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Proof of the Gauge Independence of the Conformal Anomaly of Bosonic String in the Sense of Kraemmer and Rebhan

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Abstract

Kraemmer and Rebhan claimed the gauge independence of the conformal anomaly of bosonic string for various gauge fixings in the framework of the perturbation theory of two-dimensional quantum gravity. It is pointed out that their proof is wrong. The gauge independence is proved for the gauge-fixings which reduce to the linearized de Donder gauge in the flat limit of the background metric. Similar remarks are made also for the Rebhan-Kraemmer current anomaly.

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1. Introduction

As is well known, the bosonic string can be described by the two-dimensional quantum gravity coupled with D scalar fields, where D denotes the dimension of the world where a string lives. In noncovariant gauges such as the conformal gauge, the conformal anomaly disappears if and only if $D = 26$.

About one decade ago, several authors [1–6] extended this result to the case of covariant gauges. Their method was to calculate the conformal-anomaly term in the two-point function of "energy-momentum tensor" $T_{\mu\nu}$ in the framework of perturbation theory. All of them claimed that the critical dimension $D = 26$ was obtained.

A remarkable proposition, which we call KR proposition, was put forward by Kraemmer and Rebhan [4]: They claimed the *gauge independence of the conformal anomaly*. They gave a very simple proof of it based on the BRS invariance only.

On the other hand, in 1992, the present authors [7] explicitly demonstrated the indefiniteness of the definition of $T_{\mu\nu}$ in the de Donder gauge and pointed out that the conformal-anomaly term depends on the choice of $T_{\mu\nu}$ so that $D = 26$ is not necessarily obtained.

Since the above two claims look apparently contradictory, we have examined the KR proposition closely. Although many explicit examples support the KR proposition, we have found that the proof of Kraemmer and Rebhan is *wrong*. We therefore propose a new proof of the KR proposition, which requires the detailed analysis of the conformal-anomaly term. Our analysis clarifies under what circumstances the KR proposition is valid. There is no contradiction between the KR proposition and our claim. This point will be discussed in detail in a separate paper [8].

The present paper is organized as follows. In Section 2, we briefly review the KR proposition and its proof of Kraemmer and Rebhan critically. In Section 3, we systematize the calculation of the conformal-anomaly term. In this connection, it is found convenient that the terms appearing in $T_{\mu\nu}$ are classified into several "types". In Section 4, we consider deformations of the gauge-fixing and FP-ghost Lagrangian density under certain conditions. We then find that under deformations there are four conservation laws for the numbers of terms characterized by the above-mentioned "types". From those conservation laws, the invariance of the conformal-anomaly term under deformations follows. The final section is devoted to discussions, where some remarks are made on the anomaly of the FP-ghost number current.

2. Kraemmer-Rebhan's work

In this section, we review the work of Kraemmer and Rebhan [4] concerning the gauge independence of the conformal anomaly.

The covariant-gauge two-dimensional quantum gravity is described by the gravitational field $g_{\mu\nu}$, the B-field b_ρ, the FP ghost c^σ and antighost $\bar{c}_\tau$, together with scalar fields X_M ($M = 0, 1, \ldots, D-1$). We write $g \equiv \det g_{\mu\nu}$ and $\tilde{g}^{\mu\nu} \equiv \sqrt{-g}g^{\mu\nu}$. The BRS transformation of those fields are as follows:

$$\boldsymbol{\delta}_* g_{\mu\nu} = -\partial_\mu c^\sigma \cdot g_{\sigma\nu} - \partial_\nu c^\sigma \cdot g_{\mu\sigma} - c^\sigma \partial_\sigma g_{\mu\nu}, \tag{2.1}$$

$$\boldsymbol{\delta}_* X_M = -c^\lambda \partial_\lambda X_M, \tag{2.2}$$

$$\boldsymbol{\delta}_* c^\sigma = -c^\lambda \partial_\lambda c^\sigma, \tag{2.3}$$

$$\boldsymbol{\delta}_* \bar{c}_\tau = i b_\tau, \tag{2.4}$$

$$\boldsymbol{\delta}_* b_\rho = 0. \tag{2.5}$$

In general, we should also consider the Weyl BRS transformation, but since the conformal degree of freedom is eliminated, we omit describing the Weyl gauge-fixing plus Weyl FP-ghost Lagrangian density for simplicity.

The total Lagrangian density $\mathcal{L}$ consists of the string one,

$$\mathcal{L}_{\mathrm{S}} = \frac{1}{2}\tilde{g}^{\mu\nu}\partial_\mu X^M \cdot \partial_\nu X_M, \tag{2.6}$$

the gauge-fixing one, $\mathcal{L}_{\mathrm{GF}}$, and the FP-ghost one, $\mathcal{L}_{\mathrm{FP}}$. Of course, $\mathcal{L}_{\mathrm{S}}$ and $\mathcal{L}_{\mathrm{GF}} + \mathcal{L}_{\mathrm{FP}}$ are BRS-invariant up to total-divergence terms.

Kraemmer and Rebhan [4] employ the perturbative approach *in the intrinsic way*, that is, they employ it *before* writing down the expression for $\mathcal{L}_{\mathrm{GF}} + \mathcal{L}_{\mathrm{FP}}$. They first set

$$g_{\mu\nu} = \hat{g}_{\mu\nu} + h_{\mu\nu}, \tag{2.7}$$

where $\hat{g}_{\mu\nu}$ denotes the classical background metric, while $h_{\mu\nu}$ does the "small" quantum gravitational field. After substituting (2.7) into $\mathcal{L}$, only terms quadratic in quantum fields are retained. Since this model has no higher-loop corrections, it is perturbatively justifiable to neglect higher-order terms. It is worth noting that Kraemmer and Rebhan also neglected linear terms. The reason for this is that linear terms do not contribute to the conformal anomaly in the perturbative calculation.

Since $\mathcal{L}_{\rm GF}$ is proportional to b_ρ, the above prescription implies that $\mathcal{L}_{\rm GF}$ is also proportional to $h_{\mu\nu}$, that is, we have

$$\mathcal{L}_{\rm GF} = b_\rho F^{\rho\sigma\tau} h_{\sigma\tau}, \tag{2.8}$$

where $F^{\rho\sigma\tau}$ is a first-order differential operator depending on $\hat{g}_{\mu\nu}$ and its derivatives. The condition that the conformal degree of freedom is eliminated is expressed as

$$F^{\rho\sigma\tau}\hat{g}_{\sigma\tau} = 0. \tag{2.9}$$

The FP-ghost Lagrangian density is determined by

$$\mathcal{L}_{\rm GF} + \mathcal{L}_{\rm FP} = -i\boldsymbol{\delta}_*^{(0)}(\bar{c}_\rho F^{\rho\sigma\tau} h_{\sigma\tau}). \tag{2.10}$$

Here $\boldsymbol{\delta}_*^{(0)}$ is the linearized BRS transformation obtained from (2.1)–(2.5) by substituting (2.7) and by neglecting nonlinear terms in quantum fields. We find

$$\begin{aligned}
\boldsymbol{\delta}_*^{(0)} h_{\mu\nu} &= -\hat{g}_{\mu\sigma}\partial_\nu c^\sigma - \hat{g}_{\sigma\nu}\partial_\mu c^\sigma - \partial_\sigma \hat{g}_{\mu\nu}\cdot c^\sigma \\
&= -(\hat{\nabla}_\mu c_\nu + \hat{\nabla}_\nu c_\mu), & (2.11) \\
\boldsymbol{\delta}_*^{(0)} X_M &= 0, & (2.12) \\
\boldsymbol{\delta}_*^{(0)} c^\sigma &= 0, & (2.13) \\
\boldsymbol{\delta}_*^{(0)} \bar{c}_\tau &= ib_\tau, & (2.14) \\
\boldsymbol{\delta}_*^{(0)} b_\rho &= 0, & (2.15)
\end{aligned}$$

where $\hat{\nabla}_\lambda$ denotes covariant differentiation with respect to $\hat{g}_{\mu\nu}$. Here the raising and lowering of the index of c are made by using $\hat{g}^{\mu\nu}$ and $\hat{g}_{\mu\nu}$. Because of the appearance of covariant differentiation, the expression (2.11) is essentially irrelevant to whether c^σ or c_σ is chosen as the primary field. Substituting (2.11) into (2.10), we obtain

$$\mathcal{L}_{\rm FP} = -i\bar{c}_\rho F^{\rho\sigma\tau}(\hat{\nabla}_\sigma c_\tau + \hat{\nabla}_\tau c_\sigma). \tag{2.16}$$

In order to avoid the appearance of second-order derivatives in (2.16), it is usual to make partial integration in (2.8). We then have

$$\mathcal{L}_{\rm GF} = ({F_1}^{\lambda\rho\sigma\tau}\partial_\lambda b_\rho + {F_2}^{\rho\sigma\tau} b_\rho) h_{\sigma\tau}. \tag{2.17}$$

$$\mathcal{L}_{\rm FP} = -i({F_1}^{\lambda\rho\sigma\tau}\partial_\lambda \bar{c}_\rho + {F_2}^{\rho\sigma\tau}\bar{c}_\rho)(\hat{\nabla}_\sigma c_\tau + \hat{\nabla}_\tau c_\sigma), \tag{2.18}$$

which are used in perturbative calculation.

Now, the conformal anomaly is discussed on the basis of calculating

$$\frac{\delta^2 \Gamma}{\delta \hat{g}_{\mu\nu} \delta \hat{g}_{\lambda\rho}}, \tag{2.19}$$

where Γ is the effective action. This is equivalent to calculating the two-point function of $T_{\mu\nu}$, i.e.,

$$\langle\, T_{\mu\nu}(x)\, T_{\lambda\rho}(y)\,\rangle, \tag{2.20}$$

where

$$T_{\mu\nu} \equiv 2(-\hat{g})^{-1/2} \frac{\delta}{\delta \hat{g}^{\mu\nu}} \int d^2x\, \mathcal{L} \tag{2.21}$$

and $\langle\ \cdots\ \rangle$ denotes the vacuum expectation value of a time-ordered product. Usually, we take the flat limit $\hat{g}_{\mu\nu} = \eta_{\mu\nu}$ in (2.20) to avoid unnecessary complication. $T_{\mu\nu}$ is the symmetric energy-momentum tensor if $\mathcal{L}$ is a scalar density under general coordinate transformation with respect to the background metric [1, 5–7] Kraemmer and Rebhan [4], however, do not impose this condition, whence $T_{\mu\nu}$ is no longer a conserved quantity in general.

The two-point function (2.20) is calculated by means of Feynman rules. If (2.8) with $\hat{g}_{\mu\nu} = \eta_{\mu\nu}$ reproduces the homogeneous linear part of the de Donder gauge fixing, the nonvanishing propagators are given by

$$\langle\, X_M(x) X_N(y)\,\rangle = \eta_{MN} D_{\mathrm{F}}(x-y), \tag{2.22}$$

$$\langle\, h_{\mu\nu}(x) b_\rho(y)\,\rangle = (\eta_{\mu\rho}\partial_\nu + \eta_{\rho\nu}\partial_\mu - \eta_{\mu\nu}\partial_\rho)^x D_{\mathrm{F}}(x-y), \tag{2.23}$$

$$\langle\, c^\sigma(x) \bar{c}_\tau(y)\,\rangle = i\delta^\sigma{}_\tau D_{\mathrm{F}}(x-y). \tag{2.24}$$

One-loop integrals are evaluated with the help of dimensional regularization. The result can be written as

$$\langle\, T_{\mu\nu}(x) T_{\lambda\rho}(y)\,\rangle = (D+a)\Phi_{\mu\nu\lambda\rho}(x-y) + \text{local terms}, \tag{2.25}$$

where a is a numerical constant and the Fourier transform of $\Phi_{\mu\nu\lambda\rho}$ is proportional to $p_\mu p_\nu p_\lambda p_\rho/(p^2+i0)$. The first term of the r.h.s. of (2.25) is the conformal-anomaly term. It vanishes if and only if $D + a = 0$. Many concrete examples yield $a = -26$.

The KR proposition is as follows.

The value of a is independent of the choice of $F^{\rho\sigma\tau}$.

The proof of Kraemmer and Rebhan [4] is as follows. Let $\Gamma^{(1)}$ be the one-loop contribution to Γ. Consider any infinitesimal variation $\delta F^{\rho\sigma\tau}$ of $F^{\rho\sigma\tau}$. Then we have

$$\begin{aligned} \delta\Gamma^{(1)}[\hat{g}] &= -i\langle \boldsymbol{\delta}_*^{(0)}\Big(\int d^2x\,\bar{c}_\rho \delta F^{\rho\sigma\tau} h_{\sigma\tau}\Big)\rangle \\ &= 0 \end{aligned} \tag{2.26}$$

because the BRS invariance is not violated spontaneously. From (2.26), they inferred that $\delta a = 0$.

If their proof were correct, then the same reasoning for $F^{\rho\sigma\tau}$ would imply that $a = 0$. Evidently, something must be wrong. Indeed, in order to deduce the invariance of (2.20) from (2.26), *we need the commutativity between* $\boldsymbol{\delta}_*^{(0)}$ *and* $\delta/\delta\hat{g}^{\mu\nu}$*, but they actually do not commute.* In fact, from (2.11) we have

$$\frac{\delta}{\delta\hat{g}^{\mu\nu}}\Big(\boldsymbol{\delta}_*^{(0)} h_{\lambda\rho}\Big) \neq 0 = \boldsymbol{\delta}_*^{(0)}\Big(\frac{\delta}{\delta\hat{g}^{\mu\nu}} h_{\lambda\rho}\Big). \tag{2.27}$$

Thus we can say nothing about $\langle\, T_{\mu\nu}T_{\lambda\rho}\,\rangle$ from (2.26).

To prove the KR proposition, we must make the detailed analysis of $\langle\, T_{\mu\nu}T_{\lambda\rho}\,\rangle$ itself.

3. Classification of the terms of $T_{\mu\nu}$

Before entering into the general consideration, we consider a concrete example for better understanding.

The simplest choice of gauge fixing which satisfies (2.9) and reduces to the linearized de Donder gauge for $\hat{g}_{\mu\nu} = \eta_{\mu\nu}$ is

$$\mathcal{L}_{\rm GF} = \partial^\sigma b^\tau \cdot (h_{\sigma\tau} - \frac{1}{2}\hat{g}_{\sigma\tau}\hat{g}^{\alpha\beta}h_{\alpha\beta}), \tag{3.1}$$

where $b^\tau = \eta^{\tau\rho}b_\rho$, and from (2.18) the corresponding $\mathcal{L}_{\rm FP}$ is

$$\mathcal{L}_{\rm FP} = -i\partial^\sigma\bar{c}^\tau \cdot (\hat{\nabla}_\sigma c_\tau + \hat{\nabla}_\tau c_\sigma - \hat{g}_{\sigma\tau}\hat{g}^{\alpha\beta}\hat{\nabla}_\alpha c_\beta) \tag{3.2}$$

with $\bar{c}^\tau = \eta^{\tau\rho}\bar{c}_\rho$. Then (2.21) with $\hat{g}_{\mu\nu} = \eta_{\mu\nu}$ yields

$$T_{\mu\nu} = T^{\rm S}{}_{\mu\nu} + T^{\rm GF}{}_{\mu\nu} + T^{\rm FP}{}_{\mu\nu}, \tag{3.3}$$

$$T^{\rm S}{}_{\mu\nu} = \partial_\mu X^M \cdot \partial_\nu X_M + \cdots, \tag{3.4}$$

$$T^{\rm GF}{}_{\mu\nu} = -\partial^\sigma b_\sigma \cdot h_{\mu\nu} + \cdots, \tag{3.5}$$

$$\begin{aligned} iT^{\rm FP}{}_{\mu\nu} = [&2\partial_\mu\bar{c}_\nu \cdot \partial_\sigma c^\sigma + \partial_\sigma\partial_\mu\bar{c}_\nu \cdot c^\sigma - \partial_\mu\bar{c}_\sigma \cdot \partial^\sigma c_\nu - \partial_\sigma\bar{c}_\mu \cdot \partial^\sigma c_\nu \\ &+ (\mu \leftrightarrow \nu)\,] \;+\; \cdots, \end{aligned} \tag{3.6}$$

where dots indicate terms proportional to $\eta_{\mu\nu}$, which are irrelevant to the conformal anomaly. In (3.5), the free-field equation for b_ρ,

$$\partial_\lambda b_\rho + \partial_\rho b_\lambda - \eta_{\lambda\rho}\partial^\sigma b_\sigma = 0, \tag{3.7}$$

has been used.

The two-point function,

$$\langle\, T_{\mu\nu} T_{\lambda\rho} \,\rangle = \langle\, T^{\mathrm{S}}{}_{\mu\nu} T^{\mathrm{S}}{}_{\lambda\rho} \,\rangle + \langle\, T^{\mathrm{GF}}{}_{\mu\nu} T^{\mathrm{GF}}{}_{\lambda\rho} \,\rangle + \langle\, T^{\mathrm{FP}}{}_{\mu\nu} T^{\mathrm{FP}}{}_{\lambda\rho} \,\rangle, \tag{3.8}$$

is calculated by using (2.22)–(2.24) and the formulae

$$\partial_\mu \partial_\nu D_{\mathrm{F}} \cdot \partial_\lambda \partial_\rho D_{\mathrm{F}} = \frac{1}{2}\Phi_{\mu\nu\lambda\rho} + \text{local terms}, \tag{3.9}$$

$$\partial_\mu \partial_\nu \partial_\lambda D_{\mathrm{F}} \cdot \partial_\rho D_{\mathrm{F}} = \Phi_{\mu\nu\lambda\rho} + \text{local terms}. \tag{3.10}$$

It is easy to see that the contribution to a from $\langle\, T^{\mathrm{GF}}{}_{\mu\nu} T^{\mathrm{GF}}{}_{\lambda\rho} \,\rangle$ is $+2$, while that from $\langle\, T^{\mathrm{FP}}{}_{\mu\nu} T^{\mathrm{FP}}{}_{\lambda\rho} \,\rangle$ is -28; hence $a = 2 - 28 = -26$.

It is interesting to see how -28 results from $\langle\, T^{\mathrm{FP}}{}_{\mu\nu} T^{\mathrm{FP}}{}_{\lambda\rho} \,\rangle$. We symbolically write $\partial_\mu \bar{c}_\nu \cdot \partial_\sigma c^\sigma + (\mu \leftrightarrow \nu)$ as I, $\partial_\sigma \partial_\mu \bar{c}_\nu \cdot c^\sigma + (\mu \leftrightarrow \nu)$ as II, and so on, so that the r.h.s. of (3.6) is symbolically written as $2I + II - III - IV$. The contribution to a is given by $2^2\langle\, I\, I \,\rangle + 2\cdot 2\langle\, I\, II \,\rangle + \langle\, II\, II \,\rangle + \langle\, III\, III \,\rangle \sim 2^2(-2) + 2\cdot 2(-4) + (-2) + (-2) = -28$. It is important to note that $\langle\, I\, III \,\rangle \sim 0$, $\langle\, II\, III \,\rangle \sim 0$ and IV does not contribute at all. Those facts come from whether or not each of c and $\bar{c}$ has an external index μ or ν.

Now, we consider the general situation. From the above remark, it is reasonable to classify the terms appearing in $iT^{\mathrm{FP}}{}_{\mu\nu}$ into the following four "types".

Type AB	:	Neither c nor $\bar{c}$ have an external index.
Type A	:	Only c has an external index.
Type B	:	Only $\bar{c}$ has an external index.
Type O	:	Both c and $\bar{c}$ have an external index.

Furthermore, we call the term having first-order derivatives only $Rh+$ and the one having a second-order derivative $Rh-$. We abbreviate (Type A, $Rh+$) as A^+, and so on. For example, for the terms in (3.6), we see that $I \sim B^+$, $II \sim B^-$, $III \sim A^+$, $IV \sim O^+$. Note that $2B^+ + B^-$ yields -26 precisely.

We list up the possible terms appearing in $T_{\mu\nu}$ and classify them according to the above classification.

	$Rh+$	$Rh-$
AB	$\partial_\mu\bar{c}_\sigma\cdot\partial_\nu c^\sigma+(\mu\leftrightarrow\nu)$	$2\partial_\mu\partial_\nu\bar{c}_\sigma\cdot c^\sigma,\quad 2\bar{c}_\sigma\partial_\mu\partial_\nu c^\sigma$
A	$\partial_\mu\bar{c}_\sigma\cdot\partial^\sigma c_\nu+(\mu\leftrightarrow\nu)$	$\partial_\mu\partial^\sigma\bar{c}_\sigma\cdot c_\nu+(\mu\leftrightarrow\nu)$
	$\partial^\sigma\bar{c}_\sigma\cdot\partial_\mu c_\nu+(\mu\leftrightarrow\nu)$	$\bar{c}_\sigma\partial^\sigma\partial_\mu c_\nu+(\mu\leftrightarrow\nu)$
B	$\partial_\sigma\bar{c}_\mu\cdot\partial_\nu c^\sigma+(\mu\leftrightarrow\nu)$	$\partial_\nu\partial_\sigma c_\mu\cdot c^\sigma+(\mu\leftrightarrow\nu)$
	$\partial_\mu\bar{c}_\nu\cdot\partial^\sigma c_\sigma+(\mu\leftrightarrow\nu)$	$\bar{c}_\nu\partial_\mu\partial_\sigma c^\sigma+(\mu\leftrightarrow\nu)$
O	$\partial_\sigma\bar{c}_\mu\cdot\partial^\sigma c_\nu+(\mu\leftrightarrow\nu)$	—

Some remarks are in order. O^- is nonexistent because $\Box c_\sigma=\Box\bar{c}_\sigma=0$. Type O is effectively equivalent to zero. The cross term of Type A and Type B vanishes. Type AB is effectively equivalent to Type A plus Type B.

A special care is necessary for $Rh-$. Since $\partial_\mu\partial_\nu\partial_\lambda\partial_\rho D_{\mathrm{F}}\cdot D_{\mathrm{F}}$ is infrared divergent, we must check the nonappearance of nonvanishing cross terms of the form $\langle\,(\partial\partial\bar{c}\cdot c)(\bar{c}\partial\partial c)\,\rangle$. By explicit calculation based on the FP-ghost Lagrangian density (4.5) in the next section, it is found that we encounter only the following three kinds of the $Rh-$ terms.

$$
\begin{aligned}
A^-\ &:\qquad \bar{c}_\sigma\partial^\sigma\partial_\mu c_\nu+(\mu\leftrightarrow\nu),\\
B^-\ &:\qquad \partial_\nu\partial_\sigma\bar{c}_\mu\cdot c^\sigma+(\mu\leftrightarrow\nu),\\
A^-\ &:\qquad (\bar{c}_\sigma\partial_\mu\partial_\nu c^\sigma-\bar{c}_\nu\partial_\mu\partial_\sigma c^\sigma)+(\mu\leftrightarrow\nu).
\end{aligned}
$$

The last one is Type A because it is Type AB minus Type B. Since the cross term of Type A and Type B vanish, we do not encounter infrared divergence.

Next, we classify the terms appearing in $T^{\mathrm{GF}}{}_{\mu\nu}$ in the following way.

$$
\begin{aligned}
\text{Type } A\ &:\ b \text{ has no external index,}\\
\text{Type } O\ &:\ b \text{ has an external index;}\\
Rh+\ &:\ \text{Differential operator acts on } b,\\
Rh-\ &:\ \text{Differential operator acts on } h.
\end{aligned}
$$

We encounter the following terms.

	$Rh+$	$Rh-$
Type A	$\frac{1}{2}[\partial_\mu b_\sigma\cdot h_\nu{}^\sigma+(\mu\leftrightarrow\nu)]$	$\frac{1}{2}[b_\sigma\partial_\mu h_\nu{}^\sigma+(\mu\leftrightarrow\nu)]$
	$\partial^\sigma b_\sigma\cdot h_{\mu\nu}$	$b_\sigma\partial^\sigma h_{\mu\nu}$
Type O	$\frac{1}{2}[\partial_\sigma b_\mu\cdot h_\nu{}^\sigma+(\mu\leftrightarrow\nu)]$	$\frac{1}{2}[b_\mu\partial_\sigma h_\nu{}^\sigma+(\mu\leftrightarrow\nu)]$

Type O is effectively equivalent to zero. The two terms of A^+ are effectively equal because their difference is of Type O owing to (3.7).

We denote the algebraic sum of the coefficients of the $A^{\pm}$ and $AB^{\pm}$ terms appearing in $iT^{\rm FP}{}_{\mu\nu}$ by $N(A^{\pm})$, that of the $B^{\pm}$ and $AB^{\pm}$ terms appearing in $iT^{\rm FP}{}_{\mu\nu}$ by $N(B^{\pm})$ and that of the $A^{\pm}$ terms appearing in $T^{\rm GF}{}_{\mu\nu}$ by $M(A^{\pm})$. Then we have the general formula

$$\begin{aligned} a &= [M(A^+)]^2 \times 2 + 2M(A^+)M(A^-) \times 4 + [M(A^-)]^2 \times 2 \\ &\quad + [N(A^+)]^2 \times (-2) + 2N(A^+)N(A^-) \times (-4) + [N(A^-)]^2 \times (-2) \\ &\quad + [N(B^+)]^2 \times (-2) + 2N(B^+)N(B^-) \times (-4) + [N(B^-)]^2 \times (-2). \end{aligned} \tag{3.11}$$

Therefore, if

$$N(A^+) = M(A^+), \tag{3.12}$$

$$N(A^-) = M(A^-), \tag{3.13}$$

$$N(B^+) = 2, \tag{3.14}$$

$$N(B^-) = 1, \tag{3.15}$$

then we obtain $a = -26$.

4. Deformations and conservation laws

We discuss the general case of $\mathcal{L}_{\rm GF} + \mathcal{L}_{\rm FP}$ given by (2.17) and (2.18). We require formal tensorial invariance. Then we may assume, without loss of generality, that $F_1{}^{\lambda\rho\sigma\tau}$ involves no $\partial_\kappa \hat{g}_{\alpha\beta}$ and that $F_2{}^{\rho\sigma\tau}$ is proportional to $\partial_\kappa \hat{g}_{\alpha\beta}$ because we need $T_{\mu\nu}$ only for $\hat{g}_{\mu\nu} = \eta_{\mu\nu}$. We rewrite (2.17) into the form which manifestly exhibits the satisfaction of the condition (2.9) and reduces to the linearized de Donder gauge fixing in the flat limit:

$$\mathcal{L}_{\rm GF} = \sum_{j=1}^{2} (\tilde{F}_{1j}{}^{\lambda\rho\sigma\tau}[\hat{g}]\partial_\lambda b_\rho + \tilde{F}_{2j}{}^{\rho\sigma\tau}[\hat{g},\, \partial\hat{g}] b_\rho) H^j{}_{\sigma\tau}, \tag{4.1}$$

where

$$H^1{}_{\sigma\tau} \equiv h_{\sigma\tau} - \frac{1}{2}\hat{g}_{\sigma\tau}\hat{g}^{\alpha\beta}h_{\alpha\beta}, \tag{4.2}$$

$$H^2{}_{\sigma\tau} \equiv h_{\sigma\tau} - \hat{g}_{\sigma\tau}\frac{\eta^{\alpha\beta}h_{\alpha\beta}}{\eta^{\gamma\delta}\hat{g}_{\gamma\delta}}, \tag{4.3}$$

and

$$\sum_{j=1}^{2} \tilde{F}_{1j}{}^{\lambda\rho\sigma\tau}[\eta] = \eta^{\lambda\sigma}\eta^{\rho\tau}. \tag{4.4}$$

Of course, $\tilde{F}_{2j}{}^{\rho\sigma\tau} = 0$ for $\hat{g}_{\mu\nu} = \eta_{\mu\nu}$. Correspondingly, (2.18) is rewritten as

$$\mathcal{L}_{\rm FP} = -i\sum_{j=1}^{2} (\tilde{F}_{1j}{}^{\lambda\rho\sigma\tau}[\hat{g}]\partial_\lambda \bar{c}_\rho + \tilde{F}_{2j}{}^{\rho\sigma\tau}[\hat{g}, \partial\hat{g}]\bar{c}_\rho) K^j{}_{\sigma\tau}, \tag{4.5}$$

where

$$K^1{}_{\sigma\tau} \equiv \hat{\nabla}_\sigma c_\tau + \hat{\nabla}_\tau c_\sigma - \hat{g}_{\sigma\tau}\hat{g}^{\alpha\beta}\hat{\nabla}_\alpha c_\beta, \tag{4.6}$$

$$K^2{}_{\sigma\tau} \equiv \hat{\nabla}_\sigma c_\tau + \hat{\nabla}_\tau c_\sigma - \hat{g}_{\sigma\tau}\frac{2\eta^{\alpha\beta}\hat{\nabla}_\alpha c_\beta}{\eta^{\gamma\delta}\hat{g}_{\gamma\delta}}. \tag{4.7}$$

We start with the two simplest cases:

$$\text{(i)} \qquad \tilde{F}_{11}{}^{\lambda\rho\sigma\tau} = \eta^{\lambda\sigma}\eta^{\rho\tau}, \qquad \tilde{F}_{12} = \tilde{F}_{2j} = 0; \tag{4.8}$$

$$\text{(ii)} \qquad \tilde{F}_{12}{}^{\lambda\rho\sigma\tau} = \eta^{\lambda\sigma}\eta^{\rho\tau}, \qquad \tilde{F}_{11} = \tilde{F}_{2j} = 0. \tag{4.9}$$

The first case (4.8) is nothing but the example discussed at the beginning of Section 3. From (3.5) and (3.6), we find that

$$\begin{aligned} &M(A^+) = -1, \qquad M(A^-) = 0; \\ &N(A^+) = -1, \qquad N(A^-) = 0, \qquad N(B^+) = 2, \qquad N(B^-) = 1. \end{aligned} \tag{4.10}$$

In the second case (4.9), we have

$$T^{\mathrm{GF}}{}_{\mu\nu} = 0 + \cdots, \tag{4.11}$$

$$\begin{aligned} iT^{\mathrm{FP}}{}_{\mu\nu} = [&2\partial_\mu\bar{c}_\nu \cdot \partial_\sigma c^\sigma + \partial_\sigma\partial_\mu\bar{c}_\nu \cdot c^\sigma + \partial^\sigma\bar{c}_\sigma \cdot \partial_\mu c_\nu - \partial_\mu\bar{c}_\sigma \cdot \partial^\sigma c_\nu - \partial_\sigma\bar{c}_\mu \cdot \partial^\sigma c_\nu \\ &+ (\mu \leftrightarrow \nu)\ ] \ + \ \cdots, \end{aligned} \tag{4.12}$$

where use has been made of (3.7). Accordingly,

$$\begin{aligned} &M(A^+) = M(A^-) = 0; \\ &N(A^+) = N(A^-) = 0, \qquad N(B^+) = 2, \qquad N(B^-) = 1. \end{aligned} \tag{4.13}$$

In both cases, (3.12)–(3.15) hold.

Now, we proceed to the general case. To do this, we employ *finite deformations* rather than infinitesimal variations. By combining the following three kinds of deformations, we can achieve the general expression (4.1).

D1. To take a linear combination of any two already constructed gauge fixings in such a way that the normalization condition (4.4) is kept.

D2. To insert a background metric, that is, to replace $\delta^\alpha{}_\beta$ by $\hat{g}^{\alpha\gamma}\eta_{\gamma\beta}$ or $\eta^{\alpha\gamma}\hat{g}_{\gamma\beta}$.[3]

[3] Replacement of $\delta^\alpha{}_\beta$ by $\eta^{\alpha\gamma}\hat{g}_{\gamma\beta}$ in $f(\det\delta^\alpha{}_\beta)$ implies the insertion of an arbitrary function of $-\hat{g}$. Of course, such insertion is irrelevant to the conformal anomaly.

D3. To add a term proportional to $\partial_\kappa \hat{g}_{\alpha\beta} \cdot b_\rho H^j{}_{\sigma\tau}$.

As for **D1**, it is self-evident that (3.12)–(3.15) hold if they hold for both constituents. As for **D2** and **D3**, we have only to take account of the new contribution to $T_{\mu\nu}$ due to the deformation considered, and therefore the other parts of $\mathcal{L}_{\rm GF} + \mathcal{L}_{\rm FP}$ may be reduced to their expressions in the flat limit *beforehand*.

First, we consider the Type-A conservation laws (3.12) and (3.13). They are nothing but the consequence of the BRS invariance. We observe:

1. The deformation made in $\mathcal{L}_{\rm GF}$ is exactly the same as in the one made in $\mathcal{L}_{\rm FP}$ if b_ρ is replaced by $\bar{c}_\rho$.

2. If b has an internal index, $\bar{c}$ has also an internal index, and *vice versa*. Thus Type A in $T^{\rm GF}{}_{\mu\nu}$ corresponds to Type A or Type AB in $iT^{\rm FP}{}_{\mu\nu}$, while Type O in $T^{\rm GF}{}_{\mu\nu}$ corresponds to Type O or Type B in $iT^{\rm FP}{}_{\mu\nu}$.

3. The distinction between $Rh+$ and $Rh-$ is common in $T^{\rm GF}{}_{\mu\nu}$ and in $iT^{\rm FP}{}_{\mu\nu}$.

From these properties, we can conclude that the change of $M(A^\pm)$ due to the deformation is equal to that of $N(A^\pm)$.

Next, we proceed to the Type-B conservation laws (3.14) and (3.15). In this case, of course, we have only to consider $iT^{\rm FP}{}_{\mu\nu}$. In the flat background, both $K^1{}_{\sigma\tau}$ and $K^2{}_{\sigma\tau}$ reduce to

$$\partial_\tau c_\sigma + \partial_\sigma c_\tau - \eta_{\sigma\tau}\partial^\alpha c_\alpha. \tag{4.14}$$

Type B and Type AB are characterized by the property that c has an internal index. There are four possibilities in whether or not σ and τ are replaced by μ and ν in the new terms arisen in $iT^{\rm FP}{}_{\mu\nu}$ owing to the deformation.

1. The case in which neither σ nor τ is replaced by an external index. Then since both σ and τ are internal, they must be contracted in $iT^{\rm FP}{}_{\mu\nu}$. Hence we have

$$\eta^{\sigma\tau}(\partial_\tau c_\sigma + \partial_\sigma c_\tau - \eta_{\sigma\tau}\partial^\alpha c_\alpha) = 0, \tag{4.15}$$

that is, there is no change in $N(B^\pm)$.

2. The case in which σ alone is replaced by an external index μ or ν. Then the first term of (4.14) gives no change to $N(B^{\pm})$. The changes due to the second term and the third one just cancel out.

3. The case in which τ alone is replaced by μ or ν. Similar to Case 2.

4. The case in which both σ and τ are replaced by external indices. Then, evidently, neither the first term of (4.14) nor the second term contributes to the change of $N(B^{\pm})$. Furthermore, since the third term is proportional to $\eta_{\mu\nu}$, it does not contribute.

Thus, in all cases, $N(B^{\pm})$ remains unchanged. Accordingly, (3.14) and (3.15) hold always.

From the above consideration, we have seen that (3.12)–(3.15) are valid for any $T^{\mathrm{GF}}{}_{\mu\nu}$ and $iT^{\mathrm{FP}}{}_{\mu\nu}$ obtainable from (4.1) and (4.5). This completes the proof of the KR proposition.

5. Discussions

In the present paper, we have succeeded in proving the gauge independence of the conformal anomaly in the sense of Kraemmer and Rebhan. It is important to note that the BRS invariance is relevant to the terms of Type A only. *The critical value $a = -26$ arises from the terms of Type B.* Its invariance is due to the special properties of the particular expression (4.14) rather than the BRS invariance.

Although the original form [4] of the KR proposition includes also noncovariant gauges, our proof applied only to the gauges which reduce to the linearized de Donder gauge in the flat limit. This is because we should not change the Feynman rules. Hence we cannot yet give a satisfactory answer to the question on whether it is inevitable or accidental that both covariant gauges and noncovariant gauges give a the same value -26. As will be discussed in detail in a separate paper [8], the validity of the KR proposition is dependent on the perturbative approach *in the crucial way*. Hence it is quite difficult to compare two gauges which have different Feynman rules.

Kraemmer and Rebhan [4] presented also a proof of the gauge independence of the anomaly of the so-called "Rebhan-Kraemmer current" [9, 10], which is a sum of the FP-ghost number current $j_{\mathrm{c}}{}^{\mu}$ and the Lagrangian-multiplier (or b-field) current $j_{\mathrm{L}}{}^{\mu}$.

However, as long as one considers the two-point functions $\langle\, j_{\rm c}{}^{\mu} T_{\lambda\rho}\,\rangle$ and $\langle\, j_{\rm L}{}^{\mu} T_{\lambda\rho}\,\rangle$, our criticism applies also to the proof of the gauge independence of the anomaly of the Rebhan-Kraemmer current.

Rebhan and Kraemmer [3] emphasized the difference between the expressions for $j_{\rm c}{}^{\mu}$ in Düsedau's case (background generally covariant) [1] and in their case (given by (4.9)). However, the *Noether currents* in both cases coincide in the flat limit:

$$ij_{\rm c}{}^{\mu} = \bar{c}_{\sigma}\partial^{\mu}c^{\sigma} - \partial^{\mu}\bar{c}_{\sigma}\cdot c^{\sigma} + \partial_{\sigma}(\bar{c}^{\sigma}c^{\mu} - \bar{c}^{\mu}c^{\sigma}), \tag{5.1}$$

as it should be. Düsedau [1] neglected the total-divergence terms tacitly, but *this negligence is the origin of the difference*. As emphasized in Ref. [7], *we must not neglect total-divergence terms in the anomaly calculation*. The total-divergence terms of (5.1) actually contribute to the anomaly. Indeed, if Rebhan and Kraemmer also had unconsciously neglected the total-divergence terms, they would have been led to a different conclusion.

The anomaly of $\langle\, j_{\rm c}{}^{\mu}\, T_{\lambda\rho}\,\rangle$ arises from the loop integral $\partial\partial D_{\rm F}\cdot\partial D_{\rm F}$. The general formula for the coefficient is

$$a_{\rm c} = N(A^{+}) + N(A^{-}) - N(B^{+}) - N(B^{-}). \tag{5.2}$$

On the other hand, since

$$j_{\rm L}{}^{\mu} = -b_{\sigma}h^{\mu\sigma} + \frac{1}{2}b^{\mu}h^{\sigma}{}_{\sigma} \tag{5.3}$$

in the flat limit, we have

$$a_{\rm L} = -M(A^{+}) - M(A^{-}). \tag{5.4}$$

From (3.12)–(3.15), therefore, we find

$$a_{\rm c} + a_{\rm L} = -3. \tag{5.5}$$

This establishes the gauge independence of the anomaly of the Rebhan-Kraemmer current.

References

[1] D. W. Düsedau, Phys. Lett. **B188** (1987), 51.

[2] L. Baulieu and A. Bilal, Phys. Lett. **B192** (1987), 339.

[3] A. Rebhan and U. Kraemmer, Phys. Lett. **B196** (1987), 477.

[4] U. Kraemmer and A. Rebhan, Nucl. Phys. **B315** (1989), 717.

[5] J. I. Lattore, Nucl. Phys. **B297** (1988), 171.

[6] D. Z. Freedman, J. I. Lattore and K. Pilch, Nucl. Phys. **B306** (1988), 77.

[7] M. Abe and N. Nakanishi, Mod. Phys. Lett. **A7** (1992), 1799.

[8] M. Abe and N. Nakanishi, Preprint RIMS-1164 (1997).

[9] W. Kummer, Phys. Lett. **B231** (1989), 53.

[10] O. Piguet, D. Schwarz and M. Schweda, Phys. Lett. **B245** (1990), 393.