been wondering how the real structure (such as the location of the natural boundaries of its solutions) comes about from the equation, comparing the situation with that encountered in the case of infinite order differential equations that characterizes the theta-zerovalue (up to a constant multiple); a satisfactory answer has been given for infinite order differential equations by the so-called reconstruction theorem. (See [SKK 1] for the details.) Fortunately enough, I happened to encounter an interesting and illuminating paper [JK]; it analyzes the Chazy equation

$$(2.2)\qquad \frac{\partial^3 y(t)}{\partial t^3} = 2y(t)\frac{\partial^2 y(t)}{\partial t^2} - 3(\frac{\partial y(t)}{\partial t})^2$$

by a variant of WKB analysis (the exponential expansion) to detect the location of the natural boundary of a solution, which had been studied by [C] and recently by [AC]. An important feature of exponential expansions is that the real structure of a solution becomes visible through the fact that $|\exp(n\varphi)|$ tends to $\infty$ or 0 as $n$ tends to $\infty$ according as $\mathrm{Re}\varphi$ is positive or negative. Actually the exponential expansion that Joshi and Kruskal used is an efficient method in analyzing non-linear differential equations in general, and the scope of its applicability is not restricted to the subject discussed here. As an example let me note that Takei and I benefited much from the expansion at the primitive stage of our study of Painlevé transcendent. (Cf. [KT 1]) Although the Chazy equation looks much simpler than the Jacobi equation, it is known (e.g. [O]) that the Jacobi equation is more basic in the sense that the Chazy equation controls only a sum of logarithmic derivatives of theta-zerovalues. At the same time I was not overly satisfied with [JK] as their method, although ingenious and inspiring, does not give a proof that the detected singularity forms a natural boundary. My expectation was that,

if I can do a similar job for the Jacobi equation, probably I can make use of the lacunary character of the exponential expansion to confirm that they really form a natural boundary. This expectation turned out to be true, and I report it here.

Let me begin the discussion in a heuristic manner. At first, to fix the way of thinking, let me introduce a large parameter $\eta > 0$ and set

$$z = \eta^{-1} t \tag{2.3}$$

to find the following equation (2.4) from (2.1):

$$\begin{aligned}(y^2\frac{\partial^3 y}{\partial z^3} - 15y\frac{\partial y}{\partial z}\frac{\partial^2 y}{\partial z^2} + 30(\frac{\partial y}{\partial z})^3)^2 + 32(y\frac{\partial^2 y}{\partial z^2} - 3(\frac{\partial y}{\partial z})^2)^3 \\ = \pi^2\eta^2 y^{10}(y\frac{\partial^2 y}{\partial z^2} - 3(\frac{\partial y}{\partial z})^2)^2.\end{aligned} \tag{2.4}$$

The reason for introducing the large parameter $\eta$ is, at least in this article, just for the convenience of the construction of the required 1-parameter solution. As

$$y_0 = Cz^{-1/2} \qquad (C : \text{a constant}) \tag{2.5}$$

satisfies (2.4), the function $\varphi$ defined by

$$y = y_0 + \varepsilon\varphi + O(\varepsilon^2) \qquad (|\varepsilon| \ll 1) \tag{2.6}$$

should nullify the coefficient of $\varepsilon^2$ in (2.4) if it should exist at all. [Because of the multiplicity the coefficient of $\varepsilon$ in (2.4) trivially vanishes.] Hence $\varphi$ should satisfy the following:

$$\begin{aligned}2y_0y_0^{(3)}\varphi &+ y_0^2\varphi^{(3)} - 15y_0y_0'\varphi'' - 15y_0y_0''\varphi' - 15y_0'y_0''\varphi + 90y_0'^2\varphi' \\ &= \pm\pi\eta y_0^5(y_0\varphi'' + y_0''\varphi - 6y_0'\varphi'),\end{aligned} \tag{2.7}$$

where $'$ denotes $\partial/\partial z$ and $\varphi^{(3)}$ denotes $\partial^3\varphi/\partial z^3$. Here I have used the fact that $(yy'' - 3y'^2)^3$ begins with the term of $O(\varepsilon^3)$ as $y_0y_0'' - 3y_0'^2$ identically vanishes. To fix our computation I consider the case where the $-$-sign is chosen in the right-hand side of (2.7), but I keep the constant $C$ explicitly. [The case where $+$-sign is chosen can be discussed similarly, but the case with $-$-sign makes things conform to the common usage of several notations

as will be seen below.] Then (2.7) assumes the following form:

$$\begin{aligned}(2.8)\qquad \varphi^{(3)} &+ \frac{15}{2}z^{-1}\varphi'' + \frac{45}{2}z^{-2}\varphi' + \frac{15}{8}z^{-3}\varphi \\ &= -\pi C^4\eta(z^{-2}\varphi'' + 3z^{-3}\varphi' + \frac{3}{4}z^{-4}\varphi)\end{aligned}$$

Let me now employ the WKB ansatz, that is,

$$(2.9)\qquad \varphi = \exp(\int^z S(z,\eta)dz)$$

with

$$(2.10)\qquad S = \eta S_{-1}(z) + S_0(z) + \eta^{-1}S_1(z) + \cdots.$$

This WKB ansatz entails

$$\begin{aligned}(2.11)\quad (S^3 + 3SS' + S'') &+ \frac{15}{2z}(S^2 + S') + \frac{45}{2z^2}S + \frac{15}{8z^3} \\ &= -\pi C^4\eta(\frac{1}{z^2}(S^2 + S') + \frac{3}{z^3}S + \frac{3}{4z^4}).\end{aligned}$$

Equating the coefficients of $\eta^l (l = 3, 2, 1)$, I find

$$(2.12)\qquad S_{-1} = \frac{-\pi C^4}{z^2},$$

$$\begin{aligned}(2.13)\quad 3S_{-1}^2S_0 &+ 3S_{-1}S'_{-1} + \frac{15}{2z}S_{-1}^2 \\ &= -\pi C^4\{\frac{1}{z^2}(2S_{-1}S_0 + S'_{-1}) + \frac{3}{z^3}S_{-1}\},\end{aligned}$$

$$\begin{aligned}(2.14)\quad 3S_{-1}S_0^2 &+ 3S_{-1}^2S_1 + 3S_{-1}S'_0 + 3S_0S'_{-1} + S''_{-1} \\ &+ \frac{15}{2z}(2S_{-1}S_0 + S'_{-1}) + \frac{45}{2z^2}S_{-1} \\ &= -\pi C^4\{\frac{1}{z^2}(2S_{-1}S_1 + S_0^2 + S'_0) + \frac{3}{z^3}S_0 + \frac{3}{4}z^{-4}).\end{aligned}$$

Thus I obtain the following:

$$(2.15)\qquad S_0 = -\frac{1}{2z},$$

$$(2.16)\qquad S_1 = \frac{-9z^0}{\pi C^4}, i.e., \frac{-9}{\pi C^4}.$$

Hence $\varphi$ has the form

$$A\exp(\pi C^4\eta/z)\{\frac{1}{\sqrt{z}}+O(\eta^{-1})\}, \tag{2.17}$$

where $A$ is an $\eta$-dependent constant that appears as the end-point contribution of the integral in (2.9) and each term in $O(\eta^{-1})$ has a positive power of $z$. As we have thus found an approximate solution of (2.4) that contains an exponential factor (1-instanton term in the terminology of [KT 1] etc.), we are prepared to construct a multiple-instanton solution of (2.4) in an iterative manner. As we know, however, that the theta-zerovalue or its imaginary transform satisfies (2.1), I bypass the detailed computation and try to find a solution $y(t)$ of (2.1) in the form

$$y(t)=\frac{Cu(s)}{\sqrt{t}} \text{ with } s=\exp(\frac{B}{t}), \tag{2.18}$$

where $B=\pi C^4$ and $u(s)$ is a formal power series in $s$. As I now begin the exact computation, I set $\eta=1$ and deal with (2.1) itself. The computation so far done with $\eta$ was designed to guess the correct form (2.18) of the one-parameter solution, i.e., the exponential expansion or the multiple-instanton expansion. In what follows, $u_{sss}$ etc. denote respectively $\partial^3u/\partial s^3$ etc. The relation (2.18) entails

$$y\frac{\partial^2y}{\partial t^2}-3(\frac{\partial y}{\partial t})^2=C^2B^2t^{-5}(suu_s+s^2uu_{ss}-3s^2u_s^2). \tag{2.19}$$

On the other hand, we trivially find

$$\begin{aligned} y^2\frac{\partial^3y}{\partial t^3} &- 15y\frac{\partial y}{\partial t}\frac{\partial^2y}{\partial t^2}+30(\frac{\partial y}{\partial t})^3 \\ &= y\frac{\partial}{\partial t}(y\frac{\partial^2y}{\partial t^2}-3(\frac{\partial y}{\partial t})^2)-10\frac{\partial y}{\partial t}(y\frac{\partial^2y}{\partial t^2}-3(\frac{\partial y}{\partial t})^2). \end{aligned} \tag{2.20}$$

Substituting the right-hand side of (2.19) into (2.20), I then obtain

$$\begin{aligned} y^2\frac{\partial^3y}{\partial t^3} &- 15y\frac{\partial y}{\partial t}\frac{\partial^2y}{\partial t^2}+30(\frac{\partial y}{\partial t})^3 \\ &= -C^3B^3t^{-15/2}(s^3u^2u_{sss}-15s^3uu_su_{ss}+30s^3u_s^3 \\ &\quad -15s^2uu_s^2+3s^2u^2u_{ss}+su^2u_s). \end{aligned} \tag{2.21}$$

Substitution of (2.18), (2.19) and (2.21) into (2.1) gives us

$$\begin{aligned}(2.22)\qquad (s^3u^2u_{sss} &- 15s^3uu_su_{ss} + 30s^3u_s^3 - 15s^2uu_s^2 + 3s^2u^2u_{ss}\\ &+ su^2u_s)^2 + 32(suu_s + s^2uu_{ss} - 3s^2u_s^2)^3\\ &= C^8B^{-2}\pi^2u^{10}(suu_s + s^2uu_{ss} - 3s^2u_s^2)^2\\ &= u^{10}(suu_s + s^2uu_{ss} - 3s^2u_s^2)^2,\end{aligned}$$

as $B = \pi C^4$. Dividing both sides of (2.22) by $s^2$, I finally obtain the following relation that $u(s)$ should satisfy:

$$\begin{aligned}(2.23)\qquad &(u^2u_s + s(3u^2u_{ss} - 15uu_s^2) + s^2(u^2u_{sss} - 15uu_su_{ss} + 30u_s^3))^2\\ +\ &32s(uu_s + s(uu_{ss} - 3u_s^2))^3 - u^{10}(uu_s + s(uu_{ss} - 3u_s^2))^2 = 0.\end{aligned}$$

As I shall show below, (2.23) admits a (formal) power series solution

$$(2.24)\qquad u(s) = \sum_{j\geq 0} c_j s^j \quad (c_j : \text{a constant})$$

Furthermore I shall find that, if $c_0, c_1 \neq 0$, then

$$(2.25)\qquad c_0^8 = 1,$$

$$(2.26)\qquad c_2 = 0$$

and all $c_j$ $(j \geq 3)$ are uniquely determined by $c_0$ and $c_1$. In view of (2.18) this means that a "one-parameter solution" $y(t)$ of (2.1) is found as the heuristic WKB-type analysis predicted. Still more important is the following fact: since the imaginary transform of $\vartheta_3(-it)$, i.e., $(-t)^{-1/2}\vartheta_3(-it^{-1})$ satisfies (2.1), the uniqueness assertion above implies that the power series solution for $C = 1$ and with $c_1 = 2c_0 = 2i$ should be lacunary, i.e., $c_j = 0$ if $j \neq n^2$ $(n = 0, 1, 2, 3, \cdots)$. [Logically speaking, it is much more preferable to prove this lacunary character of the series without resorting to the knowledge of the theta-zerovalue; it should be possible to prove the lacunary character of power series solution of (2.23) without using any extra information, but I have not tried it seriously.] Note that (2.26) can be regarded as a symptom of this general fact. This lacunary character of the series (2.24) proves the existence of a natural boundary by the Fabry gap theorem. (Cf. Section 1.) Since the one-parameter solution contains a parameter $\pi C^4$ in the exponential, the natural boundary is movable. Thus a precise study of the differential

equation (2.23) shows that the exponential or multiple-instanton expansion really *proves* the existence of movable natural boundaries of a solution of the Jacobi equation (2.1).

Now let me prove the above statements concerning a power series solution $u$ of the equation (2.23).

The coefficient of $s^0$ in (2.23) is

$$(c_0^2c_1)^2 - c_0^{10}(c_0c_1)^2. \tag{2.27}$$

Since $c_1 \neq 0$, vanishing of this implies (2.25). Similarly, the coefficient of $s^1$ in (2.23) is given by

$$\begin{aligned} &4c_0^3c_1^3 + 4c_0^4c_1c_2 + 12c_0^4c_1c_2 + 2c_0^3c_1^3 \\ &-(12c_0^{11}c_1^3 + 4c_0^{12}c_1c_2 + 4c_0^{12}c_1c_2 - 6c_0^{11}c_1^3). \end{aligned} \tag{2.28}$$

Using (2.25) together with the assumption $c_1 \neq 0$, I then infer that $c_2$ should vanish. To proceed further, namely, to confirm that $c_n (n \geq 3)$ is inductively determined by (2.23), let me note that $c_n$ may appear non-trivially only in the coefficient $C_p$ of $s^p$ $(p \geq n-1)$ in (2.23). Further, the part $C_{n-1}^0$ of $C_{n-1}$ that consists of terms containing $c_n$ originates only from the following differential polynomial $P$:

$$P = (u^2u_s + 3su^2u_{ss} + s^2u^2u_{sss})^2 - u^{12}u_s^2 - 2su^{12}u_su_{ss}. \tag{2.29}$$

Other terms in (2.23) such as $32s(uu_s + s(uu_{ss} - 3u_s^2))^3$ or terms relevant to $-15suu_s^2$ etc. are irrelevant in that the contribution containing $c_n$ that originates from the terms gives effects only to $C_p$ with $p \geq n$. Needless to say, $P$ still contains terms irrelevant to $C_{n-1}^0$. Disposing of such irrelevant terms, i.e.,

$$R = 9s^2u^4u_{ss}^2 + 6s^3u^4u_{ss}u_{sss} + s^4u^4u_{sss}^2, \tag{2.30}$$

I concentrate my attention on $P - R$, namely

$$u^4u_s^2 + 6su^4u_su_{ss} + 2s^2u^4u_su_{sss} - u^{12}u_s^2 - 2su^{12}u_su_{ss}. \tag{2.31}$$

Thus I find the following:

$$\begin{aligned} C_{n-1} = {} &2c_0^4c_1nc_n + 6c_0^4c_1n(n-1)c_n + 2c_0^4c_1n(n-1)(n-2)c_n \\ &-2c_0^{12}c_1nc_n - 2c_0^{12}c_1n(n-1)c_n + A_{n-1}(c_0, c_1, c_2, \cdots, c_{n-1}), \end{aligned} \tag{2.32}$$

where $A_{n-1}$ is a polynomial of $c_0, c_1, c_2, \cdots, c_{n-1}$. Since $c_0^8 = 1$ and $c_1 \neq 0$, the relation (2.32) uniquely determines $c_n$ ($n \geq 3$) in terms of $c_0, c_1, \cdots, c_{n-1}$. This completes the proof of the desired uniqueness assertion.

Before ending this report, let me note some miscellaneous observations. First of all, I should mention that I could do the same job even if I started from the ansatz

$$y(t) = Cv(s) \text{ with } s = \exp(\pi C^4 t) \tag{2.33}$$

instead of (2.18). [In fact, the computation is a little simpler in this case.] The reason I presented the argument starting from the ansatz (2.18) is just that it preserves the WKB flavor better by making the analytically interesting factor $1/\sqrt{t}$ explicit. Actually this fact is related to another important feature of the Jacobi equation, namely the $SL(2,\mathbb{C})$- invariance of (2.1). ([J, Theorem in p.186]) As the resemblance between the computation starting from the ansatz (2.18) and that starting from (2.33) [, which I omitted here] indicates, we can confirm by a straightforward computation that, if $y(t)$ satisfies (2.1), then $\delta y(1/t)/\sqrt{t}$ ($\delta^8 = 1$) also satisfies (2.1); this is the core-part of the $SL(2,\mathbb{C})$- invariance of (2.1). I also note that the more trivial part of the $SL(2.\mathbb{C})$-invariance, namely the fact that $y(t+a)$ ($a$: a constant) satisfies (2.1) if $y(t)$ is a solution (2.1), is reflected by the arbitrariness of $c_1$ in the "exponential expansion" (2.24).

As a conclusion I like to say that the exponential expansion, or the multiple-instanton expansion, has turned out to be efficient in analyzing the Jacobi equation, which is frightful in its appearance. I hope this is a good, though tiny, evidence in favor of M. Sato's program of controlling automorphic functions by differential equations, which he advocated in 1992. ([Sa])

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