WKB analysis of Painlevé transcendents with a large parameter. III. — Local reduction of 2-parameter Painlevé transcendents

Dedicated to Professor Mikio SATO on his seventieth birthday

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0 Introduction

This paper is the third of a series of articles on WKB analysis of Painlevé transcendents with a large parameter η, i.e., solutions of Painlevé equations (P_J) $(J = \mathrm{I}, \mathrm{II}, \cdots, \mathrm{VI})$ with a large parameter η, which are

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tabulated in Table 0.1 below:

Table 0.1 *Painlevé equations with a large parameter η.*

$$(P_{\mathrm{I}}) \qquad \frac{d^2\lambda}{dt^2} = \eta^2(6\lambda^2 + t),$$

$$(P_{\mathrm{II}}) \qquad \frac{d^2\lambda}{dt^2} = \eta^2(2\lambda^3 + t\lambda + c),$$

$$(P_{\mathrm{III}})$$
$$\frac{d^2\lambda}{dt^2} = \frac{1}{\lambda}\left(\frac{d\lambda}{dt}\right)^2 - \frac{1}{t}\frac{d\lambda}{dt} + 8\eta^2\left[2c_\infty\lambda^3 + \frac{c'_\infty}{t}\lambda^2 - \frac{c'_0}{t} - 2\frac{c_0}{\lambda}\right],$$

$$(P_{\mathrm{IV}})$$
$$\frac{d^2\lambda}{dt^2} = \frac{1}{2\lambda}\left(\frac{d\lambda}{dt}\right)^2 - \frac{2}{\lambda} + 2\eta^2\left[\frac{3}{4}\lambda^3 + 2t\lambda^2 + (t^2 + 4c_1)\lambda - \frac{4c_0}{\lambda}\right],$$

$$(P_{\mathrm{V}})$$
$$\begin{aligned}\frac{d^2\lambda}{dt^2} = {} & \left(\frac{1}{2\lambda} + \frac{1}{\lambda - 1}\right)\left(\frac{d\lambda}{dt}\right)^2 - \frac{1}{t}\frac{d\lambda}{dt} + \frac{(\lambda-1)^2}{t^2}\left(2\lambda - \frac{1}{2\lambda}\right) \\ & + \eta^2\frac{2\lambda(\lambda-1)^2}{t^2}\left[(c_0 + c_\infty) - c_0\frac{1}{\lambda^2} - c_2\frac{t}{(\lambda-1)^2} - c_1t^2\frac{\lambda+1}{(\lambda-1)^3}\right],\end{aligned}$$

$$(P_{\mathrm{VI}})$$
$$\begin{aligned}\frac{d^2\lambda}{dt^2} = {} & \frac{1}{2}\left(\frac{1}{\lambda} + \frac{1}{\lambda-1} + \frac{1}{\lambda-t}\right)\left(\frac{d\lambda}{dt}\right)^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{\lambda-t}\right)\frac{d\lambda}{dt} \\ & + \frac{2\lambda(\lambda-1)(\lambda-t)}{t^2(t-1)^2}\left[1 - \frac{\lambda^2 - 2t\lambda + t}{4\lambda^2(\lambda-1)^2}\right. \\ & \left. + \eta^2\left\{(c_0 + c_1 + c_t + c_\infty) - c_0\frac{t}{\lambda^2} + c_1\frac{t-1}{(\lambda-1)^2} - c_t\frac{t(t-1)}{(\lambda-t)^2}\right\}\right].\end{aligned}$$

Here and in what follows we use essentially the same symbols as those used in the preceding articles of this series ([KT1] and [AKT2]); they differ from those used in [KT1] and [AKT2] only in the following points: First, the constants, originally denoted by α_0 etc., appearing in several tables of the equations (e.g. Table 0.1 above) are denoted by c_0 etc. in this article so that they may be distinguished from the free parameters

contained in the 2-parameter formal solution $\lambda_J(t;\alpha,\beta)$ of (P_J). Second, we use the symbol $\lambda_J^{(0)}$ to denote the 0-parameter solution of (P_J) constructed in [KT1], which is denoted simply by λ_J there. [Note that the 0-parameter solution $\lambda_J^{(0)}(t)$ coincides with the 2-parameter solution $\lambda_J(t;\alpha,\beta)$ with $\alpha=\beta=0$.] Third, the function $\Phi_J(t,\eta)$ used here is different from that of [AKT2] in that it contains an extra η-dependence $2\alpha_0\beta_0\log\eta$. (Cf. (1.18).) This change of the notation makes the explicit form of the correspondence between parameters contained in $\lambda_{\rm I}$ and those in λ_J somewhat simpler. Now the principal purpose of this paper is to establish a reduction theorem (Theorem 4.1) for 2-parameter solutions $\lambda_J(t;\alpha,\beta)$, which are constructed in [AKT2]; the result is a natural generalization of the corresponding result for 0-parameter solutions $\lambda_J^{(0)}$ ([KT1], Theorem 2.3), and it validates our conjecture given in Section 4 of [AKT2]. For the reader's convenience we summarize in Section 1 some basic facts concerning the 2-parameter solutions. Our strategy of the proof of the main result given in Section 4 is to regard (P_J) as the condition that enables us to deform some relevant Schrödinger equation (SL_J) in an isomonodromic manner in the sense of [JMU]; the Schrödinger equation (SL_J) is tabulated in Table 0.2 and Table 0.3 below and the deformation equation (D_J) describing its isomonodromic deformation is given in the subsequent Table 0.4:

Table 0.2 *The Schrödinger equation (SL_J) that is relevant to (P_J).*

$$(SL_J)\qquad \left(-\frac{\partial^2}{\partial x^2}+\eta^2 Q_J(x,t,\eta)\right)\psi(x,t,\eta)=0,$$

where the potential Q_J is given below with K_J being listed in the subsequent Table 0.3.

$$Q_{\rm I}=4x^3+2tx+2K_{\rm I}-\eta^{-1}\frac{\nu}{x-\lambda}+\eta^{-2}\frac{3}{4(x-\lambda)^2},$$

$$Q_{\rm II}=x^4+tx^2+2cx+2K_{\rm II}-\eta^{-1}\frac{\nu}{x-\lambda}+\eta^{-2}\frac{3}{4(x-\lambda)^2},$$

$$Q_{\rm III}=\frac{c_0t^2}{x^4}+\frac{c_0't}{x^3}+\frac{c_\infty't}{x}+c_\infty t^2+\frac{tK_{\rm III}}{2x^2}$$

$$+\eta^{-1}\left(\frac{1}{2x^2}-\frac{1}{x(x-\lambda)}\right)\lambda\nu+\eta^{-2}\frac{3}{4(x-\lambda)^2},$$

$$Q_{\rm IV}=\frac{c_0}{x^2}+c_1+\left(\frac{x+2t}{4}\right)^2+\frac{K_{\rm IV}}{2x}-\eta^{-1}\frac{\lambda\nu}{x(x-\lambda)}+\eta^{-2}\frac{3}{4(x-\lambda)^2},$$

$$Q_{\rm V}=\frac{c_0}{x^2}+\frac{c_1t^2}{(x-1)^4}+\frac{c_2t}{(x-1)^3}+\frac{c_\infty}{(x-1)^2}+\frac{tK_{\rm V}}{x(x-1)^2}$$
$$-\eta^{-1}\frac{\lambda(\lambda-1)\nu}{x(x-1)(x-\lambda)}+\eta^{-2}\frac{3}{4(x-\lambda)^2},$$

$$Q_{\rm VI}=\frac{c_0}{x^2}+\frac{c_1}{(x-1)^2}+\frac{c_\infty}{x(x-1)}+\frac{c_t}{(x-t)^2}+\frac{t(t-1)K_{\rm VI}}{x(x-1)(x-t)}$$
$$-\eta^{-1}\frac{\lambda(\lambda-1)\nu}{x(x-1)(x-\lambda)}+\eta^{-2}\frac{3}{4(x-\lambda)^2}.$$

Table 0.3

$$K_{\rm I}=\frac{1}{2}[\nu^2-(4\lambda^3+2t\lambda)],$$

$$K_{\rm II}=\frac{1}{2}[\nu^2-(\lambda^4+t\lambda^2+2c\lambda)],$$

$$K_{\rm III}=\frac{2\lambda^2}{t}\left[\nu^2-\eta^{-1}\frac{3\nu}{2\lambda}-\left(\frac{c_0t^2}{\lambda^4}+\frac{c_0't}{\lambda^3}+\frac{c_\infty't}{\lambda}+c_\infty t^2\right)\right],$$

$$K_{\rm IV}=2\lambda\left[\nu^2-\eta^{-1}\frac{\nu}{\lambda}-\left(\frac{c_0}{\lambda^2}+c_1+\left(\frac{\lambda+2t}{4}\right)^2\right)\right],$$

$$K_{\rm V}=\frac{\lambda(\lambda-1)^2}{t}$$
$$\times\left[\nu^2-\eta^{-1}\left(\frac{1}{\lambda}+\frac{1}{\lambda-1}\right)\nu\right.$$
$$\left.-\left(\frac{c_0}{\lambda^2}+\frac{c_1t^2}{(\lambda-1)^4}+\frac{c_2t}{(\lambda-1)^3}+\frac{c_\infty}{(\lambda-1)^2}\right)\right],$$

$$K_{\rm VI}=\frac{\lambda(\lambda-1)(\lambda-t)}{t(t-1)}$$
$$\times\left[\nu^2-\eta^{-1}\left(\frac{1}{\lambda}+\frac{1}{\lambda-1}\right)\nu\right.$$

$$- \left(\frac{c_0}{\lambda^2} + \frac{c_1}{(\lambda-1)^2} + \frac{c_\infty}{\lambda(\lambda-1)} + \frac{c_t}{(\lambda-t)^2} \right) \Bigg].$$

Table 0.4 *Deformation equations.*

$$(D_J) \qquad \frac{\partial \psi}{\partial t} = A_J \frac{\partial \psi}{\partial x} - \frac{1}{2} \frac{\partial A_J}{\partial x} \psi,$$

where A_J denotes the function given below:

$$\begin{aligned} A_{\mathrm{I}} &= A_{\mathrm{II}} = \frac{1}{2(x-\lambda)}, \\ A_{\mathrm{III}} &= \frac{2\lambda x}{t(x-\lambda)} + \frac{x}{t}, \\ A_{\mathrm{IV}} &= \frac{2x}{x-\lambda}, \\ A_{\mathrm{V}} &= \frac{(\lambda-1)x(x-1)}{t(x-\lambda)}, \\ A_{\mathrm{VI}} &= \frac{(\lambda-t)x(x-1)}{t(t-1)(x-\lambda)}. \end{aligned}$$

To be more concrete, we first define another formal series ν_J so that (λ_J, ν_J) solves the following Painlevé Hamiltonian system (H_J), a compatibility condition of (SL_J) and (D_J), defined by the aid of K_J given above:

$$(H_J) \qquad \begin{cases} \dfrac{d\lambda}{dt} = \eta \dfrac{\partial K_J}{\partial \nu} \\ \\ \dfrac{d\nu}{dt} = -\eta \dfrac{\partial K_J}{\partial \lambda}, \end{cases}$$

and then we substitute (λ_J, ν_J) into (λ, ν) in Q_J to establish some WKB-theoretic local reduction of (SL_J) to (SL_{I}). Interestingly enough, the transformation that establishes this local equivalence between the Schrödinger equations tells us how to relate λ_J and λ_{I}. (See Section 4 for the precise statement.) One important ingredient in our proof is the

analysis of the canonical form of (SL_J) near its double turning point and its isomonodromic deformation. (Section 2 and Section 3.) We hope our main results given in Section 4 will turn out to be useful to describe the connection formula for the Borel resummed formal solution $\lambda_J^{(0)}$. (See [T2] for some concrete results when $J = \mathrm{I}$.)

A resumé of this paper is reported by [KT2].

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1 Basic facts about $\lambda_J(t;\alpha,\beta)$

The 2-parameter solution $\lambda_J(t;\alpha,\beta)$ of (P_J), with which we are concerned in this article, is constructed by the so-called multiple-scale analysis. The explicit scheme for the construction is given in [AKT2], and here we list up its basic properties without discussing how to construct it. (See e.g. [BO] for basic facts in multiple-scale analysis. As far as we know, [JK1] and [JK2] are the first that effectively employed multiple-scale analysis in analyzing the Painlevé transcendents.) In order to describe its concrete form let us first fix some notations.

As is clear from Table 0.1, each (P_J) has the following structure in common:

$$\frac{d^2\lambda}{dt^2} = G_J(\lambda, \frac{d\lambda}{dt}, t) + \eta^2 F_J(\lambda, t), \tag{1.1}$$

where G_J is a polynomial in $d\lambda/dt$ with degree equal to or at most 2. We then fix a holomorphic function $\lambda_0(t)$ that satisfies

$$F_J(\lambda_0(t), t) = 0 \tag{1.2}$$

near a generic but fixed point t_*. (See Poposition 1.1 of [KT1] for the precise condition which t_* is to satisfy.) Let us now introduce some more notations relevant to λ_0; to simplify the notations we often omit the subscript J, like F instead of F_J, and use the symbol μ to denote

$d\lambda/dt$.

$$F^{(j)} = \frac{\partial^j F}{\partial \lambda^j}(\lambda_0(t), t), \tag{1.3}$$

$$G^{(j,k)} = \frac{\partial^{j+k} G}{\partial \lambda^j \partial \mu^k}(\lambda_0(t), \frac{d\lambda_0(t)}{dt}, t), \tag{1.4}$$

$$\begin{aligned} \mathcal{C}_J &= \frac{5}{6}\left(\frac{F^{(2)}}{F^{(1)}}\right)^2 - \frac{5}{6}\frac{F^{(2)}}{F^{(1)}}G^{(0,2)} \\ &\quad + \frac{1}{3}(G^{(0,2)})^2 - \frac{1}{2}\frac{F^{(3)}}{F^{(1)}} + \frac{1}{2}G^{(1,2)}, \end{aligned} \tag{1.5}$$

$$\phi_J = \int_r^t \sqrt{F^{(1)}} ds, \tag{1.6}$$

where r is a simple turning point for λ_0 in the sense of Definition 2.1 (i) of [KT1]. (Although the wording "for λ_J" is used there, the definition depends only on λ_0.)

$$\theta_J = \exp\{-\frac{1}{2}\int^t \exp(\int^u G^{(0,1)} dv)\mathcal{C}_J(u) du\}. \tag{1.7}$$

As the definition of θ_J contains two indefinite integrals, we should make its normalization precise. Here we employ the following normalization which we hope to be most useful in presenting several formulas in a neat and unified manner (see [T3] for concrete evidences):

To normalize the integral

$$g(u) = \exp(\int^u G^{(0,1)} dv), \tag{1.8}$$

let us note the following equality:

$$G^{(0,1)} = -\frac{\dfrac{dC_J(\lambda_0(u), u)}{du}}{C_J(\lambda_0(u), u)}, \tag{1.9}$$

where $C_J(\lambda, t)$ is a rational function introduced in [KT1] in a completely different context; for the reader's convenience we tabulate them below.

Table 1.1

$$\begin{aligned} C_{\mathrm{I}} &= C_{\mathrm{II}} = 1, \\ C_{\mathrm{III}} &= \frac{t}{4\lambda^2}, \\ C_{\mathrm{IV}} &= \frac{1}{4\lambda}, \\ C_{\mathrm{V}} &= \frac{t}{2\lambda(\lambda-1)^2}, \\ C_{\mathrm{VI}} &= \frac{t(t-1)}{2\lambda(\lambda-1)(\lambda-t)}. \end{aligned}$$

Hence we normalize $g(u)$ so that it coincides with $1/C_J(\lambda_0(t),t)$. To normalize θ_J completely we first define a function $R_J(x,t)$ by

$$Q_{J,0}(x,t) = (x-\lambda_0)^2 C_J(x,t)^2 R_J(x,t), \tag{1.10}$$

where $Q_{J,0}$ denotes the degree 0 (in η) part of Q_J with $(\lambda,\nu) = (\lambda_0, 0)$. As one can readily verify, R_J is a quadratic polynomial in x for $J = \mathrm{II}, \cdots, \mathrm{VI}$. (When $J = \mathrm{I}, R_{\mathrm{I}}$ reduces to a linear function $4(x+2\lambda_0)$.) Next we introduce another function $D_J(t)$ to denote a fixed constant multiple of the discriminant of R_J regarded as a function of x. We conventionally choose the constant factor to be 1/16 and define D_{I} to be 1. Then it is known ([T3]) that the following holds:

$$\frac{\dfrac{d\theta_J(t)}{dt}}{\theta_J(t)} = \frac{5\dfrac{dF_J^{(1)}}{dt}}{F_J^{(1)}} + \frac{2\dfrac{dC_J(\lambda_0(t),t)}{dt}}{C_J(\lambda_0(t),t)} - \frac{2\dfrac{dD_J}{dt}}{D_J}, \tag{1.11}$$

that is, θ_J is a constant multiple of

$$F_J^{(1)5} C_J(\lambda_0(t),t)^2 D_J^{-2}. \tag{1.12}$$

We finally choose the constant factor to be 1, and this fixes a normalization of θ_J.

Again for the convenience of the reader we list up the function $D_J(t)$ below; we describe $D_J(t)$ with explicit use of λ_0, and hence the presentation is not unique.

Table 1.2

$$
\begin{aligned}
D_{\mathrm{I}}(t) &= 1, \\
D_{\mathrm{II}}(t) &= -\frac{1}{4}(2\lambda_0^2 + t), \\
D_{\mathrm{III}}(t) &= 16\left(\frac{2c_\infty c'_\infty}{t}\lambda_0 + \frac{c'^2_\infty}{t^2} + \frac{2c_\infty c'_0}{t\lambda_0}\right), \\
D_{\mathrm{IV}}(t) &= -\frac{1}{2}(\lambda_0^2 + 2t\lambda_0 + 8c_1), \\
D_{\mathrm{V}}(t) &= \frac{1}{t^2}\left(\frac{c_2^2}{(\lambda_0-1)^4} - \frac{4c_0c_1}{\lambda_0^2(\lambda_0-1)^2} + \frac{4c_1c_2t}{(\lambda_0-1)^5} + \frac{4c_1^2t^2}{(\lambda_0-1)^6}\right), \\
D_{\mathrm{VI}}(t) &= \frac{1}{t^2(t-1)^2}\left(\frac{c_0^2}{\lambda_0^4} + \frac{c_1^2}{(\lambda_0-1)^4} + \frac{c_t^2}{(\lambda_0-t)^4} - \frac{2c_0c_1}{\lambda_0^2(\lambda_0-1)^2}\right. \\
&\quad \left. - \frac{2c_0c_t}{\lambda_0^2(\lambda_0-t)^2} - \frac{2c_1c_t}{(\lambda_0-1)^2(\lambda_0-t)^2}\right).
\end{aligned}
$$

Remark 1.1 When $D_J(t) = 0$, two simple turning points merge at the point t.

The 2-parameter solution $\lambda_J(t;\alpha,\beta)$ constructed by the multiple-scale analysis in [AKT2] is now expressed in the following form:

$$\lambda_0 + \eta^{-1/2}\Lambda(t,\eta), \tag{1.13}$$

where $\Lambda(t,\eta)$ has the formal expansion (in $\eta^{-1/2}$)

$$\sum_{j=0}^{\infty} \eta^{-j/2}\Lambda_{j/2}(t,\eta) \tag{1.14}$$

and the precise description of the η-dependence of each coefficient $\Lambda_{j/2}(t,\eta)$ is given as follows:

$$\Lambda_{j/2}(t,\eta) = \sum_{k=0}^{j+1} b^{(j/2)}_{j+1-2k}(t)\exp((j+1-2k)\Phi_J(t,\eta)), \tag{1.15}$$

where

(1.16) $b_1^{(0)}(t) = \dfrac{\alpha_0}{\sqrt[4]{F^{(1)}C_J(\lambda_0(t),t)^2}}$ for some complex number α_0,

(1.17) $b_{-1}^{(0)}(t) = \dfrac{\beta_0}{\sqrt[4]{F^{(1)}C_J(\lambda_0(t),t)^2}}$ for some complex number β_0,

(1.18) $\Phi_J(t,\eta) = \phi_J(t)\eta + \alpha_0\beta_0 \log(\theta_J(t)\eta^2)$,

(1.19) $b_l^{(j/2)}(t)(l \neq \pm 1)$ is free from η, and it is determined algebraically and uniquely by $\{b_{j'+1-2k}^{(j'/2)}(t)\}_{j'<j, 0\leq k\leq j'+1}$ in a recursive manner,

(1.20) $b_{\pm 1}^{(j/2)}(t) = 0$ for an odd integer j, while for a positive even integer $j = 2n$ $(1 \leq n$: integer$)$ $b_{\pm 1}^{(j/2)}(t) = b_{\pm 1}^{(n)}(t)$ is an analytic function of t that satisfies the following differential equation:

$$\left[\frac{d}{dt} + \frac{\frac{d\sqrt{F^{(1)}}}{dt}}{2\sqrt{F^{(1)}}} + \frac{\frac{dC_J(\lambda_0(t),t)}{dt}}{2C_J(\lambda_0(t),t)} - \begin{pmatrix} \alpha_0\beta_0 & \alpha_0^2 \\ -\beta_0^2 & -\alpha_0\beta_0 \end{pmatrix} \frac{\frac{d\theta_J}{dt}}{\theta_J}\right] \begin{pmatrix} b_1^{(n)} \\ b_{-1}^{(n)} \end{pmatrix} = R_n,$$

where R_n is determined by $\{b_{j'+1-2k}^{(j'/2)}\}_{j'<j=2n, 0\leq k\leq j'+1}$ (together with $F^{(p)}$ and $G^{(p,q)}$).

Remark 1.2 Since $G^{(0,1)}$ coincides with $-\dfrac{dC_J(\lambda_0(u),u)}{du}/C_J(\lambda_0(u),u)$ as we have noted, the expressions (1.16) and (1.17) for $b_{\pm 1}^{(0)}(t)$ immediately follows from (1.25) in [AKT2]. We also find that the differential equation in (1.20) is exactly the same as the equation (1.28) in [AKT2].

Summing up, the η-dependence of $\Lambda_{j/2}(t,\eta)$ is only through the so-

called l-instanton terms $\exp(l\Phi_J(t,\eta))$ and each $\Lambda_{j/2}(t,\eta)$ contains only instanton terms with "alternating parity", namely

(1.21)

$\Lambda_{j/2}$ consists of $(j+1-2k)$-instanton terms with $0 \le k \le j+1$.

In other words, if we define a "change of parity" operator ι by

$$\iota(\eta^{-k/2}) = (-1)^k \eta^{-k/2},\ \iota(\exp(l\Phi_J)) = (-1)^l \exp(l\Phi_J) \quad (k,l = 0, \pm 1, \pm 2, \cdots), \tag{1.22}$$

then we find $\iota(\Lambda_{j/2}(t,\eta)) = (-1)^{j+1}\Lambda_{j/2}(t,\eta)$; consequently we have

$$\iota(\Lambda(t,\eta)) = -\Lambda(t,\eta), \tag{1.23}$$

$$\iota(\lambda_J(t;\alpha,\beta)) = \lambda_J(t,\alpha,\beta). \tag{1.24}$$

Note that instanton terms are regarded to be of degree 0 in η, although differentiation in t generates a term with the degree in η enhanced by 1 through multiplication by $\phi'(t)\eta$. We also note that $b_{\pm 1}^{(n)}(t)(n \ge 1)$ contains 2 free parameters, which we denote by (α_n, β_n). In order to see how $b_{\pm 1}^{(n)}(t)$ contains these free parameters we consider the equation in (1.20) with $R_n = 0$, that is,

$$\left[\frac{d}{dt} + \frac{\frac{d\sqrt{F^{(1)}}}{dt}}{2\sqrt{F^{(1)}}} + \frac{\frac{dC_J(\lambda_0(t),t)}{dt}}{2C_J(\lambda_0(t),t)} - \begin{pmatrix} \alpha_0\beta_0 & \alpha_0^2 \\ -\beta_0^2 & -\alpha_0\beta_0 \end{pmatrix} \frac{\frac{d\theta_J}{dt}}{\theta_J}\right] \begin{pmatrix} \widetilde{b}_1^{(n)} \\ \widetilde{b}_{-1}^{(n)} \end{pmatrix} = 0. \tag{1.25}$$

It is then clear that any solution of (1.25) has the form

$$(F^{(1)})^{-\frac{1}{4}}(C_J(\lambda_0(t),t))^{-\frac{1}{2}}\theta_J^{\begin{pmatrix} \alpha_0\beta_0 & \alpha_0^2 \\ -\beta_0^2 & -\alpha_0\beta_0 \end{pmatrix}} \begin{pmatrix} \alpha_n \\ \beta_n \end{pmatrix} \tag{1.26}$$

for some complex numbers α_n and β_n. We also note

$$\theta_J^{\begin{pmatrix} \alpha_0\beta_0 & \alpha_0^2 \\ -\beta_0^2 & -\alpha_0\beta_0 \end{pmatrix}} = 1 + \begin{pmatrix} \alpha_0 \\ -\beta_0 \end{pmatrix} (\beta_0\ \alpha_0) \log\theta_J. \tag{1.27}$$

Hence a solution of (1.25) is a sum of two terms, one of which contains free parameters α_n and β_n in a similar way as $b_1^{(0)}$ and $b_{-1}^{(0)}$ do, and the other of which depends on α_n and β_n through their combination $(\alpha_0\beta_n + \beta_0\alpha_n)$. This is the way how $\Lambda_{j/2} = \Lambda_n (j = 2n)$ depends on 2 free parameters (α_n, β_n).

Thus the precise meaning of "2-parameter (solution)" is "2-infinite series", reflecting the fact that (P_J) contains a large parameter η. As $\Lambda = \sum_{j\geq 0} \eta^{-j/2}\Lambda_{j/2}$, we sometimes use symbols α and β to designate $\sum_{n\geq 0} \alpha_n \eta^{-n}$ and $\sum_{n\geq 0} \beta_n \eta^{-n}$ respectively. (Note that free parameters α_n and β_n are allowed only when n is an integer.) In order to avoid some degeneracy, we assume the following condition (1.28) throughout this paper:

(1.28) $$\alpha_0\beta_0 \neq 0.$$

Although "one-parameter" case, i.e., the situation where either α or β vanishes, is also important, we omit the discussion of the case; the treatment of one-parameter case is similar, but much simpler. Our analysis in this paper uses these two parameters (α, β) to label a solution of (P_J); our goal is to establish a correspondence between $\widetilde{\lambda}_J(\widetilde{t}; \widetilde{\alpha}, \widetilde{\beta})$ and $\lambda_{\mathrm{I}}(t; \alpha, \beta)$ by suitably relating $(\widetilde{\alpha}, \widetilde{\beta})$ and (α, β). Furthermore our hope is to describe the connection formula for the Painlevé transcendents by making use of the correspondence. (Cf. [T1] and [T2].)

Remark 1.3 So far we have been concerned with the construction of $\lambda_J(t; \alpha, \beta)$; once λ_J is constructed, another function $\nu_J(t; \alpha, \beta)$ can be automatically found in the following form by the Painlevé Hamiltonian system (H_J) given in Section 0:

(1.29) $$\nu_J = \eta^{-1/2}\mathcal{N}(t, \eta)$$

with

(1.30) $$\mathcal{N} = \sum_{j\geq 0} \eta^{-j/2}\mathcal{N}_{j/2}(t, \eta),$$

where the instanton structure of $\mathcal{N}_{j/2}$ is the same as that of $\Lambda_{j/2}$, namely $\mathcal{N}_{j/2}$ is of the following form

$$\sum_{k=0}^{j+1} c_{j+1-2k}^{(j/2)}(t)\exp((j+1-2k)\Phi_J(t,\eta)). \tag{1.31}$$

Using the "change of parity" operator ι defined by (1.22), we can express the instanton structure of $\mathcal{N}$ and ν_J also in the following manner:

$$\iota(\mathcal{N}(t,\eta)) = -\mathcal{N}(t,\eta), \tag{1.32}$$

$$\iota(\nu_J(t,\alpha,\beta)) = \nu_J(t,\alpha,\beta). \tag{1.33}$$

To prove these properties of $\mathcal{N}$ and ν_J, it suffices to look at the first equation of (H_J). As a matter of fact, if we define a rational function $B_J(x,\lambda,t)$ by

$$B_J = (x-\lambda)A_J, \tag{1.34}$$

we find

$$\frac{\partial K_J}{\partial \nu} = 2B_J(\lambda,\lambda,t)\nu - \eta^{-1}\frac{\partial B_J}{\partial x}(\lambda,\lambda,t). \tag{1.35}$$

(We note that [KT1] uses the symbol B_J to denote $-\frac{1}{2}\frac{\partial}{\partial x}A_J$ and that another symbol a_J is used to denote $(x-\lambda)A_J$; in this article we spare the symbol a_J for denoting a simple turning point of (SL_J).) Since the proof of (1.35) is a straightforward one, we omit it here and just list up $\partial K_J/\partial\nu$ and B_J for the convenience of the reader:

$$\frac{\partial K_{\mathrm{I}}}{\partial \nu} = \frac{\partial K_{\mathrm{II}}}{\partial \nu} = \nu, \tag{1.36}$$

$$\frac{\partial K_{\mathrm{III}}}{\partial \nu} = \frac{4\lambda^2\nu}{t} - \frac{3\eta^{-1}\lambda}{t}, \tag{1.37}$$

$$\frac{\partial K_{\mathrm{IV}}}{\partial \nu} = 4\lambda\nu - 2\eta^{-1}, \tag{1.38}$$

$$\frac{\partial K_{\mathrm{V}}}{\partial \nu} = \frac{2\lambda(\lambda-1)^2\nu}{t} - \frac{\eta^{-1}(\lambda-1)(2\lambda-1)}{t}, \tag{1.39}$$

$$\frac{\partial K_{\mathrm{VI}}}{\partial \nu} = \frac{2\lambda(\lambda-1)(\lambda-t)\nu}{t(t-1)} - \frac{\eta^{-1}(\lambda-t)(2\lambda-1)}{t(t-1)}. \tag{1.40}$$

Table 1.3

$$\begin{aligned} B_{\mathrm{I}} &= B_{\mathrm{II}} = \frac{1}{2}, \\ B_{\mathrm{III}} &= \frac{x(x+\lambda)}{t}, \\ B_{\mathrm{IV}} &= 2x, \\ B_{\mathrm{V}} &= \frac{x(x-1)(\lambda-1)}{t}, \\ B_{\mathrm{VI}} &= \frac{x(x-1)(\lambda-t)}{t(t-1)}. \end{aligned}$$

We also note that the following relation holds:

$$2B_J(\lambda,\lambda,t)C_J(\lambda,t) = 1, \tag{1.41}$$

where $C_J(\lambda,t)$ is the rational function tabulated in Table 1.1. Hence it immediately follows from (1.35), (1.41) and the first equation of (H_J) that

$$\begin{aligned} \nu_J(t;\alpha,\beta) &= \eta^{-1}C_J(\lambda_J(t;\alpha,\beta),t)\times \\ &\quad \left\{\frac{d}{dt}\lambda_J(t;\alpha,\beta) + \frac{\partial B_J}{\partial x}(\lambda,\lambda,t)|_{\lambda=\lambda_J(t;\alpha,\beta)}\right\}. \end{aligned} \tag{1.42}$$

The structure of $\mathcal{N}$ and ν_J mentioned above can be verified by this expression (1.42).

2 Equation (*Can*) and its deformation

As we have explained in Section 0, the correspondence between $\widetilde{\lambda}_J(\widetilde{t};\widetilde{\alpha},\widetilde{\beta})$ and $\lambda_{\mathrm{I}}(t;\alpha,\beta)$ we are seeking for will be obtained by a transformation between the relevant Schrödinger equations (SL_J) and (SL_{I}). In this section we discuss the canonical form (*Can*) of (SL_J) and its isomonodromic deformation which we will use in establishing the transformation between (SL_J) and (SL_{I}) mentioned above. To distinguish variables and functions relevant to (SL_J) and (*Can*), we put the symbol $\sim$ over those relevant to (SL_J) in what follows.

Like in Section 1, we choose and fix a generic point $\widetilde{t}_*$ which satisfies the conditions in Proposition 1.1 of [KT1]. As we indicated in Section 1, multiple-scale analysis enables us to construct a 2-parameter solution $(\widetilde{\lambda}_J(\widetilde{t};\widetilde{\alpha},\widetilde{\beta}), \widetilde{\nu}_J(\widetilde{t};\widetilde{\alpha},\widetilde{\beta}))$ of (H_J) on a neighborhood $\widetilde{V}$ of $\widetilde{t}_*$. (See [AKT 2, Section 1] for details.) Substituting the 2-parameter solution into $(\widetilde{\lambda},\widetilde{\nu})$ in the coefficients of $\widetilde{Q}_J(\widetilde{x},\widetilde{t},\eta)$, we find that on a neighborhood of $\widetilde{x} = \widetilde{\lambda}^{(0)}_{J,0}(\widetilde{t})$ (SL_J) is transformed to a Schrödinger equation

$$\left(-\frac{\partial^2}{\partial x^2} + \eta^2 Q_{\mathrm{can}}(x,\widetilde{t},\eta)\right)\psi(x,\widetilde{t},\eta) = 0 \tag{2.1}$$

with the potential Q_{can} of the following form:

$$\begin{aligned} Q_{\mathrm{can}}(x,\widetilde{t},\eta) &= Q_{\mathrm{can}}(x;\widetilde{E}_J(\widetilde{t},\eta),\widetilde{\rho}_J(\widetilde{t},\eta),\widetilde{\sigma}_J(\widetilde{t},\eta),\eta) \\ &= 4x^2 + \eta^{-1}\widetilde{E}_J(\widetilde{t},\eta) \\ &\quad + \frac{\eta^{-3/2}\widetilde{\rho}_J(\widetilde{t},\eta)}{x - \eta^{-1/2}\widetilde{\sigma}_J(\widetilde{t},\eta)} + \frac{3\eta^{-2}}{4(x-\eta^{-1/2}\widetilde{\sigma}_J(\widetilde{t},\eta))^2} \end{aligned} \tag{2.2}$$

where $\widetilde{E}_J, \widetilde{\rho}_J$ and $\widetilde{\sigma}_J$ are formal series of $\eta^{-1/2}$ with holomorphic coefficients defined on $\widetilde{V}$ that may contain instanton terms $\exp(l\widetilde{\Phi}_J(\widetilde{t},\eta))$ to be regarded as terms of degree 0 in η. ([AKT 2, Theorem 3.1])

Let us describe how $\widetilde{E}_J, \widetilde{\rho}_J$ and $\widetilde{\sigma}_J$ are determined by the transformation. In what follows we abbreviate $\widetilde{\lambda}^{(0)}_{J,0}(\widetilde{t})$ to $\widetilde{\lambda}_0(\widetilde{t})$.

Theorem 2.1 *([AKT 2, Theorem 3.1]) Let $\widetilde{V}$ be a sufficiently small neighborhood of $\widetilde{t}_*$. Then there exist a neighborhood $\widetilde{U}$ of $\widetilde{x} = \widetilde{\lambda}_0(\widetilde{t}_*)$ and a formal series*

$$\begin{aligned} x_J(\widetilde{x},\widetilde{t},\eta) &= x_{J,0}(\widetilde{x},\widetilde{t},\eta) + x_{J,1/2}(\widetilde{x},\widetilde{t},\eta)\eta^{-1/2} \\ &\quad + x_{J,1}(\widetilde{x},\widetilde{t},\eta)\eta^{-1} + \cdots, \end{aligned} \tag{2.3}$$

whose coefficients $x_{J,j/2}(\widetilde{x},\widetilde{t},\eta)$ are holomorphic on $\widetilde{U}\times\widetilde{V}$ so that the

following hold:

(2.4) *The η-dependence of $x_{J,j/2}(\widetilde{x},\widetilde{t},\eta)$ $(j \geq 2)$ is only through instanton terms $\exp(l\widetilde{\Phi}_J(\widetilde{t},\eta))$ $(l = j-2-2k$ with $0 \leq k \leq j-2)$ that appear in the 2-parameter multiple-scale solution $(\widetilde{\lambda}_J(\widetilde{t};\widetilde{\alpha},\widetilde{\beta}), \widetilde{\nu}_J(\widetilde{t};\widetilde{\alpha},\widetilde{\beta}))$ of (H_J),*

(2.5) *$x_{J,0}$ contains no instanton terms, that is, $x_{J,0}$ is free from η,*

(2.6) $\dfrac{\partial x_{J,0}}{\partial \widetilde{x}}$ *never vanishes on* $\widetilde{U} \times \widetilde{V}$,

(2.7) $x_{J,0}(\widetilde{\lambda}_0(\widetilde{t}),\widetilde{t}) = 0$,

(2.8) $x_{J,1/2}$ *identically vanishes,*

(2.9)
$$\begin{aligned}\widetilde{Q}_J(\widetilde{x},\widetilde{t},\eta) &= \left(\frac{\partial x_J}{\partial \widetilde{x}}\right)^2 Q_{\mathrm{can}}(x_J(\widetilde{x},\widetilde{t},\eta);\widetilde{E}_J(\widetilde{t},\eta),\widetilde{\rho}_J(\widetilde{t},\eta),\widetilde{\sigma}_J(\widetilde{t},\eta),\eta) \\ &\quad -\frac{1}{2}\eta^{-2}\{x_J(\widetilde{x},\widetilde{t},\eta);\widetilde{x}\},\end{aligned}$$

where

(2.10) $$\widetilde{\sigma}_J(\widetilde{t},\eta) = \eta^{1/2}x_J(\widetilde{\lambda}_J(\widetilde{t},\eta),\widetilde{t},\eta),$$

(2.11) $$\widetilde{\rho}_J(\widetilde{t},\eta) = -\frac{\eta^{1/2}\widetilde{\nu}_J(\widetilde{t},\eta)}{\dfrac{\partial x_J}{\partial \widetilde{x}}(\widetilde{\lambda}_J(\widetilde{t},\eta),\widetilde{t},\eta)} - \frac{3\eta^{-1/2}\dfrac{\partial^2 x_J}{\partial \widetilde{x}^2}(\widetilde{\lambda}_J(\widetilde{t},\eta),\widetilde{t},\eta)}{4\left(\dfrac{\partial x_J}{\partial \widetilde{x}}(\widetilde{\lambda}_J(\widetilde{t},\eta),\widetilde{t},\eta)\right)^2},$$

(2.12) $$\widetilde{E}_J(\widetilde{t},\eta) = \widetilde{\rho}_J(\widetilde{t},\eta)^2 - 4\widetilde{\sigma}_J(\widetilde{t},\eta)^2.$$

Here $\{x_J;\widetilde{x}\}$ denotes the Schwarzian derivative, i.e.,

$$\{x_J;\widetilde{x}\} = \frac{\dfrac{\partial^3 x_J}{\partial \widetilde{x}}}{\dfrac{\partial x_J}{\partial \widetilde{x}}} - \frac{3}{2}\left(\frac{\dfrac{\partial^2 x_J}{\partial \widetilde{x}^2}}{\dfrac{\partial x_J}{\partial \widetilde{x}}}\right)^2.$$

We refer the reader to Section 3 of [AKT2] for the proof of Theorem 2.1. We note that $x_J(\widetilde{\lambda}_J(\widetilde{t},\eta),\widetilde{t},\eta)$ is of degree equal to or less

than $-1/2$ with respect to η because of (2.7). Similarly $\widetilde{\nu}_J$ is of degree equal to or less than $-1/2$ with respect to η by its definition (cf. Remark 1.3). Thus both $\widetilde{\sigma}_J$ and $\widetilde{\rho}_J$ are of degree 0 with respect to η in spite of their appearance. We also note that the following instanton structure of $\widetilde{\sigma}_J$ and $\widetilde{\rho}_J$ are found for the multiple-scale solution $(\widetilde{\lambda}_J, \widetilde{\nu}_J)$ constructed in [AKT2, Section 1]:

(2.13) $\widetilde{\sigma}_{J,j/2}(\widetilde{t},\eta)$ and $\widetilde{\rho}_{J,j/2}(\widetilde{t},\eta)$ cosist of $((j+1)-2k)$-instanton terms with $0 \le k \le j+1$,

that is,

$$\iota(\widetilde{\sigma}_J(\widetilde{t},\eta)) = -\widetilde{\sigma}_J(\widetilde{t},\eta), \tag{2.14}$$

$$\iota(\widetilde{\rho}_J(\widetilde{t},\eta)) = -\widetilde{\rho}_J(\widetilde{t},\eta). \tag{2.15}$$

As a matter of fact, the relations (2.14) and (2.15) together with

$$\iota(x_J(\widetilde{x},\widetilde{t},\eta)) = x_J(\widetilde{x},\widetilde{t},\eta) \tag{2.16}$$

can be proved by the induction argument employed in the proof of Theorem 3.1 of [AKT2].

Relations (2.9) and (2.10) strongly indicate that $\widetilde{\sigma}_J$ should be a solution of a "canonical Painlevé equation" that should appear as a compatibility condition of the Schrödinger equation (2.1) and a suitable deformation equation. In order to embody this indication, we first show the following

Proposition 2.1 *Let (Can) denote the following equation*

$$\left(-\frac{\partial^2}{\partial x^2} + \eta^2 Q_{\mathrm{can}}(x; E, \rho, \sigma, \eta)\right)\psi(x,t,\eta) = 0 \tag{2.17}$$

with $E = \rho^2 - 4\sigma^2$, *and let* (D_{can}) *denote the following equation*

$$\frac{\partial \psi}{\partial t} = A_{\mathrm{can}}\frac{\partial \psi}{\partial x} - \frac{1}{2}\frac{\partial A_{\mathrm{can}}}{\partial x}\psi \tag{2.18}$$

with

$$A_{\mathrm{can}} = \frac{1}{2(x - \eta^{-1/2}\sigma)}. \tag{2.19}$$

Suppose that ρ and σ satisfy the following Hamiltonian system $(H_{\rm can})$:

$$(H_{\rm can})\begin{cases}\dfrac{d\rho}{dt} = -4\eta\sigma \\ \\ \dfrac{d\sigma}{dt} = -\eta\rho.\end{cases} \tag{2.20}$$

Then (Can) and $(D_{\rm can})$ are in involution.

Proof. As one can readily verify, (*Can*) and $(D_{\rm can})$ are in involution if the following relation holds:

$$\frac{\partial Q_{\rm can}}{\partial t} = A_{\rm can}\frac{\partial Q_{\rm can}}{\partial x} + 2\frac{\partial A_{\rm can}}{\partial x}Q_{\rm can} - \frac{\eta^{-2}}{2}\frac{\partial^3 A_{\rm can}}{\partial x^3}. \tag{2.21}$$

(See [F] for example. Actually a straightforward computation shows that (2.21) entails

$$\begin{aligned}\left[-\frac{\partial^2}{\partial x^2} + \eta^2 Q_{\rm can},\ \frac{\partial}{\partial t} - A_{\rm can}\frac{\partial}{\partial x} + \frac{1}{2}\frac{\partial A_{\rm can}}{\partial x}\right] \\ = -2\frac{\partial A_{\rm can}}{\partial x}\left(-\frac{\partial^2}{\partial x^2} + \eta^2 Q_{\rm can}\right).\end{aligned} \tag{2.22}$$

Here the square bracket denotes the commutator.)

Since $E = \rho^2 - 4\sigma^2$ by the assumption, we can rewrite $Q_{\rm can}$ as follows:

$$\begin{aligned}Q_{\rm can} &= 4(x-\eta^{-1/2}\sigma)^2 + 8(x-\eta^{-1/2}\sigma)\eta^{-1/2}\sigma + \eta^{-1}\rho^2 \\ &\quad + \frac{\eta^{-3/2}\rho}{(x-\eta^{-1/2}\sigma)} + \frac{3\eta^{-2}}{4(x-\eta^{-1/2}\sigma)^2}.\end{aligned} \tag{2.23}$$

Substituting (2.23) into the right-hand side of (2.21), we compare the coefficients of like powers of $(x-\eta^{-1/2}\sigma)$; each side of (2.21) consists of terms of the form $c_j(t)(x-\eta^{-1/2}\sigma)^{-j}$ $(0 \le j \le 3)$ as terms of the form $c_4(x-\eta^{-1/2}\sigma)^{-4}$ cancel out among themselves, and hence we end

up with the following conditions:

$$\frac{3}{2}\eta^{-5/2}\frac{\partial\sigma}{\partial t} = -\frac{3}{2}\eta^{-3/2}\rho, \tag{2.24}$$

$$\eta^{-2}\rho\frac{\partial\sigma}{\partial t} = -\eta^{-1}\rho^2, \tag{2.25}$$

$$\eta^{-3/2}\frac{\partial\rho}{\partial t} = -4\eta^{-1/2}\sigma, \tag{2.26}$$

$$\eta^{-1}\left(2\rho\frac{\partial\rho}{\partial t} - 8\sigma\frac{\partial\sigma}{\partial t}\right) = 0. \tag{2.27}$$

It is clear that (2.20) validates all these requirements. Q.E.D.

3 Systems $(\mathcal{DSL}_J)$ and System $(\mathcal{DC}an)$.

As we have shown in Section 2, equation (Can) can be deformed if (ρ, σ) satisfies Hamiltonian system $(H_{\rm can})$. As (SL_J) also admits a deformation if (λ, ν) satisfies (H_J), we now try to generalize Theorem 2.1 so that equations (SL_J) and (D_J) may be simultaneously transformed to (Can) and $(D_{\rm can})$. To discuss problems of this sort we introduce the following

Definition 3.1. (i) $(\mathcal{DSL}_J)$ denotes the following simultaneous equations:

$$\begin{cases} \left(-\dfrac{\partial^2}{\partial\widetilde{x}^2} + \eta^2\widetilde{Q}_J(\widetilde{x},\widetilde{t},\eta)\right)\widetilde{\psi} = 0 \\ \dfrac{\partial\widetilde{\psi}}{\partial\widetilde{t}} = \widetilde{A}_J\dfrac{\partial\widetilde{\psi}}{\partial\widetilde{x}} - \dfrac{1}{2}\dfrac{\partial\widetilde{A}_J}{\partial\widetilde{x}}\widetilde{\psi}. \end{cases} \tag{3.1}$$

(ii) $(\mathcal{DC}an)$ denotes the following simultaneous equations:

$$\begin{cases} \left(-\dfrac{\partial^2}{\partial x^2} + \eta^2 Q_{\rm can}(x,t,\eta)\right)\psi = 0 \\ \dfrac{\partial\psi}{\partial t} = A_{\rm can}\dfrac{\partial\psi}{\partial x} - \dfrac{1}{2}\dfrac{\partial A_{\rm can}}{\partial x}\psi. \end{cases} \tag{3.2}$$

Although Theorem 2.1 guarantees that

$$\widetilde{\psi}(\widetilde{x},\widetilde{t},\eta) = \left(\frac{\partial x_J}{\partial\widetilde{x}}\right)^{-1/2}\psi(x_J(\widetilde{x},\widetilde{t},\eta),\widetilde{t},\eta) \tag{3.3}$$

satisfies (SL_J) if $\psi(x,t,\eta)$ satisfies (Can) (precisely speaking, (2.1)), $\widetilde{\psi}(\widetilde{x},\widetilde{t},\eta)$ does not necessarily satisfy $(\mathcal{DSL}_J)$ even if $\psi(x,t,\eta)$ satisfies $(\mathcal{DC}an)$; we need to find a suitable change of variables between t and $\widetilde{t}$ together with the transformation $x(\widetilde{x},\widetilde{t},\eta)$. (Cf. Corollary 3.1 below.) This change of variables is attained by requiring $(\widetilde{\sigma}_J,\widetilde{\rho}_J)$ coincides with a solution $(\sigma_{\rm can},\rho_{\rm can})$ of $(H_{\rm can})$. Although $(\sigma_{\rm can},\rho_{\rm can})$ cannot be uniquely fixed by $(\widetilde{\sigma}_J,\widetilde{\rho}_J)$, one invariant should be respected, that is, $\widetilde{E}_J = E_{\rm can}$ should hold. Thus the first ingredient we need is the following Lemma 3.1. Here and in what follows $(\rho_{\rm can},\sigma_{\rm can})$ denotes, by definition, a solution of $(H_{\rm can})$ and $E_{\rm can}$ denotes $\rho_{\rm can}^2 - 4\sigma_{\rm can}^2$.

Lemma 3.1 *(i) $\widetilde{E}_J = \widetilde{\rho}_J^2 - 4\widetilde{\sigma}_J^2$ is independent of $\widetilde{t}$.*
(ii) $E_{\rm can}$ is independent of t.

Proof. (i) Although the first claim has already been proved in [AKT2, Section 3], we show its core part here again in view of its importance. Let $\widetilde{S}_{J,\rm odd}$ and $S_{\rm can,odd}$ denote the odd part of the solution of the Riccati equation attached to (SL_J) and (2.1) respectively. Then Theorem 2.1 entails

$$
\begin{aligned}
(3.4)\quad & \oint_{|\widetilde{x}-\widetilde{\lambda}_0(\widetilde{t})|=\delta} \widetilde{S}_{J,\rm odd}(\widetilde{x},\widetilde{t},\eta)d\widetilde{x} \\
&= \oint_{|\widetilde{x}-\widetilde{\lambda}_0(\widetilde{t})|=\delta} S_{\rm can,odd}(x_J(\widetilde{x},\widetilde{t},\eta),\widetilde{t},\eta)\frac{\partial x_J(\widetilde{x},\widetilde{t},\eta)}{\partial\widetilde{x}}d\widetilde{x} \\
&= \oint_{|x|=\delta'} S_{\rm can,odd}(x,\widetilde{t},\eta)dx = \frac{\pi i}{2}\widetilde{E}_J(\widetilde{t},\eta)
\end{aligned}
$$

for sufficiently small positive numbers δ and δ'. (Cf. [AKT2, (3.33)].) As a consequence of the fact that $\widetilde{\psi}(\widetilde{x},\widetilde{t},\eta)$ satisfies (D_J), we know

$$
(3.5)\qquad \frac{\partial}{\partial\widetilde{t}}\oint \widetilde{S}_{J,\rm odd}d\widetilde{x} = 0
$$

(cf. [AKT2, (2.15)]). Hence $\widetilde{E}_J$ is independent of $\widetilde{t}$.

(ii) Since $(\sigma_{\rm can}, \rho_{\rm can})$ satisfies $(H_{\rm can})$, we find

$$\begin{aligned} \frac{\partial}{\partial t}E_{\rm can} &= 2\rho_{\rm can}\frac{\partial \rho_{\rm can}}{\partial t} - 8\sigma_{\rm can}\frac{\partial \sigma_{\rm can}}{\partial t} \\ &= -8\eta\rho_{\rm can}\sigma_{\rm can} + 8\eta\sigma_{\rm can}\rho_{\rm can} = 0. \end{aligned} \tag{3.6}$$

Therefore $E_{\rm can}$ is independent of t.

Q.E.D

As a solution of $(H_{\rm can}), (\sigma_{\rm can}, \rho_{\rm can})$ has the following simple form:

$$\begin{cases} \sigma_{\rm can}(t,\eta) = A(\eta)\exp(2\eta t) + B(\eta)\exp(-2\eta t) \\ \rho_{\rm can}(t,\eta) = -2A(\eta)\exp(2\eta t) + 2B(\eta)\exp(-2\eta t), \end{cases} \tag{3.7}$$

where $A(\eta) = \sum_{j\geq 0} A_{j/2}\eta^{-j/2}$ and $\sum_{j\geq 0} B_{j/2}\eta^{-j/2}$ with $A_{j/2}$ and $B_{j/2}$ being constants. Hence we find

$$E_{\rm can} = -16A(\eta)B(\eta), \tag{3.8}$$

and, in particular,

$$E_{{\rm can},0} = -16A_0B_0. \tag{3.9}$$

On the other hand, for the $(\widetilde{\alpha}, \widetilde{\beta})$-dependence of $\widetilde{E}_J$ we can prove the following

Lemma 3.2 *(i) The top term $\widetilde{E}_{J,0}$ of $\widetilde{E}_J = \sum_{l\geq 0}\widetilde{E}_{J,l/2}$ satisfies*

$$\widetilde{E}_{J,0} = -8\widetilde{\alpha}_0\widetilde{\beta}_0. \tag{3.10}$$

(ii) For any odd integer l, $\widetilde{E}_{J,l/2}$ vanishes.
(iii) Let $l = 2n$ $(n \geq 1)$ be a positive even integer. Then $\widetilde{E}_{J,l/2} = \widetilde{E}_{J,n}$ depends only on $\{\widetilde{\alpha}_m, \widetilde{\beta}_m\}_{0\leq m\leq n}$. Furthermore

$$\widetilde{E}_{J,n} + 8(\widetilde{\alpha}_0\widetilde{\beta}_n + \widetilde{\beta}_0\widetilde{\alpha}_n) \quad \textit{is independent of } (\widetilde{\alpha}_n, \widetilde{\beta}_n). \tag{3.11}$$

Proof. It follows from the instanton structure of $\widetilde{\sigma}_J$ and $\widetilde{\rho}_J$ (cf. (2.13)) that $\widetilde{E}_{J,l/2}$ (l: odd) consists of odd instanton terms, because $\widetilde{E}_J = \widetilde{\rho}_J^2 - 4\widetilde{\sigma}_J^2$. An odd instanton term cannot be independent of $\widetilde{t}$, unless it vanishes. Hence $\widetilde{E}_{J,l/2}$ (l: odd) should vanish, that is, (ii) holds.

Let us next prove (3.10). The equations (2.10) and (2.11) which respectively determine $\widetilde{\sigma}_J$ and $\widetilde{\rho}_J$ tell us that

$$\widetilde{\sigma}_{J,0} = \frac{\partial x_{J,0}}{\partial \widetilde{x}}(\widetilde{\lambda}_0, \widetilde{t})\widetilde{\Lambda}_0(\widetilde{t}, \eta), \tag{3.12}$$

$$\widetilde{\rho}_{J,0} = -\left(\frac{\partial x_{J,0}}{\partial \widetilde{x}}(\widetilde{\lambda}_0, \widetilde{t})\right)^{-1} \widetilde{\mathcal{N}}_0(\widetilde{t}, \eta). \tag{3.13}$$

The explicit form of $\widetilde{\Lambda}_0(\widetilde{t}, \eta)$ is given by

$$\widetilde{\Lambda}_0(\widetilde{t}, \eta) = (\widetilde{F}_J^{(1)})^{-1/4}(\widetilde{C}_J(\widetilde{\lambda}_0, \widetilde{t}))^{-1/2} \left\{\widetilde{\alpha}_0 \exp \widetilde{\Phi}_J + \widetilde{\beta}_0 \exp(-\widetilde{\Phi}_J)\right\} \tag{3.14}$$

(cf. (1.15), (1.16) and (1.17)), while (1.42) implies

$$\begin{aligned} \widetilde{\mathcal{N}}_0(\widetilde{t}, \eta) &= \widetilde{C}_J(\widetilde{\lambda}_0, \widetilde{t})\Bigg[(\widetilde{F}_J^{(1)})^{-1/4}(\widetilde{C}_J(\widetilde{\lambda}_0, \widetilde{t}))^{-1/2} \\ &\qquad \left\{\widetilde{\alpha}_0 \exp \widetilde{\Phi}_J - \widetilde{\beta}_0 \exp(-\widetilde{\Phi}_J)\right\}\Bigg] \frac{d\widetilde{\Phi}_J}{d\widetilde{t}} \\ &= (\widetilde{F}_J^{(1)})^{1/4}(\widetilde{C}_J(\widetilde{\lambda}_0, \widetilde{t}))^{1/2} \left\{\widetilde{\alpha}_0 \exp \widetilde{\Phi}_J - \widetilde{\beta}_0 \exp(-\widetilde{\Phi}_J)\right\}, \end{aligned} \tag{3.15}$$

as the top term of the right-hand side of (1.42) can be obtained through differentiation (in $\widetilde{t}$) of the instanton terms $\exp(\pm\widetilde{\Phi}_J)$ contained in $\widetilde{\Lambda}_0(\widetilde{t}, \eta)$. Since $x_{J,0}$ satisfies the top degree part of (2.9), i.e.,

$$\widetilde{Q}_{J,0} = 4\left(\frac{\partial x_{J,0}(\widetilde{x}, \widetilde{t})}{\partial \widetilde{x}}\right)^2 x_{J,0}(\widetilde{x}, \widetilde{t})^2, \tag{3.16}$$

the comparison of coefficients of the Taylor series at $\widetilde{x} = \widetilde{\lambda}_0(\widetilde{t})$ of each side of (3.16) entails

$$\begin{aligned} \frac{\partial x_{J,0}}{\partial \widetilde{x}}(\widetilde{\lambda}_0, \widetilde{t}) &= \left(\frac{1}{8}\frac{\partial^2 \widetilde{Q}_{J,0}}{\partial \widetilde{x}^2}(\widetilde{\lambda}_0, \widetilde{t})\right)^{1/4} \\ &= \frac{1}{\sqrt{2}}(\widetilde{F}_J^{(1)})^{1/4}(\widetilde{C}_J(\widetilde{\lambda}_0, \widetilde{t}))^{1/2}. \end{aligned} \tag{3.17}$$

Here we have used the relation (1.37) in [KT1]. Combining (3.12)$\sim$(3.15) and (3.17) together, we have

$$\widetilde{\sigma}_{J,0} = \frac{1}{\sqrt{2}}\left\{\widetilde{\alpha}_0 \exp\widetilde{\Phi}_J + \widetilde{\beta}_0 \exp(-\widetilde{\Phi}_J)\right\}, \tag{3.18}$$

$$\widetilde{\rho}_{J,0} = \sqrt{2}\left\{-\widetilde{\alpha}_0 \exp\widetilde{\Phi}_J + \widetilde{\beta}_0 \exp(-\widetilde{\Phi}_J)\right\}. \tag{3.19}$$

Hence $\widetilde{E}_{J,0} = \widetilde{\rho}^2_{J,0} - 4\widetilde{\sigma}^2_{J,0} = -8\widetilde{\alpha}_0\widetilde{\beta}_0$, i.e., (3.10).

We can prove (3.11) in a similar manner. As a matter of fact, by the definition of $(\widetilde{\alpha}_n, \widetilde{\beta}_n)$ $\widetilde{\Lambda}_n(\widetilde{t}, \eta)$ depends only on $\{\widetilde{\alpha}_m, \widetilde{\beta}_m\}_{0\le m\le n}$ and by (1.42) we find this is also the case for $\widetilde{\mathcal{N}}_n(\widetilde{t}, \eta)$. Furthermore the induction which was employed in [AKT2] to construct $x_J(\widetilde{x}, \widetilde{t}, \eta)$ (as well as $\widetilde{\sigma}_J, \widetilde{\rho}_J$, etc.) together with the fact that $\widetilde{Q}_{J,n}$ contains $\{\widetilde{\alpha}_m, \widetilde{\beta}_m\}_{0\le m\le n-1}$ only (, which can be verified by using the explicit form (3.5) of $\widetilde{Q}_J$ in [AKT2] and an argument similar to the proof of Lemma 2.1 there,) implies the following:

(3.20) $x_{J,n}(\widetilde{x}, \widetilde{t}, \eta)$ depends only on $\{\widetilde{\alpha}_m, \widetilde{\beta}_m\}_{0\le m\le n-1}$,

(3.21) $\widetilde{\sigma}_{J,n}(\widetilde{t}, \eta)$ and $\widetilde{\rho}_{J,n}(\widetilde{t}, \eta)$ depend only on $\{\widetilde{\alpha}_m, \widetilde{\beta}_m\}_{0\le m\le n}$.

Hence $\widetilde{E}_{J,n}$ is also determined by $\{\widetilde{\alpha}_m, \widetilde{\beta}_m\}_{0\le m\le n}$. In particular, since the $(\widetilde{\alpha}_n, \widetilde{\beta}_n)$-dependence of $\widetilde{\sigma}_{J,n}$ and $\widetilde{\rho}_{J,n}$ comes from $(\partial x_{J,0}(\widetilde{\lambda}_0, \widetilde{t})/\partial\widetilde{x})\widetilde{\Lambda}_n(\widetilde{t}, \eta)$ and $-(\partial x_{J,0}(\widetilde{\lambda}_0, \widetilde{t})/\partial\widetilde{x})^{-1}\widetilde{\mathcal{N}}_n(\widetilde{t}, \eta)$ respectively (cf. (2.10) and (2.11)), we find that

$$\widetilde{E}_{J,n} + \left\{2\widetilde{\rho}_{J,0}\frac{\widetilde{\mathcal{N}}_n(\widetilde{t}, \eta)}{\dfrac{\partial x_{J,0}}{\partial\widetilde{x}}(\widetilde{\lambda}_0, \widetilde{t})} + 8\widetilde{\sigma}_{J,0}\frac{\partial x_{J,0}}{\partial\widetilde{x}}(\widetilde{\lambda}_0, \widetilde{t})\widetilde{\Lambda}_n(\widetilde{t}, \eta)\right\} \tag{3.22}$$

is independent of $(\widetilde{\alpha}_n, \widetilde{\beta}_n)$. Now it follows from (1.26) and (1.42) that the $(\widetilde{\alpha}_n, \widetilde{\beta}_n)$-dependent part of $\widetilde{\Lambda}_n(\widetilde{t}, \eta)$ and $\widetilde{\mathcal{N}}_n(\widetilde{t}, \eta)$ is explicitly given by

$$\begin{aligned}(\widetilde{F}^{(1)}_J)^{-1/4}(\widetilde{C}_J(\widetilde{\lambda}_0, \widetilde{t}))^{-1/2}\{&(\widetilde{\alpha}_n \exp\widetilde{\Phi}_J + \widetilde{\beta}_n \exp(-\widetilde{\Phi}_J))\\ &+(\widetilde{\alpha}_0\widetilde{\beta}_n + \widetilde{\beta}_0\widetilde{\alpha}_n)(\widetilde{\alpha}_0 \exp\widetilde{\Phi}_J - \widetilde{\beta}_0 \exp(-\widetilde{\Phi}_J))\log\widetilde{\theta}_J\}\end{aligned} \tag{3.23}$$

and by

$$(3.24)\quad (\widetilde{F}_J^{(1)})^{1/4}(\widetilde{C}_J(\widetilde{\lambda}_0,\widetilde{t}))^{1/2}\{(\widetilde{\alpha}_n\exp\widetilde{\Phi}_J-\widetilde{\beta}_n\exp(-\widetilde{\Phi}_J))$$
$$+(\widetilde{\alpha}_0\widetilde{\beta}_n+\widetilde{\beta}_0\widetilde{\alpha}_n)(\widetilde{\alpha}_0\exp\widetilde{\Phi}_J+\widetilde{\beta}_0\exp(-\widetilde{\Phi}_J))\log\widetilde{\theta}_J\}$$

respectively (cf. (1.27)). Substituting (3.23) and (3.24) together with (3.17)$\sim$(3.19) into (3.22), we obtain (3.11).

Q.E.D.

Remark 3.1. The concrete form of the first few terms of $\widetilde{E}_J$ as well as those of $\widetilde{\rho}_J$ and $\widetilde{\sigma}_J$ in the case of $J=\mathrm{I}$ and II is given in Appendix.

Remark 3.2. Lemma 3.2, especially (3.10) and (3.11), means that the principal part of $\widetilde{E}_J$ is given by $-8\widetilde{\alpha}\widetilde{\beta}=-8(\sum\widetilde{\alpha}_n\eta^{-n})(\sum\widetilde{\beta}_n\eta^{-n})$. Although we conjecture that $\widetilde{E}_J$ should be exactly equal to $-8\widetilde{\alpha}\widetilde{\beta}$ (which is a motivation of our study done in [KT3]) and this is actually true in the case of $J=\mathrm{I}$ because of the homogeneity of $\widetilde{E}_{\mathrm{I}}$ with respect to η, proving it for general J is still an open problem.

As we have assumed $\widetilde{\alpha}_0\widetilde{\beta}_0\neq 0$ (cf. (1.28)), Lemma 3.1, (3.8) and Lemma 3.2 enable us to find $(A(\eta),\,B(\eta))$ so that the following hold:

$$(3.25)\qquad E_{\mathrm{can}}=\widetilde{E}_J$$

$$(3.26)\qquad A_{j/2}=B_{j/2}=0\quad\text{if}\quad j:\ \mathrm{odd}.$$

Fixing $(A(\eta),B(\eta))$ thus found, we can now find an infinite series $t_J(\widetilde{t},\eta)=t_J(\widetilde{t};\widetilde{\alpha},\widetilde{\beta},A,B;\eta)=\sum_{j\geq 0}t_{J,j/2}(\widetilde{t},\eta)\eta^{-j/2}$ in $\eta^{-1/2}$ so that

$$(3.27)\qquad \sigma_{\mathrm{can}}(t_J(\widetilde{t};\widetilde{\alpha},\widetilde{\beta},A,B;\eta),\eta)=\widetilde{\sigma}_J(\widetilde{t};\widetilde{\alpha},\widetilde{\beta};\eta)$$

and

$$(3.28)\qquad \rho_{\mathrm{can}}(t_J(\widetilde{t};\widetilde{\alpha},\widetilde{\beta},A,B;\eta),\eta)=\widetilde{\rho}_J(\widetilde{t};\widetilde{\alpha},\widetilde{\beta};\eta)$$

hold. To describe the precise structure of $t_J(\widetilde{t};\widetilde{\alpha},\widetilde{\beta},A,B;\eta)$, we summarize its property in the following

Lemma 3.3 *Let $\widetilde{t}_*$ be the same as in Theorem 2.1. Then the series $t_J(\widetilde{t};\widetilde{\alpha},\widetilde{\beta},A,B;\eta)$ that satisfies (3.27) and (3.28) can be chosen so that the following hold:*

(3.29) *$t_{J,j/2}(\widetilde{t},\eta)$ is holomorphic on a neighborhood $\widetilde{V}$ of $\widetilde{t}_*$,*

(3.30) $$t_{J,0}(\widetilde{t}) = \frac{1}{2}\widetilde{\phi}_J(\widetilde{t}),$$

(3.31) $t_{J,1/2}(\widetilde{t})$ *identically vanishes,*

(3.32) $$t_{J,1}(\widetilde{t},\eta) = \frac{1}{2}\log\frac{\widetilde{\alpha}_0}{\sqrt{2}A_0} + \frac{\widetilde{\alpha}_0\widetilde{\beta}_0}{2}\log(\widetilde{\theta}_J\eta^2),$$

(3.33) $t_{J,j/2}$ $(j \geq 3)$ *has the following instanton structure:*

$$\sum_{k=0}^{j-2} s^{(j/2)}_{j-2-2k}(\widetilde{t})\exp\left((j-2-2k)\widetilde{\Phi}_J(\widetilde{t},\eta)\right).$$

Proof. We omit the suffix J of t_J for simplicity during the proof. In view of the definition of $\widetilde{\sigma}_J(\widetilde{t},\eta)$, it is clear that we have to choose t_0, $t_{1/2}$ and t_1 by (3.30), (3.31) and (3.32) respectively. (Cf. (1.7) for the definition of $\widetilde{\theta}_J(\widetilde{t})$.) Note that the relation $E_{\mathrm{can},0} = \widetilde{E}_{J,0}$ implies $\widetilde{\alpha}_0/\sqrt{2}A_0 = \sqrt{2}B_0/\widetilde{\beta}_0$; hence (3.32) entails

(3.34) $$[\sigma_{\mathrm{can}}(t_0(\widetilde{t}) + t_1(\widetilde{t})\eta^{-1},\eta)]_0 = \widetilde{\sigma}_{J,0}(\widetilde{t},\eta)$$

and

(3.35) $$[\rho_{\mathrm{can}}(t_0(\widetilde{t}) + t_1(\widetilde{t})\eta^{-1},\eta)]_0 = \widetilde{\rho}_{J,0}(\widetilde{t},\eta).$$

(Cf. (3.18) and (3.19).) Here and in what follows $[\sigma_{\mathrm{can}}(t_0(\widetilde{t})+t_1(\widetilde{t})\eta^{-1},\eta)]_j$ etc. denote the degree j part of $\sigma_{\mathrm{can}}(t_0(\widetilde{t}) + t_1(\widetilde{t})\eta^{-1},\eta)$ etc. with respect to η^{-1}. We now construct $t_{j/2}(\widetilde{t},\eta)$ $(j \geq 3)$ by the induction on j,i.e., by assuming $t_{j'/2}$ $(j' \leq j-1)$ have been constructed. Let us recall the instanton structure of (2.13) of $\widetilde{\sigma}_J$ and $\widetilde{\rho}_J$, that is, $\widetilde{\sigma}_{J,j/2}$ and $\widetilde{\rho}_{J,j/2}$ consist of l-instanton terms with $l = j+1-2k$, $0 \leq k \leq j+1$. Using the Taylor expansion of $\exp(t_{3/2}\eta^{-1/2} + \cdots + t_{j/2}\eta^{-(j-2)/2})$, we deduce from the requirement

(3.36) $$\begin{aligned}&[\sigma_{\mathrm{can}}(t_0(t) + t_1(\widetilde{t})\eta^{-1} + \cdots + t_{j/2}(\widetilde{t},\eta)\eta^{-j/2},\eta)]_{(j-2)/2}\\ &= \widetilde{\sigma}_{J,(j-2)/2}(\widetilde{t},\eta)\end{aligned}$$

the following

(3.37)

$$t_{j/2}(\widetilde{t},\eta)=\frac{X_{j/2}}{2(A_0\exp(2\eta t_0+2t_1)-B_0\exp(-(2\eta t_0+2t_1)))},$$

where $X_{j/2}$ consists of l-instanton terms with $l=j-1-2k\quad(0\le k\le j-1)$ by the instanton structure of $\widetilde{\sigma}_J$ together with the induction hypothesis. On the other hand, the relation (3.25) implies

(3.38)

$$[\rho_{\rm can}(t_0(\widetilde{t})+\cdots+t_{j/2}(\widetilde{t},\eta)\eta^{-j/2},\eta)^2]_{(j-2)/2}=[\widetilde{\rho}_J(\widetilde{t},\eta)^2]_{(j-2)/2}.$$

It then follows from (3.35) that

(3.39)

$$[\rho_{\rm can}(t_0(\widetilde{t})+\cdots+t_{j/2}(\widetilde{t},\eta)\eta^{-j/2},\eta)]_{(j-2)/2}=\widetilde{\rho}_{J,(j-2)/2}(\widetilde{t},\eta).$$

Hence it follows from this that

(3.40)

$$t_{j/2}(\widetilde{t},\eta)=\frac{Y_{j/2}}{-4(A_0\exp(2\eta t_0+2t_1)+B_0\exp(-(2\eta t_0+2t_1)))}$$

with $Y_{j/2}$ consisting of l-instanton terms with $|l|\le j-1$ by the instanton structure of $\widetilde{\rho}_J$ and the induction hypothesis. Then the comparison of (3.37) and (3.40) shows that $X_{j/2}$ should be divisible by $(A_0\exp(2\eta t_0+2t_1)-B_0\exp(-(2\eta t_0+2t_1)))$ in the polynomial ring generated by instanton terms. Hence $t_{j/2}$ has the form specified in (3.33). Thus the induction proceeds, completing the proof of the lemma.

Q.E.D.

The infinite series $t_J(\widetilde{t},\eta)=t_J(\widetilde{t};\widetilde{\alpha},\widetilde{\beta},A,B;\eta)$ thus constructed together with the series $x_J(\widetilde{x},\widetilde{t},\eta)$ in Theorem 2.1 brings the simultaneous equations $(\mathcal{DSL}_J)$ to $(\mathcal{DC}an)$; the precise statement is as follows:

Proposition 3.1 *Let $\psi(x,t,\eta)$ be a WKB solution of (Can) that satisfies $(D_{\rm can})$ also, and let $\widetilde{\psi}(\widetilde{x},\widetilde{t},\eta)$ denote the following function determined by transformations $x=x_J(\widetilde{x},\widetilde{t},\eta)$ given by (2.3) and $t=t_J(\widetilde{t},\eta)$*

given in Lemma 3.3:

(3.41)

$$\widetilde{\psi}(\widetilde{x},\widetilde{t},\eta)=\left(\frac{\partial x_J(\widetilde{x},\widetilde{t},\eta)}{\partial\widetilde{x}}\right)^{-1/2}\psi(x_J(\widetilde{x},\widetilde{t},\eta),t_J(\widetilde{t},\eta),\eta).$$

Then $\widetilde{\psi}$ satisfies $(\mathcal{DSL}_J)$ near $\widetilde{x}=\widetilde{\lambda}_{J,0}(\widetilde{t}_*)$.

Proof. Throughout this proof, $\widetilde{S}_{\rm odd}(\widetilde{x},\widetilde{t},\eta)$ (resp., $S_{\rm odd}(x,t,\eta)$) denotes the odd part of the solution $\widetilde{S}$ (resp., S_{can}) of the Riccati equation associated with (SL_J) (resp., $(\mathcal{C}an)$). (See [AKT2, Definition 2.1] for the definition of the odd part.) We also abbreviate symbols $x_J(\widetilde{x},\widetilde{t},\eta)$ and $t_J(\widetilde{t},\eta)$ to $x(\widetilde{x},\widetilde{t},\eta)$ and $t(\widetilde{t},\eta)$ unless there is any fear of confusions.

To begin with, let us note that the t-dependence of $Q_{\rm can}$ is only through E, ρ and σ; t does not appear in a bare form. Since $S_{\rm can}$ is constructed by recursively solving algebraic equations, the t-dependence of $S_{\rm can}$ is also only through E,ρ and σ. Otherwise stated, $S_{\rm can}$ may be written as

(3.42) $$S_{\rm can}(x,t,\eta)=S_{\rm can}(x;\rho_{\rm can}(t,\eta),\sigma_{\rm can}(t,\eta);\eta).$$

(Note that E is determined by ρ and σ; $E=\rho^2-4\sigma^2$.)

Now Theorem 2.1 entails the following:

(3.43)

$$\widetilde{S}_{\rm odd}(\widetilde{x},\widetilde{t},\eta)=\frac{\partial x(\widetilde{x},\widetilde{t},\eta)}{\partial\widetilde{x}}S_{\rm odd}(x(\widetilde{x},\widetilde{t},\eta);\rho_J(\widetilde{t},\eta),\sigma_J(\widetilde{t},\eta);\eta).$$

(See [AKT1] for the proof of the fact that a relation between potentials (like (2.9)) entails a relation between solutions of Riccati equations (like (3.43)).) On the other hand, it follows from the definition of $t(\widetilde{t},\eta)$ the right-hand side of (3.43) is identical with

(3.44) $$\frac{\partial x}{\partial\widetilde{x}}S_{\rm odd}(x(\widetilde{x},\widetilde{t},\eta);\rho_{\rm can}(t(\widetilde{t},\eta),\eta),\sigma_{\rm can}(t(\widetilde{t},\eta),\eta);\eta),$$

and hence we may further rewrite it as

$$\frac{\partial x}{\partial \widetilde{x}} S_{\mathrm{odd}}(x(\widetilde{x},\widetilde{t},\eta),t(\widetilde{t},\eta),\eta) \tag{3.45}$$

in view of (3.42). Thus we have

$$\widetilde{S}_{\mathrm{odd}}(\widetilde{x},\widetilde{t},\eta) = \frac{\partial x(\widetilde{x},\widetilde{t},\eta)}{\partial \widetilde{x}} S_{\mathrm{odd}}(x(\widetilde{x},\widetilde{t},\eta),t(\widetilde{t},\eta),\eta). \tag{3.46}$$

Differentiating both sides of (3.46) with respect to $\widetilde{t}$, we find

$$\begin{aligned}\frac{\partial \widetilde{S}_{\mathrm{odd}}}{\partial \widetilde{t}} &= \frac{\partial^2 x}{\partial \widetilde{x}\partial \widetilde{t}} S_{\mathrm{odd}}(x(\widetilde{x},\widetilde{t},\eta),t(\widetilde{t},\eta),\eta) \\ &\quad + \frac{\partial x}{\partial \widetilde{x}} \left(\frac{\partial S_{\mathrm{odd}}(x(\widetilde{x},\widetilde{t},\eta),t(\widetilde{t},\eta),\eta)}{\partial x} \frac{\partial x}{\partial \widetilde{t}} \right. \\ &\qquad \left. + \frac{\partial S_{\mathrm{odd}}(x(\widetilde{x},\widetilde{t},\eta),t(\widetilde{t},\eta),\eta)}{\partial t} \frac{\partial t}{\partial \widetilde{t}} \right).\end{aligned} \tag{3.47}$$

Since $\widetilde{S}_{\mathrm{odd}}$ satisfies

$$\frac{\partial \widetilde{S}_{\mathrm{odd}}}{\partial \widetilde{t}} = \frac{\partial}{\partial \widetilde{x}} (\widetilde{A}_J \widetilde{S}_{\mathrm{odd}}) \tag{3.48}$$

(cf. [AKT2, (2.14)]), we obtain the following (3.49) from (3.46) and (3.47):

$$\begin{aligned}&\frac{\partial}{\partial \widetilde{x}} \left\{ \widetilde{A}_J \frac{\partial x}{\partial \widetilde{x}} S_{\mathrm{odd}}(x(\widetilde{x},\widetilde{t},\eta),t(\widetilde{t},\eta),\eta) \right\} \\ &- \frac{\partial}{\partial \widetilde{x}} \left\{ \frac{\partial x}{\partial \widetilde{t}} S_{\mathrm{odd}}(x(\widetilde{x},\widetilde{t},\eta),t(\widetilde{t},\eta),\eta) \right\} \\ &- \frac{\partial x}{\partial \widetilde{x}} \frac{\partial t}{\partial \widetilde{t}} \frac{\partial S_{\mathrm{odd}}(x(\widetilde{x},\widetilde{t},\eta),t(\widetilde{t},\eta),\eta)}{\partial t} = 0.\end{aligned} \tag{3.49}$$

In parallel with (3.48) we can also verify

$$\frac{\partial S_{\mathrm{odd}}}{\partial t} = \frac{\partial}{\partial x} (A_{\mathrm{can}} S_{\mathrm{odd}}) \tag{3.50}$$

and hence we obtain

(3.51)
$$\frac{\partial}{\partial\widetilde{x}}\left\{\left(\widetilde{A}_J\frac{\partial x}{\partial\widetilde{x}}-\frac{\partial x}{\partial\widetilde{t}}-A_{\rm can}\frac{\partial t}{\partial\widetilde{t}}\right)S_{\rm odd}(x(\widetilde{x},\widetilde{t},\eta),t(\widetilde{t},\eta),\eta)\right\}=0.$$

As was shown in the proof of Proposition 2.2 of [KT1], the following relation

(3.52)
$$\widetilde{A}_J\frac{\partial x}{\partial\widetilde{x}}-\frac{\partial x}{\partial\widetilde{t}}-A_{\rm can}\frac{\partial t}{\partial\widetilde{t}}=0$$

guarantees that $\widetilde{\psi}$ given by (3.41) satisfies $(\mathcal{DSL}_J)$. Hence our task is to deduce (3.52) from (3.51). To attain it, we introduce the following two symbols:

(3.53)
$$\begin{aligned}\mathcal{J}=2(x(\widetilde{x},\widetilde{t},\eta)\ &-\ \eta^{-1/2}\sigma(t(\widetilde{t},\eta),\eta))\Big(\widetilde{A}_J\frac{\partial x(\widetilde{x},\widetilde{t},\eta)}{\partial\widetilde{x}}\\ &-\ \frac{\partial x(\widetilde{x},\widetilde{t},\eta)}{\partial\widetilde{t}}-A_{\rm can}\frac{\partial t(\widetilde{t},\eta)}{\partial\widetilde{t}}\Big),\end{aligned}$$

(3.54)
$$\mathcal{K}=\frac{\eta^{-1}S_{\rm odd}(x(\widetilde{x},\widetilde{t},\eta),t(\widetilde{t},\eta),\eta)}{2(x(\widetilde{x},\widetilde{t},\eta)-\eta^{-1/2}\sigma(t(\widetilde{t},\eta),\eta))}.$$

The relation (3.51) can be rewritten as

(3.55)
$$\frac{\partial}{\partial\widetilde{x}}(\mathcal{J}\mathcal{K})=0,$$

and our task is to confirm $\mathcal{J}=0$. To prove this, we use the induction on the degree in $\eta^{-1/2}$ of $\mathcal{J}$. For clearer description of the argument we prepare the following sublemmas. Here and in what follows $\left.\frac{\partial x}{\partial\widetilde{x}}\right|_{\tilde{x}=\tilde{\lambda}_J}$ etc. respectively denote $\frac{\partial x}{\partial\widetilde{x}}(\lambda_J(\widetilde{t},\eta),\widetilde{t},\eta)$ etc.

Sublemma 3.1 *The following relation holds for* $x=x(\widetilde{x},\widetilde{t},\eta)$ *and* $t=t(\widetilde{t},\eta)$:

(3.56)
$$\begin{aligned}&\widetilde{\rho}_J(\widetilde{t},\eta)\left((2\widetilde{B}_J(\frac{\partial x}{\partial\widetilde{x}})^2)\Big|_{\tilde{x}=\tilde{\lambda}=\tilde{\lambda}_J}-\frac{\partial t}{\partial\widetilde{t}}\right)\\ &\qquad=\eta^{-1/2}\left(\frac{\partial x}{\partial\widetilde{t}}-\frac{3}{2}\widetilde{B}_J\frac{\partial^2x}{\partial\widetilde{x}^2}-\frac{\partial\widetilde{B}_J}{\partial\widetilde{x}}\frac{\partial x}{\partial\widetilde{x}}\right)\Big|_{\tilde{x}=\tilde{\lambda}=\tilde{\lambda}_J},\end{aligned}$$

where $\widetilde{B}_J$ is the rational function tabulated in Table 1.3.

Sublemma 3.2 *Let $\widetilde{X}$ denote $\widetilde{x}-\widetilde{\lambda}_J$ and let $O(X^l)$ denote a sum of terms containing a factor $\widetilde{X}^m$ $(m \geq l)$. Then we find*

$$
\begin{aligned}
(3.57)\quad \mathcal{J} \;=\; & \left((2\widetilde{B}_J(\frac{\partial x}{\partial \widetilde{x}})^2)\Big|_{\widetilde{x}=\widetilde{\lambda}=\widetilde{\lambda}_J} - \frac{\partial t}{\partial \widetilde{t}} \right) \\
& +2\left((\frac{3}{2}\widetilde{B}_J\frac{\partial^2 x}{\partial \widetilde{x}^2} + \frac{\partial \widetilde{B}_J}{\partial \widetilde{x}}\frac{\partial x}{\partial \widetilde{x}} - \frac{\partial x}{\partial \widetilde{t}})\frac{\partial x}{\partial \widetilde{x}} \right)\Big|_{\widetilde{x}=\widetilde{\lambda}=\widetilde{\lambda}_J}\widetilde{X} \\
& +O(\widetilde{X}^2).
\end{aligned}
$$

Proof of Sublemma 3.1. Differentiating both sides of (3.27) with respect to $\widetilde{t}$, we use (2.10) to find

$$
(3.58)\quad \frac{d\sigma_{\mathrm{can}}(t,\eta)}{dt}\Big|_{t=t(\widetilde{t},\eta)}\frac{dt}{d\widetilde{t}} = \eta^{1/2}\left(\frac{\partial x}{\partial \widetilde{x}}\Big|_{\widetilde{x}=\widetilde{\lambda}_J}\frac{d\widetilde{\lambda}_J}{d\widetilde{t}} + \frac{\partial x}{\partial \widetilde{t}}\Big|_{\widetilde{x}=\widetilde{\lambda}_J} \right).
$$

It then follows from (H_J) and (H_{can}) together with (1.35) that the following holds:

$$
\begin{aligned}
(3.59)\quad -\eta\rho_{\mathrm{can}}(t(\widetilde{t},\eta),\eta)\frac{dt}{d\widetilde{t}} \;=\; & \eta^{1/2}\Big(\eta\frac{\partial x}{\partial \widetilde{x}}\Big|_{\widetilde{x}=\widetilde{\lambda}_J}(2\widetilde{B}_J\Big|_{\widetilde{x}=\widetilde{\lambda}=\widetilde{\lambda}_J}\widetilde{\nu}_J \\
& -\eta^{-1}\frac{\partial \widetilde{B}_J}{\partial \widetilde{x}}\Big|_{\widetilde{x}=\widetilde{\lambda}=\widetilde{\lambda}_J}) + \frac{\partial x}{\partial \widetilde{t}}\Big|_{\widetilde{x}=\widetilde{\lambda}_J} \Big).
\end{aligned}
$$

Hence (3.28) and (2.11) entail

$$
\begin{aligned}
(3.60)\qquad \widetilde{\rho}_J(\widetilde{t},\eta)\frac{dt}{d\widetilde{t}} \;=\; & 2\Big(\widetilde{B}_J\Big(\frac{\partial x}{\partial \widetilde{x}}\Big)^2 \Big)\Big|_{\widetilde{x}=\widetilde{\lambda}=\widetilde{\lambda}_J}\widetilde{\rho}_J + \frac{3}{2}\eta^{-1/2}\Big(\widetilde{B}_J\frac{\partial^2 x}{\partial \widetilde{x}^2} \Big)\Big|_{\widetilde{x}=\widetilde{\lambda}=\widetilde{\lambda}_J} \\
& +\eta^{-1/2}\Big(\frac{\partial \widetilde{B}_J}{\partial \widetilde{x}}\frac{\partial x}{\partial \widetilde{x}} \Big)\Big|_{\widetilde{x}=\widetilde{\lambda}=\widetilde{\lambda}_J} - \eta^{-1/2}\frac{\partial x}{\partial \widetilde{t}}\Big|_{\widetilde{x}=\widetilde{\lambda}_J}.
\end{aligned}
$$

Rearranging (3.60), we finally obtain the required relation:

$$
\begin{aligned}
(3.61)\qquad & \widetilde{\rho}_J(\widetilde{t},\eta)\Big(2(\widetilde{B}_J\Big(\frac{\partial x}{\partial \widetilde{x}}\Big)^2)\Big|_{\widetilde{x}=\widetilde{\lambda}=\widetilde{\lambda}_J}-\frac{dt}{d\widetilde{t}}\Big)\\
&\quad=\eta^{-1/2}\Big(\frac{\partial x}{\partial \widetilde{t}}\Big|_{\widetilde{x}=\widetilde{\lambda}_J}-\frac{3}{2}(\widetilde{B}_J\frac{\partial^2 x}{\partial \widetilde{x}^2})\Big|_{\widetilde{x}=\widetilde{\lambda}=\widetilde{\lambda}_J}\\
&\qquad-(\frac{\partial \widetilde{B}_J}{\partial \widetilde{x}}\frac{\partial x}{\partial \widetilde{x}})\Big|_{\widetilde{x}=\widetilde{\lambda}=\widetilde{\lambda}_J}\Big).
\end{aligned}
$$

Q.E.D

Proof of Sublemma 3.2. Noting the relation (2.10), we apply the Taylor expansion in $\widetilde{X}$ to each term in $\mathcal{J}$:

$$
\begin{aligned}
(3.62)\mathcal{J} &= 2\Big\{\frac{\partial x}{\partial \widetilde{x}}\Big|_{\widetilde{x}=\widetilde{\lambda}_J}\widetilde{X}+\frac{1}{2}\frac{\partial^2 x}{\partial \widetilde{x}^2}\Big|_{\widetilde{x}=\widetilde{\lambda}_J}\widetilde{X}^2+O(\widetilde{X}^3)\Big\}\\
&\times\ \Big\{\widetilde{X}^{-1}\Big(\widetilde{B}_J\Big|_{\widetilde{x}=\widetilde{\lambda}=\widetilde{\lambda}_J}+\frac{\partial \widetilde{B}_J}{\partial \widetilde{x}}\Big|_{\widetilde{x}=\widetilde{\lambda}=\widetilde{\lambda}_J}\widetilde{X}+O(\widetilde{X}^2)\Big)\Big(\frac{\partial x}{\partial \widetilde{x}}\Big|_{\widetilde{x}=\widetilde{\lambda}_J}\\
&\quad+\frac{\partial^2 x}{\partial \widetilde{x}^2}\Big|_{\widetilde{x}=\widetilde{\lambda}_J}\widetilde{X}+O(\widetilde{X}^2)\Big)-\Big(\frac{\partial x}{\partial \widetilde{t}}\Big|_{\widetilde{x}=\widetilde{\lambda}_J}+O(\widetilde{X})\Big)\Big\}-\frac{\partial t}{\partial \widetilde{t}}\\
&= \Big((2\widetilde{B}_J\Big(\frac{\partial x}{\partial \widetilde{x}}\Big)^2)\Big|_{\widetilde{x}=\widetilde{\lambda}=\widetilde{\lambda}_J}-\frac{\partial t}{\partial \widetilde{t}}\Big)+\Big(3\widetilde{B}_J\frac{\partial x}{\partial \widetilde{x}}\frac{\partial^2 x}{\partial \widetilde{x}^2}\\
&\quad+2\frac{\partial \widetilde{B}_J}{\partial \widetilde{x}}\Big(\frac{\partial x}{\partial \widetilde{x}}\Big)^2-2\frac{\partial x}{\partial \widetilde{x}}\frac{\partial x}{\partial \widetilde{t}}\Big)\Big|_{\widetilde{x}=\widetilde{\lambda}=\widetilde{\lambda}_J}\widetilde{X}+O(\widetilde{X}^2).
\end{aligned}
$$

This shows the required results.

Q.E.D.

Let us now return to the proof of Proposition 3.1. Writing $\mathcal{J}$ as $\sum_{k\geq 0}\eta^{-k/2}\mathcal{J}_{k/2}$, we use the induction on k to confirm that $\mathcal{J}_{k/2}$ vanishes for any k. Let us first confirm $\mathcal{J}_0=0$. Since the right-hand side of (3.56) is of degree at most $(-1/2)$ in η, the degree 0 part of its left-hand side vanishes. Since we have assumed $\alpha_0\beta_0\neq 0$, this implies

$$
(3.63)\qquad \Big(2\widetilde{B}_{J,0}(\frac{\partial x_0}{\partial \widetilde{x}})^2\Big)\Big|_{\widetilde{x}=\widetilde{\lambda}=\widetilde{\lambda}_{J,0}}-\frac{\partial t_0}{\partial \widetilde{t}}=0.
$$

Here $\widetilde{B}_{J,0}$ etc. denote its degree 0 part in η, as usual. Hereafter we will abbreviate $\widetilde{\lambda}_{J,0}$ to $\widetilde{\lambda}_0$ for the brevity of the notation. On the other hand, (3.57) shows that this coincides with $\mathcal{J}_0\Big|_{\widetilde{x}=\widetilde{\lambda}_0}$, which implies $\mathcal{J}_0\mathcal{K}_0\Big|_{\widetilde{x}=\widetilde{\lambda}_0}$ also vanishes as we can readily verify $\mathcal{K}_0 = 1$. At the same time, (3.55) entails

$$\frac{\partial}{\partial\widetilde{x}}(\mathcal{J}_0\mathcal{K}_0) = 0, \tag{3.64}$$

and hence $\mathcal{J}_0\mathcal{K}_0$ is independent of $\widetilde{x}$. Thus $\mathcal{J}_0\mathcal{K}_0$ vanishes identically, that is, $\mathcal{J}_0$ vanishes identically.

Let us now suppose that $\mathcal{J}_{k/2}$ vanishes identically for $k \leq k_0$. Since $\dfrac{\partial x_0}{\partial\widetilde{x}}\Big|_{\widetilde{x}=\widetilde{\lambda}_0}$ is different from 0, $\dfrac{\partial x}{\partial\widetilde{x}}\Big|_{\widetilde{x}=\widetilde{\lambda}_J}$ is invertible and the inverse is of degree at most 0 in η. Hence we can deduce the following relation (3.65) from (3.57):

$$\begin{aligned}\Big(\ \frac{\partial\mathcal{J}}{\partial\widetilde{x}}\Big|_{\widetilde{x}=\widetilde{\lambda}_J}\Big)\Big/\Big(2\frac{\partial x}{\partial\widetilde{x}}\Big|_{\widetilde{x}=\widetilde{\lambda}_J}\Big)\\ = -\Big(\frac{\partial x}{\partial\widetilde{t}} - \frac{3}{2}\widetilde{B}_J\frac{\partial^2 x}{\partial\widetilde{x}^2} - \frac{\partial\widetilde{B}_J}{\partial\widetilde{x}}\frac{\partial x}{\partial\widetilde{x}}\Big)\Big|_{\widetilde{x}=\widetilde{\lambda}=\widetilde{\lambda}_J}.\end{aligned} \tag{3.65}$$

Then the induction hypothesis together with (3.65) implies that the right-hand side of (3.65) is of degree at most $(-(k_0+1)/2)$ in η. Since the right-hand side of (3.56) contains an extra factor $\eta^{-1/2}$, we then conclude that the degree $(-(k_0+1)/2)$ part in the left-hand side of (3.56) should vanish. In view of (3.57), we thus find

$$\mathcal{J}_{(k_0+1)/2}\Big|_{\widetilde{x}=\widetilde{\lambda}_0} = 0 \tag{3.66}$$

by the assumption $\alpha_0\beta_0 \neq 0$. On the other hand (3.55) and the induction hypothesis entail

$$\frac{\partial}{\partial\widetilde{x}}(\mathcal{J}_{(k_0+1)/2}\mathcal{K}_0) = 0. \tag{3.67}$$

Combining (3.66) and (3.67), we conclude $\mathcal{J}_{(k_0+1)/2}\mathcal{K}_0$ vanishes identically, and hence $\mathcal{J}_{(k_0+1)/2}$ vanishes identically. Thus the induction proceeds, completing the proof of Proposition 3.1.

Q.E.D.

In the course of the above proof of Proposition 3.1, we have also proved the following result.

Proposition 3.2 *Let $x = x_J(\widetilde{x}, \widetilde{t}, \eta)$ and $t = t_J(\widetilde{t}, \eta)$ be the same as in Proposition 3.1. Then each side of (3.56) vanishes, namely,*

$$\frac{\partial t}{\partial \widetilde{t}} = \Big(2\widetilde{B}_J(\frac{\partial x}{\partial \widetilde{x}})^2\Big)\Big|_{\widetilde{x}=\widetilde{\lambda}=\widetilde{\lambda}_J}, \tag{3.68}$$

$$\Big(\frac{\partial x}{\partial \widetilde{t}} - \frac{3}{2}\widetilde{B}_J\frac{\partial^2 x}{\partial \widetilde{x}^2} - \frac{\partial \widetilde{B}_J}{\partial \widetilde{x}}\frac{\partial x}{\partial \widetilde{x}}\Big)\Big|_{\widetilde{x}=\widetilde{\lambda}=\widetilde{\lambda}_J} = 0. \tag{3.69}$$

Corollary 3.1 *The pair $(\widetilde{\sigma}_J(\widetilde{t}, \eta), \widetilde{\rho}_J(\widetilde{t}, \eta))$ satisfies the following Hamiltonian system:*

$$\begin{cases} \dfrac{\partial \widetilde{\sigma}_J}{\partial \widetilde{t}} = -2\eta\Big(\widetilde{B}_J(\dfrac{\partial x}{\partial \widetilde{x}})^2\Big)\Big|_{\widetilde{x}=\widetilde{\lambda}=\widetilde{\lambda}_J}\widetilde{\rho}_J \\ \dfrac{\partial \widetilde{\rho}_J}{\partial \widetilde{t}} = -8\eta\Big(\widetilde{B}_J(\dfrac{\partial x}{\partial \widetilde{x}})^2\Big)\Big|_{\widetilde{x}=\widetilde{\lambda}=\widetilde{\lambda}_J}\widetilde{\sigma}_J. \end{cases} \tag{3.70}$$

Proof. Combination of (H_{can}), (3.27), (3.28) and (3.68) immediately entails (3.70).

Q.E.D.

Remark 3.3. (i) An important consequence of (3.68) is that $t = t_J(\widetilde{t}, \eta)$ is unique modulo additive constants. Otherwise stated, $t_J(\widetilde{t}, \eta)$ contains a free parameter (depending on η) in an additive manner. The existence of this free parameter corresponds to the fact that the relation between parameters $(\widetilde{\alpha}, \widetilde{\beta})$ and (A, B) is given by just (essentially) one relation, i.e., (3.25) and is not uniquely determined in this construction (cf. (3.32) for example).

(ii) Since (3.69) is described in terms of $x = x_J(\widetilde{x}, \widetilde{t}, \eta)$ only, it can be regarded, at least logically speaking, as a defining equation of the Painlevé transcendent $\widetilde{\lambda}_J(\widetilde{t}, \eta)$.

4 Local reduction of 2-parameter Painlevé transcendents.

The purpose of this section is to generalize our result concerning the local reduction of 0-parameter Painlevé transcendents ([KT1]) to cover arbitrary 2-parameter Painlevé transcendents, which have been constructed in [AKT2, Section 1] by multiple-scale analysis. The reduction will be shown by making use of a transformation that brings $(\mathcal{DSL}_J)$ to $(\mathcal{DSL}_{\mathrm{I}})$; although our main result in this article (Theorem 4.1 below) resembles to our previous result for 0-parameter Painlevé transcendents ([KT1, Theorem 2.1]) in its appearance, its analytic contents considerably differ from those of the previous one, reflecting the complicated singularity structure of the WKB solution of (SL_J) near its double turning point. In order to overcome the troubles originating from the complexity of the WKB solution near the double truning point, we make full use of Proposition 3.1 in our reasoning. Thus we are to handle three Schrödinger equations at once: (SL_J), (SL_{I}) and (Can). Accordingly we need to distinguish three variables, each of which is relevant to one of these three equations. If we strictly follow the notations of Section 2 and Section 3, we would use three symbols like $x, \widetilde{x}$ and $\widetilde{\widetilde{x}}$, etc. To avoid such cumbersome notations we put $\sim$ over variables and functions relevant to (SL_J), and use the symbol $(x_{\mathrm{I}}(x,t,\eta),\, t_{\mathrm{I}}(t,\eta))$ (resp., $(x_J(\widetilde{x},\widetilde{t},\eta), t_J(\widetilde{t},\eta))$) to denote the transformation that brings $(\mathcal{DSL}_{\mathrm{I}})$ (resp., $(\mathcal{DSL}_J)$) to $(\mathcal{DC}an)$. Although we seldom need it, we use the symbol $(x_{\mathrm{can}}, t_{\mathrm{can}})$ to denote the variables of (Can), if necessary. With these conventions, let us describe the geometric situation of our problem; since the geometric situation is determined by the top degree part of relevant objects, it is the same as the 0-parameter case discussed in [KT1]. We refer the reader to [KT1] for the definition of terminologies used here such as a Stokes curve for $\widetilde{\lambda}_J^{(0)}$ etc.

Let $\widetilde{t}_*$ be a point in a Stokes curve for $\widetilde{\lambda}_J^{(0)}$ that emanates from a simple turning point $\widetilde{r}$ for $\widetilde{\lambda}_J^{(0)}$. Then, unless $\widetilde{t}_* = \widetilde{r}$, there exist a simple turning point $\widetilde{a}(\widetilde{t})$ and a Stokes curve $\widetilde{\gamma}$ of (SL_J) that connects $\widetilde{a}(\widetilde{t})$ and the double turning point $\widetilde{\lambda}_0(\widetilde{t})$. See Proposition 2.1 of [KT1]

for the proof of the existence of such a Stokes curve $\widetilde{\gamma}$. Our main result below asserts that we can construct a transformation that brings (SL_J) to $(SL_{\rm I})$ on a neighborhood of $\widetilde{\gamma}$, even when we substitute a 2-parameter solution $(\widetilde{\lambda}_J, \widetilde{\nu}_J)$ (resp., $(\lambda_{\rm I}, \nu_{\rm I})$) to (λ, ν) that appears in (SL_J) (resp., $(SL_{\rm I})$). Note that we have to keep in mind that, in order to construct such a transformation, we should need a correspondence between the parameters $(\widetilde{\alpha}, \widetilde{\beta})$ and (α, β) contained in $(\widetilde{\lambda}_J, \widetilde{\nu}_J)$ and $(\lambda_{\rm I}, \nu_{\rm I})$ respectively. We also note that, although the transformation thus constructed should contain instanton terms like $\exp(\pm\phi(t)\eta)$, no x-dependent exponential function like $\exp(\chi(x,t)\eta)$ appears.

Theorem 4.1 *Let $\widetilde{t}_*$ and $\widetilde{\gamma}$ be chosen as above. Then for each 2-parameter multiple-scale solution $(\widetilde{\lambda}_J(\widetilde{t}, \eta; \widetilde{\alpha}, \widetilde{\beta}), \widetilde{\nu}_J(\widetilde{t}, \eta; \widetilde{\alpha}, \widetilde{\beta}))$ of (H_J) there exists a 2-parameter multiple-scale solution $(\lambda_{\rm I}(t, \eta; \alpha, \beta), (\nu_{\rm I}(t, \eta; \alpha, \beta))$ of $(H_{\rm I})$ for which the following holds:*

There exist a neighborhood $\widetilde{U}$ of $\widetilde{\gamma}$, a neighborhood $\widetilde{V}$ of $\widetilde{t}_*$ and holomorphic functions $x_{j/2}(\widetilde{x}, \widetilde{t}, \eta)$ $(j = 0, 1, 2, \cdots)$ on $\widetilde{U} \times \widetilde{V}$ and $t_{j/2}(\widetilde{t}, \eta)$ on $\widetilde{V}$ which satisfy the relations below:

(i) The function t_0 is independent of η and satisfies

$$\widetilde{\phi}_J(\widetilde{t}) = \phi_{\rm I}(t_0(\widetilde{t})). \tag{4.1}$$

(ii) $\dfrac{dt_0}{d\widetilde{t}}$ never vanishes on $\widetilde{V}$.

(iii) The function x_0 is also independent of η and satisfies

$$x_0(\widetilde{\lambda}_{J,0}(\widetilde{t}), \widetilde{t}) = \lambda_{{\rm I},0}(t_0(\widetilde{t})), \tag{4.2}$$

$$x_0(\widetilde{a}(\widetilde{t}), \widetilde{t}) = -2\lambda_{{\rm I},0}(t_0(\widetilde{t})) \ (= a(t_0(\widetilde{t}))). \tag{4.3}$$

(iv) $\dfrac{\partial x_0}{\partial \widetilde{x}}$ never vanishes on $\widetilde{U} \times \widetilde{V}$.

(v) $x_{1/2}$ and $t_{1/2}$ vanish identically.

(vi) The η-dependence of $x_{j/2}$ and $t_{j/2}$ $(j \geq 2)$ is only through instanton terms $\exp(l\widetilde{\Phi}_J(\widetilde{t}, \eta))$ $(l = j - 2 - 2k$ with $0 \leq k \leq j-2)$ that appears in the 2-parameter multiple-scale solution $(\widetilde{\lambda}_J(\widetilde{t}; \widetilde{\alpha}, \widetilde{\beta}), \widetilde{\nu}_J(\widetilde{t}; \widetilde{\alpha}, \widetilde{\beta}))$ of (H_J).

(vii) For $x(\widetilde{x},\widetilde{t},\eta)=\sum_{j\geq 0}x_{j/2}(\widetilde{x},\widetilde{t},\eta)\eta^{-j/2}$ *and* $t(\widetilde{t},\eta)=\sum_{j\geq 0}t_{j/2}(\widetilde{t},\eta)\eta^{-j/2}$*, the following relations hold:*

$$x(\widetilde{\lambda}_J(\widetilde{t},\eta;\widetilde{\alpha},\widetilde{\beta}),\widetilde{t},\eta)=\lambda_{\mathrm{I}}(t(\widetilde{t},\eta),\eta;\alpha,\beta), \tag{4.4}$$

$$\widetilde{Q}_J(\widetilde{x},\widetilde{t},\eta)=\left(\frac{\partial x(\widetilde{x},\widetilde{t},\eta)}{\partial\widetilde{x}}\right)^2 Q_{\mathrm{I}}(x(\widetilde{x},\widetilde{t},\eta),t(\widetilde{t},\eta),\eta) -\frac{1}{2}\eta^{-2}\{x(\widetilde{x},\widetilde{t},\eta);\widetilde{x}\}, \tag{4.5}$$

where the 2*-parameter solution of* (H_J) *and* (H_{I}) *are substituted into* (λ,ν) *in the coefficients of* $\widetilde{Q}_J$ *and* Q_{I}*, respectively, and* $\{x;\widetilde{x}\}$ *denotes the Schwarzian derivative.*

Remark 4.1 The relation (4.4) in the above describes the required local reduction of 2-parameter solutions $\widetilde{\lambda}_J$ to λ_{I}.

Proof of Theorem 4.1. Since $x_0(\widetilde{x},\widetilde{t})$ and $t_0(\widetilde{t})$, i.e., the top degree part of the transformation, satisfying (i)$\sim$(iv) have already been constructed in [KT1, Theorem 2.1], it suffices to discuss the higher degree part of the transformation. Let us first fix the correspondence of the parameters: For a given pair of parameters $(\widetilde{\alpha},\widetilde{\beta})=(\sum_{n\geq 0}\widetilde{\alpha}_n\eta^{-n},\ \sum_{n\geq 0}\widetilde{\beta}_n\eta^{-n})$ satisfying (1.28) we choose $(A(\eta),B(\eta))=(\sum_{n\geq 0}A_n\eta^{-n},\sum_{n\geq 0}B_n\eta^{-n})$ in $(\sigma_{\mathrm{can}}(t_{\mathrm{can}},\eta),\rho_{\mathrm{can}}(t_{\mathrm{can}},\eta))$ and (α,β) so that

$$E_{\mathrm{I}}=E_{\mathrm{can}}=\widetilde{E}_J \tag{4.6}$$

holds; Lemma 3.2 guarantees that such a choice of parameters is possible. Then, in view of the argument in Section 3, we can construct $t_J(\widetilde{t},\eta)$ and $t_{\mathrm{I}}(t,\eta)$ so that they satisfy

$$\sigma_{\mathrm{can}}(t_J(\widetilde{t},\eta),\eta)=\widetilde{\sigma}_J(\widetilde{t},\eta) \tag{4.7}$$

$$\rho_{\mathrm{can}}(t_J(\widetilde{t},\eta),\eta)=\widetilde{\rho}_J(\widetilde{t},\eta) \tag{4.8}$$

and

$$\sigma_{\mathrm{can}}(t_{\mathrm{I}}(t,\eta),\eta)=\sigma_{\mathrm{I}}(t,\eta) \tag{4.9}$$

$$\rho_{\mathrm{can}}(t_{\mathrm{I}}(t,\eta),\eta)=\rho_{\mathrm{I}}(t,\eta). \tag{4.10}$$

The transformations $(x_J(\widetilde{x},\widetilde{t},\eta),t_J(\widetilde{t},\eta))$ and $(x_{\rm I}(x,t,\eta),t_{\rm I}(t,\eta))$ respectively bring $(\mathcal{DSL}_J)$ and $(\mathcal{DSL}_{\rm I})$ to the same canonical system $(\mathcal{DC}an)$. Further, since

$$\frac{dt_{\rm I,0}}{dt}=\frac{1}{2}\frac{\partial\phi_{\rm I}}{\partial t}=\sqrt{3\lambda_{\rm I,0}(t)}, \tag{4.11}$$

$t_{\rm I}(t,\eta)$ is invertible near $t_0(\widetilde{t})$ where $t_{\rm I,0}(t_0(\widetilde{t}))=t_{J,0}(\widetilde{t})$ (cf. (3.30) and (4.1)) and $\widetilde{t}$ is sufficiently close to $\widetilde{t}_*$. Similarly $x_{\rm I}(x,t,\eta)$ is also invertible near $(x_0(\widetilde{\lambda}_{J,0}(\widetilde{t}),\widetilde{t}),t_0(\widetilde{t}))=(\widetilde{\lambda}_{\rm I,0}(t_0(\widetilde{t})),t_0(\widetilde{t}))$ by (2.6). Hence the transformation

$$\begin{cases} x_{\rm can}=x_{\rm I}(x,t,\eta)\\ t_{\rm can}=t_{\rm I}(t,\eta)\end{cases} \tag{4.12}$$

can be inverted as

$$\begin{cases} x=x_{\rm I}^{-1}(x_{\rm can},t_{\rm can},\eta)\\ t=t_{\rm I}^{-1}(t_{\rm can},\eta)\end{cases} \tag{4.13}$$

and we can define $(x(\widetilde{x},\widetilde{t},\eta),t(\widetilde{t},\eta))$ by the following:

$$\begin{cases} x(\widetilde{x},\widetilde{t},\eta)=x_{\rm I}^{-1}(x_J(\widetilde{x},\widetilde{t},\eta),t_J(\widetilde{t},\eta),\eta)\\ t(\widetilde{t},\eta)=t_{\rm I}^{-1}(t_J(\widetilde{t},\eta),\eta).\end{cases} \tag{4.14}$$

Note that (4.14) can be rewritten in the following way:

$$\begin{cases} x_{\rm I}(x(\widetilde{x},\widetilde{t},\eta),t(\widetilde{t},\eta),\eta)=x_J(\widetilde{x},\widetilde{t},\eta)\\ t_{\rm I}(t(\widetilde{t},\eta),\eta)=t_J(\widetilde{t},\eta).\end{cases} \tag{4.15}$$

It then follows from (3.46) that the following equality holds near $\widetilde{x}=\widetilde{\lambda}_{J,0}(\widetilde{t})$:

$$\widetilde{S}_{J,\rm odd}(\widetilde{x},\widetilde{t},\eta)=\frac{\partial x}{\partial\widetilde{x}}S_{\rm I,odd}(x(\widetilde{x},\widetilde{t},\eta),t(\widetilde{t},\eta),\eta). \tag{4.16}$$

We also note that (4.7)$\sim$(4.10) and (4.15) imply

$$\sigma_{\rm I}(t(\widetilde{t},\eta),\eta)=\widetilde{\sigma}_J(\widetilde{t},\eta), \tag{4.17}$$

$$\rho_{\rm I}(t(\widetilde{t},\eta),\eta)=\widetilde{\rho}_J(\widetilde{t},\eta). \tag{4.18}$$

The transformation $(x(\widetilde{x},\widetilde{t},\eta),t(\widetilde{t},\eta))$ thus defined is "almost the required" one. As a matter of fact, $x_0(\widetilde{x},\widetilde{t})$ and $t_0(\widetilde{t})$ were constructed in [KT1] so that they satisfy the top degree part of (4.15) (cf. (3.30) and (3.13) in [AKT2] and [KT1, Section 2]). It is, however, not yet the final one; although being holomorphic near $\widetilde{x}=\widetilde{\lambda}_{J,0}(\widetilde{t})$, $x(\widetilde{x},\widetilde{t},\eta)$ may be singular at $\widetilde{x}=\widetilde{a}(\widetilde{t})$. Note that each $x_{j/2}(\widetilde{x},\widetilde{t},\eta)$ satisfies a differential equation which has a regular singularity at both $\widetilde{x}=\widetilde{\lambda}_{J,0}(\widetilde{t})$ and $\widetilde{x}=\widetilde{a}(\widetilde{t})$ in view of (4.17) and (4.18). (Cf. (4.46) below. See also [AKT2, Section 4].) In what follows we try to adjust the free parameters that remain in $t(\widetilde{t},\eta)$ (cf. Remark 3.1 (i)) so that this transformation may become non-singular at $\widetilde{x}=\widetilde{a}(\widetilde{t})$. This is a kind of "matching problem" (cf. e.g. [AKT1, Section 3], [KT1, Section 2], etc.). (In the sense that the transformation defined by (4.14) is not the final one, using the symbol $x(\widetilde{x},\widetilde{t},\eta)$ is a slight abuse of notations, but we hope there is no fear of confusions.)

To make the argument run smoothly, we introduce the following auxiliary transformation

$$y(\widetilde{x},\widetilde{t},\eta)=\sum_{j\geq 0}y_{j/2}(\widetilde{x},\widetilde{t},\eta)\eta^{-j/2} \tag{4.19}$$

that brings (SL_J) to (SL_{I}) near the simple turning point $\widetilde{x}=\widetilde{a}(\widetilde{t})$ provided that the transformation $t(\widetilde{t},\eta)=t_{\mathrm{I}}^{-1}(t_J(\widetilde{t},\eta),\eta)$ in the t-variable is substituted into t in the coefficients of Q_{I}. Otherwise stated, $y(\widetilde{x},\widetilde{t},\eta)$ satisfies the following relation near $\widetilde{x}=\widetilde{a}(\widetilde{t})$:

$$\widetilde{S}_{J,\mathrm{odd}}(\widetilde{x},\widetilde{t},\eta)=\frac{\partial y}{\partial\widetilde{x}}S_{\mathrm{I},\mathrm{odd}}(y(\widetilde{x},\widetilde{t},\eta),t(\widetilde{t},\eta),\eta). \tag{4.20}$$

As $\widetilde{Q}_J$ and Q_{I} contain instanton terms in their coefficients, our reduction theorem near a simple turning point ([AKT1]) cannot be applied immediately to this case. But, the reasoning can be given in exactly the same manner, and in the course of the proof we can readily confirm that its instanton structure is the same as that of x_J (cf. (2.4)), i.e., $y_{j/2}$ consists of l-instaton terms with $l=j-2-2k$, $0\leq k\leq j-2$.

Now, to attain the "matching", we introduce the following functions:

$$R(x,t,\eta)=\int_{-2\lambda_{\mathrm{I},0}(t)}^{x}\eta^{-1}S_{\mathrm{I,odd}}(z,t,\eta)dz, \tag{4.21}$$

$$\mathcal{F}(\widetilde{x},\widetilde{t},\eta)=R(x(\widetilde{x},\widetilde{t},\eta),t(\widetilde{t},\eta),\eta), \tag{4.22}$$

$$\mathcal{G}(\widetilde{x},\widetilde{t},\eta)=R(y(\widetilde{x},\widetilde{t},\eta),t(\widetilde{t},\eta),\eta). \tag{4.23}$$

Here the factor η^{-1} in the right-hand side of (4.21) is just for convenience. It is clear by (4.16) and (4.20) that we find the following:

$$\frac{\partial\mathcal{F}}{\partial\widetilde{x}}=\eta^{-1}\widetilde{S}_{J,\mathrm{odd}}(\widetilde{x},\widetilde{t},\eta), \tag{4.24}$$

$$\frac{\partial\mathcal{G}}{\partial\widetilde{x}}=\eta^{-1}\widetilde{S}_{J,\mathrm{odd}}(\widetilde{x},\widetilde{t},\eta). \tag{4.25}$$

Thus we obtain

$$\frac{\partial(\mathcal{F}-\mathcal{G})}{\partial\widetilde{x}}=0. \tag{4.26}$$

Next we try to prove

$$\frac{\partial(\mathcal{F}-\mathcal{G})}{\partial\widetilde{t}}=0. \tag{4.27}$$

Once this is proved, it implies that $\mathcal{F}-\mathcal{G}$ is independent of both $\widetilde{x}$ and $\widetilde{t}$; this is clearly the first step to attain the "matching". To prove (4.27), we first note the following

Sublemma 4.1 *For $x(\widetilde{x},\widetilde{t},\eta)$ and $t(\widetilde{t},\eta)$ constructed above, we find*

$$\frac{\partial x}{\partial\widetilde{t}}=\widetilde{A}_J(\widetilde{x},\widetilde{t})\frac{\partial x}{\partial\widetilde{x}}-A_{\mathrm{I}}(x(\widetilde{x},\widetilde{t},\eta),t(\widetilde{t},\eta))\frac{\partial t}{\partial\widetilde{t}}. \tag{4.28}$$

Proof of Sublemma 4.1. It follows from (3.52) that the following equalities hold:

$$\widetilde{A}_J(\widetilde{x},\widetilde{t})\frac{\partial x_J}{\partial\widetilde{x}}-\frac{\partial x_J}{\partial\widetilde{t}}-A_{\mathsf{can}}(x_J,t_J)\frac{\partial t_J}{\partial\widetilde{t}}=0, \tag{4.29}$$

$$A_{\mathrm{I}}(x,t)\frac{\partial x_{\mathrm{I}}}{\partial x}-\frac{\partial x_{\mathrm{I}}}{\partial t}-A_{\mathsf{can}}(x_{\mathrm{I}},t_{\mathrm{I}})\frac{\partial t_{\mathrm{I}}}{\partial t}=0. \tag{4.30}$$

On the other hand, the relation (4.15) entails

$$\frac{\partial x_{\rm I}}{\partial x}\bigg|_{\substack{x=x(\widetilde{x},\widetilde{t},\eta)\\ t=t(\widetilde{t},\eta)}}\frac{\partial x}{\partial\widetilde{x}}=\frac{\partial x_J}{\partial\widetilde{x}},\quad \frac{\partial t_{\rm I}}{\partial t}\bigg|_{t=t(\widetilde{t},\eta)}\frac{\partial t}{\partial\widetilde{t}}=\frac{\partial t_J}{\partial\widetilde{t}}. \tag{4.31}$$

Differentiating (4.15) with respect to $\widetilde{t}$, we also find

$$\frac{\partial x_{\rm I}}{\partial x}\bigg|_{\substack{x=x(\widetilde{x},\widetilde{t},\eta)\\ t=t(\widetilde{t},\eta)}}\frac{\partial x}{\partial\widetilde{t}}+\frac{\partial x_{\rm I}}{\partial t}\bigg|_{\substack{x=x(\widetilde{x},\widetilde{t},\eta)\\ t=t(\widetilde{t},\eta)}}\frac{\partial t}{\partial\widetilde{t}}=\frac{\partial x_J}{\partial\widetilde{t}}. \tag{4.32}$$

Substituting (4.29) and (4.30) into (4.32), we obtain

$$\begin{aligned}&\frac{\partial x_{\rm I}}{\partial x}\bigg|_{\substack{x=x(\widetilde{x},\widetilde{t},\eta)\\ t=t(\widetilde{t},\eta)}}\frac{\partial x}{\partial\widetilde{t}}+\bigg(A_{\rm I}(x,t)\frac{\partial x_{\rm I}}{\partial x}\\ &-A_{\rm can}(x_{\rm I}(x,t,\eta),t_{\rm I}(t,\eta))\frac{\partial t_{\rm I}}{\partial t}\bigg|_{\substack{x=x(\widetilde{x},\widetilde{t},\eta)\\ t=t(\widetilde{t},\eta)}}\bigg)\frac{\partial t}{\partial\widetilde{t}}\\ &=\widetilde{A}_J(\widetilde{x},\widetilde{t})\frac{\partial x_J}{\partial\widetilde{x}}-A_{\rm can}(x_J,t_J)\frac{\partial t_J}{\partial\widetilde{t}}.\end{aligned} \tag{4.33}$$

It then follows from (4.15) and (4.31) that

$$\Big(\frac{\partial x_{\rm I}}{\partial x}\Big(\frac{\partial x}{\partial\widetilde{t}}+A_{\rm I}\Big(\frac{\partial t}{\partial\widetilde{t}}\Big)\Big)\Big)\bigg|_{\substack{x=x(\widetilde{x},\widetilde{t},\eta)\\ t=t(\widetilde{t},\eta)}}=\widetilde{A}_J(\widetilde{x},\widetilde{t})\frac{\partial x_J}{\partial\widetilde{x}}. \tag{4.34}$$

Hence (4.31) implies

$$\Big(\frac{\partial x}{\partial\widetilde{t}}+A_{\rm I}(x(\widetilde{x},\widetilde{t},\eta),t(\widetilde{t},\eta))\Big)\frac{\partial t}{\partial\widetilde{t}}=\widetilde{A}_J\frac{\partial x}{\partial\widetilde{x}}. \tag{4.35}$$

This completes the proof of Sublemma 4.1. Q.E.D.

Let us resume the proof of Theorem 4.1. It follows from (4.28) that

$$\begin{aligned}\frac{\partial\mathcal{F}}{\partial\widetilde{t}}&=\frac{\partial R}{\partial x}\frac{\partial x}{\partial\widetilde{t}}+\frac{\partial R}{\partial t}\frac{\partial t}{\partial\widetilde{t}}\\ &=\eta^{-1}S_{\rm I,odd}\Big(\widetilde{A}_J\frac{\partial x}{\partial\widetilde{x}}-A_{\rm I}(x(\widetilde{x},\widetilde{t},\eta),t(\widetilde{t},\eta))\frac{\partial t}{\partial\widetilde{t}}\Big)+\frac{\partial R}{\partial t}\frac{\partial t}{\partial\widetilde{t}}.\end{aligned} \tag{4.36}$$

Since $S_{\mathrm{I,odd}}$ obeys the deformation equation (3.48), and since the integral (4.21) should be understood as a contour integral at the lower end-point by the square-root character of the integrand (cf. [KT1, Proof of Proposition 2.2]), we find

$$\begin{aligned} \frac{\partial R}{\partial t} &= \eta^{-1} \int_{-2\lambda_{\mathrm{I},0}(t)}^{x} \frac{\partial}{\partial t} S_{\mathrm{I,odd}}(z,t,\eta)dz \\ &= \eta^{-1} A_{\mathrm{I}}(x,t) S_{\mathrm{I,odd}}(x,t,\eta). \end{aligned} \tag{4.37}$$

Hence (4.36) and (4.37) together with (4.16) imply

$$\frac{\partial \mathcal{F}}{\partial \widetilde{t}} = \eta^{-1} \widetilde{A}_J \frac{\partial x}{\partial \widetilde{x}} S_{\mathrm{I,odd}} = \eta^{-1} \widetilde{A}_J \widetilde{S}_{J,\mathrm{odd}}. \tag{4.38}$$

Since the relation (4.28) is valid even if we replace $x(\widetilde{x},\widetilde{t},\eta)$ with $y(\widetilde{x},\widetilde{t},\eta)$ (cf. [KT1, Proof of Proposition 2.2]), the same reasoning leads to

$$\frac{\partial \mathcal{G}}{\partial \widetilde{t}} = \eta^{-1} \widetilde{A}_J \widetilde{S}_{J,\mathrm{odd}}. \tag{4.39}$$

Thus we obtain (4.27).

Combining (4.26) and (4.27), we conclude

$$\mathcal{F} - \mathcal{G} = \sum_{j \geq 0} C_{j/2} \eta^{-j/2} \tag{4.40}$$

holds with a genuine constant $C_{j/2}$.

Let us now prove the following claim $(\mathcal{C})_j$ for any j by using the induction on j:

$(\mathcal{C})_j$ A correct choice of $t_{j/2}$ entails the vanishing of $C_{j/2}$ and the coincidence of $x_{j/2}$ and $y_{j/2}$.

It is clear that $(\mathcal{C})_0$ holds. Since $x_{1/2} = y_{1/2} = 0$ and $t_{1/2} = 0$, $(\mathcal{C})_1$ is also valid. Let us suppose $(\mathcal{C})_k$ holds for every $k < j$ and suppose

that j is even. It follows from the definition that

$$\begin{aligned}(4.41)\qquad \mathcal{F}_{j/2}-\mathcal{G}_{j/2} \;=\; & \eta^{-1}\int_{-2\lambda_{\mathrm{I},0}(t_0(\widetilde{t}))}^{x(\widetilde{x},\widetilde{t},\eta)} S_{\mathrm{I},-1}(z,t(\widetilde{t},\eta))dz \\ & -\eta^{-1}\int_{-2\lambda_{\mathrm{I},0}(t_0(\widetilde{t}))}^{y(\widetilde{x},\widetilde{t},\eta)} S_{\mathrm{I},-1}(z,t(\widetilde{t},\eta))dz \\ & +\mathcal{H}(j/2),\end{aligned}$$

where $\mathcal{H}(j/2)$ consists of terms relevant to $S_{\mathrm{I},l}$ with $l\geq 0$. As $\mathcal{H}(j/2)$ contributes to the term of degree $(-j/2)$ in η, it follows from the definition of $\mathcal{H}(j/2)$ that only $x_{l'/2}$ and $y_{l'/2}$ with $l'<j$ are relevant to $\mathcal{H}(j/2)$; by the induction hypothesis, then, $\mathcal{H}(j/2)$ should vanish. Let $f(x,t)$ denote

$$(4.42)\qquad \int_{-2\lambda_{\mathrm{I},0}(t)}^{x} S_{\mathrm{I},-1}(z,t)dz.$$

Then by the Taylor expansion we find

$$\begin{aligned}(4.43)\qquad & f(x(\widetilde{x},\widetilde{t},\eta),t)-f(y(\widetilde{x},\widetilde{t},\eta),t) \\ & = S_{\mathrm{I},-1}(x_0,t)\{\eta^{-1}x_1+\cdots+\eta^{-j/2}x_{j/2}+\cdots) \\ & \qquad\qquad -(\eta^{-1}y_1+\cdots+\eta^{-j/2}y_{j/2}+\cdots)\} \\ & +\frac{1}{2!}\frac{\partial S_{\mathrm{I},-1}}{\partial x}(x_0,t)\{(\eta^{-1}x_1+\cdots)^2-(\eta^{-1}y_1+\cdots)^2\} \\ & +\cdots .\end{aligned}$$

Hence terms that contain $x_{j/2}$ or $y_{j/2}$ and that contribute to $\mathcal{F}_{j/2}-\mathcal{G}_{j/2}$ originate solely from

$$(4.44)\qquad S_{\mathrm{I},-1}(x_0,t)(\eta^{-j/2}x_{j/2}-\eta^{-j/2}y_{j/2});$$

other terms like $\frac{1}{2!}\frac{\partial S_{\mathrm{I},-1}}{\partial x}(2\eta^{-1-j/2}x_1x_{j/2})$ do not contribute to $\mathcal{F}_{j/2}-\mathcal{G}_{j/2}$. Thus we obtain

$$\begin{aligned}(4.45)\qquad & 2\sqrt{x_0+2\lambda_{\mathrm{I},0}(t_0(\widetilde{t}))}(x_0-\lambda_{\mathrm{I},0}(t_0(\widetilde{t})))(x_{j/2}(\widetilde{x},\widetilde{t},\eta) \\ & -y_{j/2}(\widetilde{x},\widetilde{t},\eta))=C_{j/2}.\end{aligned}$$

On the other hand, the same argument together with the deformation equation (3.48) for $S_{\mathrm{I,odd}}$ also shows that the relations (4.16) and (4.20) respectively imply the following differential equations for $x_{j/2}$ and $y_{j/2}$:

$$Lx_{j/2} = R(x_0, \cdots, x_{(j-1)/2}, t_0, \cdots, t_{(j-1)/2}), \tag{4.46}$$

$$Ly_{j/2} = R(y_0, \cdots, y_{(j-1)/2}, t_0, \cdots, t_{(j-1)/2}), \tag{4.47}$$

where L is a differential operator defined by

$$\begin{aligned} Lw &= S_{\mathrm{I},-1}(x_0(\widetilde{x},\widetilde{t}), t_0(\widetilde{t}))\frac{\partial w}{\partial \widetilde{x}} \\ &\quad + \frac{\partial x_0}{\partial \widetilde{x}}\frac{\partial S_{\mathrm{I},-1}}{\partial x}(x_0(\widetilde{x},\widetilde{t}), t_0(\widetilde{t}))w \\ &\quad + \frac{\partial x_0}{\partial \widetilde{x}}\frac{\partial S_{\mathrm{I},-1}}{\partial t}(x_0(\widetilde{x},\widetilde{t}), t_0(\widetilde{t}))t_{j/2}(\widetilde{t},\eta) \\ &= \frac{\partial}{\partial \widetilde{x}}\{2\sqrt{x_0 + 2\lambda_{\mathrm{I},0}(t_0(\widetilde{t}))}(x_0 - \lambda_{\mathrm{I},0}(t_0(\widetilde{t})))w\} \\ &\quad + \frac{\partial}{\partial \widetilde{x}}\{\sqrt{x_0 + 2\lambda_{\mathrm{I},0}(t_0(\widetilde{t}))}\}t_{j/2}(\widetilde{t},\eta) \end{aligned} \tag{4.48}$$

and $R(x_0, \cdots, x_{(j-1)/2}, t_0, \cdots, t_{(j-1)/2})$ is determined by $x_{k/2}$ and $t_{k'/2}$ with $k, k' \leq j-1$. The induction hypothesis tells us that the right-hand side of (4.47) coincides with that of (4.46) and, since $x_{j/2}$ is non-singular near $\widetilde{x} = \widetilde{\lambda}_{J,0}(\widetilde{t})$ by its definition, it must be holomorphic at $\widetilde{x} = \widetilde{\lambda}_{J,0}(\widetilde{t})$. Hence $y_{j/2}$ has an at most simple pole near $\widetilde{x} = \widetilde{\lambda}_{J,0}(\widetilde{t})$. To be more precise, $y_{j/2}$ has the form

$$\frac{d_{j/2} - t_{j/2}(\widetilde{t},\eta)}{2(x_0 - \lambda_{\mathrm{I},0}(t_0(\widetilde{t})))} + (\text{regular function near } \widetilde{x} = \widetilde{\lambda}_{J,0}(\widetilde{t})), \tag{4.49}$$

where $d_{j/2} = d_{j/2}(\widetilde{t},\eta)$ is determined by $x_{k/2}$ and $t_{k'/2}$ with $k, k' \leq j-1$ and, in particular, independent of $t_{j/2}(\widetilde{t},\eta)$. Substituting (4.49) into (4.45) and evaluating the resulting function at $\widetilde{x} = \widetilde{\lambda}_{J,0}(\widetilde{t})$, we find

$$\sqrt{3\lambda_{\mathrm{I},0}(t_0(\widetilde{t}))}(t_{j/2}(\widetilde{t},\eta) - d_{j/2}(\widetilde{t},\eta)) = C_{j/2}. \tag{4.50}$$

We now recall that $t(\widetilde{t},\eta)$ contains free parameters which are contained in $t_{\mathrm{I}}(t,\eta)$; if we express t_{I}^{-1} as

$$s_0+\eta^{-1}s_1+\eta^{-3/2}s_{3/2}+\cdots, \tag{4.51}$$

then we have

$$\begin{aligned} t &= s_0(t_{\mathrm{I},0}+\eta^{-1}t_{\mathrm{I},1}+\cdots) \\ &\quad +\eta^{-1}s_1(t_{\mathrm{I},0}+\eta^{-1}t_{\mathrm{I},1}+\cdots)+\cdots \\ &= s_0(t_{\mathrm{I},0}(t))+\cdots+\eta^{-j/2}\{s_0'(t_{\mathrm{I},0}(t))t_{\mathrm{I},j/2}(t) \\ &\quad +s_{j/2}(t_{\mathrm{I},0}(t))+r(j/2)\}+\cdots, \end{aligned} \tag{4.52}$$

where $r(j/2)$ are determined by $s_{k/2}$ and $t_{k'/2}$ with $k,k'\le j-1$. As a direct computation shows

$$t_{\mathrm{I},0}(t)=\frac{1}{2}\phi_{\mathrm{I}}(t)=2^{\frac{7}{4}}3^{\frac{1}{4}}5^{-1}(-1)^{\frac{1}{4}}t^{\frac{5}{4}} \tag{4.53}$$

and hence

$$\frac{ds_0}{dt_{\mathsf{can}}}=2^{\frac{3}{5}}3^{-\frac{1}{5}}5^{-\frac{1}{5}}(-1)^{-\frac{1}{5}}t_{\mathsf{can}}^{-\frac{1}{5}}, \tag{4.54}$$

each $s_{j/2}$ (j: even) contains a free parameter C in the form $Ct^{-1/4}$ in an additive manner and so does $t_{j/2}(\widetilde{t},\eta)$. Therefore a suitable choice of the free parameter C makes $C_{j/2}$ vanish at some fixed $\widetilde{t}$, say, at $\widetilde{t}_*$. Hence (4.45) proves

$$x_{j/2}=y_{j/2}. \tag{4.55}$$

Since j is even by the assumption, $j+1$ is odd. This implies $C_{(j+1)/2}$ consists of odd instanton terms. But, then $C_{(j+1)/2}$ cannot be a constant unless it vanishes. Hence $C_{(j+1)/2}$ vanishes, and the above reasoning guarantees

$$x_{(j+1)/2}=y_{(j+1)/2}. \tag{4.56}$$

Thus the induction proceeds; $(\mathcal{C})_j$ is valid for every j. Otherwise stated, the transformation $x(\widetilde{x},\widetilde{t},\eta)$ for the adjusted constant now extends analytically onto a neighborhood of the simple turning point $\widetilde{x}=\widetilde{a}(\widetilde{t})$.

Since all free parameters in $t(\widetilde{t},\eta)$ have been fixed, the correspondence of constants $(\widetilde{\alpha}(\eta),\widetilde{\beta}(\eta))$ and $(\alpha(\eta),\beta(\eta))$ is also fixed. The properties (v) and (vi) of $x_{j/2}$ and $t_{j/2}$ have been confirmed in the course of their construction, and since (4.5) is another version of (4.16) (cf. [AKT1, Section 3]), what remains to be proved is (4.4). This is actually another consequence of Sublemma 4.1; in fact, (4.28) reads as follows:

$$
\begin{aligned}
(4.57)\qquad & 2\widetilde{B}_J(\widetilde{x},\widetilde{\lambda}_J,\widetilde{t})\frac{x(\widetilde{x},\widetilde{t},\eta)-\lambda_{\mathrm{I}}(t(\widetilde{t},\eta),\eta)}{\widetilde{x}-\widetilde{\lambda}_J(\widetilde{t},\eta)}\frac{\partial x}{\partial\widetilde{x}} \\
& =\frac{\partial t}{\partial\widetilde{t}}+2(x(\widetilde{x},\widetilde{t},\eta)-\lambda_{\mathrm{I}}(t(\widetilde{t},\eta),\eta))\frac{\partial x}{\partial\widetilde{t}}.
\end{aligned}
$$

Since the right-hand side of (4.57) is non-singular at $\widetilde{x}=\widetilde{\lambda}_J(\widetilde{t},\eta)$, we find

$$
(4.58)\qquad x(\widetilde{\lambda}_J(\widetilde{t},\eta),\widetilde{t},\eta)=\lambda_{\mathrm{I}}(t(\widetilde{t},\eta),\eta),
$$

completing the proof of Theorem 4.1. Q.E.D.

Appendix. Some explicit formulas for E_J,ρ_J and σ_J.

In this Appendix we list up the concrete form of the first few terms of E_J,ρ_J and σ_J in the case of $J=\mathrm{I}$ and II for the reader's convenience, although a part of them has already been listed in [AKT2]. (Using this chance, we should like to correct some misprints in [AKT2]: In the formula (A.13) in [AKT2, Appendix] $(\frac{1}{12}\pm\frac{3}{2}\alpha_0\beta_0+\frac{17}{2}\alpha_0^2\beta_0^2)$ should be replaced by $(-\frac{7}{48}\pm\frac{3}{2}\alpha_0\beta_0+\frac{17}{2}\alpha_0^2\beta_0^2)$.) As in the other sections we use

the following notation in what follows:

$$
\begin{aligned}
\alpha_0, \beta_0 \ &: \quad \text{arbitrary complex numbers,} \\
\phi_J &= \int^t \sqrt{\frac{\partial F_J}{\partial \lambda}(\lambda_{J,0}(s), s)}\, ds \qquad (J = \mathrm{I}, \mathrm{II}), \\
e^{\Phi_J} &= (\theta_J(t)\eta^2)^{\alpha_0\beta_0} e^{\phi_J \eta} \\
&= \begin{cases} \left((12\lambda_{\mathrm{I},0})^5 \eta^2\right)^{\alpha_0\beta_0} e^{\phi_{\mathrm{I}}\eta} & (J = \mathrm{I}) \\ \left(\dfrac{16\Delta^5\eta^2}{(2\lambda_{\mathrm{II},0}^2 + t)^2}\right)^{\alpha_0\beta_0} e^{\phi_{\mathrm{II}}\eta} & (J = \mathrm{II}). \end{cases}
\end{aligned}
$$

Here Δ denotes $6\lambda_{\mathrm{II},0}^2 + t$. We may omit the subscript J if there is no fear of confusions. We also assume all α_n and β_n except α_0 and β_0 are equal to zero in the following description of E, ρ and σ.

$$
\begin{aligned}
[\text{For} J &= \mathrm{I}] \\
E_0 &= -8\alpha_0\beta_0 \\
E_{1/2} &= E_1 = 0 \\
\rho_0 &= \sqrt{2}(-\alpha_0 e^{\Phi} + \beta_0 e^{-\Phi}) \\
\rho_{1/2} &= \frac{8\sqrt{2}}{3}(12\lambda_0)^{-5/4}\Big(-\alpha_0^2 e^{2\Phi} + \beta_0^2 e^{-2\Phi}\Big) \\
\rho_1 &= \sqrt{2}(12\lambda_0)^{-5/2}\Big[-\frac{89}{18}\alpha_0^3 e^{3\Phi} + \Big(\frac{16}{3} + \frac{167}{18}\alpha_0\beta_0 + 282\alpha_0^2\beta_0^2\Big)\alpha_0 e^{\Phi} \\
&\quad -\Big(-\frac{16}{3} + \frac{167}{18}\alpha_0\beta_0 - 282\alpha_0^2\beta_0^2\Big)\beta_0 e^{-\Phi} + \frac{89}{18}\beta_0^3 e^{-3\Phi}\Big] \\
\sigma_0 &= \frac{\sqrt{2}}{2}(\alpha_0 e^{\Phi} + \beta_0 e^{-\Phi}) \\
\sigma_{1/2} &= \frac{4\sqrt{2}}{3}(12\alpha_0)^{-5/4}(\alpha_0^2 e^{2\Phi} - 2\alpha_0\beta_0 + \beta_0^2 e^{-2\Phi}) \\
\sigma_1 &= \frac{\sqrt{2}}{2}(12\lambda_0)^{-5/2}\Big[\frac{89}{18}\alpha_0^3 e^{3\Phi} - \Big(\frac{16}{3} + \frac{89}{18}\alpha_0\beta_0 + 282\alpha_0^2\beta_0^2\Big)\alpha_0 e^{\Phi}
\end{aligned}
$$

$$
\begin{aligned}
& -\Big(-\frac{16}{3} + \frac{89}{18}\alpha_0\beta_0 - 282\alpha_0^2\beta_0^2 \Big)\beta_0 e^{-\Phi} + \frac{89}{18}\beta_0^3 e^{-3\Phi}\Big] \\
[\mathrm{For}J &= \mathrm{II}] \\
E_0 &= -8\alpha_0\beta_0 \\
E_{1/2} &= E_1 = 0 \\
\rho_0 &= \sqrt{2}(-\alpha_0 e^{\Phi} + \beta_0 e^{-\Phi}) \\
\rho_{1/2} &= \frac{8\sqrt{2}}{3}\lambda_0\Delta^{-5/4}(-\alpha_0^2 e^{2\Phi} + \beta_0^2 e^{-2\Phi}) \\
\rho_1 &= \sqrt{2}\lambda_0^2\Delta^{-5/2}\Big[-\left(\frac{89}{18} + \frac{3}{8}\frac{\Delta}{\lambda_0^2}\right)\alpha_0^3 e^{3\Phi} \\
& \quad +\Big\{ \left(\frac{16}{3} + \frac{167}{18}\alpha_0\beta_0 + 282\alpha_0^2\beta_0^2\right) + \left(-\frac{1}{2} - \frac{3}{8}\alpha_0\beta_0 - \frac{9}{2}\alpha_0^2\beta_0^2\right)\frac{\Delta}{\lambda_0^2} \\
& \qquad + \left(-\frac{1}{48} + \alpha_0^2\beta_0^2\right)\left(\frac{\Delta}{\lambda_0^2}\right)^2\left(1 - \frac{\Delta}{4\lambda_0^2}\right)^{-1}\Big\}\alpha_0 e^{\Phi} \\
& \quad -\Big\{ \left(-\frac{16}{3} + \frac{167}{18}\alpha_0\beta_0 - 282\alpha_0^2\beta_0^2\right) + \left(\frac{1}{2} - \frac{3}{8}\alpha_0\beta_0 + \frac{9}{2}\alpha_0^2\beta_0^2\right)\frac{\Delta}{\lambda_0^2} \\
& \qquad + \left(\frac{1}{48} - \alpha_0^2\beta_0^2\right)\left(\frac{\Delta}{\lambda_0^2}\right)^2\left(1 - \frac{\Delta}{4\lambda_0^2}\right)^{-1}\Big\}\beta_0 e^{-\Phi} \\
& \quad + \left(\frac{89}{18} + \frac{3}{8}\frac{\Delta}{\lambda_0^2}\right)\beta_0^3 e^{-3\Phi}\Big] \\
\sigma_0 &= \frac{\sqrt{2}}{2}(\alpha_0 e^{\Phi} + \beta_0^{-\Phi}) \\
\sigma_{1/2} &= \frac{4\sqrt{2}}{3}\lambda_0\Delta^{-5/4}(\alpha_0^2 e^{2\Phi} - 2\alpha_0\beta_0 + \beta_0^2 e^{-2\Phi}) \\
\sigma_1 &= \frac{\sqrt{2}}{2}\lambda_0^2\Delta^{-5/2}\Big[\left(\frac{89}{18} + \frac{3}{8}\frac{\Delta}{\lambda_0^2}\right)\alpha_0^3 e^{3\Phi} \\
& \quad -\Big\{ \left(\frac{16}{3} + \frac{89}{18}\alpha_0\beta_0 + 282\alpha_0^2\beta_0^2\right) + \left(-\frac{1}{2} + \frac{3}{8}\alpha_0\beta_0 - \frac{9}{2}\alpha_0^2\beta_0^2\right)\frac{\Delta}{\lambda_0^2} \\
& \qquad + \left(-\frac{1}{48} + \alpha_0^2\beta_0^2\right)\left(\frac{\Delta}{\lambda_0^2}\right)^2\left(1 - \frac{\Delta}{4\lambda_0^2}\right)^{-1}\Big\}\alpha_0 e^{\Phi}
\end{aligned}
$$

$$
\begin{aligned}
&-\Big\{\left(-\frac{16}{3}+\frac{89}{18}\alpha_0\beta_0-282\alpha_0^2\beta_0^2\right)+\left(\frac{1}{2}+\frac{3}{8}\alpha_0\beta_0+\frac{9}{2}\alpha_0^2\beta_0^2\right)\frac{\Delta}{\lambda_0^2}\\
&\qquad+\left(\frac{1}{48}-\alpha_0^2\beta_0^2\right)\left(\frac{\Delta}{\lambda_0^2}\right)^2\left(1-\frac{\Delta}{4\lambda_0^2}\right)^{-1}\Big\}\beta_0 e^{-\Phi}\\
&+\left(\frac{89}{18}+\frac{3}{8}\frac{\Delta}{\lambda_0^2}\right)\beta_0^3 e^{-3\Phi}\Big]
\end{aligned}
$$

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