The early development of the representation theory of semisimple Lie groups: A. Hurwitz, I. Schur, H. Weyl

by

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Any historical development takes place inside a complicated network of interactions of the most various kinds. This is especially true of Hermann Weyl's celebrated early work on the representation theory of compact and complex Lie groups ([Weyl 1925]). There is already an excellent account on this and later work of Hermann Weyl on representation theory by Armand Borel ([Borel 1986]) which deals with almost all the strands that came together in Weyl's papers. Also there is forthcoming work by Thomas Hawkins on Weyl's shift of interest from relativity theory to representation theory ([Hawkins 1998]). The aim of this note is to pick up one of these strands leading from the works of A. Hurwitz ([Hurwitz 1897]) and I. Schur ([Schur 1924, I, II, III]) to Weyl's celebrated "character formula". In our presentation we shall adhere to actual mathematical terminology and we have not made any research into sources beyond published mathematical articles. Thus, from the historical side our presentation may well leave open numerous questions.

1. The character formula

Herman Weyl's formula for the character of the irreducible representation $L(\lambda)$ with highest weight λ of a semisimple complex Lie group G reads:

$$\chi_\lambda = \frac{\sum\limits_{w\in W} \det(w) e^{w(\lambda+\rho)}}{\sum\limits_{w\in W} \det(w) e^{w(\rho)}} \quad .$$

It may be viewed as the culmination point of a series of mathematical developments starting in the last century. Before we are going into details let us explain the meaning of this formula in present–day terminology (as to be found e.g. in the textbooks [Adams 1969], [Bourbaki 1968, 1982], [Bröcker–tom Dieck 1985], [Serre 1965]). To simplify our approach and to link it more to the essential technical developments in the works of A. Hurwitz, I. Schur, and H. Weyl, let us assume that G is a compact Lie group with simply connected semisimple part, for example the unitary group $G = U(n)$.

Let $T \subset G$ be a maximal torus of G. Then $T \cong (S^1)^r$ where $S^1 = U(1)$ is the circle group and r the (reductive) rank of G, e.g. $T =$ subgroup of diagonal matrices in $U(n)$,$T \cong (U(1))^n$. Any element of G may be conjugated into T (e.g. any element in $U(n)$ can be diagonalized by means of a unitary base change). Let $P = \mathrm{Hom}(T, S^1)$ denote the group of characters of T into S^1. Following established tradition we shall identify P (via differentials at e) as a lattice $\mathbb{Z}^r$ inside $\mathbb{R}^r = (\mathrm{Lie}\, T)^*$ the dual $\mathrm{Hom}(\mathrm{Lie}\, T, \mathbb{R})$ of the Lie algebra of T. Then the global character corresponding to $\mu \in P$ is written in (formal) exponential form

$$e^\mu : T \to S^1.$$

Let us illustrate this in the case of $G = U(n)$. We identify $\mathrm{Lie}\, T$ with $\mathbb{R}^n$ by

$$\begin{array}{ccc} \mathbb{R}^n & \longrightarrow & \mathrm{Lie}\, T \\ (\varphi_1, \ldots, \varphi_n) & \longmapsto & \begin{pmatrix} 2\pi i\varphi_1 & & \\ & \ddots & \\ & & 2\pi i\varphi_n \end{pmatrix}. \end{array}$$

Then $P = \mathbb{Z}^n \subset \mathbb{R}^n \cong (\mathbb{R}^n)^*$, and for

$$\mu = (m_1, \ldots, m_n)\ ,\ t = \begin{pmatrix} \exp(2\pi i\varphi_1) & & \\ & \ddots & \\ & & \exp(2\pi i\varphi_n) \end{pmatrix},$$

we have

$$e^\mu(t) = \exp(2\pi i(m_1\varphi_1 + \ldots + m_n\varphi_n)).$$

Let $N_G(T)$ denote the normalizer of T in G. Then $W = N_G(T)/T$ is a finite group, the "Weyl group" of (G,T) which acts naturally and faithfully as a euclidean reflection group on $P \subset \mathbb{R}^r$. The reflection hyperplanes ("mirrors") are defined as the orthogonals to the roots of T in G which are defined as the characters $\alpha \in P\backslash\{0\}$ such that the T–eigenspace

$$(\mathrm{Lie}\, G)_\alpha := \{X \in \mathrm{Lie}\, G | Ad(t)X = e^\alpha(t)X \quad \text{for all} \quad t \in T\}$$

with respect to the adjoint representation

$$Ad : G \to \mathrm{Aut}\,(\mathrm{Lie}\, G)$$

is non–trivial.

Let us consider the example $U(n)$. Let $\varepsilon_i \in P$ denote the i–th standard basis vector of $\mathbb{Z}^n, i = 1, \ldots, n$. Then $P = \bigoplus_{i=1}^{n} \mathbb{Z}\varepsilon_i$, and the set of roots is

$$\Sigma = \{\varepsilon_i - \varepsilon_j | i \neq j\} \subset P.$$

The Weyl group W is isomorphic to the symmetric group S_n acting by permutation of the n coordinates on P.

In general, the set Σ of all roots can be partitioned, in many ways,

$$\Sigma = \Sigma^+ \cup \Sigma^-$$

into "positive" and "negative" roots, $\Sigma^- = -\Sigma^+$ (e.g. $\Sigma^+ = \{\varepsilon_i - \varepsilon_j | i < j\}$ in case $U(n)$), and, correspondingly, a fundamental domain P_+ for the action of W on P can be specified. The set P_+ is called the set of dominant weights and may be described as follows:

$$P_+ = \left\{\mu \in P | \frac{2(\alpha,\mu)}{(\alpha,\alpha)} \in \mathbb{N}_{\geq 0} \quad \text{for all} \quad \alpha \in \Sigma^+\right\}$$

where $(\quad,\quad)$ denotes a W–invariant scalar product on $\mathbb{R}^r = P \otimes_{\mathbb{Z}} \mathbb{R}$.

Let $\rho \in P$ be an element of the form

$$\rho = \left(\frac{1}{2} \sum_{\alpha \in \sum^+} \alpha \right) + \omega$$

where ω is an arbitrary element in P^W, i.e. fixed by W, or such that $(\alpha, \omega) = 0$ for all $\alpha \in \Sigma$. For all $\lambda \in P$ one has

$$\lambda \in P_+ \Longleftrightarrow \lambda + \rho \in P_{++}$$

where $P_{++} = \left\{ \mu \in P | \frac{2(\alpha,\mu)}{(\alpha,\alpha)} \in \mathbb{N}_{>0} \text{ for all } \alpha \in \Sigma^+ \right\}$ is the "interior" of the fundamental domain P_+.

In case $G = U(n)$, we have with respect to the choices made and the usual scalar product on $\mathbb{R}^n$

$$P_+ = \left\{ \sum_{i=1}^{n} m_i \varepsilon_i | m_i \geq m_{i+1} \right\} \ , \ P_{++} = \left\{ \sum_{i=1}^{n} m_i \varepsilon_i | m_i > m_{i+1} \right\}.$$

A convenient choice is

$$\rho = (n-1)\varepsilon_1 + (n-2)\varepsilon_2 + \ldots + \varepsilon_{n-1}.$$

Let now $L(\lambda)$ be the irreducible $\mathbb{C} - G$–module with highest weight $\lambda \in P^+$. Then $L(\lambda)$ decomposes with respect to T into a direct sum of eigenspaces

$$\begin{aligned} L(\lambda) &= \bigoplus_{\mu \in P} L(\lambda)_\mu, \\ L(\lambda)_\mu &= \{ v \in L(\lambda) | t.v = e^\mu(t) v \ \text{ for all } t \in T \}. \end{aligned}$$

The character value $\chi_\lambda(t) = \text{trace}\,(t, L(\lambda))$ of t on $L(\lambda)$ may be written

$$\chi_\lambda(t) = \sum_{\mu \in P} \dim\, L(\lambda)_\mu e^\mu(t)$$

and Weyl's formula states

$$\sum_{\mu \in P} \dim\, L(\lambda)_\mu e^\mu = \frac{\sum\limits_{w \in W} \det(w) e^{w(\lambda+\rho)}}{\sum\limits_{w \in W} \det(w) e^{w(\rho)}}$$

as functions on T. Here we still have to add that $\det(w)$ is the determinant of w as a transformation on P ($\det(w) = \pm 1$).

Let us come back to our example $G = U(n)$. For any $\mu = \sum_{i=1} m_i \varepsilon_i \in P$ and

$$t = \begin{pmatrix} x_1 & & & & & \\ & \cdot & & & & \\ & & \cdot & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ & & & & & x_n \end{pmatrix} \in T$$

we have

$$e^{\mu}(t) = x_1^{m_1} \cdot \cdots \cdot x_n^{m_n}.$$

Let $w = \sigma$ be an element of $W = S_n$. Then

$$e^{w(\mu)}(t) = x_{\sigma(1)}^{m_1} \cdot \cdots \cdot x_{\sigma(n)}^{m_n}$$

and

$$\left(\sum_{w \in W} \det(w) e^{w(\mu)} \right)(t) = \sum_{\sigma \in S_n} \operatorname{sign}(\sigma) x_{\sigma(1)}^{m_1} \cdot \cdots \cdot x_{\sigma(n)}^{m_n} = \begin{vmatrix} x_1^{m_1} & \cdots & x_n^{m_1} \\ \vdots & & \vdots \\ x_1^{m_n} & \cdots & x_n^{m_n} \end{vmatrix}.$$

In particular, for $\rho = (n-1)\varepsilon_1 + \cdots + \varepsilon_{n-1}$ we obtain the discriminant (Vandermonde)

$$\left(\sum_{w \in W} \det(w) e^{w(\rho)} \right)(t) = \Delta(x_1, \ldots, x_n)$$

$$\Delta(x_1, \ldots, x_n) = \begin{vmatrix} x_1^{n-1} & \cdots & x_n^{n-1} \\ \vdots & & \vdots \\ x_1 & \cdots & x_n \\ 1 & \cdots & 1 \end{vmatrix} = \prod_{i<j} (x_i - x_j).$$

Thus, for a dominant weight $\lambda = \sum_{i=1}^{n} c_i \varepsilon_i$, we have the following form of Weyl's character formula

$$\chi_\lambda(t) = \frac{\begin{vmatrix} x_1^{c_1+(n-1)} & \cdots & x_n^{c_1+(n-1)} \\ \vdots & & \vdots \\ x_1^{c_n} & \cdots & x_n^{c_n} \end{vmatrix}}{\begin{vmatrix} x_1^{(n-1)} & \cdots & x_n^{(n-1)} \\ \vdots & & \vdots \\ x_1^{0} & \cdots & x_n^{0} \end{vmatrix}} \quad ,$$

a formula which had already been found by I. Schur in his dissertation (Berlin, 1901, cf. [Schur 1901] §23, formula (44)) where he had determined the irreducible polynomial representations of $GL_n(\mathbb{C})$ as well as their characters (the above formula is just a rewriting of "Schur's character formula" involving the characters of $GL_n(\mathbb{C})$ on the symmetric powers $S^d(\mathbb{C}^n)$ by means of the Jacobi–Trudy–formula in the theory of symmetric functions).

2. Adolf Hurwitz: The invariant integral on Lie groups and the construction of invariants

The main objects of invariant theory in the last century were polynomial invariants and covariants. In present–day–terminology these relate to actions of a group G on finite–dimensional complex vector spaces V and W or to linear representations

$$\rho : G \to GL(V) \quad \text{and} \quad \mu : G \to GL(W).$$

An *invariant* $f : V \to \mathbb{C}$ is a polynomial function $f \in \mathbb{C}[V]$ satisfying

$$f(gv) = f(v) \quad \text{for all} \quad v \in V \quad \text{and all} \quad g \in G.$$

A *covariant* $\phi : V \to W$ is a G–invariant polynomial map $\phi \in \mathbb{C}[V] \otimes_{\mathbb{C}} W$, i.e. ϕ satisfies

$$\phi(gv) = g\phi(v) \quad \text{for all} \quad v \in V \quad \text{and all} \quad g \in G.$$

In particular, any covariant ϕ may also be viewed as an invariant in the G–module $\mathbb{C}[V] \otimes_{\mathbb{C}} W$,

$$\phi \in (\mathbb{C}[V] \otimes_{\mathbb{C}} W)^G .^{(*)}$$

For a finite group G, the most obvious way to construct G–invariants from arbitrary elements (in a G–Module M) is given by averaging over the group:

$$I(P) = \frac{1}{|G|} \sum_{g \in G} (g.P)$$

for any $P \in M$.

For certain infinite discrete groups an analogous summation procedure had occurred in the works of Eisenstein, Weierstraß and Poincaré (elliptic and modular functions, Poincaré series for automorphic functions and forms).

Adolf Hurwitz (1859 - 1919) had been familiar with this construction, as a student of Felix Klein as well as a temporary student at Berlin. In particular, in his thesis ([Hurwitz 1881]) he had made essential use of it in the foundations of the theory of elliptic modular functions. If not through his immediate mathematical interests, at least through his long–term friendship with David Hilbert (1862 - 1943), dating back to common years at Königsberg, Hurwitz may have been attracted to the field of invariant theory proper. After a first clarifying article on the basic setting of invariant theory ([Hurwitz 1894] containing, among others, a clear presentation of the representations of linear groups on symmetric and alternating tensors as well as a generalization of Hermite's reciprocity law) he submits a note to the "Königliche Gesellschaft

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$^{(*)}$In most classical situations, the spaces V and W are themselves spaces of (homogeneous) polynomials on some vector spaces U with G–action,

$$V = \mathbb{C}[U]_m \ , \ W = \mathbb{C}[U]_n .$$

A covariant φ attaches thus to any form $F \in \mathbb{C}[U]_m$ a form $\varphi(F) \in \mathbb{C}[U]_n$ in a polynomial and "covariant" way. If $x_1, \ldots, x_N$ are coordinates on U and $a_1, \ldots, a_M$ are coordinates on V (= coefficients of the forms of degree m in the x_i) then $\varphi(F)$ can be expressed as a polynomial in the a_j and x_i satisfying the appropriate invariance condition. Since the geometry of the zero–set of F and $\varphi(F)$ is untouched by a scalar factor, invariance is also often defined up to a character of G, i.e. a power of the determinant in case $G = GL(U)$. We shall neglect this modification without any harm.

der Wissenschaften zu Göttingen" with the title "Über die Erzeugung der Invarianten durch Integration" ([Hurwitz 1897]). In it he proposes to replace the summation in the averaging process for finite, or suitable infinite groups by an integration procedure in the case of "continuous" groups. His main concern is the classical situation, i.e. the study of polynomial invariants under $SL_n(\mathbb{C})$ on the various spaces "of algebraic forms in n variables", i.e. on the symmetric powers $S^d(\mathbb{C}^n)$ of the dual of $\mathbb{C}^n$. However, by his approach he is naturally led to the consideration of "closed" subgroups of $SL_n(\mathbb{C})$.

As a start, he restricts to the real field and the special orthogonal group $G = SO_n(\mathbb{R})$ acting on $\mathbb{R}^n$ and the various symmetric powers. He views G as a compact submanifold ("Gebilde") of the euclidean space $\mathbb{R}^{n^2} = M_n(\mathbb{R})$ which thus inherits an induced Riemannian metric and associated volume form. Since, by right multiplication, G acts orthogonally on $M_n(\mathbb{R})$, preserving the submanifold G, the metric and volume form on G are also preserved. Thus he obtains, in actual terminology, a right–invariant volume form dg on G:

$$\int_G f(gh)dg = \int_G f(g)dg \quad \text{for all} \quad h \in G,\ f \in C^0(G, \mathbb{R}),$$

$$\int_G dg = M > 0.$$

If $F : S \to \mathbb{R}$ is an arbitrary polynomial function on $S = S^d(\mathbb{R}^n)$, then the function $I(F) = J : S \to \mathbb{R}$ defined by

$$J(a) = \int_G F(ga)dg$$

is a G–invariant function on S.(*)

Of course, the volume dg is easily normalized (by dividing by M), and then any invariant F is reproduced

$$I(F) = F \quad \text{for all} \quad F \in \mathbb{R}[S]^G.$$

(*)Unfortunately, Hurwitz gives only a verbal proof of this result. It seems to us that his set–up would require a left–invariant volume. Anyhow, the dg constructed is left–invariant, too, and also transposition (or inversion) on G are trivially metric and volume preserving.

Using Hilbert's procedure ([Hilbert 1890, V]) in his proof for the finite generation of polynomial $SL_n(\mathbb{C})$–invariants and replacing his differential Cayley operator Ω by I this gives a generalization of his proof to the situation at hand as well as to all situations where a form dg with the above properties exists.

Before Hurwitz goes on to these other situations he first gives a completely explicit formula for dg in terms of Euler–angles:

For $\nu = 1, \ldots, n-1$ and $\varphi \in \mathbb{R}$ (or $\varphi \in [0, 2\pi]$) let $E_\nu(\varphi)$ denote the orthogonal transformation of $\mathbb{R}^n$ fixing all coordinates $x_i, i \neq \nu, \nu+1$, and rotating the $(\nu, \nu+1)$–plane by the angle φ. Then, with respect to some ordering of the $\frac{n(n-1)}{2}$ pairs $i < j, i, j \in \{1, \ldots, n\}$, there is a function $\nu : \{(i,j) | i < j\} \to \{1, \ldots, n-1\}$ such that

$$\begin{aligned} E : \mathbb{R}^{\frac{n(n-1)}{2}} &\longrightarrow G \\ (\varphi_{ij}) &\longmapsto \prod_{i<j} E_{\nu(i,j)}(\varphi_{ij}) \end{aligned}$$

gives a parametrization of G (injective in a suitable subset of $[0, 2\pi]^{\frac{n(n-1)}{2}}$).

Neglecting a constant factor, Hurwitz obtains the following form for the pull back of the volume form

$$E^*(dg) = \prod_{i<j} \sin(\varphi_{ij})^{i-1} d\varphi_{ij} \ ,$$

in particular, for $n = 2$ we get

$$E^*(dg) = d\varphi_{12}$$

and for $n = 3$

$$E^*(dg) = \sin(\varphi_{23})\, d\varphi_{12}\, d\varphi_{23}\, d\varphi_{13}.$$

The evaluation of an integral

$$I(F), \quad \text{for} \quad F \in S^d(\mathbb{R}^n)$$

is thus reduced to elementary trigonometric integrals.

Hurwitz now turns to the case of main interest, polynomial $SL_n(\mathbb{C})$–invariants on $S^d(\mathbb{C}^n)$. He observes that, due to the non–compactness of $SL_n(\mathbb{C})$, the averaging procedure

$$\int F(ga)dg$$

(for a suitable invariant volume dg) will diverge, in the general situation. However, looking at $\mathbb{C}^n$ with its standard hermitian form as euclidean $\mathbb{R}^{2n}$ Hurwitz concentrates on that subgroup G of $SL_n(\mathbb{C})$ which acts orthogonally on $\mathbb{R}^{2n}$, i.e. the special unitary group $SU_n(\mathbb{C})$ which is of real dimension n^2-1. As a closed subgroup of $SO_{2n}(\mathbb{R})$ it admits a (right–) invariant metric and volume. Moreover, there is an analogue of the Euler–parametrization for G

$$E : \mathbb{R}^{n^2-1} \to G$$

which extends to a complex–analytic parametrization

$$E_{\mathbb{C}} : \mathbb{C}^{n^2-1} \to SL_n(\mathbb{C}).$$

Thus, there is an explicit invariant integral

$$I(F)(a) := \int_G F(ga)dg$$

for $G = SU_n(\mathbb{C})$, and any G–invariant polynomial on $S = S^d(\mathbb{C}^n)$ is automatically $SL_n(\mathbb{C})$–invariant (by analytic continuation; note that the matrix coefficients of $g \in SL_n(\mathbb{C})$ on S are polynomials in those of g).

As in the case of the orthogonal group, Hurwitz determines the concrete form of $E^*(dg)$ reducing the evaluation of $I(F)$ to the computation of elementary integrals and, in the case of the action of $SL_2(\mathbb{C})$ on binary forms, he derives formulas for $I(F)$ in terms of the coefficients of the form F.

In the final section the construction of an invariant integral[(*)] is extended to arbitrary Lie groups. From our present understanding this is done by

[(*)] Hurwitz' development may consistently be read as dealing either with right– or with left–invariant volumes. Employing present–day conventions (on the composition order of maps) one should consider right–invariant volumes.

right or left translation of a volume form on the Lie algebra of G. Due to the conceptual limitations of his day Hurwitz has to restrict to Lie groups diffeomorphic to $\mathbb{R}^n$ or, alternatively, to "local" Lie groups. However, his arguments could well be extended to arbitrary group manifolds. Of course, for non–compact groups the effectiveness of the integral procedure

$$\int_G F(ga)dg$$

to produce invariants depends on the convergence of that integral which may fail, in most cases.

3. Issai Schur: Characters and the enumeration of invariants

Representation theory of finite groups as we understand it today originated mainly in the work of G. Frobenius (1849 - 1917) at the end of last century (cf. [Hawkins 1971, 1974]). In his efforts he was soon joined by his student, Issai Schur (1875 - 1941), who, in the first decade of his mathematical career (1901 - 1911), contributed a series of fundamental investigations to the new field. In his dissertation at Berlin University (1901) Schur determined the irreducible polynomial representations of GL_n (more or less explicitly over the base field $\mathbb{C}$ of complex numbers but essentially valid over an arbitrary field of characteristic zero). In particular, he developed a formula for their characters (cf. our first section) and a relation to the representation theory of the symmetric groups. Other papers, partly in collaboration with Frobenius, deal with basic questions on projective, real, or integral representations, and special attention is given to comprehensive explicit results in case of the symmetric, alternating, and 2×2–linear groups over finite fields. Whereas in the period (1911 - 1923) his mathematical interests spread out into various directions of analytic, algebraic or arithmetic nature Schur's entrance into the Prussian Academy of Sciences (since 1922 as a "successor" to his teacher, G. Frobenius, cf. [Schur 1922]) may have brought him back to his origins(*) so that in 1924 he was able to submit to the academy three extensive reports

(*)It is very hard to get any information directly out of Schur's papers about his motivations. Besides his relation to Frobenius also the book by E. Study on invariant theory

on a "new application of integral calculus to problems in invariant theory" ([Schur 1924]).

Schur begins his first paper (1. Mitteilung) with some improvements on Hurwitz' treatment of "projective invariants", i.e. polynomial invariants of $SL_n(\mathbb{C})$ (a more explicit proof of the "Zariski–denseness" of $SU_n(\mathbb{C})$ in $SL_n(\mathbb{C})$, a simplification of Hurwitz' integral construction of $SU_n(\mathbb{C})$–invariants, and an identification of that procedure with Hilbert's original differential Ω–process), and then turns to the representation theory and the counting problem ("Abzählungsproblem") of the special orthogonal group $SO_n(\mathbb{R})$. He recalls Hurwitz' construction of the right–invariant integral on $SO_n(\mathbb{R})$ by means of the Euler parametrization, and in analogy to the development for finite groups, he uses it to establish the basic properties of continuous complex representations $\rho : SO_n(\mathbb{R}) \to GL(V)$ and their associated characters $\chi : SO_n(\mathbb{R}) \to \mathbb{C}, \chi(g) = \operatorname{trace} \rho(g)$, as there are:

- any such ρ is unitary with respect to a suitable hermitian scalar product on V,
- as a consequence, any such ρ is completely reducible, and
- up to "similarity" (conjugation in $GL(V)$) ρ is completely determined by its character χ,
- with respect to the hermitian product

$$\begin{aligned} \langle f, h \rangle \qquad &:= \int\limits_G f(g)\overline{h}(g)dg \\ \int\limits_G dg = 1, \qquad &G = SO_n(\mathbb{C}), \end{aligned}$$

  on the space $C^0(G, \mathbb{C})$ of continuous functions we have the orthonormality relations for irreducible ("einfache") characters χ and χ':

$$\langle \chi, \chi' \rangle = \delta_{\chi\chi'} \quad \text{(Kronecker–Delta)} .$$

([Study 1923]) containing some polemic against his contemporaries may have had an effect on Schur (Study and Schur had been colleagues at Bonn University for a short time, 1913 - 1916). At least in the case of Hermann Weyl the rôle of Study's book is acknowledged (cf. [Borel 1986], [Hawkins 1998]). See however [Hawkins 1987] for the history of the counting problem and Schur's position.

Quite generally, the last point gives a way to calculate the dimension of the space V^G of fixed vectors in a representation $\rho : G \to GL(V)$ with character χ, i.e.

$$\dim V^G = \langle \chi, 1 \rangle = \int_G \chi(g) dg.$$

In particular, if one wants to "count" the linearly independent G–invariant homogeneous polynomials of degree k on V one has to compute

$$\int_G \chi^{(k)}(g) dg$$

where $\chi^{(k)}$ is the character of the induced action of G on the k–th symmetric power of V^* (or V since G acts orthogonally). Using Molien's generating series (in the formal variable z)

$$\sum_{k=0}^{\infty} \chi^{(k)}(g) z^k = \frac{1}{\det(1 - z\rho(g))}$$

Schur gives some reformulations of that integral and finally some explicit computations for $G = SO_3(\mathbb{R})$ and multihomogeneous invariants of m–tuples of bilinear forms on $\mathbb{R}^3$, an example studied before by Study ([Study 1889, 1923]). Before closing his article Schur extends Hurwitz' right–invariant integral to the full orthogonal groups and remarks that the general results of his paper apply to all cases of groups where Hurwitz' construction of the integral works. He underlines that the orthogonal groups are not only quite relevant for applications but that the "Abzählungsproblem" admits of a "praktisch wirklich brauchbare" solution by means of his integral calculus.

More than an application of integral calculus to problems in invariant theory, Schur's second paper ("II: Über die Darstellung der Drehungsgruppe durch lineare homogene Substitutionen") is an application of invariant theory to integral calculus. In it he succeeds in determining the (continuous) irreducible representations of all orthogonal groups, $O_n(\mathbb{R})$ and $SO_n(\mathbb{R})$, their degrees and, apart form the "two–sided" ("zweiseitige") representations of $SO_{2\nu}(\mathbb{R})$, their characters.

The basic result from invariant theory used is the later so called "First Fundamental Theorem" (cf. [Weyl 1946]) for the orthogonal group going back to Study ([Study 1897]) but reproved by Schur:

Let $V = \mathbb{R}^n$ denote the natural representation space of the orthogonal group $O_n(\mathbb{R})$ with invariant scalar product

$$\begin{array}{rcl} V \times V & \longmapsto & \mathbb{R} \\ (v, w) & \longmapsto & v \cdot w. \end{array}$$

Then the first fundamental theorem describes the $O_n(\mathbb{R})$–invariant polynomials of a set of r vectors $(v_1, v_2, \ldots, v_r), v_i \in V$, or, in today's terminology, the multi–graded ring

$$\mathbb{R}[V \times \ldots \times V]^G = (\mathbb{R}[V] \otimes \ldots \otimes \mathbb{R}[V])^G = \bigoplus_{d_1, \ldots, d_r} (\mathbb{R}[V]_{d_1} \otimes \ldots \otimes \mathbb{R}[V]_{d_r})^G$$

where $G = O_n(\mathbb{R})$ and $\mathbb{R}[V]_d$ denotes the homogeneous real polynomials of degree d on V (alternatively, the k–th symmetric power on $V^* \cong V$). The result is that for $r \leq n$ this $\mathbb{R}$–algebra is generated by the $\frac{r(r+1)}{2}$ algebraically independent contractions $J_{\alpha\beta}, 1 \leq \alpha \leq \beta \leq r$,

$$\begin{array}{rcl} J_{\alpha\beta} : V \times \ldots \times V & \longrightarrow & \mathbb{R}, \\ J_{\alpha\beta}(v_1, \ldots, v_r) & = & v_\alpha \cdot v_\beta \end{array}$$

(scalar product of v_α and v_β). By means of Molien's formula and in terms of generating functions this can be expressed as follows ($f(z, g) = \det(1 - zg)$ will denote the characteristic polynomial of g on V and p_d the character of G on $\mathbb{R}[V]_d$)$^{(*)}$:

$$(MS) \quad \int_G \frac{dg}{f(z_1, g) \cdots f(z_r, g)} = \sum_{d_1, \ldots, d_r} \left(\int_G p_{d_1} \cdots p_{d_r} dg \right) z_1^{d_1} \cdots z_r^{d_r} =$$

$$\sum_{d_1, \cdots, d_r} \dim (\mathbb{R}[V]_{d_1} \otimes \cdots \otimes \mathbb{R}[V]_{d_r})^G z_1^{d_1} \cdots z_r^{d_r} = \prod_{1 \leq \alpha \leq \beta \leq r} \frac{1}{1 - z_\alpha z_\beta}.$$

This formula, evaluating all the integrals $\int_{O_n(\mathbb{R})} p_{d_1}(g) \cdots p_{d_r}(g) dg$, $d_1, \ldots, d_r \in \mathbb{N}_0$, is the cornerstone for Schur's subsequent reasoning.

In his thesis (1901) Schur had expressed the irreducible polynomial characters of $GL_n(\mathbb{C})$ as determinants in selected symmetric powers p_k of the natural

$^{(*)}$This formula also holds for $SO_n(\mathbb{R})$ as long as $r < n$. For $r = n$ one has to take into account the determinant of n vectors as an additional $SO_n(\mathbb{R})$–invariant.

representation. Since the restriction of p_k to $O_n(\mathbb{R})$ decomposes, Schur singles out the spherical harmonics

$$q_k = p_k - p_{k-2}$$

(corresponding to the subspace of $S^k(V)$ orthogonal to $q.S^{k-2}(V), q$ denoting the invariant symmetric bilinear form) and proposes once more a determinantal form for the irreducible characters of $SO_n(\mathbb{R})$ (we shall assume $n = 2\nu+1$ odd from now on[(**)]).

For any set of integers $\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_\nu \geq 0$ he puts

$$\chi_{\alpha_1,\ldots,\alpha_\nu} = \begin{vmatrix} q_{\alpha_1} & q_{\alpha_1+1} + q_{\alpha_1-1} & \cdots & q_{\alpha_1+\nu-1} + q_{\alpha_1-\nu+1} \\ q_{\alpha_2-1} & q_{\alpha_2} + q_{\alpha_2-2} & \cdots & q_{\alpha_2+\nu-2} + q_{\alpha_2-\nu} \\ \vdots & \vdots & & \vdots \\ q_{\alpha_\nu-\nu+1} & q_{\alpha_\nu-\nu+2} + q_{\alpha_\nu-\nu} & \cdots & q_{\alpha_\nu} + q_{\alpha_\nu-2\nu+2} \end{vmatrix}$$

(where $q_k = 0$ if $k < 0, q_0 = 1, q_1 = p_1$), and by extremely skillful manipulations with determinants he is able to deduce the orthogonality relations

$$\int_{SO_n(\mathbb{R})} \chi_{\alpha_1,\ldots,\alpha_\nu} \cdot \chi_{\beta_1,\ldots,\beta_\nu} = \delta_{\alpha_1,\beta_1} \cdots \delta_{\alpha_\nu,\beta_\nu}$$

from formula (MS) in a purely algebraic way. A computation of the values of $\chi_{\alpha_1,\ldots\alpha_\nu}$ at the identity gives a positive number $M_{\alpha_1,\ldots,\alpha_\nu}$ thus showing that they are irreducible characters of that degree.

To prove the completeness of his set of characters Schur has to show that there is no character $\gamma : SO_n(\mathbb{R}) \to \mathbb{R}$ such that

$$\int_{SO_n(\mathbb{R})} \gamma(g)\chi_{\alpha_1,\ldots,\alpha_\nu}(g)dg = 0$$

for all $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_\nu \geq 0$ (we still keep the assumption n odd, $n = 2\nu + 1$). For that he employs a method of dimensional reduction which,

[(**)] Actually, Schur proceeds by a subtle interplay between representations of $O_n(\mathbb{R})$ and $SO_n(\mathbb{R})$ for arbitrary n. Whereas this is nontrivial for even n, the direct product structure $O_n(\mathbb{R}) = SO_n(\mathbb{R}) \times \{\pm 1\}$, for n odd, allows to read Schur's presentation along our somewhat simplified account. In this case, all characters are real valued, in addition!

in a modified form, should become central in Hermann Weyl's approach and may be described, in present terminology, as follows.

Any class function on $G = SO_n(\mathbb{R})$ is determined by its restriction to the maximal torus $T \cong (S^1)^\nu$ of 2×2–blockdiagonal rotations. In particular, a restricted character $\gamma|_T$ decomposes as a finite sum of terms of the form

$$t_{\alpha_1,\dots,\alpha_\nu}(g) = \sum_{(\beta_1,\dots,\beta_\nu)} e^{i(\beta_1\varphi_1+\cdots+\beta_\nu\varphi_\nu)}$$

where $\varphi_1, \dots, \varphi_\nu$ are the rotation angles of $g \in T$ and where $(\beta_1, \dots, \beta_\nu)$ runs over the orbit of $(\alpha_1, \dots, \alpha_\nu)$ under the action of the group $W = S_\nu \ltimes (\mathbb{Z}/2\mathbb{Z})^\nu$ of permutations and sign changes of the α_i. Conversely, he shows that all such orbit sums may be obtained as finite linear combinations of the restricted characters $\chi_{\alpha_1,\dots,\alpha_\nu}|_T, \alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_\nu \geq 0$. Without providing further details he claims that this leads to the completeness statement ("Dies ist gewiß richtig ..."). (*)

The paper concludes with remarks on the rationality of the representations constructed, the exterior powers of the natural representation, the group $SO_4(\mathbb{R})$ and some properties of the degrees of the irreducible representations.

Schur's third paper ("III. Vereinfachung des Integralkalküls. Realitätsfragen") takes up a problem discussed already in his second paper. (In the following, we shall assume once more $n = 2\nu + 1$, for simplicity).

Any continuous class function γ on $G = SO_n(\mathbb{R})$ is completely determined by its restriction $\gamma|_T$ to the maximal torus T and yields an element in $C^0(T, \mathbb{R})^W$, i.e. as a function of the rotation variables $\varphi_1, \dots, \varphi_\nu$ the restriction is invariant with respect to permutations and sign–changes of the

(*) The argument may be easily completed as follows: For the restrictions to T one has a linear expression

$$\gamma|_T = \sum_{(\alpha)} c_{(\alpha)}\chi_{(\alpha)}|_T,\ c_{(\alpha)} \in \mathbb{C},$$

$(\alpha) = (\alpha_1, \dots, \alpha_\nu)$. This implies the equality

$$\gamma = \sum_{(\alpha)} c_{(\alpha)}\chi_{(\alpha)}$$

for the class functions on G. Orthonormality then gives $c_{(\alpha)} = 0$ for all (α), i.e. $\gamma = 0$.

φ_i. Already in his separate discussion of the case $n = 3$, Schur had found a function $M \in C^0(T, \mathbb{R})^W$ such that

$$\int_G \gamma(g)dg = \int_T \gamma(\varphi)M(\varphi)d\varphi$$

reducing the three–dimensional integral on the left–hand–side to a one–dimensional integral on the right. He had clearly seen the advantages of such a formula in the general case (reducing an $\frac{n(n-1)}{2}$–dimensional integral to a $\nu = \left[\frac{n}{2}\right]$–dimensional one) but had also been unable to deduce such a formula in a direct way. Now he provides such a derivation as a consequence of his previous work on the irreducible characters. Starting from a proposed formula for M, e.g.

$$M(\varphi_1, \ldots, \varphi_\nu) = \prod_{\alpha=1}^{\nu}(1 - \cos\,\varphi_\alpha) \prod_{1\le\alpha<\beta\le\nu} (\cos\,\varphi_\alpha - \cos\,\varphi_\beta)^2$$

for $G = SO_{2\nu+1}(\mathbb{R})$, he has to check the formula

$$\int_G \gamma(g)dg = \int_T \gamma(t)M(t)dt = c_0 \int_0^{2\pi} \ldots \int_0^{2\pi} \gamma(\varphi_1, \ldots, \varphi_\nu)M(\varphi_1, \ldots, \varphi_\nu)d\varphi_1 \ldots d\varphi_\nu$$

for all class functions γ (and a "natural" constant c_0). However, in his arguments for the completeness of his system of irreducible characters he had seen that a dense subspace of $C^0(T, \mathbb{R})^W$ was provided by the span of all products $p_{d_1} \cdots p_{d_\nu}$ (characters of symmetric powers restricted to T). Thus, by using once more the identity (MS), he has to show

$$\int_T \frac{M(t)dt}{f(z_1, t) \cdots f(z_\nu, t)} = \int_G \frac{dg}{f(z_1, g) \cdots f(z_\nu, g)} = \prod_{1\le\alpha\le\beta\le\nu} \frac{1}{1 - z_\alpha\, z_\beta}\ ,$$

and he achieves this by essentially elementary means (Cauchy's determinant formula and an evaluation of one–dimensional trigonometric integrals).

For $n = 2\nu + 1$ this reduction can also be used for the full orthogonal group $O_{2\nu+1}(\mathbb{R})$, i.e. for the integration over the second component of this group. In the case $n = 2\nu$, the second component of $O_n(\mathbb{R})$ requires a modification of the function M (there is a reduction to $\nu - 1$ angular variables, now).(*)

(*) As found out later (cf. [Weyl 1946]) the structure of M is the same as that for the symplectic group in $n - 2$ dimensions.

As an application of the "simplified" integral calculus Schur rederives some of Study's enumerative numbers for the quaternary orthogonal group by means of a quick integration. The article concludes with a precise determination of the reality type of the irreducible representations of $SO_n(\mathbb{R})$ (non–trivial only for $n = 2\nu$ and "two–sided" representations).

Already in the introduction to his second paper Schur comments on the related work of E. Cartan [Cartan 1913, 1914]) on the representation theory of the "infinitesimal" semisimple Lie groups, pointed out to him by Hermann Weyl, and in an appended footnote he acknowledges recent work of Weyl ([Weyl 1924c]) providing a new approach to the representation theory of the group $SO_n(\mathbb{R})$ in a development of Cartan's method. Moreover, in a "note added in proof" at the same place he signals recent success of Weyl in deriving Schur's result in a shorter way at the same time applicable to a larger class of groups. Actually, Schur's third paper is already preceded by a note of Hermann Weyl to the Prussian academy ([Weyl 1924c]) in which he sketches these new methods and results. Thus, in the introduction to his third paper, Schur has to admit that Weyl has found a direct geometric way to prove the integral formula for class functions and to derive the whole representation theory of the rotation group in a direct analytic way without recourse to invariant theory (as Schur had already done for $n = 2$ and 3). Thus, Schur's work had, by then, been completely superseded by Weyl's simpler and further–reaching methods. Nonetheless, when later Weyl undertook the construction of the irreducible representations of the orthogonal groups in a purely algebraic way ([Weyl 1946]) his development was based on the same invariant theoretic results (the "First Fundamental Theorem") as was Schur's. In this context, also the indebtness of our two authors to R. Brauer should not go unnoticed: Schur acknowledges Brauer's help in the completeness proof of his second paper (cf. footnote 2 on p. 316 mentioning "Herrn stud. math. R. Brauer") and Weyl gives Brauer a great share for having obtained these results in [Weyl 1946] (cf. the introduction of loc. cit.).

4. Hermann Weyl: The integral and the character formula

Hermann Weyl (1885–1955) was educated as a student at Göttingen. With his doctoral dissertation (1908) and his subsequent research topics he mainly

joined Hilbert's and his school's investigations on the spectral theory of integral and differential operators. As an outgrowth of his activity as a Göttingen–Privatdozent he already published his first, highly influential book "Die Idee der Riemannschen Fläche" ([Weyl 1913]) which, among other things, provided a first "rigorous" treatment of abstract Riemann surfaces.

In 1913 Weyl took up a chair of mathematics at the ETH Zürich and soon, under the presence of Einstein's developing theory of general relativity switched his mathematical interests to the foundation, elaboration and generalization of Riemannian geometry and relativity. Again, this led once more to a celebrated book "Raum, Zeit, Materie" ([Weyl 1918]) as well as to more specialized publications like "Riemann's Habilitationsvortrag" ([Weyl 1919]) or "Vorlesungen über das Raumproblem" ([Weyl 1923]). The essence of the last mentioned "Raumproblem" is a differential–geometric characterization of the orthogonal groups. As a result of E. Cartan's independent contributions to this problem ([Cartan 1923]) Weyl familiarized himself with Cartan's and Killing's fundamental work on the structure and representations of the semisimple Lie algebras (e.g. [Killing 1888–90], [Cartan 1894], [Cartan 1913], [Cartan 1914], cf. also [Borel 1986] and [Hawkins 1998] for further details).(*)

As clearly stated by Weyl himself ([Weyl 1949]) and convincingly elaborated by Hawkins ([Hawkins 1998], but, cf. also [Borel 1986]) his attempt to understand the nature of "fields on space–time" or of "natural" or "covariant quantities" on (differentiable) manifolds M constituted his basic motivation for the study of representation theory. In present–day terminology such fields are just sections of vector bundles associated to linear representations of the structure group G of the frame bundle $\mathcal{F}(M)$ of M. By combining Cartan's infinitesimal constructions ([Cartan 1913]) with Schur's argument for complete reducibility ([Schur 1924, I]) and his own topological arguments for the simple connectivity of the special unitary groups $SU(n), n \geq 2$, Weyl could show in [Weyl 1924b] that all irreducible (differentiable) linear representations of SL_n (and, essentially also of GL_n) could be realized by their induced

(*)In that context it should be pointed out that one of Weyl's basic techniques in his own proof (e.g. [Weyl 1923], 8. Vorl. u. Anhang 12) is the decomposition of a Lie algebra L into eigenspaces ("Länder") with respect to the adjoint action of a diagonalisable element $h \in L$ ("Haupttransformation"). In case of a semisimple L (which is Weyl's case, as easily derived e.g. like in [Cartan 1923] and a "generic" h this leads to the Killing–Cartan decomposition of L into the direct sum of a Cartan subalgebra and one–dimensional root spaces. In that sense, Weyl was familiar with one of the fundamental techniques of the Killing–Cartan theory right from the start of his own investigations on the space problem.

actions on subspaces of tensors of the natural representation given by distinguished symmetry conditions. It is this fact which constituted for him "the true mathematical basis of tensor calculus" ([Weyl 1924b]).(*) Already before, in [Weyl 1924a] he had reduced these symmetry conditions to those given by the Young–Frobenius idempotents, thus making contact with the Frobenius–Schur methods in representation theory (cf. [Hawkins 1998] for more details).

In view of this development it may have been quite natural for Schur to provide Weyl with his subsequent papers [Schur 1924, II, III] before publication. As discussed above, Schur's third paper had been dedicated to a derivation of the following fundamental "class integral formula"

$$\int_G \gamma(g)dg = \int_T \gamma(\varphi)M(\varphi)d\varphi$$

for a class function γ on $SO_n(\mathbb{R})$ and a certain auxiliary function M on the maximal torus $T \subset SO_n(\mathbb{R})$. It is this integral formula (including the explicit form of M) for which Weyl was now able to give a completely lucid geometric derivation, uniformly valid for all semisimple (complex or compact) groups. He first sketched this in a letter to Schur ([Weyl 1924c]) and then finally elaborated this, together with other basic results in his three great papers [Weyl 1925].

Weyl's letter consists of four sections:

1) an introduction,

2) the derivation of the integral formula for the unitary groups,

3) the derivation of the character and dimension formula for the unitary groups,

4) a sketch of 3) (and 2)) for the general semisimple case.

(*) As the history of physics has shown, the "naturality" of the fields associated to the full frame bundle $\mathcal{F}(M)$ has been superseded quickly. Dirac's relativistic wave equation deals already with spinor fields associated to the finer spin structure underlying a (semi–) Riemannian manifold and thus with the "spin representations" discovered by Cartan ([Cartan 1913]). Moreover, today's particle physics deals with fields associated with abstract principal bundles P on M, the structure group G being now the group of "internal symmetries".

In addition, the formulas for the classical groups ("orthogonale und Komplexgruppe") are explicitly stated in sections 2) and 3).

Weyl's specific arguments in sections 2) and 3) exhibit already the general structure in such a transparency that we shall describe them, once more as in our section 1 above, from our present–day point of view.(*) In particular, we assume his basic results of [Weyl 1925] on general semisimple groups established, like e.g. the existence of a simply–connected compact real form G with maximal torus T, character lattice P and Weyl group W for every semisimple type.

The derivation of the class integral formula is obtained in three steps:

- Any element in G is conjugate to an element in T. This fact is well–known for the unitary and orthogonal groups, easily provided for the symplectic group (in [Weyl 1925] II, §1), but requires some differential–geometric arguments (as under the subsequent third point) in the general case (cf. [Weyl 1925] IV, §1).

- Weyl regards the coset space G/T as an abstract manifold admitting a (G–left invariant) volume element. The basic fact is here the existence of a T–invariant volume element on the tangent space of G/T at T. Weyl identifies this space with the sum of the root spaces

$$\bigoplus_{\alpha\in\sum} (\mathrm{Lie}\, G)_\alpha \qquad (**)$$

  and derives the invariant volume from the occurence of positive and negative roots in pairs. The measure $d\,\overline{g}$ on G/T may henceforth be normalized to

$$\int_{G/T} d\,\overline{g} = 1.$$

(*) For some readers not familiar with the theory of semisimple Lie algebras even Weyl's elaborated treatment of the unitary case in [Weyl 1925] may seem somewhat sketchy. For those we may recommend Weyl's exposition in [Weyl 1946], Chap VII, which, to our knowledge, is the most detailed one given by himself.

(**) This is, strictly speaking, the complexified tangent space.

- Finally, Weyl compares the volume element $(d\,\overline{g}) \times (dt)$ on $G/T \times T$ (here, dt denotes the normalized "angular" measure on the torus T) with the normalized volume dg on G under the conjugating map

$$\kappa : (gT, t) \longmapsto gtg^{-1},$$

or, in other words, he computes the Jacobian $J(gT,t)$ of this map. He obtains the expression

$$J(gT,t) = \prod_{\alpha \in \sum} (e^{\alpha}(t) - 1),$$

or,

$$J(t) = \Delta(t).\,\overline{\Delta}(t),$$

where

$$\Delta = \prod_{\alpha \in \sum^{+}} (e^{\alpha/2} - e^{-\alpha/2})^{(*)}$$

is the "Weyl–denominator".
Integration over a class function $f : G \to \mathbb{C}$ gives now

$$\int_G f(g)dg = \frac{1}{|W|} \int_T f(t)\Delta(t)\overline{\Delta(t)}dt.^{(**)}$$

In view of this formula and Schur's orthogonality relations, the determination of the set of (potential) irreducible characters turns out to be an elementary combinatorial task. By definition, any such character (when restricted to T) is a finite W–invariant Fourier series

$$\chi \in \mathbb{Z}[P]^W \subset C^0(T, \mathbb{C})^W.$$

To find an orthonormal system of such functions with respect to the hermitian product

$$(f,h) = \int_G f(g)\overline{h(g)}dg$$

$^{(*)}$Note that Δ is still an element of $\mathbb{Z}[P]$ (cf. [Weyl 1925] IV, §3).

$^{(**)}$The factor $1/|W|$ is nowadays interpreted in terms of the mapping degree of κ. Weyl's treatment of it remains formal.

is the same as to find an orthonormal system in $\mathbb{Z}[P]^W$ with respect to

$$(f,h)_\Delta := \frac{1}{|W|}\int\limits_T f(t)\Delta(t)\overline{h(t)\Delta(t)}dt.$$

However, multiplication by Δ induces a $\mathbb{Z}$–linear isomorphism between $\mathbb{Z}[P]^W$ and $\mathbb{Z}[P]^{W,-}$, the W–anti–invariant Fourier series (for the derivation in the general case, cf. [Weyl 1925] IV, §3), and an orthonormal basis with respect to the "usual " product $(f,h) \longmapsto \frac{1}{|W|}\int\limits_T f(t)\overline{h(t)}dt$ on $\mathbb{Z}[P]^{W,-}$ is given by the elementary anti–invariant sums

$$A(\mu) = \sum_{w\in W} \det(w)\, e^{w(\mu)},$$

μ running through the interior P^{++} of a fundamental domain for W on P. Thus, a potential set of irreducible characters is given by the set of

$$\chi_\lambda = \frac{A(\lambda+\rho)}{\Delta}\ , \quad \text{where}$$

λ runs through P^+ and $\Delta = A(\rho)$ is the "smallest" anti–invariant ("Weyl–denominator identity").

It remains to be shown that the χ_λ describe actual characters. For that, Weyl has to refer to [Cartan 1913], where for any $\lambda \in P^+$ an irreducible representation with highest weight λ (corresponding to the "leading term" e^λ in χ_λ) is constructed. The coincidence of the characters is then quickly deduced.

As a final result, by specializing the character formula, Weyl obtains a formula for the dimension of the module corresponding to λ

$$\dim\, L(\lambda) = \frac{\prod\limits_{\alpha\in\sum^+}(\lambda+\rho,\alpha)}{\prod\limits_{\alpha\in\sum^+}(\rho,\alpha)}\ ,$$

where $(\ ,\ )$ denotes a W–invariant scalar product on P.

In contrast to the elegance of his basic arguments, Weyl was not pleased about his dependence on Cartan's explicit constructions. Thus, already in a

footnote to [Weyl 1924c] he envisages a uniform construction of all irreducible representations by establishing a "completeness relation" for the characters. In [Weyl 1925] IV, §4, he elaborates on that. However, the definite breakthrough had to wait until his joint work with F. Peter ([Weyl 1927]) on the decomposition of the regular representation of G, another cornerstone in the development of the representation theory of Lie groups.

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