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D=26 and Exact Solution to the Conformal-Gauge Two-Dimensional Quantum Gravity

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Abstract

The conformal-gauge two-dimensional quantum gravity is formulated in the framework of the BRS quantization and solved completely in the Heisenberg picture: All n-point Wightman functions are explicitly obtained. The field-equation anomaly is shown to exist as in other gauges, but there is no other subtlety. At the critical dimension $D = 26$ of the bosonic string, the field-equation anomaly is shown to be absent. However, this result is not equivalent to the statement that the conformal anomaly is proportional to $D - 26$. The existence of the FP-ghost number current anomaly is seen to be an illusion.

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1. Introduction

The theory of the bosonic string of a finite length can be consistently formulated only at the critical dimension $D = 26$.[1] On the other hand, the two-dimensional quantum gravity coupled with D scalar fields can be regarded as the string theory in the D-dimensional spacetime, but in this case the string is not of finite length. In the conformal gauge, the conformal anomaly proportional to $D - 26$ is obtained in the functional-integral formalism by Fujikawa's method based on the functional-integral measure.[2] For covariant gauges (de Donder gauge and some other gauge fixings involving differential operator), however, perturbative approach only has been available to deduce the conformal anomaly: The two-point function of the "energy-momentum tensor" exhibits a nonlocal term proportional to $D-26$, which is identified with the conformal anomaly.[3–8] Such a term is obtained also in the perturbative approach to the conformal-gauge case.

In our previous paper,[9] we have thoroughly reexamined the derivations of the conformal anomaly in the covariant gauges. Our conclusions are as follows. The proper framework of the two-dimensional quantum gravity formulated in the Heisenberg picture has no anomaly for any particular symmetry. Instead, it has a new-type anomaly, called "field-equation anomaly",[10] whose existence is confirmed also in some other two-dimensional models.[11,12] By making use of the field-equation anomaly, one *can* encounter an anomaly for any particular symmetry such as the conformal anomaly *at one's will*. Especially, the conformal anomaly proportional to $D - 26$ is shown to be obtained by employing a particular perturbative approach based on the conventional choice of the B-field, but of course this result has no intrinsic meaning in the Heisenberg picture.

The purpose of the present paper is to extend the consideration made in Ref.9 to the case of conformal gauge. This is of particular interest because in the conformal gauge the conformal anomaly is shown to be proportional to $D - 26$ not only perturbatively but also nonperturbatively as stated above. What are found in the present paper are as follows. In the conformal gauge, we can explicitly construct all n-point Wightman functions, which are consistent with the BRS invariance and the FP-ghost number conservation. The field-equation anomaly is shown to exist, and it is proportional to $D-26$ though it is not equivalent to the conformal anomaly. There is no such ambiguity of the critical dimension as was found in the covariant-gauge case. The perturbative approach is shown to be inadequate, but it happens to yield the same value owing to the speciality of the conformal gauge.

The present paper is organized as follows. In Sec. 2, we present the BRS formulation of the two-dimensional quantum gravity in the conformal gauge. In Sec. 3, we show that the theory becomes much transparent by rewriting traceless symmetric tensors into vector-like quantities. In Sec. 4, it is shown that further simplification is achieved by introducing light-cone coordinates. In Sec. 5, all n-point Wightman functions are explicitly constructed. In Sec. 6, the existence of the field-equation anomaly is demonstrated. In Sec. 7, the conformal anomaly is considered and its connection with the field-equation anomaly is discussed. The final section is devoted to the discussion.

2. Basic formulation

We present the BRS formulation of the conformal-gauge two-dimensional quantum gravity.[13]

The contravariant gravitational field $g^{\mu\nu}$ is parametrized as

$$g^{\mu\nu} = e^{-\theta}(\eta^{\mu\nu} + h^{\mu\nu}), \tag{2.1}$$

where $\eta^{\mu\nu}$ denotes the Minkowski metric and $h^{\mu\nu}$ is a traceless symmetric tensor:

$$\eta_{\mu\nu} h^{\mu\nu} = 0. \tag{2.2}$$

Let $g^{-1} \equiv \det g^{\mu\nu}$ and $\tilde{g}^{\mu\nu} \equiv (-g)^{1/2} g^{\mu\nu}$; then

$$\tilde{g}^{\mu\nu} = (\eta^{\mu\nu} + h^{\mu\nu})(1 - \det h^{\sigma\tau})^{-1/2}. \tag{2.3}$$

It is important to note that $\det h^{\sigma\tau}$ is quadratic in $h^{\mu\nu}$.

Let $\boldsymbol{\delta}_*$ be the conventional BRS transformation and c^μ be the FP ghost. From

$$\boldsymbol{\delta}_*\, g^{\mu\nu} = g^{\mu\sigma}\partial_\sigma c^\nu + g^{\nu\sigma}\partial_\sigma c^\mu - \partial_\sigma g^{\mu\nu} \cdot c^\sigma \tag{2.4}$$

together with

$$\eta_{\mu\nu}\, \boldsymbol{\delta}_* h^{\mu\nu} = 0, \tag{2.5}$$

we obtain[a]

$$\begin{aligned} \boldsymbol{\delta}_* h^{\mu\nu} &= \partial^\mu c^\nu + \partial^\nu c^\mu + h^{\mu\sigma}\partial_\sigma c^\nu + h^{\nu\sigma}\partial_\sigma c^\mu - \partial_\sigma h^{\mu\nu} \cdot c^\sigma \\ &\quad -(\eta^{\mu\nu} + h^{\mu\nu})(\partial_\sigma c^\sigma + h^{\sigma\tau}\partial_\sigma c_\tau), \end{aligned} \tag{2.6}$$

$$\boldsymbol{\delta}_* \theta = -(\partial_\sigma c^\sigma + h^{\sigma\tau}\partial_\sigma c_\tau + \partial_\sigma \theta \cdot c^\sigma). \tag{2.7}$$

[a] The fifth term of (2.6) is missing in Ref.13.

For the FP ghost c^μ and scalar fields ϕ_M ($M = 0, 1, \cdots, D-1$), we have

$$\boldsymbol{\delta}_* c^\mu = -c^\sigma \partial_\sigma c^\mu, \tag{2.8}$$

$$\boldsymbol{\delta}_* \phi_M = -c^\sigma \partial_\sigma \phi_M. \tag{2.9}$$

Let $\tilde{b}_{\mu\nu}$ and $\bar{c}_{\mu\nu}$ be the B field and the FP antighost, respectively; they are both traceless symmetric tensors. As usual, we have

$$\boldsymbol{\delta}_* \bar{c}_{\mu\nu} = i\tilde{b}_{\mu\nu}, \tag{2.10}$$

$$\boldsymbol{\delta}_* \tilde{b}_{\mu\nu} = 0. \tag{2.11}$$

Now, the Lagrangian density of the conformal-gauge two-dimensional quantum gravity is given by

$$\mathcal{L} = \mathcal{L}_{\mathrm{S}} + \mathcal{L}_{\mathrm{GF}} + \mathcal{L}_{\mathrm{FP}}, \tag{2.12}$$

where

$$\mathcal{L}_{\mathrm{S}} = \frac{1}{2}\tilde{g}^{\mu\nu}\partial_\mu\phi_M \cdot \partial_\nu\phi^M, \tag{2.13}$$

$$\begin{aligned}\mathcal{L}_{\mathrm{GF}} + \mathcal{L}_{\mathrm{FP}} &= \frac{1}{2}i\boldsymbol{\delta}_* \left(\bar{c}_{\mu\nu}h^{\mu\nu}\right)\\ &= -\frac{1}{2}\tilde{b}^{\mu\nu}h_{\mu\nu} - \frac{i}{2}\bar{c}_{\mu\nu}\boldsymbol{\delta}_* h^{\mu\nu},\end{aligned} \tag{2.14}$$

which does not involve the conformal degree of freedom, θ.

The field equations which follows from (2.12) are as follows. First, we have

$$h^{\mu\nu} = 0. \tag{2.15}$$

The other field equations can be simplified by using (2.15); especially, $\det h^{\sigma\tau}$ does not contribute to field equations. Taking the tracelessness of $h^{\mu\nu}$, $\tilde{b}_{\mu\nu}$ and $\bar{c}_{\mu\nu}$ into account, we have

$$\begin{aligned}&-\tilde{b}_{\mu\nu} - i[\bar{c}_{\mu\sigma}\partial_\nu c^\sigma + \bar{c}_{\nu\sigma}\partial_\mu c^\sigma + \partial_\sigma\bar{c}_{\mu\nu}\cdot c^\sigma - \eta_{\mu\nu}\bar{c}_{\sigma\tau}\partial^\sigma c^\tau]\\ &\qquad +\partial_\mu\phi_M\cdot\partial_\nu\phi^M - \frac{1}{2}\eta_{\mu\nu}\partial_\sigma\phi_M\cdot\partial^\sigma\phi^M = 0,\end{aligned} \tag{2.16}$$

$$\partial^\mu \bar{c}_{\mu\nu} = 0, \tag{2.17}$$

$$\partial^\mu c^\nu + \partial^\nu c^\mu - \eta^{\mu\nu}\partial_\sigma c^\sigma = 0, \tag{2.18}$$

$$\Box\phi_M = 0. \tag{2.19}$$

Next, we carry out the canonical quantization. The canonical variables are c^μ and ϕ_M. Their canonical conjugates are

$$\pi_{c\,\mu} \equiv \frac{\partial}{\partial(\partial_0 c^\mu)}\mathcal{L} = i\bar{c}_{\mu 0}, \tag{2.20}$$

$$\pi_\phi{}^M \equiv \frac{\partial}{\partial(\partial_0 \phi_M)}\mathcal{L} = \partial_0 \phi^M. \tag{2.21}$$

Hence, nontrivial canonical commutation relations are

$$\{i\bar{c}_{\mu 0}(x),\ c^\lambda(y)\}_{x^0=y^0} = -i\delta_\mu{}^\lambda \delta(x^1 - y^1), \tag{2.22}$$

$$[\partial_0\phi_M(x),\ \phi^N(y)]_{x^0=y^0} = -i\delta_M{}^N \delta(x^1 - y^1). \tag{2.23}$$

The fields c^μ, $\bar{c}_{\mu\nu}$ and ϕ_M are thus free fields The two-dimensional commutators are given by

$$\{\bar{c}_{\mu\nu}(x),\ c^\lambda(y)\} = (\delta_\mu{}^\lambda\partial_\nu + \delta_\nu{}^\lambda\partial_\mu - \eta_{\mu\nu}\partial^\lambda)D(x-y), \tag{2.24}$$

$$[\phi_M(x),\ \phi^N(y)] = i\delta_M{}^N D(x-y), \tag{2.25}$$

where

$$D(x) \equiv -\frac{1}{2}\epsilon(x^0)\theta(x^2). \tag{2.26}$$

3. Rewriting traceless symmetric tensors

We encounter three traceless symmetric tensors $h^{\mu\nu}$, $\tilde{b}_{\mu\nu}$ and $\bar{c}_{\mu\nu}$, but their treatment is rather inconvenient because of their tracelessness. It is more convenient to rewrite them as if they were vectors.

Generically, let $X_{\mu\nu}$ be a traceless symmetric tensor:

$$X_{\mu\nu} = X_{\nu\mu}, \qquad \eta^{\mu\nu}X_{\mu\nu} = 0. \tag{3.1}$$

Then it has only two independent components $X_{00} = X_{11}$ and $X_{01} = X_{10}$. We introduce a constant, totally symmetric rank-3 tensorlike quantity $\xi^{\mu\nu\lambda}$ by

$$\begin{aligned} \xi^{\mu\nu\lambda} &= 1 \quad \text{for } \mu+\nu+\lambda \equiv 0 \ (\text{mod.}2) \\ &= 0 \quad \text{for } \mu+\nu+\lambda \equiv 1 \ (\text{mod.}2). \end{aligned} \tag{3.2}$$

Then we define X^λ by

$$X^\lambda \equiv \frac{1}{\sqrt{2}}\xi^{\lambda\mu\nu}X_{\mu\nu}. \tag{3.3}$$

We have the following formulae:

$$\xi_{\mu\nu\lambda}\xi^{\lambda\sigma\tau} = \delta_\mu{}^\sigma\delta_\nu{}^\tau + \delta_\mu{}^\tau\delta_\nu{}^\sigma - \eta_{\mu\nu}\eta^{\sigma\tau}, \tag{3.4}$$

$$\xi_{\lambda\mu\nu}\xi^{\mu\nu\rho} = 2\delta_\lambda{}^\rho, \tag{3.5}$$

$$\eta^{\mu\nu}\xi_{\mu\nu\lambda} = 0; \tag{3.6}$$

$$X_{\mu\nu} = \frac{1}{\sqrt{2}}\xi_{\mu\nu\lambda}X^\lambda, \tag{3.7}$$

$$\sqrt{2}\partial^\nu X_{\mu\nu} = \xi_{\mu\nu\lambda}\partial^\nu X^\lambda, \tag{3.8}$$

$$\sqrt{2}\xi^{\sigma\tau\mu}\partial^\nu X_{\mu\nu} = (\delta_\lambda{}^\tau\partial^\sigma + \delta_\lambda{}^\sigma\partial^\tau - \eta^{\sigma\tau}\partial_\lambda)X^\lambda. \tag{3.9}$$

Applying the above consideration to $h^{\mu\nu}$, $\tilde{b}_{\mu\nu}$ and $\bar{c}_{\mu\nu}$, we introduce h_λ, $\tilde{b}^\lambda$ and $\bar{c}^\lambda$. Then (2.6), (2.10) and (2.11) are rewritten as

$$\boldsymbol{\delta}_* h_\lambda = \sqrt{2}\xi_{\lambda\mu\nu}\partial^\mu c^\nu + \xi_{\lambda\mu\nu}\xi^{\mu\sigma\tau}h_\sigma\partial_\tau c^\nu - \partial_\nu(h_\lambda c^\nu) - \frac{1}{\sqrt{2}}h_\lambda\xi^{\nu\sigma\tau}h_\nu\partial_\sigma c_\tau, \tag{3.10}$$

$$\boldsymbol{\delta}_*\bar{c}^\lambda = i\tilde{b}^\lambda, \tag{3.11}$$

$$\boldsymbol{\delta}_*\tilde{b}^\lambda = 0, \tag{3.12}$$

respectively. Correspondingly, (2.14) is rewritten as

$$\mathcal{L}_{\mathrm{GF}} + \mathcal{L}_{\mathrm{FP}} = -\frac{1}{2}\tilde{b}^\lambda h_\lambda - \frac{i}{2}\bar{c}^\lambda\boldsymbol{\delta}_* h_\lambda. \tag{3.13}$$

Field equations (2.15), (2.16) and (2.17) become

$$h_\mu = 0. \tag{3.14}$$

$$-\tilde{b}^\mu - i[\xi_{\sigma\tau\rho}\xi^{\rho\mu\lambda}\bar{c}^\sigma\partial_\lambda c^\tau + \partial_\sigma\bar{c}^\mu\cdot c^\sigma] + \frac{1}{\sqrt{2}}\xi^{\mu\sigma\tau}\partial_\sigma\phi_M\cdot\partial_\tau\phi^M = 0, \tag{3.15}$$

$$\xi_{\lambda\mu\nu}\partial^\mu\bar{c}^\nu = 0, \tag{3.16}$$

respectively. On the other hand, (2.18) is rewritten as

$$\xi_{\lambda\mu\nu}\partial^\mu c^\nu = 0. \tag{3.17}$$

From (3.15), with the help of field equations, we obtain

$$\xi_{\lambda\mu\nu}\partial^\mu\tilde{b}^\nu = 0. \tag{3.18}$$

Evidently. (3.18) is the BRS transform of (3.16). Furthermore, the identity (3.6) is rewritten as

$$\xi_{\lambda\mu\nu}\partial^\mu x^\nu = 0. \tag{3.19}$$

As in the de Donder-gauge quantum Einstein gravity,[14,15] it is natural to introduce "supercoordinate" $X^\nu \equiv (x^\nu,\ \tilde{b}^\nu,\ c^\nu,\ \bar{c}^\nu)$; then (3.16)-(3.19) are unified into

$$\xi_{\lambda\mu\nu}\partial^\mu X^\nu = 0. \tag{3.20}$$

Accordingly, we have many conservation laws:

$$\partial^\mu(\xi_{\mu\nu\lambda}X^\nu) = 0, \tag{3.21}$$

$$\partial^\mu(\xi_{\mu\nu\lambda}X^\nu X^\lambda) = 0. \tag{3.22}$$

From (3.21) and (3.22), we obtain many symmetry generators, most of which are spontaneously broken. Unfortunately, it is very difficult to find the corresponding symmetry transformations which leave the action invariant, because in the action we must not use the field equations, especially (3.14).[16]

The two-dimensional anticommutation relation (2.24) is rewritten as

$$\{c^\rho(x),\ \bar{c}^\lambda(y)\} = \sqrt{2}\xi^{\rho\lambda\nu}\partial_\nu D(x-y). \tag{3.23}$$

The two-dimensional commutators involving $\tilde{b}^\mu$ can be calculated by expressing $\tilde{b}^\mu$ in terms of $\bar{c}^\rho$, c^λ and ϕ_M from (3.15). For example, we have

$$[\tilde{b}^\lambda(x),\ \phi_M(y)] = i\sqrt{2}\xi^{\lambda\mu\nu}\partial_\mu\phi_M(x)\cdot\partial_\nu D(x-y). \tag{3.24}$$

Unfortunately, if $[\tilde{b}^\lambda(x),\ \tilde{b}^\rho(y)]$ is calculated by this method or by using the BRS transform, we obtain various expressions which apparently look different, depending on the ways of calculation. This problem is resolved in next section.

4. Use of light-cone coordinates

Two-dimensional commutators are much simplified if we use light-cone coordinates $x^\pm \equiv (x^0 \pm x^1)/\sqrt{2}$. This is because in the light-cone coordinates we have $\xi_{\mu\nu\lambda} = 0$ only except

$$\xi_{+++} = \xi_{---} = \sqrt{2}. \tag{4.1}$$

Therefore, (3.20) reduces to

$$\partial_\mp X^\pm = 0, \tag{4.2}$$

that is, $X^\pm$ is a function of $x^\pm$ only. Furthermore, (2.26) yields

$$\partial_\pm D(x) = -\frac{1}{2}\delta(x^\pm). \tag{4.3}$$

Hence we always have

$$[X^+(x),\ Y^-(y)] = 0 \tag{4.4}$$

because there is no mixing of x^+ and y^-. Furthermore, since there is a complete symmetry in $+ \leftrightarrow -$, it is sufficient to calculate $[X^+(x),\ Y^+(y)]$ only.

First, the two-dimensional commutation relations (3.23) and (2.25) are simplified into

$$\{c^+(x),\ \bar{c}^+(y)\} = -\delta(x^+ - y^+), \tag{4.5}$$

$$[\partial_+\phi_M(x),\ \phi^N(y)] = -\frac{i}{2}\delta_M{}^N\delta(x^+ - y^+), \tag{4.6}$$

respectively. Using

$$\tilde{b}^+(x) = -i(2\bar{c}^+\partial_+c^+ + \partial_+\bar{c}^+\cdot c^+) + \partial_+\phi_M\cdot\partial_+\phi^M, \tag{4.7}$$

which follows from (3.15), we calculate the commutators involving $\tilde{b}^+$. We have

$$[\tilde{b}^+(x),\ c^+(y)] = -i[c^+(x)\delta'(x^+ - y^+) + 2\partial_+c^+(x)\cdot\delta(x^+ - y^+)], \tag{4.8}$$

$$\begin{aligned}[\tilde{b}^+(x),\ \bar{c}^+(y)] &= i[\bar{c}^+(x) + \bar{c}^+(y)]\delta'(x^+ - y^+) \\ &= i[2\bar{c}^+(x)\delta'(x^+ - y^+) + \partial_+\bar{c}^+(x)\cdot\delta(x^+ - y^+)],\end{aligned} \tag{4.9}$$

$$[\tilde{b}^+(x),\ \phi_M(y)] = -i\partial_+\phi_M(x)\cdot\delta(x^+ - y^+), \tag{4.10}$$

$$[\tilde{b}^+(x),\ \tilde{b}^+(y)] = i[\tilde{b}^+(x) + \tilde{b}^+(y)]\delta'(x^+ - y^+). \tag{4.11}$$

Evidently, (4.11) is obtained also as the BRS transform of (4.9).

5. Wightman functions

Since all two-dimensional commutation relations have explicitly been obtained, we can calculate all multiple commutators. Then, according to the prescription given in our previous papers,[18,17,10,b] we can construct n-point Wightman functions $\langle\,\varphi_1(x_1)\cdots\varphi_n(x_n)\,\rangle_{\mathrm{W}}$, where $\varphi_j(x)$ denotes a generic field.

[b] For a summary, see Ref.9.

It is natural to set all 1-point functions equal to zero. Then, as is seen from multiple commutators, nonvanishing truncated n-point Wightman functions are those which consist of $(n-2)$ $\tilde{b}^+$'s and of c^+ and $\bar{c}^+$ or two ϕ_M's.

The nonvanishing 2-point functions are as follows. From (2.25) we have

$$\langle\, \phi_M(x_1)\phi^N(x_2)\,\rangle_{\mathrm{W}} = \delta_M{}^N D^{(+)}(x_1-x_2), \tag{5.1}$$

where $D^{(+)}(x)$ denotes that positive-energy part of $iD(x)$. Since

$$\partial_+ D^{(+)}(x) = -\frac{1}{4\pi}\frac{1}{x^+ - i0}, \tag{5.2}$$

(5.1) implies that

$$\partial_+{}^{x_1}\langle\, \phi_M(x_1)\phi^N(x_2)\,\rangle_{\mathrm{W}} = -\frac{1}{4\pi}\delta_M{}^N\frac{1}{x_1{}^+ - x_2{}^+ - i0}, \tag{5.3}$$

which is deduced directly from (4.6). On the other hand, (4.5) implies that

$$\langle\, \bar{c}^+(x_1)c^+(x_2)\,\rangle_{\mathrm{W}} = \langle\, c^+(x_1)\bar{c}^+(x_2)\,\rangle_{\mathrm{W}} = \frac{i}{2\pi}\frac{1}{x_1{}^+ - x_2{}^+ - i0}. \tag{5.4}$$

We proceed to the 3-point functions. From the double commutators,

$$[\,[\phi_M(x_1),\ \tilde{b}^+(x_2)],\ \phi^N(x_3)] = \frac{1}{2}\delta_M{}^N\delta(x_1{}^+ - x_2{}^+)\delta(x_2{}^+ - x_3{}^+), \tag{5.5}$$

$$\begin{aligned}&\{\,[c^+(x_1),\ \tilde{b}^+(x_2)],\ \bar{c}^+(x_3)\}\\ &\quad = i[\delta'(x_1{}^+ - x_2{}^+)\delta(x_2{}^+ - x_3{}^+) - 2\delta(x_1{}^+ - x_2{}^+)\delta'(x_2{}^+ - x_3{}^+)],\end{aligned} \tag{5.6}$$

together with $[\,[\phi_M,\ \phi^N],\ \tilde{b}^+] = [\,\{c^+,\ \bar{c}^+\},\ \tilde{b}^+] = 0$, we deduce[c]

$$\langle\, \phi_M(x_1)\tilde{b}^+(x_2)\phi^N(x_3)\,\rangle_{\mathrm{W}} = -\frac{1}{2(2\pi)^2}\delta_M{}^N\frac{1}{x_1{}^+ - x_2{}^+ - i0}\frac{1}{x_2{}^+ - x_3{}^+ - i0}, \tag{5.7}$$

$$\begin{aligned}\langle\, c^+(x_1)\tilde{b}^+(x_2)\bar{c}^+(x_3)\,\rangle_{\mathrm{W}} = \frac{i}{(2\pi)^2}\Big[&\frac{1}{(x_1{}^+ - x_2{}^+ - i0)^2}\cdot\frac{1}{x_2{}^+ - x_3{}^+ - i0}\\ &-2\frac{1}{x_1{}^+ - x_2{}^+ - i0}\cdot\frac{1}{(x_2{}^+ - x_3{}^+ - i0)^2}\Big].\end{aligned} \tag{5.8}$$

We extend the above analysis to the n-point functions. As $(n-1)!$ independent $(n-1)$-ple commutators, we adopt those whose first member is $\phi_M(x_1)$ or $c^+(x_1)$. Then those which do not have $\phi^N(x_k)$ or $\bar{c}^+(x_k)$ as the last member $(k=n)$ vanish.

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[c] For Wightman functions of other orderings, $-i0$ is changed into $+i0$ appropriately (and a minus sign is inserted for the exchange of c and $\bar{c}$).

It is easy to prove by mathematical induction that

$$[\cdots [\,[\phi_M(x_1),\ \tilde{b}^+(x_2)],\ \tilde{b}^+(x_3)],\ \cdots,\ \phi^N(x_n)]$$
$$= -\frac{1}{2}\delta_M{}^N i^{n-1}\partial_2{}^{\mathrm{R}}\cdots\partial_{n-2}{}^{\mathrm{R}}[\delta(x_1{}^+ - x_2{}^+)\cdots\delta(x_{n-1}{}^+ - x_n{}^+)] \quad (n \geqq 3), \tag{5.9}$$

where the superscript R of $\partial_k{}^{\mathrm{R}}$ means that ∂_k acts only on the right-hand factor among the ones involving $x_k{}^+$; for instance,

$$\partial_2{}^{\mathrm{R}}[\delta(x_1{}^+ - x_2{}^+)\delta(x_2{}^+ - x_3{}^+)\delta(x_3{}^+ - x_4{}^+)]$$
$$= \delta(x_1{}^+ - x_2{}^+)\delta'(x_2{}^+ - x_3{}^+)\delta(x_3{}^+ - x_4{}^+). \tag{5.10}$$

From (5.9), we deduce

$$\langle\, \phi_M(x_1)\tilde{b}^+(x_2)\cdots\tilde{b}^+(x_{n-1})\phi^N(x_n)\,\rangle_{\mathrm{W}}$$
$$= -\frac{1}{2(2\pi)^{n-1}}\delta_M{}^N \sum_{P(j_2,\,\cdots,\,j_{n-1})}^{(n-2)!} \partial_{j_2}{}^{\mathrm{R}}\cdots\partial_{j_{n-2}}{}^{\mathrm{R}}\Big[\frac{1}{x_1{}^+ - x_{j_2}{}^+ - i0}\frac{1}{x_{j_2}{}^+ - x_{j_3}{}^+ \mp i0}$$
$$\cdots\frac{1}{x_{j_{n-2}}{}^+ - x_{j_{n-1}}{}^+ \mp i0}\frac{1}{x_{j_{n-1}}{}^+ - x_n{}^+ - i0}\Big] \quad (n \geqq 3), \tag{5.11}$$

where $P(j_2,\,\cdots,\,j_{n-1})$ stands for a permutation of $(2,\,3,\,\cdots,\,n-1)$, and

$$\begin{aligned} x_j{}^+ - x_k{}^+ \mp i0 \quad &= \quad x_j{}^+ - x_k{}^+ - i0 \quad \text{for } j<k \\ &= \quad x_j{}^+ - x_k{}^+ + i0 \quad \text{for } j>k. \end{aligned} \tag{5.12}$$

We proceed to the n-point function involving c^+ and $\bar{c}^+$. First, we rewrite (4.8) as

$$[c^+(x_1),\ \tilde{b}^+(x_2)] = i(\partial_2{}^{\mathrm{L}} + 2\partial_2{}^{\mathrm{R}})[\delta(x_1{}^+ - x_2{}^+)c^+(x_2)], \tag{5.13}$$

where $\partial_k{}^{\mathrm{L}}$ acts only on the left-hand factor among the ones involving $x_k{}^+$. By making use of (5.13), it is easy to show that

$$\{\cdots [\,[c^+(x_1),\ \tilde{b}^+(x_2)],\ \tilde{b}^+(x_3)],\ \cdots,\ \bar{c}^+(x_n)\}$$
$$= -i^{n-2}(\partial_2{}^{\mathrm{L}} + 2\partial_2{}^{\mathrm{R}})\cdots(\partial_{n-1}{}^{\mathrm{L}} + 2\partial_{n-1}{}^{\mathrm{R}})[\delta(x_1{}^+ - x_2{}^+)\cdots\delta(x_{n-1}{}^+ - x_n{}^+)]. \tag{5.14}$$

Hence, as above, we have

$$\langle\, c^+(x_1)\tilde{b}^+(x_2)\cdots\tilde{b}^+(x_{n-1})\bar{c}^+(x_n)\,\rangle_{\mathrm{W}}$$
$$= \frac{i}{(2\pi)^{n-1}} \sum_{P(j_2,\,\cdots,\,j_{n-1})}^{(n-2)!} (\partial_{j_2}{}^{\mathrm{L}} + 2\partial_{j_2}{}^{\mathrm{R}})\cdots(\partial_{j_{n-1}}{}^{\mathrm{L}} + 2\partial_{j_{n-1}}{}^{\mathrm{R}})$$
$$\Big[\frac{1}{x_1{}^+ - x_{j_2}{}^+ - i0}\frac{1}{x_{j_2}{}^+ - x_{j_3}{}^+ \mp i0}\cdots\frac{1}{x_{j_{n-2}}{}^+ - x_{j_{n-1}}{}^+ \mp i0}\frac{1}{x_{j_{n-1}}{}^+ - x_n{}^+ - i0}\Big]. \tag{5.15}$$

Of course, the completely analogous formulae hold for $\langle\, \phi_M \tilde{b}^- \cdots \tilde{b}^- \phi^N \,\rangle_{\rm W}$ and $\langle\, c^- \tilde{b}^- \cdots \tilde{b}^- \bar{c}^- \,\rangle_{\rm W}$. All mixed Wightman functions vanish.

If the BRS invariance is not broken, the following Ward-Takahashi identities must hold:

$$\begin{aligned} 0 &= \langle \boldsymbol{\delta}_* [\phi_M(x_1)\tilde{b}^+(x_2)\cdots\tilde{b}^+(x_{n-2})\bar{c}^+(x_{n-1})\phi^N(x_n)]\,\rangle_{\rm W} \\ &= i\langle\, \phi_M(x_1)\tilde{b}^+(x_2)\cdots\tilde{b}(x_{n-1})\phi^N(x_n)\,\rangle_{\rm W} \\ &\quad -\langle\, c^+(x_1)\partial_+\phi_M(x_1)\cdot\tilde{b}^+(x_2)\cdots\tilde{b}^+(x_{n-2})\bar{c}^+(x_{n-1})\phi^N(x_n)\,\rangle_{\rm W} \\ &\quad +\langle\, \phi_M(x_1)\tilde{b}^+(x_2)\cdots\tilde{b}^+(x_{n-2})\bar{c}^+(x_{n-1})c^+(x_n)\partial_+\phi^N(x_n)\,\rangle_{\rm W}\,, \end{aligned} \tag{5.16}$$

$$\begin{aligned} 0 &= \langle \boldsymbol{\delta}_* [c^+(x_1)\tilde{b}^+(x_2)\cdots\tilde{b}^+(x_{n-2})\bar{c}^+(x_{n-1})\bar{c}^+(x_n)]\,\rangle_{\rm W} \\ &= -i\langle\, c^+(x_1)\tilde{b}^+(x_2)\cdots\tilde{b}^+(x_{n-2})\tilde{b}^+(x_{n-1})\bar{c}^+(x_n)\,\rangle_{\rm W} \\ &\quad +i\langle\, c^+(x_1)\tilde{b}^+(x_2)\cdots\tilde{b}^+(x_{n-2})\bar{c}^+(x_{n-1})\tilde{b}^+(x_n)\,\rangle_{\rm W} \\ &\quad -\langle\, c^+(x_1)\partial_+ c^+(x_1)\cdot\tilde{b}^+(x_2)\cdots\tilde{b}^+(x_{n-2})\bar{c}^+(x_{n-1})\bar{c}^+(x_n)\,\rangle_{\rm W}\,. \end{aligned} \tag{5.17}$$

We have explicitly confirmed for $n = 3,\,4$ that (5.11) and (5.15) are indeed consistent with (5.16) and (5.17). For example, (5.16) for $n = 3$ becomes as follows:

$$\begin{aligned} & i\langle\, \phi_M(x_1)\tilde{b}^+(x_2)\phi^N(x_3)\,\rangle_{\rm W} \\ & -\langle\, c^+(x_1)\bar{c}^+(x_2)\,\rangle_{\rm W}\,\langle\, \partial_+\phi_M(x_1)\phi^N(x_3)\,\rangle_{\rm W} \\ & +\langle\, \phi_M(x_1)\partial_+\phi^N(x_3)\,\rangle_{\rm W}\,\langle\, \bar{c}^+(x_2)c^+(x_3)\,\rangle_{\rm W} \\ &\quad = -\frac{i}{2(2\pi)^2}\delta_M{}^N \frac{1}{x_1{}^+ - x_2{}^+ - i0}\,\frac{1}{x_2{}^+ - x_3{}^+ - i0} \\ &\qquad -\frac{i}{2\pi}\cdot\frac{1}{x_1{}^+ - x_2{}^+ - i0}\cdot\left(-\frac{1}{4\pi}\right)\delta_M{}^N\frac{1}{x_1{}^+ - x_3{}^+ - i0} \\ &\qquad +\frac{1}{4\pi}\delta_M{}^N\frac{1}{x_1{}^+ - x_3{}^+ - i0}\cdot\frac{i}{2\pi}\,\frac{1}{x_2{}^+ - x_3{}^+ - i0} \\ &\quad = 0. \end{aligned} \tag{5.18}$$

The confirmation of (5.17) for $n = 4$ needs rather long calculation.

6. Field-equation anomaly

The existence of the field-equation anomaly was found in the various massless two-dimensional models[10–12]: One of field equations is broken at the level of the representation in terms of state vectors, *modulo* a field equation which is obtained from the original field equation by differentiating it once or twice but has the same degree of freedom as its. In this section we discuss the field-equation anomaly in the conformal-gauge two-dimensional quantum gravity.

We write (2.16) as

$$2\frac{\delta}{\delta h^{\mu\nu}}\int d^2x\,\mathcal{L} \equiv \mathcal{T}_{\mu\nu} \equiv -\tilde{b}_{\mu\nu} + \tilde{\mathcal{T}}_{\mu\nu} = 0. \tag{6.1}$$

We introduce $\mathcal{T}^\lambda$ and $\tilde{\mathcal{T}}^\lambda$ according to (3.3), so that

$$\mathcal{T}^\lambda \equiv -\tilde{b}^\lambda + \tilde{\mathcal{T}}^\lambda = 0. \tag{6.2}$$

We note that (3.18) is rewritten as

$$\xi_{\lambda\mu\nu}\partial^\mu \mathcal{T}^\nu = 0. \tag{6.3}$$

For calculating Wightman functions, we should not use (4.7), but write

$$\tilde{\mathcal{T}}^+ \equiv -i(2\bar{c}^+\partial_+ c^+ + \partial_+\bar{c}^+\cdot c^+) + \partial_+\phi_M\cdot\partial_+\phi^M. \tag{6.4}$$

Then from the formulae given in Sec. 5, we obtain

$$\langle\, \tilde{b}^+(x_1)\tilde{b}^+(x_2)\,\rangle_{\mathrm{W}} = 0, \tag{6.5}$$

$$\langle\, \tilde{b}^+(x_1)\tilde{\mathcal{T}}^+(x_2)\,\rangle_{\mathrm{W}} = (D-26)\Phi^{++}, \tag{6.6}$$

$$\langle\, \tilde{\mathcal{T}}^+(x_1)\tilde{\mathcal{T}}^+(x_2)\,\rangle_{\mathrm{W}} = (D-26)\Phi^{++}, \tag{6.7}$$

where

$$\Phi^{++} \equiv \frac{1}{2(2\pi)^2}\cdot\frac{1}{(x_1{}^+ - x_2{}^+ - i0)^4} \neq 0. \tag{6.8}$$

We thus see that

$$\langle\, \tilde{b}^+(x_1)\mathcal{T}^+(x_2)\,\rangle_{\mathrm{W}} = (D-26)\Phi^{++}, \tag{6.9}$$

$$\langle\, \mathcal{T}^+(x_1)\mathcal{T}^+(x_2)\,\rangle_{\mathrm{W}} = -(D-26)\Phi^{++}, \tag{6.10}$$

that is, we encounter the field-equation anomaly because (6.2) is violated at the level of Wightman functions. Of course, the once-differentiated equation (6.3) is not broken at all.

The field-equation anomaly has arisen because $\langle\, \tilde{b}^+\tilde{b}^+\,\rangle_{\mathrm{W}}$ is not equal to $\langle\, \tilde{b}^+\tilde{\mathcal{T}}^+\,\rangle_{\mathrm{W}} = \langle\, \tilde{\mathcal{T}}^+\tilde{\mathcal{T}}^+\,\rangle_{\mathrm{W}}$. From (4.11), it is impossible to make the former equal to

$(D-26)\Phi^{++}$ even if we dare to violate the BRS invariance (or FP-ghost number) so as to have $\langle\, \tilde{b}^+ \,\rangle_{\rm W} \neq 0$. Thus the appearance of the field-equation anomaly is unavoidable.

We can extend the consideration given in (6.5)-(6.10) to the 3-point functions. We find that

$$\langle\, \tilde{b}^+(x_1)\tilde{b}^+(x_2)\tilde{b}^+(x_3)\,\rangle_{\rm W} = 0, \tag{6.11}$$

$$\begin{aligned}&\langle\, \tilde{b}^+(x_1)\tilde{b}^+(x_2)\tilde{\mathcal{T}}^+(x_3)\,\rangle_{\rm W} = \langle\, \tilde{b}^+(x_1)\tilde{\mathcal{T}}^+(x_2)\tilde{\mathcal{T}}^+(x_3)\,\rangle_{\rm W} = \langle\, \tilde{\mathcal{T}}^+(x_1)\tilde{\mathcal{T}}^+(x_2)\tilde{\mathcal{T}}^+(x_3)\,\rangle_{\rm W} \\ &\quad = \frac{-1}{(2\pi)^3}\cdot\frac{D-26}{(x_1{}^+ - x_2{}^+ - i0)^2(x_2{}^+ - x_3{}^+ - i0)^2(x_1{}^+ - x_3{}^+ - i0)^2}.\end{aligned} \tag{6.12}$$

The appearance of $D-26$ is quite stable: For any linear or quadratic local operators $F_j{}^+(x)$, we see that $\langle\, F_1{}^+(x_1)\mathcal{T}^+(x_2)\,\rangle_{\rm W}$ and $\langle\, F_1{}^+(x_1)F_2{}^+(x_2)\mathcal{T}^+(x_3)\,\rangle_{\rm W}$ are either zero or proportional to $D-26$.

In spite of the presence of the field-equation anomaly, we can define various symmetry generators so as to be free of its trouble. We here present the anomaly-free definitions of the translation generator P_ν, the BRS generator Q_b and the FP-ghost number generator Q_c.

The Noether currents of translation, BRS and FP-ghost number are

$$J_\nu{}^\mu = -\frac{i}{\sqrt{2}}\xi^\mu{}_{\lambda\rho}\bar{c}^\lambda\partial_\nu c^\rho + \partial_\nu\phi_M\cdot\partial^\mu\phi^M - \frac{1}{2}\delta_\nu{}^\mu\partial_\sigma\phi_M\cdot\partial^\sigma\phi^M, \tag{6.13}$$

$$j_b{}^\mu = -\frac{i}{\sqrt{2}}\xi^\mu{}_{\lambda\rho}\bar{c}^\lambda c^\sigma\partial_\sigma c^\rho - c^\sigma\partial_\sigma\phi_M\cdot\partial^\mu\phi^M + \frac{1}{2}c^\mu\partial_\sigma\phi_M\cdot\partial^\sigma\phi^M, \tag{6.14}$$

$$j_c{}^\mu = -\frac{i}{\sqrt{2}}\xi^\mu{}_{\lambda\rho}\bar{c}^\lambda c^\rho, \tag{6.15}$$

respectively. In the light-cone representation, they reduce to

$$J_+{}^- = -i\bar{c}^+\partial_+ c^+ + \partial_+\phi_M\cdot\partial_+\phi^M, \tag{6.16}$$

$$j_b{}^- = -\bar{c}^+c^+\partial_+c^+ - c^+\partial_+\phi_M\cdot\partial_+\phi^M, \tag{6.17}$$

$$j_c{}^- = -i\bar{c}^+c^+. \tag{6.18}$$

Noting (6.4), we rewrite (6.16) and (6.17) as

$$J_+{}^- = \tilde{b}^+ + \mathcal{T}^+ + i\partial_+(\bar{c}^+c^+), \tag{6.19}$$

$$j_b{}^- = -\tilde{b}^+c^+ + i\bar{c}^+c^+\partial_+c^+ - \mathcal{T}^+c^+, \tag{6.20}$$

respectively.

Since the terms involving $\mathcal{T}^+$ suffer from the field-equation anomaly, we drop them. Thus the anomaly-free generators are defined by

$$P_\pm \equiv \int dx^\pm \tilde{b}^\pm, \tag{6.21}$$

$$Q_b \equiv \int dx^+(-\tilde{b}^+ c^+ + i\bar{c}^+ c^+ \partial_+ c^+) + \int dx^-(-\tilde{b}^- c^- + i\bar{c}^- c^- \partial_- c^-), \tag{6.22}$$

$$iQ_c \equiv \int dx^+ \bar{c}^+ c^+ + \int dx^- \bar{c}^- c^-. \tag{6.23}$$

7. Perturbative approach to the conformal anomaly

In this section, we review the perturbative approach to the conformal anomaly in order to compare it with our exact results. Since $\mathcal{L}_{\mathrm{GF}}$ contains no differentiation, the B-field $\tilde{b}_{\mu\nu}$ is nonpropagating in perturbation theory, and therefore it is customary to discard it. Then the conformal-gauge two-dimensional quantum gravity reduces to a fee field theory. Nevertheless, one wishes to encounter the conformal anomaly. The procedure for this[4] is as follows.

From (2.12) with (2.13), (2.14) and (2.6), the free Lagrangian density is given by

$$\begin{aligned}\mathcal{L}^{(0)} &= \frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi_M \cdot \partial_\nu\phi^M - \frac{1}{2}\tilde{b}_{\mu\nu}h^{\mu\nu} \\ &\quad -\frac{1}{2}i\bar{c}_{\mu\nu}(\eta^{\mu\sigma}\partial_\sigma c^\nu + \eta^{\nu\sigma}\partial_\sigma c^\mu - \eta^{\mu\nu}\partial_\sigma c^\sigma).\end{aligned} \tag{7.1}$$

One then introduces a background field $\hat{g}^{\mu\nu}$ and makes (7.1) background-covariant by replacing $\eta^{\mu\nu}$ by $\tilde{\hat{g}}^{\mu\nu}$ and ∂_μ by background-covariant differentiation $\hat{\nabla}_\mu$. In this way, one obtains

$$\begin{aligned}\hat{\mathcal{L}} &= \frac{1}{2}\tilde{\hat{g}}^{\mu\nu}\partial_\mu\phi_M \cdot \partial_\nu\phi^M - \frac{1}{2}\tilde{b}_{\mu\nu}h^{\mu\nu} \\ &\quad -\frac{1}{2}i\bar{c}_{\mu\nu}[\tilde{\hat{g}}^{\mu\sigma}\,\hat{\nabla}_\sigma c^\nu + \tilde{\hat{g}}^{\nu\sigma}\,\hat{\nabla}_\sigma c^\mu - \hat{\nabla}_\sigma(\tilde{\hat{g}}^{\mu\nu}c^\sigma)].\end{aligned} \tag{7.2}$$

Here it is very interesting to note that the quantity in the square bracket of (7.2) can be rewritten as

$$\tilde{\hat{g}}^{\mu\sigma}\partial_\sigma c^\nu + \tilde{\hat{g}}^{\nu\sigma}\partial_\sigma c^\mu - \partial_\sigma(\tilde{\hat{g}}^{\mu\nu}c^\sigma) \tag{7.3}$$

identically.

The "energy-momentum tensor" $T_{\mu\nu}$ is defined by

$$\begin{aligned} T_{\mu\nu} &\equiv 2\frac{\delta}{\delta\hat{g}^{\mu\nu}}\int d^2x\,\hat{\mathcal{L}}\Big|_{\hat{g}^{\mu\nu}=\eta^{\mu\nu}} \\ &= -i[\bar{c}_{\mu\sigma}\partial_\nu c^\sigma + \bar{c}_{\nu\sigma}\partial_\mu c^\sigma + \partial_\sigma\bar{c}_{\mu\nu}\cdot c^\sigma - \eta_{\mu\nu}\bar{c}_{\sigma\tau}\partial^\sigma c^\tau] \\ &\quad +\partial_\mu\phi_M\cdot\partial_\nu\phi^M - \frac{1}{2}\eta_{\mu\nu}\partial_\sigma\phi_M\cdot\partial^\sigma\phi^M. \end{aligned} \tag{7.4}$$

In contrast with the covariant-gauge case, $T_{\mu\nu}$ is traceless. One calculates the 2-point functions of $T_{\mu\nu}$ by using Feynman propagators

$$\langle\,\bar{c}_{\mu\nu}(x)c^\lambda(y)\,\rangle = -i(\delta_\mu{}^\lambda\partial_\nu + \delta_\nu{}^\lambda\partial_\mu - \eta_{\mu\nu}\partial^\lambda)D_{\rm F}(x-y), \tag{7.5}$$

$$\langle\,\phi_M(x)\phi^N(y)\,\rangle = \delta_M{}^N D_{\rm F}(x-y). \tag{7.6}$$

One obtains

$$\langle\,T_{\mu\nu}(x)T_{\lambda\rho}(y)\,\rangle = (D-26)\Phi_{\mu\nu\lambda\rho}(x-y) \ + \ \text{local terms}, \tag{7.7}$$

where the Fourier transform of $\Phi_{\mu\nu\lambda\rho}$ is proportional to $p_\mu p_\nu p_\lambda p_\rho/(p^2+i0)$. The nonlocal term of (7.7) is called the conformal anomaly.

Owing to the identity noted in (7.3), we have a remarkable equality[9]

$$T_{\mu\nu} = \tilde{\mathcal{T}}_{\mu\nu}, \tag{7.8}$$

where $\tilde{\mathcal{T}}_{\mu\nu}$ is the quantity defined in (6.1). Hence (7.7) is essentially nothing but (6.7). It should be noted that $\tilde{\mathcal{T}}_{\mu\nu}$ is *different* from $\mathcal{T}_{\mu\nu}$; the latter only is the sensible quantity in the exact theory. It is an accidental coincidence that both (6.7) and (6.10) are proportional to $D-26$.

The inadequacy of the perturbative approach can clearly be seen by considering other anomalies. The FP-ghost number current anomaly[6] is obtained by calculating the nonlocal term of $\langle\,j_c{}^\mu(x)T_{\lambda\rho}(y)\,\rangle$, which is found to be nonvanishing. It should be noted, however, that

$$\begin{aligned} \langle\,j_c{}^\mu(x)T_{\lambda\rho}(y)\,\rangle &= \langle\,j_c{}^\mu(x)\tilde{\mathcal{T}}_{\lambda\rho}(y)\,\rangle \\ &\neq \langle\,j_c{}^\mu(x)\mathcal{T}_{\lambda\rho}(y)\,\rangle = 0. \end{aligned} \tag{7.9}$$

Indeed, explicit calculation shows that

$$\begin{aligned} \langle\,j_c{}^-(x)\mathcal{T}^+(y)\,\rangle_{\rm W} &= -\langle\,j_c{}^-(x)\tilde{b}^+(y)\,\rangle_{\rm W} + \langle\,j_c{}^-(x)\tilde{\mathcal{T}}^+(y)\,\rangle_{\rm W} \\ &= -\frac{3}{(2\pi)^2}\cdot\frac{1}{(x^+-y^+-i0)^3} + \frac{3}{(2\pi)^2}\cdot\frac{1}{(x^+-y^+-i0)^3} = 0. \end{aligned} \tag{7.10}$$

The *non-existence* of the FP-ghost number current anomaly is quite consistent with the exact solution given in Sec. 5. Thus perturbative approach is quite misleading.

8. Discussion

In his paper entitled "Quantum gravity in two dimensions",[19] Polyakov wrote "The most simple form this formula (i.e., Polyakov's nonlocal action) takes is in the conformal gauge, where $g_{ab} = e^{\varphi}\delta_{ab}$ where it becomes a free field action. Unfortunately this simplicity is an illusion." And he adopted the light-cone gauge. Even if we employ the BRS quantization, the conformal-gauge two-dimensional quantum gravity becomes a free field theory if the B-field is discarded. Nevertheless, the critical dimension $D = 26$ is obtained in this model. In the present paper, we have clarified why such a paradoxical phenomenon happens. Our conclusion is as follows.

The field equation (6.1) for the B-field suffers from the field-equation anomaly. That is, (6.1) is valid at the level of operator algebra, but it is violated *modulo* (6.3) at the level of the representation in terms of state vectors. Therefore, if one discards the B-field under the understanding that (6.1) is nothing more than a definition of the B-field, one necessarily misses the existence of the field-equation anomaly. We have found that (6.1) is *not* a mere defining equation: It is not a trivial statement to set up an equality between a fundamental field, which is a BRS transform of the FP-antighost, and a certain composite operator, which has no linear term of fundamental fields. The anomalous behaviors of the conformal-gauge two-dimensional quantum gravity are the consequence of the field-equation anomaly for (6.1). The reason why people have never been aware of this fact is that they always adopted the path-integral type approach so that they could not clearly distinguish the operator level and the representation level. Indeed, for instance, Fujikawa[2] eliminates the B-field at the first step, so that the relevance of (6.1) to the anomaly is not explicitly recognized in his calculation.

In the conformal gauge, the field-equation anomaly seems to be always proportional to $D - 26$. This is a very special situation of the conformal gauge. What is obtained from perturbative approach, however, is not identical with this "$D - 26$". That is, the formula (7.7) which gives the "conformal anomaly" precisely proportional to $D - 26$ is, owing to (7.8), nothing but (6.7) *but not (6.10)*. As discussed in our previous papers,[9,20] the nonlocal term called "conformal anomaly" is *produced* by the operation $\delta/\delta\hat{g}^{\mu\nu}$. This fact is more clearly seen in the FP-ghost number current anomaly: While the exact solution given in Sec. 5 is completely consistent with the FP-ghost number conservation, the perturbative approach implies the existence of its anomaly.[6] This paradoxical result can be explained by recognizing that the FP-ghost number current anomaly has been *produced* by the operation $\delta/\delta\hat{g}^{\mu\nu}$,[9] that is, it is a consequence of using the artificial quantity $T_{\mu\nu}$.

Anyway, the conformal-gauge two-dimensional quantum gravity implies the existence of the filed-equation anomaly proportional to $D-26$. Thus at the critical dimension $D = 26$ the field-equation anomaly is absent in the conformal gauge. This fact is quite natural because the conformal-gauge two-dimensional quantum gravity, which is definable only in the *strictly* two-dimensional case, is a theory lying in between the string theory of finite length and the covariant (de Donder)-gauge two-dimensional quantum gravity.

Finally, we note that in the conformal gauge perturbative approach is stable in contrast with the case of covariant gauge. This is due to the fact that the gauge-fixing Lagrangian density in the conformal gauge contains no linear term in the sense of perturbation theory. In the de Donder-gauge case,[9] such a redefinition of the B-field as

$$\tilde{b}_\rho = b_\rho + ic^\sigma \partial_\sigma \bar{c}_\rho \tag{8.1}$$

changes the quadratic part of the FP-ghost Lagrangian density, which contributes to the "conformal anomaly".[21] In the conformal-gauge case, (8.1) brings no change to the quadratic part.

The B-field b_ρ appearing in the right-hand side of (8.1) is the *intrinsic* B-field, which is regarded as the primary B-field more natural that $\tilde{b}_\rho$.[9] It should be noted, however, that in contrast with the de Donder-gauge case, it is impossible to define the intrinsic B-field in the conformal-gauge case because in the latter $\sqrt{-g}$ is not available so that we cannot define the action invariant under the intrinsic BRS transformation (ib_ρ is the intrinsic BRS transform of the FP antighost). All such circumstances are quite consistent with the stability of $D-26$ in the conformal gauge.

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