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Question on D=26
— String Theory *versus* Quantum Gravity —

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Abstract

In the covariant-gauge two-dimensional quantum gravity, various derivations of the critical dimension $D = 26$ of the bosonic string are critically reviewed, and their interrelations are clarified. It is shown that the string theory is not identical with the proper framework of the two-dimensional quantum gravity, but the former should be regarded as a particular aspect of the latter. The appearance of various anomalies is shown to be explainable in terms of a new type of anomaly in a unified way.

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1. Introduction

The bosonic string theory can be described by the two-dimensional quantum gravity coupled with D scalar fields, where D is the dimension of the world in which a string lives. If one calculates the conformal anomaly in the noncovariant-gauge (e.g., conformal-gauge) two-dimensional quantum gravity, one finds that it is proportional to $D-26$. Hence the conformal anomaly is absent if and only if $D=26$. Thus the two-dimensional quantum gravity reproduces the critical dimension of the bosonic string of finite length.

About one decade ago, Düsedau[1] and several other authors[2−6] calculated the conformal anomaly in the covariant-gauge two-dimensional quantum gravity in the framework of perturbation theory. They all claimed that the conformal anomaly is proportional to $D-26$ also in the covariant-gauge cases. Especially, Kraemmer and Rebhan[4] considered a large class of gauge fixings and claimed the gauge independence of the conformal anomaly.

On the other hand, the present authors[7] found that Düsedau's result[1] of the conformal anomaly is *not* necessarily obtained if we make field redefinition before applying perturbation theory. That is, the way of calculating the conformal anomaly is ambiguous and $D=26$ is not the unique result.

Recently, Takahashi[8] has proposed a new way of deriving $D=26$ in such a way that it is free of the above ambiguity problem. He has obtained a BRS anomaly proportional to $D-26$. He has then converted this anomaly into the conformal anomaly by adding the conformal degree of freedom to the effective action.

The violation of the BRS invariance contradicts all previous work, especially, our exact solution to the de Donder-gauge two-dimensional quantum gravity, which is completely BRS-invariant. We have therefore analyzed why such discrepancy can arise[9] and found that there is a very delicate problem in evaluating massless Feynman integrals. We believe that if there are two regularizations, gauge invariant and non-invariant, one should adopt the former in the calculation of anomaly. If this principle is accepted, Takahashi's BRS-violating result must be abandoned, and therefore the subsequent derivation of the conformal anomaly is not acceptable.

The problem which we discuss in the present paper is whether or not the covariant-gauge two-dimensional quantum gravity can be identified with the string theory, that is, whether or not the critical dimension $D=26$ is an indispensable consequence of the two-dimensional quantum gravity. Our conclusion is that *the string theory is a particular aspect of the two-dimensional quantum gravity*. The proper framework of the latter is free of BRS anomaly, conformal anomaly, FP-ghost number current anomaly, etc., but one *can* encounter them at one's will. We clarify why such a paradoxical matter happens.

The present paper is organized as follows. In Sec. 2, we critically review the various derivations of $D = 26$ based on the conformal anomaly in the covariant-gauge two-dimensional quantum gravity. In Sec. 3, we describe the proper framework of the de Donder-gauge two-dimensional quantum gravity. It is quite a healthy theory except for one peculiar feature, which is called "field-equation anomaly". In Sec. 4, it is pointed out that the result reviewed in Sec. 2 is a consequence of a particular approach to the two-dimensional quantum gravity. In Sec. 5, Takahashi's calculation of the BRS anomaly is reinterpreted in the BRS-invariant framework, and it is shown that the reinterpreted one is essentially equivalent to one of the Kraemmer-Rebhan class discussed in Sec. 2. Furthermore, we show that various anomalies encountered so far are what one can construct by using the field-equation anomaly. The final section is devoted to discussion.

2. Perturbative calculations of conformal anomaly

We discuss the conformal anomaly of bosonic string theory in terms of covariant-gauge two-dimensional quantum gravity. The string coordinates are represented by D scalar fields ϕ_M ($M = 0, 1, \ldots, D-1$). We introduce the gravitational field $g_{\mu\nu}$, the gravitational B-field $\tilde{b}_\rho$, the gravitational FP-ghost c^σ and antighost $\bar{c}_\tau$. The conformal degree of freedom is eliminated so as to avoid the introduction of the Weyl B-field, FP-ghost and antighost for simplicity. We write $g \equiv \det g_{\mu\nu}$ and $\tilde{g}^{\mu\nu} \equiv (-g)^{1/2} g^{\mu\nu}$; $\det \tilde{g}^{\mu\nu} = -1$ and $\tilde{g}^{\mu\nu}$ has no conformal degree of freedom.

The conventional BRS transformation is denoted by $\boldsymbol{\delta}_*$. We have

$$\boldsymbol{\delta}_* g_{\mu\nu} = -\partial_\mu c^\lambda \cdot g_{\lambda\nu} - \partial_\nu c^\lambda \cdot g_{\mu\lambda} - c^\lambda \partial_\lambda g_{\mu\nu}, \tag{2.1}$$

$$\boldsymbol{\delta}_* \tilde{g}^{\mu\nu} = \partial_\lambda c^\mu \cdot \tilde{g}^{\lambda\nu} + \partial_\lambda c^\nu \cdot \tilde{g}^{\mu\lambda} - \partial_\lambda (c^\lambda \tilde{g}^{\mu\nu}), \tag{2.2}$$

$$\boldsymbol{\delta}_* c^\sigma = -c^\lambda \partial_\lambda c^\sigma, \tag{2.3}$$

$$\boldsymbol{\delta}_* \bar{c}_\tau = i\tilde{b}_\tau, \tag{2.4}$$

$$\boldsymbol{\delta}_* \tilde{b}_\rho = 0, \tag{2.5}$$

$$\boldsymbol{\delta}_* \phi_M = -c^\lambda \partial_\lambda \phi_M. \tag{2.6}$$

The Lagrangian density $\mathcal{L}$ consists of the string one

$$\mathcal{L}_{\rm S} = \frac{1}{2} \tilde{g}^{\mu\nu} \partial_\mu \phi^M \cdot \partial_\nu \phi_M, \tag{2.7}$$

the gauge-fixing one $\mathcal{L}_{\rm GF}$ and the FP-ghost one $\mathcal{L}_{\rm FP}$. Here $\mathcal{L}_{\rm S}$ is BRS-invariant and $\mathcal{L}_{\rm GF} + \mathcal{L}_{\rm FP}$ is BRS-exact.[a] In the de Donder gauge, we have

$$\mathcal{L}_{\rm GF} = -\partial_\mu \tilde{b}_\nu \cdot \tilde{g}^{\mu\nu}, \tag{2.8}$$

$$\mathcal{L}_{\rm FP} = -i\partial_\mu \bar{c}_\nu \cdot [\tilde{g}^{\mu\lambda} \partial_\lambda c^\nu + \tilde{g}^{\lambda\nu} \partial_\lambda c^\mu - \partial_\lambda (\tilde{g}^{\mu\nu} c^\lambda)]. \tag{2.9}$$

2.1. *Düsedau's approach*

Düsedau[1] initiated anomaly calculation in the covariant gauge. To apply perturbative approach, one sets

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \tag{2.10}$$

and therefore

$$\tilde{g}^{\mu\nu} = \eta^{\mu\nu} - \eta^{\mu\sigma}\eta^{\nu\tau} h_{\sigma\tau} + \frac{1}{2}\eta^{\mu\nu}\eta^{\sigma\tau} h_{\sigma\tau} + \cdots. \tag{2.11}$$

Substituting (2.11) into a sum of (2.7)–(2.9), and then neglecting higher order terms with respect to quantum fields and also a linear total-divergence term, we obtain

$$\begin{aligned}\mathcal{L}^{(0)\,\rm D} \ &= \ \frac{1}{2}\eta^{\mu\nu} \partial_\mu \phi^M \cdot \partial_\nu \phi_M \\ &\quad + \partial_\mu \tilde{b}_\nu \cdot (\eta^{\mu\sigma}\eta^{\nu\tau} h_{\sigma\tau} - \frac{1}{2}\eta^{\mu\nu}\eta^{\sigma\tau} h_{\sigma\tau}) \\ &\quad - i\partial_\mu \bar{c}_\nu \cdot (\eta^{\mu\lambda}\partial_\lambda c^\nu + \eta^{\lambda\nu}\partial_\lambda c^\mu - \eta^{\mu\nu}\partial_\lambda c^\lambda).\end{aligned} \tag{2.12}$$

We introduce a background (i.e., c-number) metric $\hat{g}_{\mu\nu}$ by replacing $\eta_{\mu\nu}$ by $\hat{g}_{\mu\nu}$ and ∂_μ by background-covariant differentiation $\hat{\nabla}_\mu$ and then by multiplying the resultant by $(-\hat{g})^{1/2}$. Then (2.12) becomes

$$\begin{aligned}\mathcal{L}^{\rm D} \ &= \ \frac{1}{2}\tilde{\hat{g}}^{\mu\nu} \partial_\mu \phi^M \cdot \partial_\nu \phi_M \\ &\quad + \hat{\nabla}_\mu \tilde{b}_\nu \cdot \tilde{\hat{g}}^{\mu\sigma} \hat{g}^{\nu\tau} (h_{\sigma\tau} - \frac{1}{2}\hat{g}_{\sigma\tau}\hat{g}^{\alpha\beta} h_{\alpha\beta}) \\ &\quad - i\hat{\nabla}_\mu \bar{c}_\nu \cdot [\tilde{\hat{g}}^{\mu\lambda} \hat{\nabla}_\lambda c^\nu + \tilde{\hat{g}}^{\lambda\nu} \hat{\nabla}_\lambda c^\mu - \hat{\nabla}_\lambda (\tilde{\hat{g}}^{\mu\nu} c^\lambda)].\end{aligned} \tag{2.13}$$

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[a] "BRS-exact" means that it can be written as $\boldsymbol{\delta}_*(\ \cdot\ )$. Because of $\boldsymbol{\delta}_*^2 = 0$, BRS invariance follows from BRS exactness.

This expression is a scalar density under general coordinate transformations. As is well known, the Belinfante symmetric energy-momentum tensor is given by

$$T^{\mathrm{D}}{}_{\mu\nu} = 2(-\hat{g})^{-1/2} \frac{\delta}{\delta \hat{g}^{\mu\nu}} \int d^2x \, \mathcal{L}^{\mathrm{D}} \Big|_{\hat{g}_{\sigma\tau}=\eta_{\sigma\tau}}. \tag{2.14}$$

Explicitly, we have

$$T^{\mathrm{D}}{}_{\mu\nu} = T_{\mathrm{S}\,\mu\nu} + T^{\mathrm{D}}_{\mathrm{GF}\,\mu\nu} + T^{\mathrm{D}}_{\mathrm{FP}\,\mu\nu} \tag{2.15}$$

with

$$T_{\mathrm{S}\,\mu\nu} = \partial_\mu \phi^M \cdot \partial_\nu \phi_M + \eta_{\mu\nu}\text{-term}, \tag{2.16}$$

$$T^{\mathrm{D}}_{\mathrm{GF}\,\mu\nu} = \partial_\mu \tilde{b}_\sigma \cdot h_\nu{}^\sigma + \partial_\nu \tilde{b}_\sigma \cdot h_\mu{}^\sigma + \tilde{b}_\sigma \partial^\sigma h_{\mu\nu} + \eta_{\mu\nu}\text{-terms}, \tag{2.17}$$

$$\begin{aligned} iT^{\mathrm{D}}_{\mathrm{FP}\,\mu\nu} = {} & [\partial_\mu \bar{c}_\sigma \cdot \partial_\nu c^\sigma + \partial_\sigma(\partial_\mu \bar{c}_\nu \cdot c^\sigma) + \partial_\sigma(\bar{c}^\sigma \partial_\mu c_\nu) \\ & -\partial^\sigma \bar{c}_\mu \cdot \partial_\sigma c_\nu + (\mu \leftrightarrow \nu)] + \eta_{\mu\nu}\text{-terms}. \end{aligned} \tag{2.18}$$

We calculate the two-point function of $T^{\mathrm{D}}{}_{\mu\nu}$ in the one-loop order. The free propagators are

$$\langle\, \phi^M(x)\phi^N(y)\,\rangle = \eta^{MN} D_{\mathrm{F}}(x-y), \tag{2.19}$$

$$\langle\, h_{\mu\nu}(x)\tilde{b}_\rho(y)\,\rangle = (\eta_{\mu\rho}\partial_\nu + \eta_{\rho\nu}\partial_\mu - \eta_{\mu\nu}\partial_\rho) D_{\mathrm{F}}(x-y), \tag{2.20}$$

$$\langle\, c^\sigma(x)\bar{c}_\tau(y)\,\rangle = i\delta^\sigma{}_\tau D_{\mathrm{F}}(x-y), \tag{2.21}$$

where $\langle \cdots \rangle$ and D_{F} denote the vacuum expectation value of a time-ordered product and a massless Feynman propagator, respectively. The one-loop formulae

$$\partial_\mu\partial_\nu D_{\mathrm{F}}(x-y) \cdot \partial_\lambda\partial_\rho D_{\mathrm{F}}(x-y) = \frac{1}{2}\Phi_{\mu\nu\lambda\rho}(x-y) + \cdots, \tag{2.22}$$

$$\partial_\mu\partial_\nu\partial_\lambda D_{\mathrm{F}}(x-y) \cdot \partial_\rho D_{\mathrm{F}}(x-y) = \Phi_{\mu\nu\lambda\rho}(x-y) + \cdots, \tag{2.23}$$

are derived by using dimensional regularization, where

$$\Phi_{\mu\nu\lambda\rho}(\xi) \equiv \frac{i}{12\pi} \int \frac{d^2p}{(2\pi)^2} \frac{p_\mu p_\nu p_\lambda p_\rho}{p^2 + i0} e^{-ip\xi} \tag{2.24}$$

and dots indicate (divergent) local terms. We then obtain

$$\langle\, T^{\mathrm{D}}{}_{\mu\nu}(x) T^{\mathrm{D}}{}_{\lambda\rho}(y)\,\rangle = (D + 26 - 52)\Phi_{\mu\nu\lambda\rho}(x-y) + \cdots. \tag{2.25}$$

The conformal-anomaly term proportional to $\Phi_{\mu\nu\lambda\rho}$ vanishes if and only if $D = 26$. Thus the critical dimension is found to be $D = 26$.

2.2. *Baulieu-Bilal's approach*

Since there is an interesting identity

$$\begin{aligned}&\tilde{\hat{g}}^{\mu\lambda}\hat{\nabla}_\lambda c^\nu + \tilde{\hat{g}}^{\lambda\nu}\hat{\nabla}_\lambda c^\mu - \hat{\nabla}_\lambda(\tilde{\hat{g}}^{\mu\nu}c^\lambda)\\ &= \tilde{\hat{g}}^{\mu\lambda}\partial_\lambda c^\nu + \tilde{\hat{g}}^{\lambda\nu}\partial_\lambda c^\mu - \partial_\lambda(\tilde{\hat{g}}^{\mu\nu}c^\lambda),\end{aligned} \tag{2.26}$$

the FP-ghost term of $\mathcal{L}^{\mathrm{D}}$ in (2.13) is quite similar to the exact expression for $\mathcal{L}_{\mathrm{FP}}$ in (2.9) if $\hat{g}_{\mu\nu}$ is replaced by $g_{\mu\nu}$. This similarity is complete in the case of conformal gauge because $\partial_\mu \bar{c}_\nu$ is replaced by $\bar{c}_{\mu\nu}$. This fact has induced some confusion.

Baulieu and Bilal[2] introduced a background metric, but it was inessential; their Lagrangian density is essentially nothing but the exact one:

$$\mathcal{L}^{\mathrm{BB}}{}_{\mathrm{GF}} = \tilde{b}_\mu \partial_\nu \tilde{g}^{\mu\nu}, \tag{2.27}$$

$$\mathcal{L}^{\mathrm{BB}}{}_{\mathrm{FP}} = -i\partial_\mu \bar{c}_\nu \cdot (\tilde{g}^{\mu\lambda}\partial_\lambda c^\nu - \partial_\lambda \tilde{g}^{\nu\lambda}\cdot c^\mu), \tag{2.28}$$

where $\mathcal{L}^{\mathrm{BB}}{}_{\mathrm{GF}} - \mathcal{L}_{\mathrm{GF}}$ and $\mathcal{L}^{\mathrm{BB}}{}_{\mathrm{FP}} - \mathcal{L}_{\mathrm{FP}}$ are total divergences. They eliminated $\mathcal{L}^{\mathrm{BB}}{}_{\mathrm{GF}}$ by substituting the field equation $\partial_\nu \tilde{g}^{\mu\nu} = 0$ (This is an unjustifiable procedure). Then they considered the second variational-derivative of the effective action Γ with respect to the *quantum* gravitational field, that is, the two-point function which they considered is

$$\langle\, \mathcal{T}^{\mathrm{BB}}{}_{\mu\nu}(x)\mathcal{T}^{\mathrm{BB}}{}_{\lambda\rho}(y)\,\rangle \tag{2.29}$$

where we use, in general, a script letter "$\mathcal{T}$" for the derivative with respect to the *quantum* field $g_{\mu\nu}$, that is,

$$\mathcal{T}_{\mu\nu} = 2(-g)^{-1/2}\frac{\delta}{\delta g^{\mu\nu}}\int d^2x\,\mathcal{L} \tag{2.30}$$

generally. It should satisfy

$$\mathcal{T}_{\mu\nu} = 0 \tag{2.31}$$

as an *exact* field equation.

Now, the lowest-order expression for $\mathcal{T}^{\mathrm{BB}}{}_{\mu\nu}$ is given by

$$\begin{aligned}\mathcal{T}^{\mathrm{BB}\,(0)}{}_{\mu\nu} \;=\;& \partial_\mu\phi^M\cdot\partial_\nu\phi_M - i[\partial_\mu\bar{c}_\sigma\cdot\partial_\nu c^\sigma + \partial_\mu(\partial_\sigma\bar{c}_\nu\cdot c^\sigma) + (\mu\leftrightarrow\nu)]\\ &+\eta_{\mu\nu}\text{-terms.}\end{aligned} \tag{2.32}$$

From (2.32), the coefficient of $\Phi_{\mu\nu\lambda\rho}$ becomes $D-28$,[4,7] though they claimed $D-26$.

2.3. *Kraemmer-Rebhan's approach*

Kraemmer and Rebhan[4] discussed a class of gauge fixings by extending Rebhan-Kraemmer's example.[3] Although they explicitly considered conformal degree of freedom, we here employ the form in which conformal degree of freedom is already eliminated.

Their approach starts with

$$g_{\mu\nu} = \hat{g}_{\mu\nu} + h_{\mu\nu} \tag{2.33}$$

instead of (2.10), that is, the background metric is directly introduced. Accordingly, background general covariance is *not* required. The linearized BRS transform of $h_{\mu\nu}$ is obtained by substituting (2.33) into (2.1) and by neglecting higher-order terms with respect to quantum fields; we have

$$\boldsymbol{\delta}_* h_{\mu\nu} = -\hat{g}_{\lambda\nu}\partial_\mu c^\lambda - \hat{g}_{\mu\lambda}\partial_\nu c^\lambda - \partial_\lambda \hat{g}_{\mu\nu} \cdot c^\lambda. \tag{2.34}$$

Since only one-loop order is relevant, the gauge-fixing term is chosen to be linear in $h_{\mu\nu}$ from the outset, and moreover the zeroth-order term is also neglected because it does not contribute in lowest order. Then the gauge-fixing term can generally be written as

$$\mathcal{L}^{\mathrm{KR}}{}_{\mathrm{GF}} = (F_1{}^{\mu\nu\sigma\tau}\partial_\mu \tilde{b}_\nu + F_2{}^{\nu\sigma\tau}\tilde{b}_\nu)h_{\sigma\tau}, \tag{2.35}$$

where $F_1{}^{\mu\nu\sigma\tau}$ and $F_2{}^{\nu\sigma\tau}$ are arbitrary functions of $\hat{g}_{\lambda\rho}$ and $\partial_\kappa \hat{g}_{\lambda\rho}$, but they must satisfy

$$F_1{}^{\mu\nu\sigma\tau}\hat{g}_{\sigma\tau} = 0, \qquad F_2{}^{\nu\sigma\tau}\hat{g}_{\sigma\tau} = 0 \tag{2.36}$$

because conformal degree of freedom is eliminated. The linearized BRS invariance of $\mathcal{L}^{\mathrm{KR}}{}_{\mathrm{GF}} + \mathcal{L}^{\mathrm{KR}}{}_{\mathrm{FP}}$ yields

$$\mathcal{L}^{\mathrm{KR}}{}_{\mathrm{FP}} = -i(F_1{}^{\mu\nu\sigma\tau}\partial_\mu \bar{c}_\nu + F_2{}^{\nu\sigma\tau}\bar{c}_\nu)(\hat{\nabla}_\sigma c_\tau + \hat{\nabla}_\tau c_\sigma), \tag{2.37}$$

where use has been made of an identity

$$\hat{g}_{\lambda\nu}\partial_\mu c^\lambda + \hat{g}_{\mu\lambda}\partial_\nu c^\lambda + \partial_\lambda \hat{g}_{\mu\nu} \cdot c^\lambda = \hat{\nabla}_\mu c_\nu + \hat{\nabla}_\nu c_\mu \tag{2.38}$$

with $c_\mu \equiv \hat{g}_{\mu\nu}c^\nu$.

Kraemmer and Rebhan defined "energy-momentum tensor" $T^{\mathrm{KR}}{}_{\mu\nu}$ analogously to (2.14), though, in general, $T^{\mathrm{KR}}{}_{\mu\nu}$ is no longer conserved. The coefficient of $\Phi_{\mu\nu\lambda\rho}$ in $\langle T^{\mathrm{KR}}{}_{\mu\nu} T^{\mathrm{KR}}{}_{\lambda\rho} \rangle$ is seen to be $D - 26$ for various concrete examples. Kraemmer and

Rebhan[4] claimed that this is true in general, but their proof cannot be regarded as correct. According to them, the gauge independence of the anomaly term would be a simple consequence of the linearized BRS exactness of the variation of the effective action under the infinitesimal variation of gauge fixing. But if the same reasoning were applied to the effective action itself, the anomaly coefficient would be shown to be D instead of $D-26$. The reason why their proof is wrong is the *noncommutativity* between $\boldsymbol{\delta}_*$ and $\delta/\delta\hat{g}^{\mu\nu}$, which is seen from (2.34) and the requirement

$$\frac{\delta h_{\sigma\tau}}{\delta\hat{g}^{\mu\nu}} = 0. \tag{2.39}$$

As we show in a separate paper[10], however, the gauge independence of the anomaly term is correct, as long as $\mathcal{L}^{\mathrm{KR}}{}_{\mathrm{GF}}$ reduces to the linearized de Donder gauge fixing (second line of (2.12)) in the flat limit $\hat{g}_{\mu\nu} = \eta_{\mu\nu}$. The proof is given by directly analyzing $\langle\, T^{\mathrm{KR}}{}_{\mu\nu} T^{\mathrm{KR}}{}_{\lambda\rho}\,\rangle$ in detail. Kraemmer and Rebhan claimed their proposition with more generality; for example, according to them, conformal gauge would be also included. It is very difficult, however, to compare generally the two cases having different Feynman rules in the direct analysis of $\langle\, T^{\mathrm{KR}}{}_{\mu\nu} T^{\mathrm{KR}}{}_{\lambda\rho}\,\rangle$.

2.4. *Kraemmer-Rebhan's version of Baulieu-Bilal's approach*

Kraemmer-Rebhan's general treatment[4] includes Düsedau's case,[1] Rebhan-Kraemmer's example,[3] etc. Very interesting is the fact that Baulieu-Bilal's analysis[2] can be reinterpreted as an example of Kraemmer-Rebhan's framework.[4]

The Lagrangian density $\mathcal{L}^{\mathrm{KRBB}}$ is defined by replacing $\hat{\nabla}_\mu \tilde{b}_\nu$ and $\hat{\nabla}_\mu \bar{c}_\nu$ of $\mathcal{L}^{\mathrm{D}}$ in (2.13) by $\partial_\mu \tilde{b}_\nu$ and $\partial_\mu \bar{c}_\nu$, respectively. Owing to the correspondence noted in Subsec. 2.2, $\mathcal{L}^{\mathrm{KRBB}}$ has the same form as $\mathcal{L}^{\mathrm{BB}}$, except for the gauge-fixing term, if $g_{\mu\nu}$ is replaced by $\hat{g}_{\mu\nu}$.

Calculating $T^{\mathrm{KRBB}}{}_{\mu\nu}$, we obtain

$$\begin{aligned} T^{\mathrm{KRBB}}{}_{\mu\nu} \;\; = \;\; & \partial_\mu\phi^M \cdot \partial_\nu\phi_M + \partial^\sigma \tilde{b}_\sigma \cdot h_{\mu\nu} \\ & -i[\partial_\mu\bar{c}_\sigma \cdot \partial_\nu c^\sigma + \partial_\mu(\partial_\sigma\bar{c}_\nu \cdot c^\sigma) + (\mu \leftrightarrow \nu)] \\ & +\eta_{\mu\nu}\text{-terms.} \end{aligned} \tag{2.40}$$

Note that (2.40) reproduces (2.32), though it contains the contribution from the gauge-fixing term. The coefficient of the conformal-anomaly term is now $D+2-28$ in conformity with the Kraemmer-Rebhan's proposition.[4]

3. Exact treatment

All discussions made in Sec. 2 are based on perturbation theory. But one should note that it is quite artificial to apply perturbative approach to the two-dimensional quantum gravity because it contains no expansion parameter. One may say that perturbation theory is a loop expansion rather than a parameter expansion, but one must recognize the fact that the division of the action into its free part and its interaction one is artificial and *nonunique*. It is discussed in next section that this fact is really troublemaking in the perturbative approach to the conformal anomaly.

Even apart from the string theory, the de Donder gauge two-dimensional quantum gravity is a very interesting model. It can be regarded as the *two-dimensional version of the quantum Einstein gravity* because the two-dimensional Einstein-Hilbert action is trivial. More importantly, the de Donder-gauge two-dimensional quantum gravity is, apart from its dimensionality, nothing but the *zeroth-order approximation of the de Donder-gauge quantum Einstein gravity*[11] in the expansion in powers of the Einstein gravitational constant κ [The conventional perturbative approach to quantum Einstein gravity starts with a wrong zeroth order.[12]]. Furthermore, the two-dimensional quantum gravity is *exactly solvable*.

Before entering into the discussion of the two-dimensional quantum gravity, we briefly review the covariant operator formalism of quantum Einstein gravity[13,14] in the n-dimensional spacetime.

First, we introduce the notion of the *intrinsic* BRS transformation,[15] which is denoted by $\boldsymbol{\delta}$. Just like the angular momentum, the BRS transformation $\boldsymbol{\delta}_*$ consists of its intrinsic part $\boldsymbol{\delta}$ and its orbital part:

$$\boldsymbol{\delta}_*\Phi = \boldsymbol{\delta}\Phi - c^\lambda \partial_\lambda \Phi, \tag{3.1}$$

where Φ is *any* quantum field. If Φ is a tensor field, $\boldsymbol{\delta}\Phi$ is determined by its general linear transformation property, e.g.,

$$\boldsymbol{\delta} g_{\mu\nu} = -\partial_\mu c^\lambda \cdot g_{\lambda\nu} - \partial_\nu c^\lambda \cdot g_{\mu\lambda}, \tag{3.2}$$

$$\boldsymbol{\delta}\phi_M = 0. \tag{3.3}$$

On the other hand, for ghost fields, we have

$$\boldsymbol{\delta} c^\sigma = 0, \tag{3.4}$$

$$\boldsymbol{\delta}\bar{c}_\tau = ib_\tau, \tag{3.5}$$

$$\boldsymbol{\delta}b_\rho = 0. \tag{3.6}$$

Just as $\boldsymbol{\delta}_*$ is, $\boldsymbol{\delta}$ is nilpotent, but $\boldsymbol{\delta}$ does not commute with ∂_μ: $[\boldsymbol{\delta},\ \partial_\mu] = -\partial_\mu c^\lambda \partial_\lambda$. Substituting (3.5) and (3.6) into (3.1), we have

$$\boldsymbol{\delta}_*\bar{c}_\tau = ib_\tau - c^\lambda\partial_\lambda\bar{c}_\tau, \tag{3.7}$$

$$\boldsymbol{\delta}_* b_\rho = -c^\lambda\partial_\lambda b_\rho. \tag{3.8}$$

The B-field b_ρ is different from the $\tilde{b}_\rho$ of Sec. 2. If we set[16]

$$\tilde{b}_\rho \equiv b_\rho + ic^\lambda\partial_\lambda\bar{c}_\rho, \tag{3.9}$$

we find that the conventional rules (2.1)–(2.6) of $\boldsymbol{\delta}_*$ are reproduced. Thus the effect of considering $\boldsymbol{\delta}$ is to regard b_ρ, rather than $\tilde{b}_\rho$, as the primary B-field. Contrary to its first impression, this problem is not so trivial.

Just as in constructing a Lorentz-invariant action, it is conceptually simper to require the *intrinsic* BRS invariance of the action, that is,[15]

$$\boldsymbol{\delta}[(-g)^{-1/2}\mathcal{L}] = 0. \tag{3.10}$$

The Lagrangian density $\mathcal{L}$ of the quantum Einstein gravity is given by

$$\mathcal{L} = \mathcal{L}_{\mathrm{E}} + \mathcal{L}_{\mathrm{GF}} + \mathcal{L}_{\mathrm{FP}} + \mathcal{L}_{\mathrm{matter}} \tag{3.11}$$

with

$$\mathcal{L}_{\mathrm{E}} = \frac{1}{2\kappa}(-g)^{1/2}R, \tag{3.12}$$

R being the scalar curvature,

$$\mathcal{L}_{\mathrm{GF}} = -\tilde{g}^{\mu\nu}\partial_\mu b_\nu, \tag{3.13}$$

$$\mathcal{L}_{\mathrm{FP}} = -i\tilde{g}^{\mu\nu}\partial_\mu\bar{c}_\sigma\cdot\partial_\nu c^\sigma, \tag{3.14}$$

and if we consider free scalar fields ϕ_M as matter, $\mathcal{L}_{\mathrm{matter}} = \mathcal{L}_{\mathrm{S}}$. Compared with (2.9), the simplicity of (3.14) is quite remarkable. The appearance of the simple derivative for c^σ is the manifestation of the abelian nature of translation group which is the global version of general coordinate transformation group.

The field equations which follow from (3.11) are

$$R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R=\kappa(E_{\mu\nu}-\frac{1}{2}g_{\mu\nu}g^{\sigma\tau}E_{\sigma\tau}-T_{\mathrm{S}\,\mu\nu}) \tag{3.15}$$

with $R_{\mu\nu}$ being the Ricci tensor and

$$E_{\mu\nu}\equiv\partial_\mu b_\nu+i\partial_\mu\bar{c}_\sigma\cdot\partial_\nu c^\sigma+(\mu\leftrightarrow\nu), \tag{3.16}$$

$$\partial_\mu\tilde{g}^{\mu\nu}=0, \tag{3.17}$$

$$\partial_\mu(\tilde{g}^{\mu\nu}\partial_\nu c^\sigma)=0, \tag{3.18}$$

$$\partial_\mu(\tilde{g}^{\mu\nu}\partial_\nu\bar{c}_\tau)=0, \tag{3.19}$$

$$\partial_\mu(\tilde{g}^{\mu\nu}\partial_\nu\phi_M)=0. \tag{3.20}$$

Taking the covariant derivative of (3.15) and using (3.17)–(3.19), we obtain

$$\partial_\mu(\tilde{g}^{\mu\nu}\partial_\nu b_\rho)=0. \tag{3.21}$$

This beautiful result holds only for b_ρ but not for $\tilde{b}_\rho$.

Introducing a $(4n+D)$-dimensional "supercoordinate"[17]

$$X=\{\,x^\lambda,\ b_\rho,\ c^\sigma,\ \bar{c}_\tau,\ \phi_M\,\}, \tag{3.22}$$

we can summarize (3.17)–(3.21) as

$$\partial_\mu(\tilde{g}^{\mu\nu}\partial_\nu X)=0. \tag{3.23}$$

From (3.23), we see that the $(4n+D)$-dimensional "supermomentum"

$$\mathcal{P}^\mu(X)\equiv\tilde{g}^{\mu\nu}\partial_\nu X \tag{3.24}$$

and "angular supermomentum"

$$\mathcal{M}^\mu(X,\,Y)\equiv\tilde{g}^{\mu\nu}(X\partial_\nu Y-\partial_\nu X\cdot Y) \tag{3.25}$$

are conserved, where Y denotes another supercoordinate. After canonical quantization, the corresponding charge operators form $(4n+D)$-dimensional Poincaré-like superalgebra $IOSp(2n+D,\,2n)$.[17,13,14] The existence of this remarkable symmetry justifies the assertion that b_ρ, but not $\tilde{b}_\rho$, should be regarded as the primary field.

Now, we proceed to considering the two-dimensional case $n=2$.[18,19] As is well known, in this case, $\mathcal{L}_{\mathrm{E}}$ is a total divergence and should be omitted. Then $\mathcal{L}$ reduces to

$$\mathcal{L}=-\frac{1}{2}\tilde{g}^{\mu\nu}(E_{\mu\nu}-\partial_\mu\phi^M\cdot\partial_\nu\phi_M). \tag{3.26}$$

Because of $\det \tilde{g}^{\mu\nu} = -1$, (3.26) contains only two degrees of freedom of $g_{\mu\nu}$. We can directly deal with (3.26), but it is not convenient to do so. More convenient is to add the conformal degree of freedom to (3.26) and then extract the part independent of the conformal degree of freedom; the results are the same. So, we add $(-g)^{1/2} R\, b$ to (3.26). After deriving field equations and equal-time commutation relations, we set $b = 0$.

The field equations are[b]

$$\mathcal{T}_{\mu\nu} \equiv T_{\mathrm{S}\,\mu\nu} - E_{\mu\nu} + \frac{1}{2}\tilde{g}_{\mu\nu}\tilde{g}^{\sigma\tau}E_{\sigma\tau} = 0 \tag{3.27}$$

and (3.17)–(3.20). Since (3.21) is again derived, we have the $IOSp(4+D,\,4)$ symmetry. It should be noted that (3.27) is nothing but (2.31).

It is quite remarkable that all two-dimensional commutation relations are explicitly obtained in closed form if we introduce the q-number D function, $\mathcal{D}(x,\,y)$, defined by a q-number Cauchy problem, which involves $\tilde{g}^{\mu\nu}$ only. Some of the exact two-dimensional commutation relations are as follows:

$$\{c^\sigma(x),\ \bar{c}_\tau(y)\} = -\delta^\sigma{}_\tau \mathcal{D}(x,\,y), \tag{3.28}$$

$$[\tilde{g}^{\mu\nu}(x),\ b_\rho(y)] = i\{\boldsymbol{\delta}_*[\tilde{g}^{\mu\nu}(x)],\ \bar{c}_\rho(y)\}, \tag{3.29}$$

$$[b_\lambda(x),\ b_\rho(y)] = i[\partial_\rho b_\lambda(x) + \partial_\lambda b_\rho(y)]\cdot \mathcal{D}(x,\,y) \tag{3.30}$$

and $\tilde{g}^{\mu\nu}(x)$ commutes with any field but $b_\rho(y)$. Note that all those relations are form-independent of matter field. From the two-dimensional commutation relations, we can explicitly calculate all multiple commutators in closed form.

The representation of the above field algebra is constructed by giving all Wightman functions (i.e., vacuum expectation values of simple product of fields).[20−24] With one-point Wightman functions

$$\langle\, \tilde{g}^{\mu\nu}(x_1)\,\rangle_{\mathrm{W}} = \eta^{\mu\nu}, \qquad \langle\, \Phi(x_1)\,\rangle_{\mathrm{W}} = 0, \tag{3.31}$$

where Φ denotes any field other than $\tilde{g}^{\mu\nu}$, N-point Wightman functions $W(x_1,\,\cdots,\,x_N)$ are constructed so as to satisfy the following requirements.

W1 They are consistent with the vacuum expectation values of all multiple commutators.

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[b] $\tilde{g}_{\mu\nu}$ is the inverse of $\tilde{g}^{\mu\nu}$.

W2 [Energy-positivity requirement] They are boundary values of analytic functions of ${x_1}^\mu - {x_2}^\mu, \cdots, {x_{N-1}}^\mu - {x_N}^\mu$ from the lower half-planes of their $\mu = 0$ components.

W3 [Generalized normal-product rule] $W(x_1, \cdots, x_N)$ with ${x_i}^\mu = {x_{i+1}}^\mu = \cdots = {x_j}^\mu$ $(i < j)$ is defined from $W(x_1, \cdots, x_N)$ by setting ${x_i}^\mu = {x_{i+1}}^\mu = \cdots = {x_j}^\mu$ and by deleting the resulting divergent terms in such a way that it be independent of the ordering of i, $i+1$, $\ldots$, j.

Because of the self-commutativity of $\tilde{g}^{\mu\nu}$, we have

$$\langle\, F(\tilde{g}^{\mu\nu})\,\rangle_{\mathrm{W}} = F(\eta^{\mu\nu}), \tag{3.32}$$

for any function F of $\tilde{g}^{\mu\nu}$. Especially, we have

$$\langle\, \mathcal{D}(x,\, y)\,\rangle = D(x-y), \tag{3.33}$$

which is the usual two-dimensional D-function. It is decomposed into positive-energy and negative-energy parts:

$$iD(x-y) = D^{(+)}(x-y) - D^{(+)}(y-x). \tag{3.34}$$

According to **W2**, $W(x_1, \cdots, x_N)$ is expressed only in terms of $D^{(+)}(x_i - x_j)$ $(i<j)$.

All Wightman functions are explicitly constructed.[20−24] We here present some examples of them:

$$\langle\, b_\lambda(x_1) b_\rho(x_2)\,\rangle_{\mathrm{W}} = {\partial_\lambda}^{x_2} D^{(+)}(x_1 - x_2) \cdot {\partial_\rho}^{x_1} D^{(+)}(x_1 - x_2), \tag{3.35}$$

$$\langle\, \tilde{g}^{\mu\nu}(x_1) b_\rho(x_2)\,\rangle_{\mathrm{W}} = -({\delta^\mu}_\rho \partial^\nu + {\delta^\nu}_\rho \partial^\mu - \eta^{\mu\nu}\partial_\rho)^{x_1} D^{(+)}(x_1 - x_2), \tag{3.36}$$

$$\begin{aligned}\langle\, c^\sigma(x_1)\bar{c}_\tau(x_2) b_\rho(x_3)\,\rangle_{\mathrm{W}} = i{\delta^\sigma}_\tau[&{\partial_\rho}^{x_1} D^{(+)}(x_1 - x_2)\cdot D^{(+)}(x_1 - x_3) \\ &+ {\partial_\rho}^{x_2} D^{(+)}(x_1 - x_2)\cdot D^{(+)}(x_2 - x_3)].\end{aligned} \tag{3.37}$$

The corresponding τ-functions (i.e., vacuum expectation values of time-ordered products) are obtained by simply replacing $D^{(+)}$ by D_{F}.

Our exact solution is consistent with the field equations (3.17)–(3.21) and with all symmetry generators of $IOSp(4+D, 4)$, provided that they are constructed from (3.24) and (3.25). However, our exact solution is *not* consistent with the field equation (3.27). Indeed,from (3.35)–(3.37) and $\langle\, \phi_M b_\rho\,\rangle_{\mathrm{W}} = 0$, we find that[21,22]

$$\langle\, \mathcal{T}_{\mu\nu}(x_1) b_\rho(x_2)\,\rangle_{\mathrm{W}} \neq 0. \tag{3.38}$$

We call this phenomenon "field-equation anomaly". We note that the violation is very little in the sense that the degree of freedom of (3.27) is the same as that of (3.21). The existence of field-equation anomaly is demonstrated also in some simpler models.[25, 26]

Our exact solution is completely BRS invariant. The corresponding BRS charge is given by

$$Q_b \equiv \int dx^1\, \tilde{g}^{0\nu}(b_\rho \partial_\nu c^\rho - \partial_\nu b_\rho \cdot c^\rho). \tag{3.39}$$

However, the BRS *Noether* charge is anomalous because in order to reduce it to (3.39), one has to use $\mathcal{T}_{\mu\nu} = 0$.

4. Nonuniqueness of the conformal anomaly

Now, we go back to the problem of the conformal anomaly. As discussed in Sec. 3, it is more natural to regard b_ρ, rather than $\tilde{b}_\rho$, as the primary field in the framework of the n-dimensional quantum Einstein gravity and therefore in its $n = 2$ case.

We first write the gauge-fixing plus FP-ghost part of (3.26):

$$\mathcal{L}_{\rm GF} + \mathcal{L}_{\rm FP} = -\tilde{g}^{\mu\nu}\partial_\mu b_\nu - i\tilde{g}^{\mu\nu}\partial_\mu \bar{c}_\sigma \cdot \partial_\nu c^\sigma. \tag{4.1}$$

We rewrite it by using (3.9) into

$$\mathcal{L}_{\rm GF} + \mathcal{L}_{\rm FP} \;=\; -\tilde{g}^{\mu\nu}\partial_\mu \tilde{b}_\nu - i\tilde{g}^{\mu\nu}\partial_\mu(\partial_\lambda \bar{c}_\nu \cdot c^\lambda) - i\tilde{g}^{\mu\nu}\partial_\mu \bar{c}_\sigma \cdot \partial_\nu c^\sigma. \tag{4.2}$$

Comparing (4.2) with (2.8) plus (2.9), we find that the difference between them is a total divergence

$$i\partial_\lambda(\tilde{g}^{\mu\nu}\partial_\mu \bar{c}_\nu \cdot c^\lambda). \tag{4.3}$$

Both are, therefore, equivalent in the usual sense. They are *not*, however, in the problem of the conformal anomaly.[7]

The free Lagrangian density which follows from (3.26) is

$$\begin{aligned}\mathcal{L}^{(0)} \;=\; & \frac{1}{2}\eta^{\mu\nu}\partial_\mu \phi^M \cdot \partial_\nu \phi_M \\ & +\partial_\mu b_\nu \cdot (\eta^{\mu\sigma}\eta^{\nu\tau}h_{\sigma\tau} - \frac{1}{2}\eta^{\mu\nu}\eta^{\sigma\tau}h_{\sigma\tau}) \\ & -i\eta^{\mu\nu}\partial_\mu \bar{c}_\sigma \cdot \partial_\nu c^\sigma,\end{aligned} \tag{4.4}$$

which should be compared with (2.12). The Feynman propagators implied by (4.4) are the same as those which follows from (2.12) if $\tilde{b}_\rho$ is replaced by b_ρ. More generally, we may consider[c]

$$\mathcal{L}^\alpha = \mathcal{L}^{(0)} - i\alpha\partial_\lambda(\eta^{\lambda\nu}\partial_\mu\bar{c}_\nu \cdot c^\mu - \eta^{\mu\nu}\partial_\mu\bar{c}_\nu \cdot c^\lambda), \tag{4.5}$$

α being an arbitrary constant. Of course, $\mathcal{L}^{\alpha=0} = \mathcal{L}^{(0)}$ and $\mathcal{L}^{\alpha=1}$ is identifiable with $\mathcal{L}^{(0)\,\mathrm{D}}$. In spite of the α independence of the action, the corresponding symmetric energy-momentum tensor $T^\alpha{}_{\mu\nu}$ is nontrivially α-dependent. Indeed,

$$\begin{aligned} iT^\alpha{}_{\mathrm{FP}\,\mu\nu} \;=\; & \partial_\mu\bar{c}_\sigma \cdot \partial_\nu c^\sigma + \frac{1}{2}\partial_\sigma[-\partial_\mu\bar{c}^\sigma \cdot c_\nu + \bar{c}^\sigma\partial_\mu c_\nu - \bar{c}_\nu\partial_\mu c^\sigma + \partial_\mu\bar{c}_\nu \cdot c^\sigma + (\mu \leftrightarrow \nu)] \\ & + \frac{1}{2}\alpha\partial_\sigma[\partial_\mu(\bar{c}^\sigma c_\nu + \bar{c}_\nu c^\sigma) - \partial^\sigma(\bar{c}_\mu c_\nu) + (\mu \leftrightarrow \nu)] \\ & + \eta_{\mu\nu}\text{-terms.} \end{aligned} \tag{4.6}$$

We calculate the coefficient of $\Phi_{\mu\nu\lambda\rho}$ as done in Sec. 2. Unfortunately, we generally encounter infrared divergence I arising from $\partial_\mu\partial_\nu\partial_\lambda\partial_\rho D_{\mathrm{F}} \cdot D_{\mathrm{F}}$.[d] We find that

$$\langle\, T^\alpha{}_{\mu\nu} T^\alpha{}_{\lambda\rho} \,\rangle = [D + 26 - 6 - 24\alpha - 22\alpha^2 + 4(1-\alpha^2)I]\Phi_{\mu\nu\lambda\rho} + \cdots. \tag{4.7}$$

The conformal anomaly is finite if $\alpha = 1$ or $\alpha = -1$. Anyway, the important fact is the *nonuniqueness* of the conformal anomaly. Though the nonuniqueness of the symmetric energy-momentum tensor is contrary to the common sense, one should note that $T^\alpha{}_{\mu\nu}$ is *not* an observable for any value of α.

Now, we discuss the compatibility between the above observation of the nonuniqueness and Kraemmer-Rebhan's gauge independence[4] of the conformal anomaly. As noted in Subsec. 2.2, Kraemmer and Rebhan employed perturbative approach from the outset, regarding $\tilde{b}_\rho$ as the primary field as a matter of course. Within such a perturbative framework, the conformal anomaly is unique and the critical dimension is $D = 26$. However, such a nonlinear transformation of Heisenberg fields as (3.9) is beyond their scope. *Field redefinition generally changes the division of the total Lagrangian density into the free part and the interaction one.* That is, given a theory, its perturbative expansion is *not* uniquely determined. Thus the conformal anomaly is dependent on the

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[c] Note that (4.5) contains no second-order derivatives.

[d] This fact was overlooked in Ref. 7. Eq. (27) of that paper is reproduced if I is replaced by 1/2.

choice of perturbation theory, as long as we define the conformal anomaly by using such a quantity as $T_{\mu\nu}$.

5. Takahashi's anomaly calculation and field-equation anomaly

Recently, Takahashi[8] has proposed a new approach to the problem of the critical dimension $D = 26$. In order to avoid the ambiguity problem stated in Sec. 4, $T_{\mu\nu}$ is not considered in his approach. Since his approach is very interesting, we first review it briefly.

He positively admits the transformation (3.9), i.e.,

$$\tilde{b}_\rho = -i\boldsymbol{\delta}_*(\bar{c}_\rho) = b_\rho + ic^\sigma\partial_\sigma\bar{c}_\rho. \tag{5.1}$$

In the theory defined by (3.26), he calculates

$$\begin{aligned}\langle\,\tilde{b}_\lambda(x)\tilde{b}_\rho(y)\,\rangle \;&=\; \langle\,b_\lambda(x)b_\rho(y)\,\rangle + i\langle\,b_\lambda(x)c^\sigma(y)\partial_\sigma\bar{c}_\rho(y)\,\rangle \\ &\quad + i\langle\,c^\sigma(x)\partial_\sigma\bar{c}_\lambda(x)\cdot b_\rho(y)\,\rangle \\ &\quad - \langle\,c^\sigma(x)\partial_\sigma\bar{c}_\lambda(x)\cdot c^\tau(y)\partial_\tau\bar{c}_\rho(y)\,\rangle\end{aligned} \tag{5.2}$$

perturbatively in one-loop approximation, which is actually exact. It should be noted that the perturbative orders of the first, second, third, and fourth terms are 2, 1, 1, and 0, respectively. By employing the lightcone-coordinate method (or dimensional regularization), he obtains

$$\mathcal{F}\langle\,\tilde{b}_\lambda(x)\tilde{b}_\rho(y)\,\rangle = \frac{D-26}{48\pi}\,\frac{ip_\lambda p_\rho}{p^2+i0} + \cdots, \tag{5.3}$$

where $\mathcal{F}$ denotes the Fourier transformation and dots represent divergent local terms proportional to $\eta_{\lambda\rho}$. Because $\tilde{b}_\lambda\tilde{b}_\rho$ is BRS-exact, $\langle\,\tilde{b}_\lambda\tilde{b}_\rho\,\rangle$ should vanish unless the BRS invariance is anomalous or spontaneously broken.

Takahashi[8] calculates the effective action by taking the inverse of the matrix formed by the two-point functions $\{\mathcal{F}\langle\,h_{\mu\nu}h_{\sigma\tau}\,\rangle = 0,\, \mathcal{F}\langle\,h_{\mu\nu}\tilde{b}_\rho\,\rangle,\, \mathcal{F}\langle\,\tilde{b}_\lambda h_{\sigma\tau}\,\rangle,\, \mathcal{F}\langle\,\tilde{b}_\lambda\tilde{b}_\rho\,\rangle\}$. It also contains the nonlocal terms proportional to $D-26$. He then recovers the BRS invariance by explicitly introducing the conformal degree of freedom. The nonlocal term is converted into the conformal anomaly. Indeed, he obtains the action of Polyakov's "induced" quantum gravity[27] at the one-loop level, though the theory is no longer exact at the one-loop level. Thus he claims that $D = 26$ is obtained without ambiguity.

It is very surprising that Takahashi has found the violation of the BRS invariance in the original framework of the two-dimensional quantum gravity. So far, nobody has claimed the violation of its BRS invariance. Kraemmer-Rebhan's gauge independence of the conformal anomaly was based on the BRS invariance. More importantly, as explained in Sec. 3, the exact solution to the de Donder-gauge two-dimensional quantum gravity is completely consistent with the BRS invariance. Takahashi's result is evidently inconsistent with our exact solution.

We have therefore investigated the reason for the discrepancy.[9] In order to concentrate our attention only to resolving this problem, we compare both results by modifying them into the ones on a background as common as possible, rather than respecting the original standpoints of his work and ours. Concretely, we have made the following.

1. We extract Takahashi's calculation of (5.3) only, neglecting his consideration on the effective action.

2. We reproduce our BRS-invariant result by means of perturbation theory.

3. All loop integrals are evaluated by using dimensional regularization.

We have found that the qualitative reason for the discrepancy is the ambiguity of the massless Feynman integrals themselves: Because of the presence of both ultraviolet and infrared divergences, there is no analytic domain of the complex dimension n. The quantitative reason for the discrepancy is explained in the following way.

The first term of (5.2), $\langle\, b_\lambda b_\rho \,\rangle$, is calculated in the second-order perturbation theory. The relevant Feynman diagram consists of two internal lines and two external lines. The loop, of course, consists of the former only. Likewise, $\langle\, b_\lambda \cdot c^\sigma \partial_\sigma \bar{c}_\rho \,\rangle$ and $\langle\, c^\sigma \partial_\sigma \bar{c}_\lambda \cdot b_\rho \,\rangle$ have one external line, in addition to a loop consisting of two internal lines. If one calculates the loop integrals by applying the dimensional method, *keeping the external lines strictly two-dimensional*, then one obtains Takahashi's result (5.3). On the other hand, if one does by applying the dimensional method to *all lines including external lines*, then one obtains the BRS-invariant result. Thus the discrepancy arises from whether the dimension of external lines is 2 or $n\,(\to 2)$.

The reason why the dimension of external lines is relevant is as follows. The propagator of an external line is a Fourier transform of (2.20), that is,

$$\frac{\eta_{\mu\rho}p_\nu + \eta_{\nu\rho}p_\mu - \eta_{\mu\nu}p_\rho}{p^2 + i0}. \tag{5.4}$$

From the loop-integral calculation, we obtain a term proportional to $\eta^{\mu\nu}$. Then the last term of (5.4) yields a trace of $\eta_{\mu\nu}$, which equals the dimension of the external line, i.e., 2 or n. Usually, since $\varepsilon \equiv n-2 \to 0$, the difference is trivial. In the present case, however, the loop integral consists of a convergent nonlocal term and divergent local terms. The term proportional to $\eta^{\mu\nu}$, which is one of the latter, is of order ε^{-1}. Hence a divergent local term becomes a finite nonlocal term if it is multiplied by $\varepsilon p_\rho/(p^2+i0)$. This explains the discrepancy in the evaluation of $\langle\, \tilde{b}_\lambda \tilde{b}_\rho\,\rangle$.

Takahashi asserts that the right way is to keep external lines strictly two-dimensional because the effective action, on which anomaly should be based, has no external lines. We cannot agree with him because the nonuniform application of the dimensional method violates gauge invariance. Although Takahashi asserts that his regularization method is consistent with the Ward-Takahashi identities for the effective action, their validity itself is *not* guaranteed if the Ward-Takahashi identity for the τ-functions (Green's functions) are violated. Indeed, it is logically impossible to obtain a BRS-violating result from a gauge-invariant regularization. We believe that it is natural to apply the dimensional method *to all lines equally so as to keep gauge invariance* and that it is not the right way to regard an anomalous term which is avoidable consistently as the anomaly. We thus conclude that the BRS invariance is *not* violated in the de Donder-gauge two-dimensional quantum gravity.

Then a question arises: Why has Takahashi[8] obtained $D=26$ precisely by his reasoning on the violation of the BRS invariance? To answer this, we reformulate Takahashi's calculation so as to be able to relate it to the discussion made in Sec. 2.

First, we assume that the BRS invariance is not violated. As discussed above, the crucial point arises from external lines. Moreover, external lines are encountered only when one considers higher-order perturbative corrections. That is, it is crucial in Takahashi's calculation to consider the term *linear* in quantum fields, which is always neglected in the discussions made in Sec. 2. We therefore eliminate the linear term b_ρ involved in $\tilde{b}_\rho$ by making use of the field equation (3.27). That is, we consider

$$\begin{aligned}
\tilde{\mathcal{T}}_{\mu\nu} &\equiv \mathcal{T}_{\mu\nu} + (\partial_\mu \tilde{b}_\nu + \partial_\nu \tilde{b}_\mu) \\
&= \partial_\mu \phi^M \cdot \partial_\nu \phi_M - i[\partial_\mu \bar{c}_\sigma \cdot \partial_\nu c^\sigma + \partial_\mu(\partial_\sigma \bar{c}_\nu \cdot c^\sigma) + (\mu \leftrightarrow \nu)] \\
&\quad + \tilde{g}_{\mu\nu}\tilde{g}^{\sigma\tau}\partial_\sigma b_\tau - \tilde{g}_{\mu\nu}\tilde{g}^{\sigma\tau}\Big(\frac{1}{2}\partial_\sigma \phi^M \cdot \partial_\tau \phi_M - i\partial_\sigma \bar{c}_\lambda \cdot \partial_\tau c^\lambda\Big)
\end{aligned} \qquad (5.5)$$

instead of $\tilde{b}_\nu$. We then substitute (2.11) into $\tilde{g}^{\mu\nu}$ and neglect higher-order terms:

$$\begin{aligned}\tilde{\mathcal{T}}_{\mu\nu} &= \partial_\mu\phi^M\cdot\partial_\nu\phi_M - i[\partial_\mu\bar{c}_\sigma\cdot\partial_\nu c^\sigma + \partial_\mu(\partial_\sigma\bar{c}_\nu\cdot c^\sigma) + (\mu\leftrightarrow\nu)] \\ &\quad + h_{\mu\nu}\partial^\sigma b_\sigma + \eta_{\mu\nu}\text{-terms.}\end{aligned} \tag{5.6}$$

This expression is precisely the same as $T^{\rm KRBB}{}_{\mu\nu}$ given in (2.40)! Therefore, as we already know, the nonlocal term of $\langle\,\tilde{\mathcal{T}}_{\mu\nu}(x)\tilde{\mathcal{T}}_{\lambda\rho}(y)\,\rangle$ is proportional to $D-26$. Thus Takahashi's calculation can be *reinterpreted* as the conformal anomaly of $\langle\,\tilde{\mathcal{T}}_{\mu\nu}\tilde{\mathcal{T}}_{\lambda\rho}\,\rangle$ in the *BRS-invariant* framework.

Since $\tilde{\mathcal{T}}_{\mu\nu}$ consists of $\mathcal{T}_{\mu\nu}=0$ and a BRS-exact quantity $\partial_\mu\tilde{b}_\nu+\partial_\nu\tilde{b}_\mu$, one may wonder why it can have the anomaly in the BRS-invariant framework. The reason is that the field equation $\mathcal{T}_{\mu\nu}=0$ suffers from the field-equation anomaly,[e] as explained at the end of Sec. 3. Explicitly, we have[f]

$$\langle\,\mathcal{T}_{\mu\nu}\,\mathcal{T}_{\lambda\rho}\,\rangle = (D+10)\Phi_{\mu\nu\lambda\rho}+\cdots, \tag{5.7}$$

$$\langle\,\mathcal{T}_{\mu\nu}(\partial_\lambda\tilde{b}_\rho+\partial_\rho\tilde{b}_\lambda)\,\rangle = -18\Phi_{\mu\nu\lambda\rho}+\cdots, \tag{5.8}$$

$$\langle\,(\partial_\mu\tilde{b}_\nu+\partial_\nu\tilde{b}_\mu)(\partial_\lambda\tilde{b}_\rho+\partial_\rho\tilde{b}_\lambda)\,\rangle = 0, \tag{5.9}$$

so that

$$\langle\,\tilde{\mathcal{T}}_{\mu\nu}\,\tilde{\mathcal{T}}_{\lambda\rho}\,\rangle = [D+10+2\times(-18)]\Phi_{\mu\nu\lambda\rho}+\cdots. \tag{5.10}$$

Thus we see that the field-equation anomaly underlies behind the conformal anomaly.

It should be noted further that by setting $D=26$, $\langle\,\tilde{\mathcal{T}}_{\mu\nu}\tilde{\mathcal{T}}_{\lambda\rho}\,\rangle$ becomes anomaly-free, but $\langle\,\tilde{\mathcal{T}}_{\mu\nu}\mathcal{O}\,\rangle$ generally remain anomalous. If, for example, the FP-ghost number current $j_c{}^\lambda$ is chosen as $\mathcal{O}$, the coefficient of the anomaly term of $\langle\,\tilde{\mathcal{T}}_{\mu\nu}j_c{}^\lambda\,\rangle$ is known to be -2 (independently of D).[4]

On the other hand, as is well known in the Kugo-Ojima formalism,[28] not only the BRS invariance but also the FP-ghost number conservation should not be violated so as to maintain the physical unitarity of the theory. As noted in Sec. 3, the FP-ghost number Q_c is *not* broken in the de Donder-gauge two-dimensional quantum gravity. *What is really anomalous is the field equation* $\mathcal{T}_{\mu\nu}=0$.

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[e] Therefore, those who want to believe neither BRS violation nor field-equation anomaly cannot explain the conformal anomaly of $\langle\,\tilde{\mathcal{T}}_{\mu\nu}\tilde{\mathcal{T}}_{\lambda\rho}\,\rangle$.

[f] Unfortunately, in Ref. 22, $D+8$ is given instead of $D+10$ in (5.7) because the contribution from $g_{\mu\nu}g^{\sigma\tau}\partial_\sigma b_\tau$ has been overlooked there.

To see this fact more clearly, we demonstrate the following proposition. *By using the field-equation anomaly we can eliminate the conformal anomaly of* $\langle\, T_{\mu\nu}\, T_{\lambda\rho}\,\rangle$ *considered in Sec. 2*, provided that no perturbative approximation is made at the Lagrangian level. The decomposition (2.33) is characterized by the requirements that $\hat{g}_{\mu\nu}$ is a c-number and that $h_{\mu\nu}$ is independent of $\hat{g}_{\mu\nu}$. Then, as pointed out in Sec. 2, $\delta/\delta\hat{g}_{\mu\nu}$ does *not* commute with the BRS transformation $\boldsymbol{\delta}_*$. But we can prove that

$$\left[\frac{\delta}{\delta\hat{g}_{\mu\nu}}-\frac{\delta}{\delta g_{\mu\nu}},\,\boldsymbol{\delta}_*\right]=0. \tag{5.11}$$

We *redefine* "energy-momentum tensor" by

$$T_{\mu\nu}{}'=-2(-\hat{g})^{-1/2}\hat{g}_{\mu\sigma}\hat{g}_{\nu\tau}\left(\frac{\delta}{\delta\hat{g}_{\sigma\tau}}-\frac{\delta}{\delta g_{\sigma\tau}}\right)\int d^2x\,\mathcal{L} \tag{5.12}$$

at the flat limit. The extra part is proportional to the field equation. Then $T_{\mathrm{GF}\,\mu\nu}{}'+T_{\mathrm{FP}\,\mu\nu}{}'$ is BRS-exact because of (5.11) and the BRS exactness of $\mathcal{L}_{\mathrm{GF}}+\mathcal{L}_{\mathrm{FP}}$. Furthermore, the main terms of $T_{\mathrm{S}\,\mu\nu}{}'$ are absent owing to cancellation. Hence apart from higher-order terms, $T_{\mu\nu}{}'$ is BRS-exact. Thus $\langle\, T_{\mu\nu}{}' T_{\lambda\rho}{}'\,\rangle$ has no conformal anomaly. More generally, for any BRS-invariant operator $\mathcal{O}$, $\langle\, T_{\mu\nu}{}'\,\mathcal{O}\,\rangle$ has no anomaly.

We thus find that various anomaly-like pathologies of the two-dimensional quantum gravity are originated from the field-equation anomaly, whose trouble we can bypass *without* adjusting the value of D.

6. Discussion

The two-dimensional quantum gravity has two faces: The one is the bosonic string theory, while the other is the $n=2$ version of quantum Einstein gravity. The former implies the existence of the critical dimension $D=26$, while the latter should not have such peculiar feature. Thus the two-dimensional quantum gravity is something like "Dr. Jekyll and Mr. Hyde". On the other hand, it is a mathematically well-defined object at least at the physicist's level of rigor. In the present paper, we have investigated how those paradoxical aspects are reconciled explicitly.

The de Donder gauge two-dimensional quantum gravity is exactly solvable in the Heisenberg picture. It is not only the $n=2$ case of the de Donder-gauge quantum Einstein gravity but also its zeroth-order approximation in the κ expansion apart from its dimensionality. The two-dimensional quantum gravity is a beautiful theory having many symmetry generators forming $IOSp(4+D,4)$, none of which is anomalous. In particular, the BRS invariance and the FP-ghost number conservation are unbroken.

However — a big "however" —, the two-dimensional quantum gravity has the field-equation anomaly, that is, the field equation (3.27), i.e.,

$$\mathcal{T}_{\mu\nu} = 0, \tag{6.1}$$

is violated at the level of the representation in terms of state vectors. Taking covariant derivative of (6.1) and using some other field equations, one obtains

$$\partial_\mu(\tilde{g}^{\mu\nu}\partial_\nu b_\rho) = 0. \tag{6.2}$$

Although both (6.1) and (6.2) have two degrees of freedom, it is impossible to reproduce (6.1) from (6.2) and other field equations. The healthy results stated above are the consequences of the framework obtained by replacing (6.1) by (6.2). Thus all the anomalous features of the two-dimensional quantum gravity are confined into (6.1) *modulo* (6.2). This part is totally foreign to quantum Einstein gravity.

Now, we proceed to seeing how the characteristic feature of the string theory arise from the two-dimensional quantum gravity. As we have seen in Sec. 5, the conformal anomaly, the FP-ghost number current anomaly, etc. are the consequence of the field-equation anomaly for (6.1). The key equation is

$$\begin{aligned}\tilde{\mathcal{T}}_{\mu\nu} &\equiv \mathcal{T}_{\mu\nu} - i\boldsymbol{\delta}_*(\partial_\mu\bar{c}_\nu + \partial_\nu\bar{c}_\mu) \\ &= T^{\mathrm{KRBB}}{}_{\mu\nu}\end{aligned} \tag{6.3}$$

apart from $\eta_{\mu\nu}$-terms and higher-order ones. Here $T^{\mathrm{KRBB}}{}_{\mu\nu}$, given in (2.40), is one of "energy-momentum tensors" derivable by the Kraemmer-Rebhan procedure. The two-point function of it gives the conformal anomaly proportional to $D-26$ according to the gauge independence of the conformal anomaly in the sense of Kraemmer and Rebhan.

The critical dimension $D=26$ is obtained as long as one follows the Kraemmer-Rebhan procedure, which is characterized by adopting $\tilde{b}_\rho$ as the primary field and by employing perturbative approach from the outset. As shown in Sec. 4, the two-point function of "energy-momentum tensor" $T_{\mu\nu}$ no longer gives $D=26$ if the above setting-up is abandoned. Thus the critical dimension $D=26$ is *not* the indispensable consequence of the two-dimensional quantum gravity, but a consequence of a *particular approach to it*. That is, the string theory is the two-dimensional quantum gravity *plus* *something* (choice of a particular perturbation theory). This consideration suggests that it is impossible to formulate the string theory covariantly in the Heisenberg picture.

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