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On Global Generation of Adjoint Linear Systems

Stefan Helmke

Abstract. In [5] it was shown, that for an ample divisor L on a smooth projective variety X of dimension n, with $L^n > n^n$ and $L^d.Z \geqslant m_x(Z)n^d$ for all subvarieties $Z \subset X$ of dimension d with multiplicity $m_x(Z) \leqslant \binom{n-1}{d-1}$ at some point $x \in X$, the line bundle $\mathcal{O}_X(K_X + L)$ has a section which is nonvanishing at x, where K_X denotes the canonical divisor. The aim of this paper is to show that, if we replace the assumption $L^d.Z \geqslant m_x(Z)n^d$ for $d = n-1$ by $L^d.Z \geqslant n^d$, the conclusion still holds. The proof is based on an idea of T. Fujita [4].

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Introduction

In 1985, T. Fujita [3] conjectured, that for every smooth projective variety X over an algebraically closed field of characteristic zero, a linear system of the form $|K_X + mH|$, where K_X is the canonical divisor and H is an ample divisor on X, should be globally generated, if m is greater than the dimension of X. In the case, when X is a curve, this is an easy consequence of Serre duality. Then, in 1988, I. Reider [10] proved the conjecture for surfaces S, by showing that if x is a base point of the linear system $|K_S + L|$ and L is nef with $L^2 > 4$, then there is a curve C on S containing x such that either

$$L.C = 1 \quad \text{and} \quad C^2 = 0,$$
$$\text{or} \quad L.C = 0 \quad \text{and} \quad C^2 = -1.$$

In particular, if $L = mH$ for an ample divisor H and $m > 2$, this is impossible. If $m = 2$ and $H^2 = 1$, we can not apply the theorem, because then $L^2 = 4$. In fact, the example of the projective plane with a hyperplane section H shows, that $|K_S + 2H|$ is in general not globally generated. Reider's proof of the theorem is a beautiful application of rank two vector bundle techniques. But unfortunately, this approach is limited to surfaces.

An alternative approach to the problem uses $\mathbb{Q}$-divisor techniques. Roughly speaking, one tries to find certain divisors which are very singular at one point, and not too singular at neighbor points. So far, this is quite successful in low dimensions. For example under the same hypotheses as in Reider's theorem, one can easily prove the existence of a curve C_0 containing x, with $L.C_0 \leqslant 1$. In this case, C_0 is irreducible and smooth at x. In Reider's theorem, C^2 is either -1 or 0. But this is in general only possible, if C is not irreducible, and therefore, the self-intersection number of the curve C_0 is in general different from -1 or 0.

With a little more work, it was possible to prove Fujita's freeness conjecture also for threefolds by Ein and Lazarsfeld [2] and for 4-dimensional varieties by Kawamata [7]. In the 3-dimensional case the results are even stronger (cf. [5], [7]).

This approach and also Reider's theorem suggests, that one should prove global generation of a linear system $|K_X + L|$ at a point $x \in X$, if the intersection numbers $L^d.Z$ for every subvariety Z of dimension d containing the point x are big enough. A necessary condition is $L^d.Z > d^d$. To see this, consider the following simple example.

EXAMPLE. Let X be the product of two projective spaces $\mathbb{P}^k \times \mathbb{P}^{n-k}$ and denote by $p : X \to \mathbb{P}^k$ and $q : X \to \mathbb{P}^{n-k}$ the two projections. Every line bundle on X is isomorphic to $\mathcal{O}_X(a,b) := p^*\mathcal{O}(a) \otimes q^*\mathcal{O}(b)$, for some integers a and b. In particular the canonical sheaf of X is $\mathcal{O}_X(-k-1, -n+k-1)$. Let L be the divisor of a global section of $\mathcal{O}_X(k,m)$ with m very large. Then, for every subvariety $Z \subset X$ of dimension d we have $L^d.Z \geqslant k^d$, but $|K_X + L|$ is of course empty. Furthermore, by choosing m very large, we can achieve that the intersection numbers $L^d.Z$ are arbitrary large for all subvarieties of dimension $d > k$.

But the condition $L^d.Z > d^d$ for all $Z \subset X$ of dimension d seems to be to weak, to prove global generation of $|K_X + L|$. It is not easy to get good examples, but the techniques suggest that one needs at least $L^d.Z \geqslant n^d$.

On the other hand, we would like to apply the global generation also to pluri canonical systems $|mK_X|$ of varieties X of general type. Hence, we should allow X to have some singularities, and in this context it is quite natural to assume that (X, Δ) is Kawamata log terminal (KLT) for some effective $\mathbb{R}$-divisor Δ. But later we will assume, that X is smooth, for some other technical reasons. In the following, we will fomulate the general case, but the reader may assume, that X is smooth and $\Delta = 0$.

Conjecture. *Let X be a normal projective variety of dimension n and Δ be an effective $\mathbb{R}$-divisor on X such that (X, Δ) is KLT at a point $x \in X$. Let L be a nef $\mathbb{Q}$-Cartier divisor on X such that $K_X + \Delta + L$ is Cartier. Assume that*

$$L^n > n^n \quad \text{and}$$
$$L^d.Z \geqslant n^d \quad \text{for all } Z \subset X \text{ with } x \in Z \text{ and } d = \dim Z < n.$$

Then $\mathcal{O}_X(K_X + \Delta + L)$ is globally generated at x.

For example, if X is smooth and $L = mH$ for an ample divisor H such that either $m > \dim X$ or $H^n > 1$ and $m = \dim X$, the assumptions of the conjecture are satisfied and hence, the conjecture would imply Fujita's freeness conjecture.

Angehrn and Siu [1] have shown, that the conclusion of the conjecture is true, if X is smooth and

$$(*) \qquad L^d.Z > \left(\tfrac{1}{2}n(n+1)\right)^d \quad \text{for all } Z \subset X \text{ with } x \in Z \text{ and } d = \dim Z.$$

Then, Kollár [9] generalized this to the case, where (X, Δ) is KLT.

By a different method [5], it was possible to show that $\mathcal{O}_X(K_X + L)$ is globally generated at x, if X is smooth and

$$(**) \qquad \begin{aligned} L^n &> n^n, \\ L^d.Z &\geqslant m_x(Z)n^d \quad \text{for all } Z \subset X \text{ with } m_x(Z) \leqslant \tbinom{n-1}{d-1}, \end{aligned}$$

where $m_x(Z)$ denotes the multiplicity of Z at x. The bound $(**)$ for the intersection numbers $L^d.Z$ is for all dimensions d better than Angehrn and Siu's bound $(*)$. In particular for very small dimensions d and for small codimensions $n - d$, the bound is much better, as illustrated in Fig. 1 for $n = 100$.

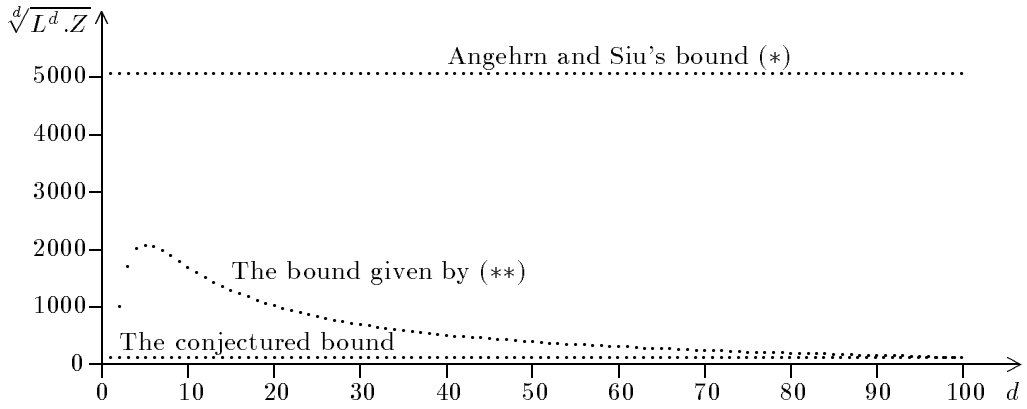

Fig. 1. Bounds on intersection numbers for $n = \dim X = 100$.

Moreover, the method has the advantage, that one can use an idea of Fujita [4] to improve the result further. In particular the conjecture could be proved in this way for smooth threefolds. The aim of this paper is to show, that one can use Fujita's trick to replace the condition $L^d.Z \geqslant m_x(Z)n^d$ by $L^d.Z \geqslant n^d$ for hypersurfaces, i.e. $d = n - 1$ (Theorem 4.4). It turns out, that for singular hypersurfaces, one needs in fact only a weaker condition. But for smooth subvarieties Z, the bound on $L^d.Z$ cannot be improved. For example for smooth threefolds, we will show

Theorem. *Let X be a smooth projective threefold and L be an ample divisor on X. Assume that for some point $x \in X$*

$$\begin{aligned} L^3 &> 27, \\ L^2.S &\geqslant 9 \quad \textit{for all surfaces } S \textit{ which are smooth at } x, \\ L^2.S &\geqslant 3 \quad \textit{for all surfaces } S \textit{ with a RDP at } x, \\ L.C &\geqslant 3 \quad \textit{for all curves } C \textit{ which are smooth at } x. \end{aligned}$$

Then $\mathcal{O}_X(K_X + L)$ is globally generated at x.

The assumption $L^2.S \geqslant 3$ for surfaces S with a rational double point at x is surprisingly weak. In fact, even if L^3 would be arbitrary large, we could not improve this bound.

The basic idea of the proof is, that the condition $L^n > n^n$ implies the existence of divisors $D \in |kL|$ with $m_x(D) > nk$ for $k \gg 0$. If x would be an isolated base-point of

the linear system $\{ D \in |kL| \mid m_x(D) > nk \}$, one could use the Kawamata Viehweg vanishing theorem, to show that x is not a base-point of $|K_X + L|$. In general, the vanishing theorem implies, that the restriction of global sections of $\mathcal{O}(K_X + L)$ to 'the most singular part Z of D' is surjective, and one tries to proceed by induction. The problem is, that Z is usally singular and we need $L^d.Z \geqslant m_x(Z)n^d$ instead of $L^d.Z \geqslant n^d$ for the next induction step. Fujita's observation was, that if all $D \in |kL|$ with $m_x(D) > nk$ contain Z with high multiplicity, one can find such divisors D with considerably larger multiplicity at x. If X is a smooth threefold, the only possible singular Z is a surface with a rational double point at x. In this case, Fujita's observation leads to a proof of the conjecture. But the general case is much more involved. In fact, it is not only the multiplicity of D at x which is important, but rather a certain ratio of this multiplicity to the multiplicity of D along the subvariety Z. Here we will consider only the case, where Z is a hypersurface. In this case, the proof consists of four steps. The first is a slight refinement of an inequality, which was used to prove the global generation of $|K_X + L|$ at x, under the assumptions $(**)$ mentioned above (Theorem 2.2). The second is an elementary bound on multiplicities of hypersurface singularities (Corollary 3.7). The third step is crucial. We show the existence of certain $\mathbb{Q}$-divisors, which are not too singular in codimension one (Proposition 4.2). The final step shows, that the divisors constructed in the former step can be used to improve the bound $(**)$ for hypersurfaces (Lemma 4.3).

Altogether gives a clear picture for minimal centers of codimension one. I hope that this helps to understand also the general case and leads to a proof of Fujita's freeness conjecture in the future.

1. Preliminaries

We will recall some notation and useful results about $\mathbb{Q}$-divisors and $\mathbb{R}$-divisors (cf. [8] for more details).

A $\mathbb{Q}$- (resp. $\mathbb{R}$-) divisor on a normal variety X, is a finite formal sum $D = \sum r_i D_i$ of distinct Weil prime divisors D_i with rational (resp. real) coefficients r_i. The round-up of such a $\mathbb{R}$-divisor is defined by $\ulcorner D \urcorner = \sum \ulcorner r_i \urcorner D_i$, where $\ulcorner r_i \urcorner$ is the smallest integer greater or equal to r_i. The fractional part of D is $\ulcorner D \urcorner - D$. A $\mathbb{R}$-divisor D is called $\mathbb{Q}$-Cartier, if mD is a Cartier divisor, for some integer $m > 0$. Let $f : X \to S$ be a proper morphism. A $\mathbb{Q}$-Cartier divisor D on X is called f-nef (resp. f-big) if for some positive integer m the divisor mD is Cartier and nef (resp. big) on $f^{-1}(U)$ for each open affine subset $U \subset S$. The following vanishing theorem is extremely useful.

Theorem 1.1 (Kawamata [6], Viehweg [11]). *Let X be smooth and $f : X \to S$ a proper morphism. If D is a f-big and f-nef $\mathbb{Q}$-divisor on X, such that the fractional part of D has normal crossings, then*

$$R^i f_* \mathcal{O}_X(K_X + \ulcorner D \urcorner) = 0 \quad \textit{for} \quad i > 0. \tag{1.1}$$

Consider a birational proper morphism $f : Y \to X$ of normal varieties and let $D = \sum r_i D_i$ be a $\mathbb{R}$-divisor on X such that $K_X + D$ is $\mathbb{Q}$-Cartier. Write

$$K_Y \equiv f^*(K_X + D) + F \quad \text{with} \quad F = \sum b_j F_j. \tag{1.2}$$

Here we assume that if F_j is the strict transform of D_i, we have $b_j = -r_i$ and all the other components of F are f-exceptional. Then F is uniquely defined by these

conditions. We call F the discrepancy and b_j the discrepancy coefficient of F_j. A subvariety $Z \subset X$ containing x is called a *center of a log canonical singularity at* x, if there is a birational morphism $f : Y \to X$ and a component F_j of the discrepancy with $Z = f(F_j)$ and $b_j \leqslant -1$. The pair (X, D) is *log canonical* (LC) (resp. *Kawamata log terminal* (KLT)) at $x \in X$, if D is effective and for every birational morphism $f : Y \to X$ the discrepancy coefficients satisfy $b_j \geqslant -1$ (resp. $b_j > -1$) for all j with $x \in f(F_j)$. Kawamata [7] has shown, that if X is KLT and (X, D) is LC at x, there is a unique minimal center at x.

Now we can state the key proposition for the construction of sections in adjoint line bundles $\mathcal{O}_X(K_X + L)$, with L ample (or big and nef). It is an easy consequence of the vanishing Theorem 1.1 (cf. [5], [7] or [9]).

Proposition 1.2. *Let X be a projective variety, Δ be an effective $\mathbb{R}$-divisor on X and L a nef and big $\mathbb{Q}$-Cartier divisor such that $K_X + \Delta + L$ is Cartier. Assume that (X, Δ) is KLT at a point $x \in X$ and that there is an effective $\mathbb{Q}$-divisor $D \equiv cL$ with $c < 1$ such that $(X, \Delta + D)$ is LC at x. If $\{x\}$ is a center of a log canonical singularity of $(X, \Delta + D)$, then there is a section of $\mathcal{O}_X(K_X + \Delta + L)$ which is nonvanishing at x.*

2. The Local Discrepancy

The general strategy to prove global generation of $|K_X + \Delta + L|$ at a point $x \in X$ is to construct $\mathbb{Q}$-divisors $D \equiv cL$, such that $(X, \Delta + D)$ is LC but not KLT at x, by using the Riemann Roch theorem and then reducing the dimension of the minimal center of $(X, \Delta + D)$ by induction. If the minimal center is the point x and $c < 1$, we are done by Proposition 1.2. This would be easy, if we assume that the intersection numbers $L^d.Z$ for subvarieties Z containing x are very large. But we try to prove global generation under very weak assumptions and hence, we have to be very careful with the choice of D. As a first step, we need an invariant of D, which measures how far D is away from having x as its minimal center. One possibility is the local discrepancy introduced in [5]. Here we give a different but equivalent formulation.

Definition. Let (X, D) be log canonical at $x \in X$. The *local discrepancy* of D at x is the number

$$b_x(X, D) := \sup \left\{ m_x(D_1) \;\middle|\; \begin{array}{l} \text{where } D_1 \text{ is an effective } \mathbb{Q}\text{-divisor} \\ \text{such that } (X, D + D_1) \text{ is LC at } x \end{array} \right\}.$$

A simple way to reduce the dimension of the minimal center of (X, D) is described in the following

Proposition 2.1 ([5, Prop. 3.2]). *Let L be an ample divisor on X and $D \geqslant 0$ a $\mathbb{Q}$-divisor with $D \equiv cL$ for some rational number $0 \leqslant c < 1$. Assume that X is smooth at x and (X, D) is log canonical with local discrepancy $b = b_x(X, D)$ at x. Let Z be the minimal center of (X, D) at x and $d = \dim Z > 0$. If*

$$L^d.Z \;>\; p^d \cdot m_x(Z), \quad \text{where} \quad p = \frac{b}{1 - c}, \tag{2.1}$$

then there is a $\mathbb{Q}$-divisor $D' \equiv c'L$ with $c < c' < 1$ such that (X, D') is log canonical at x with minimal center Z' properly contained in Z and

$$p' \;=\; \frac{b'}{1 - c'} \;<\; p, \quad \text{where} \quad b' = b_x(X, D'). \tag{2.2}$$

The divisor D' is constructed as follows. First we use (2.1) and the Riemann Roch theorem, to construct an effective $\mathbb{Q}$-divisor $D_0 \equiv L|_Z$ with $m_x(D_0) > p$. The rest of the proof uses only that (X, Δ) is KLT and $(X, \Delta + D)$ is LC at x, for some effective $\mathbb{R}$-divisor Δ. By Bertini's first theorem and Serre's vanishing theorem, we may extend D_0 to an effective $\mathbb{Q}$-divisor D_1 on X in such a way, that D_1 is in general position away from Z, i.e. it is smooth away from Z and intersects the discrepancy of a resolution of $(X, \Delta + D)$ transversal over the complement of Z. Now, D' is defined by

$$D' = D + c_1 D_1, \quad \text{where} \quad c_1 = \sup\{\, t \mid (X, \Delta + D + tD_1) \text{ is LC at } x \,\}. \tag{2.3}$$

Then we have automaticly $c' = c + c_1 < 1$ and (2.2) is just a formal consequence of this. The inequality (2.2) is crucial to proceed by induction. But unfortunately, if Z' is more singular than Z in a certain sense, the next induction step works only under a stronger assumption (2.1) for Z', since $p - p'$ is arbitrary small. The problem is, that we have to be more careful with the choice of D_0. More precisely, if we can not avoid the singular center Z', we should use this fact, to improve the bound on p' essentially. So far, this works only at the first induction step, when $Z = X$ and Z' is a hypersurface. But the idea is quite general. The first step is to improve (2.2) in the case, when the multiplicity of D_0 at x is considerably larger than just p.

Theorem 2.2. *Let (X, Δ) be KLT at x and $D \equiv cL$ be an effective nef and big $\mathbb{Q}$-Cartier divisor on X such that $(X, \Delta + D)$ is log canonical at x with minimal center Z. Assume, that there is a rational number q and a $\mathbb{Q}$-divisor $D_1 \equiv L$ such that*

$$m_x(D_1|_Z) \;>\; q \;>\; p \;=\; \frac{b_x(X, \Delta + D)}{1 - c} \tag{2.4}$$

and D_1 is in general position away from Z. Define D' by (2.3). Then $(X, \Delta + D')$ is LC at x with minimal center Z' properly contained in Z and

$$\frac{p - p'}{q - p} \;>\; \frac{c_1}{1 - c'}, \quad \textit{where} \quad p' \;=\; \frac{b_x(X, \Delta + D')}{1 - c'}. \tag{2.5}$$

Proof. The essential step in the proof of Proposition 2.1 is to show, that the local discrepancy drops by

$$b_x(X, \Delta + D') \;<\; b_x(X, \Delta + D) \;-\; c_1 q. \tag{2.6}$$

This would be clear, if the multiplicity of D_1 at x is itself greater than q. Later, we will only use this case. In the general case one can use the vanishing Theorem 1.1 in a very sophisticated way, to show that (2.6) still holds. But we will not repeat the argument here (cf. [5]). Using the definition of p and p' we see that (2.6) is equivalent to

$$p'(1 - c - c_1) \;<\; p(1 - c - c_1) \;-\; c_1(q - p)$$

and this is equivalent to (2.5). □

Later, we will use the inequality (2.5) only in a special situation. Since, the notation might be confusing, we will state this special case as a corollary.

Corollary 2.3. *Let X be a smooth projective variety of dimension n and D_1 be an effective $\mathbb{Q}$-Cartier divisor on X with $m_x(D_1) > q > n$. Define $D := cD_1$, where $c = \sup\{\, t \mid (X, tD_1)$ is LC at $x\,\}$. Then $c < 1$ and*

$$\frac{n-p}{q-n} \;>\; \frac{c}{1-c}, \quad \text{where} \quad p \;=\; \frac{b_x(X,D)}{1-c}. \tag{2.7}$$

Proof. Apply the Theorem 2.2 with $\Delta = D = 0$ and $c = 0$. □

3. The Wildness

If Z is a subvariety of X which is so singular at x, that it can not occur as a minimal center of a log canonical singularity, we may ignore this subvariety for our purpose and in this sense, we consider Z as 'tame'. But if it may occur as a minimal center, we need some measure for the singularity at x. The following is probably not the right thing, but it is enough for hypersurfaces and for our general discussion.

Definition. Let (X, D) be log canonical at $x \in X$ and $Z \subset X$ be a subvariety of dimension d containing x. The *wildness* of Z in D at x is the real number

$$w_x(Z,D) := \sup\left\{\, b_x(X, D+D_1) \;\middle|\; \begin{array}{l} D_1 \geqslant 0 \text{ such that } (X, D+D_1) \text{ is} \\ \text{LC with minimal center } Z \text{ at } x \end{array} \right\}.$$

If there is no $D_1 \geqslant 0$ such that $(X, D+D_1)$ is LC with minimal center Z at x, we define $w_x(Z,D) = 0$. In the case $D = \emptyset$ we just write $w_x(Z) = w_x(Z, \emptyset)$.

There are two basic ideas concerning the wildness. The first is, that if Z is the minimal center of a pair (X, D), the wildness $w_x(Z, D)$ is nothing else but the local discrepancy $b_x(X, D)$. Hence, we can use bounds for the wildness to get good bounds for the local discrepancy which are necessary for the application of Theorem 2.2. As an illustration, let us improve the bound on the number p from Proposition 2.1, if L^n is considerably larger than n^n.

Proposition 3.1. *Let L be an ample divisor on a smooth projective variety X with $L^n > \big((1+\alpha)n\big)^n$ for some $\alpha > 0$ and $x \in X$. Then there is an effective $\mathbb{Q}$-divisor $D \equiv cL$ with $c < 1$ such that (X, D) is LC but not KLT at x, and the wildness $w = w_x(Z)$ of the minimal center Z satisfies*

$$\frac{1+\alpha}{p} \;>\; \frac{1}{n} \;+\; \frac{\alpha}{w} \quad \text{where} \quad p \;=\; \frac{b_x(X,D)}{1-c}. \tag{3.1}$$

Proof. By Riemann Roch, there is an effective divisor $D_1 \equiv kL$ with multiplicity $m_x(D_1) > (1+\alpha)nk$ at x. Hence, we can apply Corollary 2.3 with $q = (1+\alpha)n$ to get an effective $\mathbb{Q}$-divisor $D \equiv cL$ with $c < 1$ such that (X, D) is LC but not KLT at x and furthermore

$$\frac{c}{1-c} \;<\; \frac{1}{\alpha} \;-\; \frac{p}{n\alpha}. \tag{3.2}$$

On the other hand, we have the following bound for p in terms of the wildness

$$p \;=\; \frac{b_x(X,D)}{1-c} \;=\; \frac{w_x(X,D)}{1-c} \;\leqslant\; \frac{w_x(X)}{1-c}. \tag{3.3}$$

Both inequalities together imply

$$p \;<\; w_x(X) \cdot \left(1 + \frac{1}{\alpha} \;-\; \frac{p}{n\alpha}\right) \tag{3.4}$$

and this is equivalent to (3.1). □

Proposition 3.1 can be used, to prove some results on global generation of adjoint linear systems in low dimensions (cf. Example 3.8). But in general, the results are far from being optimal and the calculations are not quiet conceptual, because 3.1 can be only applied at the first induction step and the second step is already more involved. The problem is inequality (3.3). For a proof of Fujita's freeness conjecture we need better bounds for $w_x(X,D)$. To obtain such bounds, one can try to use the following

Lemma 3.2. *Let (X,D) be LC at $x \in X$ and $x \in Z \subset X$. Assume that D_1 is an effective $\mathbb{Q}$-Cartier divisor such that Z is not contained in its support and $w_x(Z, D+D_1) > 0$. Then*

$$w_x(Z, D+D_1) \leqslant w_x(Z,D) - m_x(D_1). \tag{3.5}$$

Proof. Let $\epsilon > 0$. Then there is an effective $\mathbb{Q}$-Cartier divisor D_2, such that the pair $(X, D+D_1+D_2)$ is LC with minimal center Z at x and

$$w_x(Z, D+D_1) \leqslant b_x(X, D+D_1+D_2) + \epsilon. \tag{3.6}$$

Furthermore, there is an effective $\mathbb{Q}$-Cartier divisor D_3, such that

$$b_x(X, D+D_1+D_2) \leqslant m_x(D_3) + \epsilon \tag{3.7}$$

and $(X, D+D_1+D_2+D_3)$ is LC at x. On the other hand, Z is also the minimal center of $(X, D+D_2)$, since Z is not contained in the support of D_1. Hence,

$$w_x(Z,D) \geqslant b_x(X, D+D_2) \geqslant m_x(D_1+D_3). \tag{3.8}$$

Altogether implies $w_x(Z, D+D_1) \leqslant w_x(Z,D) - m_x(D_1) + 2\epsilon$ for all $\epsilon > 0$ and this shows (3.5). □

Lemma 3.2 is very useful, if Z is a hypersurface, as we will see in the next section. But in the higher codimensional case, there is no way to guarantee, that Z is not contained in the support of D_1. In this case we need a more general statement.

The second idea concerning the wildness is, that it can be used, to bound the multiplicity of the minimal center of a log canonical singularity.

Theorem 3.3 ([5, Theo. 4.3]). *If X is smooth and Z is the minimal center of (X,D) at x, for some effective $\mathbb{Q}$-divisor D, then its multiplicity is bounded by*

$$m_x(Z) \leqslant \binom{n - \lceil w_x(Z) \rceil}{n-d}. \tag{3.9}$$

In particular the multiplicity of a minimal center of dimension d is bounded by $\binom{n-1}{d-1}$. This together with Proposition 2.1 implies the global generation of $|K_X+L|$ at x under the assumptions $(**)$ mentioned in the introduction.

Example 3.4. If the dimension of the minimal center Z is 1, the bound (3.9) is equivalent to the smoothness of Z and this is equivalent to the normality of Z in this case. It is known, that the minimal center of a log canonical singularity is always normal and the proof of (3.9) reduces to the standard normality proof in the one dimensional case.

EXAMPLE 3.5. In the other extreme case, when Z is a hypersurface, the wildness $w_x(Z)$ is nothing else but the integer $n - m_x(Z)$. Therefore, (3.9) is the optimal bound also in this case.

EXAMPLE 3.6. The case, when Z is a surface is especially interesting. In this case Kawamata [7] has shown, that Z has a rational singularity at x. Hence $m_x(Z)$ is the local embedding dimension of Z at x minus one, which implies (3.9). But already in this case, the bound (3.9) seems to be not quiet optimal. Consider for example the cone S over the Veronese of $\mathbb{P}^1$ of degree e in $\mathbb{P}^e$. The multiplicity of S at its vertex is e and the local discrepancy of S at this point is $b = 2/e$. Of course the round-up of b is just 1 and the multiplicity is embedding dimension minus one which coincides with (3.9), even if S is here not a minimal center of a log canonical singularity inside a smooth variety, but just a log terminal variety itself. Now, b tends to zero for large e, but (3.9) as a bound for the local discrepancy gives only $b \leqslant 1$.

However, for hypersurfaces the bound (3.9) is more than enough for our purpose. In fact, we will only need the following.

Corollary 3.7. *If X is smooth of dimension n at x and (X, D) is LC at x such that the minimal center is a hypersurface H, then*

$$m_x(H) \;\leqslant\; \left(\frac{n-1}{w_x(H)}\right)^{n-1}. \tag{3.10}$$

PROOF. Let $w = w_x(H)$ be the wildness, $d = n - 1$ the dimension and $m = m_x(H)$ be the multiplicity of H. Since $d = w + m - 1$ by Example 3.5, we get

$$\left(\frac{d}{w}\right)^d \;=\; \left(1 + \frac{m-1}{w}\right)^d \;\geqslant\; 1 + \frac{d}{w}\,(m-1), \tag{3.11}$$

by Bernoulli's inequality. Because $w \leqslant d$, this proves (3.10). □

EXAMPLE 3.8. Let X be a smooth 4-manifold and H an ample divisor on X. Kawamata [7] has shown, that $K_X + 5H$ is globally generated. As an application of Proposition 3.1, let us consider the case, were the minimal center Z after the first induction step, is a surface with a tripel point at x, the most singular case by (3.9). For this, let $L = 5H$. Since $L^4 \geqslant 5^4$, we may apply Proposition 3.1 with any $\alpha < \frac{1}{4}$. To be concrete, let $\alpha = \frac{1}{5}$. If the minimal center Z of (X, D) is a surface with a tripel point at x, the bound (3.9) implies $w \leqslant 1$ and hence, inequality (3.1) is equivalent to $p < \frac{8}{3}$. Finally, we check (2.1) of Proposition 2.1 for Z

$$p^2 \cdot m_x(Z) \;<\; \frac{64}{3} \;<\; 25.$$

Therefore, we may apply Proposition 2.1 to reduce the dimension of the minimal center further. The other cases are quite similar.

We can not use Lemma 3.2 at this moment. For this, we need another concept, which will be introduced in the next section.

4. The Mobility in Codimension One

In this section, we assume that X is a smooth projective variety and L is an ample divisor on X with $L^n > n^n$. We want to find an effective $\mathbb{Q}$-divisor $D \equiv cL$ with $c < 1$, such that either the minimal center of (X, D) has at least codimension 2, or if this is not possible, it is a hypersurface H and the invariant p of (X, D), defined in Proposition 2.1, satisfies the inequality $m_x(H) \cdot p^{n-1} < n^{n-1}$. To find such a $\mathbb{Q}$-divisor we consider the vector spaces

$$\mathcal{L}_k(q) \;=\; \big\{\, f \in H^0\big(X, \mathcal{O}_X(kL)\big) \;\big|\; m_x(f) \geqslant qk \,\big\}. \tag{4.1}$$

Denote by $\mathcal{F}_k(q)$ the fixed part of the linear system $\mathbb{P}\mathcal{L}_k(q)$ and define

$$\varphi_k(q) \;=\; q \;-\; \tfrac{1}{k}\, m_x\big(\mathcal{F}_k(q)\big), \tag{4.2}$$

if $\mathcal{L}_k(q) \neq 0$ and $\varphi_k(q) = -\infty$ otherwise.

Definition. The function $\varphi(q) = \sup\{\, \varphi_k(q) \mid k \in \mathbb{N} \,\}$ is called the *mobility in codimension one* of L at x.

The first important property of the mobility is its concavity.

Lemma 4.1. *The mobility in codimension one $\varphi(q)$ is a concave function of q.*

Proof. For any q_1, q_2 and $\epsilon > 0$ we can find large integers k such that

$$m_x\big(\mathcal{F}_k(q_i)\big) \;<\; \big(q_i - \varphi(q_i) + \epsilon\big) \cdot k$$

for $i = 1, 2$. This implies

$$m_x\big(\mathcal{F}_{2k}(q)\big) \;<\; \Big(q - \tfrac{1}{2}\,\big(\varphi(q_1) + \varphi(q_2)\big) + \epsilon\Big) \cdot 2k, \quad \text{where} \quad q = \tfrac{1}{2}\,(q_1 + q_2),$$

since the fixed part $\mathcal{F}_{2k}(q)$ is contained in $\mathcal{F}_k(q_1) + \mathcal{F}_k(q_2)$. Hence, for all $\epsilon > 0$ we have $\varphi(q) > \frac{1}{2}\big(\varphi(q_1) + \varphi(q_2)\big) - \epsilon$ and this proves the lemma. □

The assumption $L^n > n^n$ implies the existence of effective $\mathbb{Q}$-divisors $D \equiv L$ with $m_x(D) > n$. But if the mobility of L at x is small, we are able to find effective $\mathbb{Q}$-divisors $D \equiv L$ with considerably larger multiplicity at x.

Proposition 4.2. *There exists a rational number $q > n$ such that*

$$\varphi(q) \;>\; w \;-\; \frac{1}{n}\,\frac{w}{n-1-w} \cdot (q - n) \quad \textit{for all} \quad 0 \leqslant w < n - 1. \tag{4.3}$$

In particular $\mathcal{L}_k(q) \neq 0$ for $k \gg 0$.

Proof. Assume, that there is no $q > n$ satisfying (4.3). We claim that then, there is a real number $0 \leqslant w < n - 1$, such that for all q

$$\varphi(q) \;\leqslant\; l(q) \;:=\; w - \delta(w) \cdot (q - n), \quad \text{where} \quad \delta(w) = \frac{1}{n}\,\frac{w}{n-1-w}. \tag{4.4}$$

To see this, define the function $\psi(q)$ by the formula

$$\psi(q) \;=\; \sup\{\, w - \delta(w)\cdot(q-n) \;\mid\; 0 \leqslant w < n-1 \,\}. \tag{4.5}$$

Then $\psi(q)$ is a convex function of q with $\psi(n) = n-1$ and $\psi(q) = 0$ for $q \geqslant n^2$. But for $q < n$ the term $w - \delta(w)(q-n)$ is arbitrary large, i.e. $\psi(q) = \infty$ for $q < n$. For completeness let us note, that

$$\psi(q) \;=\; n - 2 + \frac{q}{n} - 2\cdot\sqrt{\frac{1}{n}\,(n-1)(q-n)} \quad \text{for all} \quad n \leqslant q \leqslant n^2,$$

but we will not need this formula. By our assumption, $\varphi(q) \leqslant \psi(q)$ for all $q > n$. Since $\varphi(q)$ is concave by Lemma 4.1 and $\psi(q)$ is convex, this implies that $\varphi(q)$ and $\psi(q)$ are separated by a tangent line of $\psi(q)$, and this proves the claim (4.4) (cf. Fig. 2).

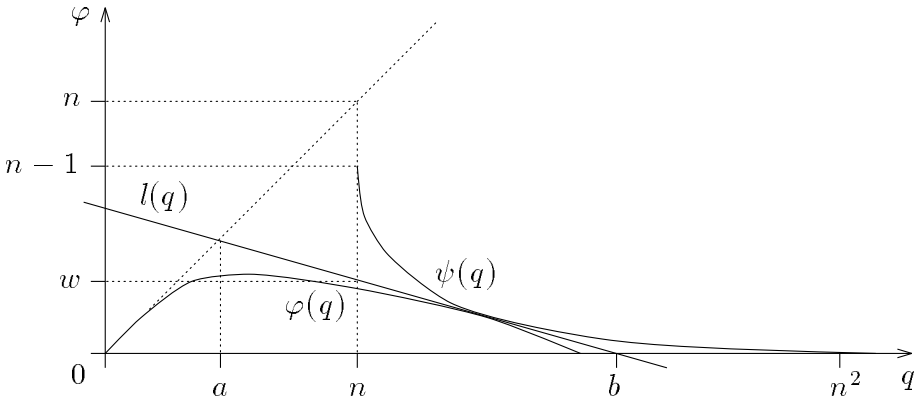

Fig. 2. The mobility under the assumption (4.4).

Define a by $l(a) = a$ and b by $l(b) = 0$ and let $W_k(q)$ be the vector space defined by the following exact sequence

$$0 \longrightarrow \mathcal{L}_k(q) \longrightarrow H^0\big(X, \mathcal{O}_X(kL)\big) \longrightarrow W_k(q) \longrightarrow 0. \tag{4.6}$$

From the Riemann Roch theorem and the assumption $L^n > (1+\epsilon)n^n$ for some small positive number ϵ, we know that

$$h^0\big(X, \mathcal{O}_X(kL)\big) \;>\; (1+\epsilon)\,\frac{(nk)^n}{n!} \quad \text{for} \quad k \gg 0. \tag{4.7}$$

The crucial claim is now, that the dimension of $W_k(b)$ is bounded by

$$\dim W_k(b) \;\leqslant\; \frac{(nk)^n}{n!} \;+\; O(k^{n-1}). \tag{4.8}$$

From (4.7) and (4.8) we get a contradiction, since then $\mathcal{L}_k(b) \neq 0$ for $k \gg 0$ by (4.6). But the mobility $\varphi(q)$ would be zero for $q = b$ and even negative for q a little greater than b by (4.4), which is impossible.

To prove the claim (4.8), consider the linear system $\{\, D - \mathcal{F}_k(q) \mid D \in \mathbb{P}\mathcal{L}_k(q) \,\}$ and let $\mathcal{L}'_k(q)$ be the associated vector space. Then, by (4.1), the multiplicity of the

general element of $\mathcal{L}'_k(q)$ at x is equal to $e = \ulcorner qk \urcorner - m_x\big(\mathcal{F}_k(q)\big)$, as long as $\mathcal{L}_k(q) \neq 0$, and hence we have the following commutative diagram

$$\begin{array}{ccccc} & & \mathcal{L}'_k(q) & \longrightarrow & \mathcal{I}^e_{X,x}/\mathcal{I}^{e+1}_{X,x} \\ & & \wr\downarrow & & \downarrow\sigma \\ 0 & \longrightarrow \mathcal{L}_k\big(q+\tfrac{1}{k}\big) & \xrightarrow{\ \iota\ } \mathcal{L}_k(q) & \xrightarrow{\ \rho\ } & \mathcal{I}^{\ulcorner qk \urcorner}_{X,x}/\mathcal{I}^{\ulcorner qk \urcorner+1}_{X,x} \end{array} \tag{4.9}$$

where $\mathcal{I}_{X,x}$ is the maximal ideal of the local ring $\mathcal{O}_{X,x}$ and the map σ is given by multiplication with a representative of the fixed part $\mathcal{F}_k(q)$. The essential point is, that ρ is not surjective if $\mathcal{F}_k(q) \neq \emptyset$, since the left vertical map is of course an isomorphism, so that the image of ρ is contained in the image of σ. More precisely, by (4.4) and the definition of the mobility $\varphi(q)$ we have

$$e \;=\; \ulcorner qk \urcorner - m_x\big(\mathcal{F}_k(q)\big) \;\leqslant\; \ulcorner \varphi(q)\cdot k \urcorner \;\leqslant\; \ulcorner l(q)\cdot k \urcorner \quad \text{if} \quad a \leqslant q \leqslant b \tag{4.10}$$

and therefore, the dimension of the cokernel of the map ι in the commutative diagram (4.9) is bounded by

$$\dim\, \mathcal{L}_k(q)/\mathcal{L}_k\big(q+\tfrac{1}{k}\big) \;\leqslant\; \dim \mathcal{I}^e_{X,x}/\mathcal{I}^{e+1}_{X,x} \;\leqslant\; \frac{\big(\ulcorner l(q)\cdot k \urcorner\big)^{n-1}}{(n-1)!} \;+\; O(k^{n-2}) \tag{4.11}$$

for $a \leqslant q \leqslant b$. By induction, we see that the dimension of $W_k(q)$ is bounded by

$$\dim W_k(q) \;\leqslant\; \sum_{s\,=\,1}^{\ulcorner ak \urcorner -1} \frac{s^{n-1}}{(n-1)!} \;+\; \sum_{s=\ulcorner ak \urcorner}^{\ulcorner qk \urcorner} \frac{\big(\,l(s/k)\cdot k\,\big)^{n-1}}{(n-1)!} \;+\; O(k^{n-1}), \tag{4.12}$$

if $q > a$. The first sum is equal to $(ak)^n/n!$ up to higher order terms in k. To calculate the second sum for $q = b$, we use the substitution $t = s - \ulcorner bk \urcorner$ and $l(q) = a(b-q)/(b-a)$ to get

$$\sum_{s=\ulcorner ak \urcorner}^{\ulcorner bk \urcorner} \frac{\big(\,l(s/k)\cdot k\,\big)^{n-1}}{(n-1)!} \;=\; \left(\frac{a}{b-a}\right)^{n-1} \cdot \sum_{t\,=\,0}^{\ulcorner bk \urcorner - \ulcorner ak \urcorner} \frac{t^{n-1}}{(n-1)!} \;+\; O(k^{n-1}).$$

The last sum is equal to $(b-a)^n k^n/n!$ up to higher order terms in k and hence we find

$$\dim W_k(b) \;\leqslant\; \frac{1}{n!}\, a^{n-1} b \cdot k^n \;+\; O(k^{n-1}). \tag{4.13}$$

The numbers a and b can be calculated from their definition as

$$a \;=\; \frac{w + n\delta(w)}{1+\delta(w)} \;=\; \frac{nw}{n-1} \quad \text{and} \quad b \;=\; \frac{w+n\delta(w)}{\delta(w)} \;=\; n(n-w). \tag{4.14}$$

In particular $(n-1)a + b = n^2$. Finally we use the elementary fact, that the geometric mean is bounded by the arithmetic mean to get

$$a^{n-1} b \;\leqslant\; \left[\frac{(n-1)\,a \;+\; b}{n}\right]^n \;=\; n^n. \tag{4.15}$$

This proves the inequality (4.8) and hence the assumption, that there is no $q > n$ satisfying (4.3) contradicts our assumption on L^n. □

Now let $q > n$ be any rational number such that $\mathcal{L}_k(q) \neq 0$ and D_1 be a general member of the linear system $\mathbb{P}\mathcal{L}_k(q)$ for $k \gg 0$. Define

$$D := c_1 D_1 \quad \text{where} \quad c_1 = \sup\{\, t \mid (X, tD_1) \text{ is LC at } x \,\}. \tag{4.16}$$

Then $D \equiv cL$ with $c = c_1 k < 1$ and (X, D) is LC but not KLT at x. This is the first induction step to reduce the dimension of the minimal center described by Proposition 2.1. The problem is, that if the minimal center Z of (X, D) is singular at x, we need at least $L^d.Z \geqslant m_x(Z)n^d$ for the next induction step. But if we choose q as in Proposition 4.2 we are able to eliminate bad centers of codimension one, i.e. if Z is a hypersurface we only need $L^{n-1}.Z \geqslant n^{n-1}$. To see this, we need another lemma.

Lemma 4.3. *Assume that the minimal center of (X, D) is a hypersurface $H \subset X$ with wildness $w = w_x(H)$ and that*

$$\varphi_k(q) \;>\; w \;-\; \delta(w) \cdot (q - n), \tag{4.17}$$

where $\delta(w)$ is defined by (4.4). Then it follows

$$p \;<\; \frac{nw}{n-1}, \quad \textit{where} \quad p = \frac{b_x(X, D)}{1 - c}. \tag{4.18}$$

Proof. Since D_1 is general, $D_1 - \mathcal{F}_k(q)$ will not contain the hypersurface H and hence, by Lemma 3.2 the wildness of H drops by

$$w_x(H, D) \;\leqslant\; w_x\big(H, c_1\mathcal{F}_k(q)\big) \;-\; c_1 \cdot m_x\big(D_1 - \mathcal{F}_k(q)\big) \tag{4.19}$$

and $w_x\big(H, c_1\mathcal{F}_k(q)\big) \leqslant w_x(H)$. The multiplicity of the divisor $D_1 - \mathcal{F}_k(q)$ is at least $k \cdot \varphi_k(q)$, by the definition (4.2) of $\varphi_k(q)$. Hence,

$$w_x(H, D) \;\leqslant\; w_x(H) \;-\; c \cdot \varphi_k(q) \;=\; (1 - c) \cdot w \;+\; c \cdot \big(w - \varphi_k(q)\big). \tag{4.20}$$

The wildness $w_x(H, D)$ is of course nothing else but the local discrepancy $b_x(X, D)$ and therefore, dividing (4.20) by $1 - c$ gives

$$p \;\leqslant\; w \;+\; \frac{c}{1-c} \cdot \big(w - \varphi_k(q)\big). \tag{4.21}$$

The inequality (2.7) of Corollary 2.3 shows, that the term $c/(1 - c)$ is lesser than $(n - p)/(q - n)$ and together with the assumption (4.17), this implies

$$p \;<\; w \;+\; \frac{n-p}{q-n} \cdot \delta(w) \cdot (q - n) \;=\; w \;+\; (n - p) \cdot \delta(w). \tag{4.22}$$

This inequality is equivalent to

$$p \;<\; \frac{w + n\delta(w)}{1 + \delta(w)} \;=\; \frac{nw}{n-1} \tag{4.23}$$

by (4.14), which proves the lemma. □

In the proof of Lemma 4.3, we used only a very special case of Theorem 2.2. However, the general statement of Theorem 2.2 should be useful for the induction, if one could solve the technical problems concerning higher codimensional minimal centers. So far, this is out of reach. Therefore, in our main theorem, we can only improve the bound on $L^d.Z$ for hypersurfaces.

Theorem 4.4. *Let X be a smooth projective variety of dimension n and L an ample divisor on X. Let $x \in X$ and assume that*

$$L^n > n^n, \tag{4.24}$$

$$L^{n-1}.H \geqslant n^{n-1} \quad \text{for all hypersurfaces } H \text{ containing } x, \tag{4.25}$$

$$L^d.Z \geqslant m_x(Z) \cdot n^d \quad \text{for all } Z \subset X \text{ with } d = \dim Z \leqslant n-2 \text{ and multiplicity } m_x(Z) \leqslant \tbinom{n-1}{d-1} \text{ at } x. \tag{4.26}$$

Then $\mathcal{O}_X(K_X + L)$ is globally generated at x.

Proof. First we choose q as in Proposition 4.2. In other words, we may assume, that the mobility of L at x satisfies $\varphi(q) > \psi(q)$ for some $q > n$, where $\psi(q)$ is defined by (4.5). Then, for k very large, we still have $\varphi_k(q) > \psi(q)$ and therefore, for such q and k, the assumption (4.17) of Lemma 4.3 is satisfied for all $0 \leqslant w < n-1$.

Now let D_1 be a general member of the linear system $\mathbb{P}\mathcal{L}_k(q)$ and define the $\mathbb{Q}$-divisor D by (4.16). If the codimension of the minimal center Z is at least two, we use Proposition 2.1 and (3.9) to proceed by induction. But if Z has codimension one, by Corollary 3.7 and Lemma 4.3 we have

$$p^{n-1} \cdot m_x(Z) \; < \left(\frac{n w_x(Z)}{n-1}\right)^{n-1} \cdot \left(\frac{n-1}{w_x(Z)}\right)^{n-1} \; = \; n^{n-1}.$$

Hence, in this case the assumption (4.25) is enough for the next induction step. Finally we will end up with a $\mathbb{Q}$-divisor satisfying all the conditions of Proposition 1.2 and therefore, the linear system $|K_X + L|$ is free at x. □

Remark 4.5. If H in the proof of Theorem 4.4 is smooth, both inequalities (3.10) and (4.15) are equalities. In fact, in the limit case $w = n-1$, one has $a = b = n$ in the proof of Proposition 4.2. Hence, in this case the assumption (4.25) is necessary. But, if H is singular, we can do a little better as the following example shows.

Example 4.6. To prove the theorem for threefolds from the introduction, we show that if X is a smooth projective threefold and L is an ample divisor on X with $L^3 > 27$, then there is a rational number $q > 3$, such that

$$\varphi(q) \; > \; 1 \; - \; \lambda(q-3), \quad \text{where} \quad \lambda = \frac{2\sqrt{3}-3}{9}. \tag{4.27}$$

In fact, if not, there would be a number $\mu \geqslant \lambda$ such that $\varphi(q) \leqslant 1 - \mu(q-3)$ for all q, since $\varphi(q)$ is concave (cf. Fig. 3). But in this case, instead of using the inequality (4.15), we can compute the estimate (4.13) for the dimension of $W_k(b)$ explicitly as

$$\frac{1}{3!}\, a^2 b \cdot k^3 \; = \; \frac{1}{3!}\, \frac{(1+3\mu)^3}{(1+\mu)^2 \mu} \cdot k^3 \; \leqslant \; \frac{1}{3!} \cdot 27 \cdot k^3.$$

The last inequality follows from $27\mu^2 + 18\mu - 1 = 27(\mu - \lambda)(\mu + \lambda + \frac{2}{3}) \geqslant 0$ and this gives the desired contradiction.

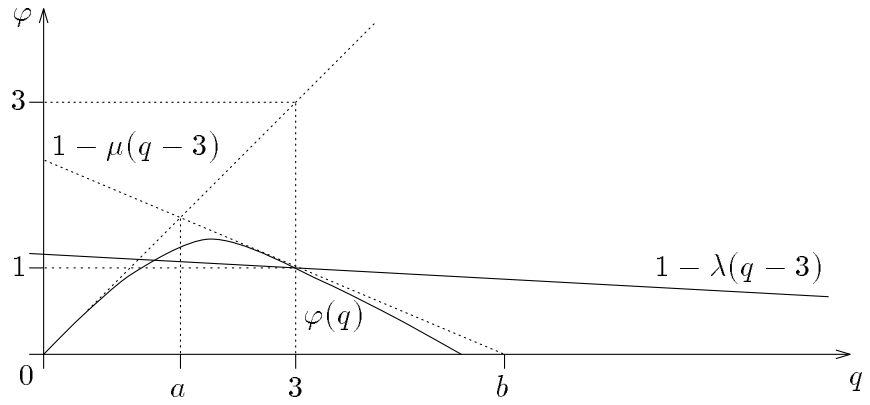

Fig. 3. The mobility under the assumption $\varphi(q) \leqslant 1 - \lambda(q-3)$ for all $q > 3$.

Now choose $q > 3$ with (4.27) and D as above. The minimal center of (X, D) is either the point x, a curve, or a surface which is smooth at x, or a surface S with a rational double point at x. In the first three cases, there is no problem. But in the last case, the wildness of S is $w = 1$ and inequality (4.23) gives now

$$p \;<\; \frac{1+3\lambda}{1+\lambda} \;=\; \frac{3}{2}\left(\sqrt{3}-1\right).$$

Because $m_x(S) = 2$, we find the bound

$$p^2 \cdot m_x(S) \;<\; 9\left(2-\sqrt{3}\right) \;<\; 3,$$

so that we can apply Proposition 2.1 as long as $L^2.S \geqslant 3$. □

Remark 4.7. What is missing, is a general concept of mobility, which works also for higher codimensional minimal centers. Another problem is, that the bound (3.9) for the multiplicity of minimal centers is too weak in general. But the multiplicity is not exacly the right object. In fact, Anghern and Siu's proof works without any bound on the multiplicity of the minimal center. Nevertheless, the main ideas leading to the proof of Theorem 4.4 are quite general and they should help, to prove Fujita's freeness conjecture.

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Research Institute for Mathematical Sciences
Kyoto University, Kitashirakawa, Sakyo-ku, Kyoto 606-01, Japan
Email Address: helmke@kurims.kyoto-u.ac.jp