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Construction of an Identically Nilpotent BRS Charge in the Kato-Ogawa String Theory

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Abstract

In previous work, the conformal-gauge two-dimensional quantum gravity in the BRS formalism has been solved completely in terms of Wightman functions. In the present paper, this result is extended to the closed and open bosonic strings of finite length; the open-string case is nothing but the Kato-Ogawa string theory. The field-equation anomaly found previously, which means a slight violation of a field equation at the level of Wightman functions, remains existent also in the finite-string cases. By using this fact, a BRS charge nilpotent even for $D \neq 26$ is explicitly constructed in the framework of the Kato-Ogawa string theory. The FP-ghost vacuum structure of the Kato-Ogawa theory is made more transparent; the appearance of half-integral ghost numbers and the artificial introduction of indefinite metric are avoided.

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1. Introduction

In 1983, Kato and Ogawa[1] published a fundamental paper on the BRS quantization of the bosonic string of finite length based on the Lagrangian density of the conformal-gauge two-dimensional quantum gravity together with open-string boundary conditions. According to their conclusion, the square of the normal-ordered BRS charge written in terms of the creation and annihilation operators of string and FP-ghosts is *nonvanishing unless* $D = 26$ *and* $\alpha_0 = 1$, where D stands for the dimension of the world in which the string lives and α_0 denotes a regularization parameter of the Hamiltonian (interpreted as the zero intercept of the leading Regge trajectory). Then they established no-ghost theorem for the string only for $D = 26$ and $\alpha_0 = 1$.

The purpose of the present paper is to point out that Kato-Ogawa's conclusion on the BRS charge is not intrinsic; more precisely speaking, we show that *it is possible to construct explicitly an identically nilpotent BRS charge in the Kato-Ogawa framework.* One may wonder why two different BRS charges exist in one particular theory. The key word for answering this question is "field-equation anomaly". So, we first explain what the field-equation anomaly is.

In the Heisenberg picture, field equations and equal-time (anti)commutation relations uniquely determine the full-dimensional (anti)commutation relations at least in principle. This problem can be explicitly worked out in the two-dimensional quantum gravity in various gauges such as de Donder gauge,[2] light-cone gauge,[3] conformal gauge,[4] etc.[5] That is, in each of those models, the algebra of field operators is completely found in closed form. The next problem is to represent this algebra in terms of state vectors so as to be consistent with certain physically natural requirements. The representation can be explicitly constructed by giving the set of all Wightman functions, i.e., vacuum expectation values of field-operator products. This representation is, of course, consistent with the full-dimensional (anti)commutation relations, but not always consistent with field equations owing to the presence of singular products in them. That is, we encounter a kind of anomaly, which we call "field-equation anomaly". The existence of the field-equation anomaly has been found in each model stated above. One should note that the violation of a field equation is very slight in the sense that by differentiating it once or twice we can find an anomaly-free field equation having the same degrees of freedom as that of the original equation.

Now, one can understand the essence of the anomaly problem of the BRS charge by the following remarks.

1. The expression for the square of the BRS charge given by Kato and Ogawa[1] *cannot be obtained without using the Fock representation*, though they wrote as if it had been obtained by straightforward operator calculation.

2. Kato and Ogawa[1] eliminated the B field (after its field redefinition) by regarding its field equation as an identity, that is, in their theory the B field equation is an equality which holds even at the representation level.

3. In our previous paper,[4] we have shown in the BRS formalism of the conformal-gauge two-dimensional quantum gravity that the B field equation suffers from the field-equation anomaly, which disappears if and only if $D = 26$. By using the B field equation at the operator level, therefore, it is possible to rewrite the BRS charge into the one which is anomaly-free, and hence identically nilpotent, at the representation level.

4. As is shown in the present paper, the essential results obtained in our previous paper can be transcribed into the case of finite string with some minor modifications.

In the framework of the Kato-Ogawa theory, we rewrite their BRS charge by using the original form of the B field equation just as has been done in our previous paper. We then obtain a BRS charge which is completely nilpotent even at the representation level. Thus one can no longer claim that the critical dimension $D = 26$ is a consequence of the requirement of the BRS invariance in the Kato-Ogawa theory. Rather, we should say the $D = 26$ is the condition for the absence of the field-equation anomaly in conformal gauge (but it is not so in de Donder gauge[2]).

As a side remark, we discuss the FP-ghost vacuum structure. Although Kato and Ogawa[1] artificially introduced a rather complicated FP-ghost vacuum structure, we show that it can be reformulated into a more natural one. As a consequence, we can avoid the appearance of half-integral ghost numbers and also the introduction of the indefinite metric which is inconsistent with the hermiticity of the original action.

The present paper is organized as follows. In Sec.2, we review the main results of the conformal-gauge two-dimensional quantum gravity obtained in our previous paper. In Sec.3, we extend them into the case of the closed string of finite length. In Sec.4, we reformulate the Kato-Ogawa open-string theory into the formalism similar to ours. In Sec.5, the FP-ghost vacuum structure of the Kato-Ogawa theory is shown to be made more transparent. In Sec.6, we explicitly construct an identically nilpotent BRS charge in the framework of the Kato-Ogawa theory. In Sec.7, we make a unified treatment of the BRS charges for both infinite and finite strings. The final section is devoted to discussion.

2. Review of our previous paper

We briefly review our previous work[4] on the conformal-gauge two-dimensional quantum gravity coupled with D scalar fields, which represent the coordinates of an infinite string.

In the conformal gauge, the gravitational field $g^{\mu\nu}$ is parametrized as

$$g^{\mu\nu} = e^{-\theta}(\eta^{\mu\nu} + h^{\mu\nu}) \tag{2.1}$$

with $\eta_{\mu\nu}h^{\mu\nu} = 0$ ($\eta_{00} = -\eta_{11} = 1,\ \eta_{01} = 0$). Then the conformal degree of freedom, θ, disappears from the action. Corresponding to the fact that $h^{\mu\nu}$ is a traceless symmetric tensor, the B field $\tilde{b}_{\mu\nu}$ and the FP antighost $\bar{c}_{\mu\nu}$ are also traceless symmetric tensors, while the FP ghost c^μ is a vector. It is, therefore, convenient to rewrite a traceless symmetric tensor, which is generically denoted by $X_{\mu\nu}$, into a vectorlike quantity

$$X^\lambda \equiv \frac{1}{\sqrt{2}}\xi^{\lambda\mu\nu}X_{\mu\nu}, \tag{2.2}$$

where $\xi^{\mu\nu\lambda}$ is a constant, totally symmetric rank-3 tensorlike quantity, defined by $\xi^{\mu\nu\lambda} = 1$ for $\mu+\nu+\lambda =$even, $\xi^{\mu\nu\lambda} = 0$ otherwise. According to (2.2), we introduce $h_\lambda,\ \tilde{b}^\lambda,\ \bar{c}^\lambda$.

Let ϕ_M ($M = 0, 1, \dots, D-1$; $\eta^{MN} = (-1)^{\delta_{M0}}\delta_{MN}$) be scalar fields, which represent the coordinates of a string. The BRS transforms of the field operators are as follows:

$$\boldsymbol{\delta}_* h_\lambda = \sqrt{2}\xi_{\lambda\mu\nu}\partial^\mu c^\nu + \xi_{\lambda\mu\nu}\xi^{\mu\sigma\tau}h_\sigma\partial_\tau c^\nu - \partial_\nu(h_\lambda c^\nu) - \frac{1}{\sqrt{2}}h_\lambda\xi^{\nu\sigma\tau}h_\nu\partial_\sigma c_\tau, \tag{2.3}$$

$$\boldsymbol{\delta}_* c^\lambda = -c^\sigma\partial_\sigma c^\lambda, \tag{2.4}$$

$$\boldsymbol{\delta}_* \bar{c}^\lambda = i\tilde{b}^\lambda, \tag{2.5}$$

$$\boldsymbol{\delta}_* \tilde{b}^\lambda = 0, \tag{2.6}$$

$$\boldsymbol{\delta}_* \phi_M = -c^\sigma\partial_\sigma\phi_M. \tag{2.7}$$

The BRS-invariant Lagrangian density is given by

$$\mathcal{L} = -\frac{1}{2}\tilde{b}^\lambda h_\lambda - \frac{i}{2}\bar{c}^\lambda \boldsymbol{\delta}_* h_\lambda + \frac{1}{2}(1 - \det h^{\sigma\tau})^{-1/2}(\eta^{\mu\nu} + h^{\mu\nu})\partial_\mu \phi_M \cdot \partial_\nu \phi^M. \quad (2.8)$$

The field equations are as follows:

$$h_\mu = 0, \quad (2.9)$$

$$\tilde{b}^\mu = -i[\xi_{\sigma\tau\rho}\xi^{\rho\mu\lambda}\bar{c}^\sigma \partial_\lambda c^\tau + \partial_\sigma \bar{c}^\mu \cdot c^\sigma] + \frac{1}{\sqrt{2}}\xi^{\mu\sigma\tau}\partial_\sigma \phi_M \cdot \partial_\tau \phi^M, \quad (2.10)$$

$$\xi_{\lambda\mu\nu}\partial^\mu X^\nu = 0 \quad \text{for } X^\nu = c^\nu,\ \bar{c}^\nu,\ \tilde{b}^\nu, \quad (2.11)$$

$$\Box \phi_M = 0, \quad (2.12)$$

where (2.11) for $X^\nu = \tilde{b}^\nu$ follows from (2.10).

From the canonical (anti)commutation relations and the field equations given above, we can explicitly calculate the two-dimensional (anti)commutation relations. We find

$$\{c^\rho(x),\ \bar{c}^\lambda(y)\} = \sqrt{2}\xi^{\rho\lambda\nu}\partial_\nu D(x-y), \quad (2.13)$$

$$[\phi_M(x),\ \phi^N(y)] = i\delta_M{}^N D(x-y), \quad (2.14)$$

where $D(x) \equiv -\frac{1}{2}\epsilon(x^0)\theta(x^2)$. Hence if (2.10) were discarded by regarding it merely as the definition of $\tilde{b}^\mu$, then the model considered would be a free field theory. This is not the right way, however, because the nonlinearity of (2.10) is the origin of anomaly.

The two-dimensional commutation relations involving the B field can be calculated by using (2.10) together with (2.13) and (2.14). The results are much simplified if we employ light-cone coordinates $x^\pm \equiv (x^0 \pm x^1)/\sqrt{2}$, with which $\xi_{\mu\nu\lambda} = 0$ except $\xi_{+++} = \xi_{---} = \sqrt{2}$. Then (2.10) and (2.11) reduce to

$$\tilde{b}^\pm = -i(2\bar{c}^\pm \partial_\pm c^\pm + \partial_\pm \bar{c}^\pm \cdot c^\pm) + \partial_\pm \phi_M \cdot \partial_\pm \phi^M \equiv \tilde{\mathcal{T}}^\pm, \quad (2.15)$$

$$\partial_\mp X^\pm = 0 \quad \text{for} \quad X^\pm = c^\pm,\ \bar{c}^\pm,\ \tilde{b}^\pm, \quad (2.16)$$

respectively. Furthermore, since

$$\partial_\pm D(x) = -\frac{1}{2}\delta(x^\pm), \quad (2.17)$$

(2.13) and (2.14) reduce to

$$\{c^\pm(x),\ \bar{c}^\pm(y)\} = -\delta(x^\pm - y^\pm), \quad (2.18)$$

$$[\partial_\pm \phi_M(x),\ \phi^N(y)] = -\frac{i}{2}\delta_M{}^N \delta(x^\pm - y^\pm), \quad (2.19)$$

respectively. Except for $[\phi_M(x),\ \phi^N(y)]$, the $+$ coordinate and the $-$ one never coexist in the right-hand side. From (2.15) together with (2.18) and (2.19), we obtain

$$
\begin{aligned}
[\tilde{b}^{\pm}(x),\ c^{\pm}(y)] &= -i[c^{\pm}(x)\delta'(x^{\pm}-y^{\pm}) + 2\partial_{\pm}c^{\pm}(x)\cdot\delta(x^{\pm}-y^{\pm})], && (2.20)\\
[\tilde{b}^{\pm}(x),\ \bar{c}^{\pm}(y)] &= i[\bar{c}^{\pm}(x)+\bar{c}^{\pm}(y)]\delta'(x^{\pm}-y^{\pm})\\
&= i[2\bar{c}^{\pm}(x)\delta'(x^{\pm}-y^{\pm}) + \partial_{\pm}\bar{c}^{\pm}(x)\cdot\delta(x^{\pm}-y^{\pm})], && (2.21)\\
[\tilde{b}^{\pm}(x),\ \phi_M(y)] &= -i\partial_{\pm}\phi_M(x)\cdot\delta(x^{\pm}-y^{\pm}), && (2.22)\\
[\tilde{b}^{\pm}(x),\ \tilde{b}^{\pm}(y)] &= i[\tilde{b}^{\pm}(x)+\tilde{b}^{\pm}(y)]\delta'(x^{\pm}-y^{\pm}). && (2.23)
\end{aligned}
$$

The totality of (2.18)$\sim$(2.23) constitutes the field algebra of the conformal-gauge two-dimensional quantum gravity.

The representation of this algebra in terms of state vectors is given by constructing all (truncated[a]) n-point Wightman functions explicitly. All 1-point Wightman functions vanish. Nonvanishing 2-point Wightman functions are[b]

$$
\langle\, 0 \,|\, \bar{c}^{\pm}(x_1)c^{\pm}(x_2)\,|\, 0\,\rangle = \langle\, 0 \,|\, c^{\pm}(x_1)\bar{c}^{\pm}(x_2)\,|\, 0\,\rangle = \frac{i}{2\pi}\frac{1}{x_1{}^{\pm}-x_2{}^{\pm}-i0}, \qquad (2.24)
$$

$$
\partial_{\pm}{}^{x_1}\langle\, 0 \,|\, \phi_M(x_1)\phi^N(x_2)\,|\, 0\,\rangle = -\frac{1}{4\pi}\delta_M{}^N\frac{1}{x_1{}^{\pm}-x_2{}^{\pm}-i0}. \qquad (2.25)
$$

Nonvanishing truncated n-point Wightman functions are those which consist of $(n-2)$ $\tilde{b}^{\pm}$'s and of either $c^{\pm}$ and $\bar{c}^{\pm}$ or two ϕ_M's. For simplicity, we present the expressions for those of particular orderings:[c]

$$
\begin{aligned}
&\langle\, 0 \,|\, c^{\pm}(x_1)\tilde{b}^{\pm}(x_2)\cdots\tilde{b}^{\pm}(x_{n-1})\bar{c}^{\pm}(x_n)\,|\, 0\,\rangle\\
&\quad = -i^{-n}\sum_{P(j_2,\cdots,j_{n-1})}^{(n-2)!}\left[\prod_{s=2}^{n-1}(\partial_{j_s}{}^{\mathrm{L}}+2\partial_{j_s}{}^{\mathrm{R}})\right]\langle\, 1,\, j_2\,\rangle^{\pm}\langle\, j_2,\, j_3\,\rangle^{\pm}\\
&\qquad\qquad \cdots\langle\, j_{n-2},\, j_{n-1}\,\rangle^{\pm}\langle\, j_{n-1},\, n\,\rangle^{\pm}, \qquad (2.26)\\
&\langle\, 0 \,|\, \phi_M(x_1)\tilde{b}^{\pm}(x_2)\cdots\tilde{b}^{\pm}(x_{n-1})\phi^N(x_n)\,|\, 0\,\rangle\\
&\quad = -\frac{i^{-n+1}}{2}\delta_M{}^N\sum_{P(j_2,\cdots,j_{n-1})}^{(n-2)!}\left[\prod_{s=2}^{n-2}\partial_{j_s}{}^{\mathrm{R}}\right]\langle\, 1,\, j_2\,\rangle^{\pm}\langle\, j_2,\, j_3\,\rangle^{\pm}\\
&\qquad\qquad \cdots\langle\, j_{n-2},\, j_{n-1}\,\rangle^{\pm}\langle\, j_{n-1},\, n\,\rangle^{\pm} \quad (n\geqq 3), \qquad (2.27)
\end{aligned}
$$

---

[a] Truncation means to drop the contributions from vacuum intermediate states.

[b] Without differentiation in (2.25), we must introduce an infrared cutoff.

[c] Those of the other orderings are obtained by changing $-i0$ into $+i0$ appropriately (and the overall sign is changed if c and $\bar{c}$ are exchanged).

where $P(j_2, \cdots, j_{n-1})$ is a permutation of $(2, 3, \cdots, n-1)$, $\partial_j{}^{\mathrm{L}}$ and $\partial_j{}^{\mathrm{R}}$ denote differentiations with respect to $x_j{}^{\pm}$ acting only on the left factor involving $x_j{}^{\pm}$ and only on the right one, respectively, and

$$\langle\, j,\, k \,\rangle^{\pm} \equiv \frac{i}{2\pi} \frac{1}{x_j{}^{\pm} - x_k{}^{\pm} - (k-j)i0}. \tag{2.28}$$

A composite-field operator is a product of field operators of the same spacetime point. The Wightman function involving a composite field is obtained from the (non-truncated) Wightman function by setting the spacetime coordinates of consecutive field operators coincident and by discarding the infinities which appear as a consequence in such a way that the result be independent of the ordering of the constituent fields of the composite field. The latter procedure is called "generalized normal product" because it reduces to Wick's normal product in the free-field case.

The representation of the field algebra in terms of Wightman functions is, of course, consistent with all two-dimensional (anti)commutation relations and also with all linear field equations (including (2.16) for $X^{\pm} = \tilde{b}^{\pm}$). However, it is *not* consistent with the B-field equation (2.15). Indeed, we have

$$\langle\, 0 \,|\, \tilde{b}^{\pm}(x_1)\tilde{b}^{\pm}(x_2) \,|\, 0 \,\rangle = 0, \tag{2.29}$$

$$\langle\, 0 \,|\, \tilde{b}^{\pm}(x_1)\tilde{\mathcal{T}}^{\pm}(x_2) \,|\, 0 \,\rangle = \langle\, 0 \,|\, \tilde{\mathcal{T}}^{\pm}(x_1)\tilde{\mathcal{T}}^{\pm}(x_2) \,|\, 0 \,\rangle = -\frac{1}{2}(D-26)[\partial_{\pm}{}^{x_1}\langle\, 1, 2 \,\rangle^{\pm}]^2, \tag{2.30}$$

where $\tilde{\mathcal{T}}^{\pm}$ denotes the right-hand side of (2.15). Thus the B-field equation (2.15), but not (2.16) for $X^{\pm} = \tilde{b}^{\pm}$, is *violated at the representation level.* We call this situation "field-equation anomaly". The field-equation anomaly disappears for $D = 26$ in the conformal gauge, but this property does not remain valid in the de Donder gauge.[2]

The BRS Noether current is given by

$$j_b{}^{\mp} = -i\bar{c}^{\pm}c^{\pm}\partial_{\pm}c^{\pm} - c^{\pm}\partial_{\pm}\phi_M \cdot \partial_{\pm}\phi^M, \tag{2.31}$$

and the corresponding BRS charge is defined by

$$Q_b = \frac{1}{\sqrt{2}} \int dx^1 [j_b{}^{-}(x^{+}) + j_b{}^{+}(x^{-})]. \tag{2.32}$$

This BRS charge is, however, anomalous. We rewrite (2.31) as

$$j_b^{\mp} = \hat{j}_b{}^{\mp} + (\tilde{b}^{\pm} - \tilde{\mathcal{T}}^{\pm})c^{\pm} \tag{2.33}$$

with

$$\hat{j}_b{}^{\mp} \equiv -\tilde{b}^{\pm} c^{\pm} + i\bar{c}^{\pm} c^{\pm} \partial_{\pm} c^{\pm}. \tag{2.34}$$

Because of (2.15), $\hat{j}_b{}^{\mp}$ is the same operator as $j_b{}^{\mp}$ at the operator level. They no longer coincide, however, at the representation level because of field-equation anomaly. The BRS charge defined by $\hat{j}_b{}^{\mp}$, i.e.,

$$\hat{Q}_b = \frac{1}{\sqrt{2}} \int dx^1 [\hat{j}_b{}^{-}(x^+) + \hat{j}_b{}^{+}(x^-)], \tag{2.35}$$

is *free of anomaly for any value of D*. Of course, any vacuum expectation value involving $\hat{Q}_b{}^2$ is zero independently of D (see Sec.7).

3. Closed String

In this section, we consider how the formulae presented in Sec.2 are modified if the string is not of infinite length but a finite ring.

Let the length of the string be 2π. That is, every function of x^μ must be periodic in x^1 with a period 2π. It is well known in the curved-spacetime quantum field theory[6] how to treat field operators and Green's functions in such a situation.

The two-dimensional commutator D-function is modified into

$$D_f(x) \equiv -\frac{1}{2}\epsilon(x^0) \sum_{m=-\infty}^{\infty} \theta((x^0)^2 - (x^1 - 2\pi m)^2), \tag{3.1}$$

whence

$$\partial_{\pm} D_f(x) = -\frac{1}{2} \sum_{m=-\infty}^{\infty} \delta(x^{\pm} - \sqrt{2}\pi m). \tag{3.2}$$

Accordingly, (2.18)~(2.23) are replaced by

$$\{c^{\pm}(x),\ \bar{c}^{\pm}(y)\} = 2\partial_{\pm} D_f(x-y), \tag{3.3}$$

$$[\partial_{\pm}\phi_M(x),\ \phi^N(y)] = i\delta_M{}^N \partial_{\pm} D_f(x-y), \tag{3.4}$$

$$[\tilde{b}^{\pm}(x),\ c^{\pm}(y)] = 2i[c^{\pm}(x)\partial_{\pm} + 2\partial_{\pm} c^{\pm}(x)]\partial_{\pm} D_f(x-y), \tag{3.5}$$

$$[\tilde{b}^{\pm}(x),\ \bar{c}^{\pm}(y)] = -2i[2\bar{c}^{\pm}(x)\partial_{\pm} + \partial_{\pm}\bar{c}^{\pm}(x)]\partial_{\pm} D_f(x-y), \tag{3.6}$$

$$[\tilde{b}^{\pm}(x),\ \phi_M(y)] = 2i\partial_{\pm}\phi_M(x) \cdot \partial_{\pm} D_f(x-y), \tag{3.7}$$

$$[\tilde{b}^{\pm}(x),\ \tilde{b}^{\pm}(y)] = -2i[\tilde{b}^{\pm}(x) + \tilde{b}^{\pm}(y)](\partial_{\pm})^2 D_f(x-y). \tag{3.8}$$

The Wightman functions are, therefore, obtained from (2.24)~(2.27) by the replacement

$$\partial_\pm D^{(+)}(x) = -\frac{1}{4\pi}\frac{1}{x^\pm - i0} \quad\Longrightarrow\quad \partial_\pm D_f^{(+)}(x) \equiv -\frac{1}{4\pi}\sum_{m=-\infty}^{\infty}\frac{1}{x^\pm - \sqrt{2}\pi m - i0}. \tag{3.9}$$

The proof of the BRS invariance for Wightman functions, done in our previous paper[4], can be straightforwardly extended to the present case.

The summation in (3.9) can be explicitly carried out to obtain

$$\begin{aligned}\partial_\pm D_f^{(+)}(x) &= -\frac{1}{4\pi}\frac{1}{\sqrt{2}}\cot\left(\frac{x^\pm}{\sqrt{2}} - i0\right)\\ &= -\frac{i}{2\sqrt{2}\pi}\left(\frac{1}{2} + \sum_{n=1}^{\infty} e^{-in\sqrt{2}x^\pm}\right).\end{aligned} \tag{3.10}$$

Hence, we have

$$\partial_\pm D_f(x) = i^{-1}[\partial_\pm D_f^{(+)}(x) + \partial_\pm D_f^{(+)}(-x)] = -\frac{1}{2\sqrt{2}\pi}\sum_{n=-\infty}^{\infty} e^{-in\sqrt{2}x^\pm}. \tag{3.11}$$

It is interesting to introduce mode expansions. For $X^\lambda = c^\lambda,\ \bar{c}^\lambda,\ \tilde{b}^\lambda$, we write

$$X^\pm(x) = \frac{1}{\sqrt{2\pi}}\sum_{n=-\infty}^{\infty} \boldsymbol{X}_n^\pm e^{-in\sqrt{2}x^\pm} \tag{3.12}$$

with $\boldsymbol{X}_n^\pm = \boldsymbol{c}_n^\pm,\ \bar{\boldsymbol{c}}_n^\pm,\ \tilde{\boldsymbol{b}}_n^\pm$. Then (3.3), (3.5), (3.6) and (3.8) are rewritten in terms of mode operators:

$$\{\boldsymbol{c}_n^\pm,\ \bar{\boldsymbol{c}}_m^\pm\} = -\sqrt{2}\,\delta_{n,\,-m}, \tag{3.13}$$

$$[\tilde{\boldsymbol{b}}_n^\pm,\ \boldsymbol{c}_m^\pm] = -\sqrt{\frac{2}{\pi}}(2n+m)\,\boldsymbol{c}_{n+m}^\pm, \tag{3.14}$$

$$[\tilde{\boldsymbol{b}}_n^\pm,\ \bar{\boldsymbol{c}}_m^\pm] = \sqrt{\frac{2}{\pi}}(n-m)\,\bar{\boldsymbol{c}}_{n+m}^\pm, \tag{3.15}$$

$$[\tilde{\boldsymbol{b}}_n^\pm,\ \tilde{\boldsymbol{b}}_m^\pm] = \sqrt{\frac{2}{\pi}}(n-m)\,\tilde{\boldsymbol{b}}_{n+m}^\pm, \tag{3.16}$$

respectively. Of course, $\boldsymbol{X}_n^\pm$ and $\boldsymbol{Y}_m^\mp$ (anti)commute.

4. Kato-Ogawa string theory

Kato and Ogawa[1] presented the BRS formalism of the open string of length π. Since their starting Lagrangian density is not the same as ours, we here start with their formulae of the mode expansions of field operators. To make the comparison easier, we translate their notation into ours in the following way.

2-dimensional coordinates:	$\sigma \Rightarrow x^1, \quad \tau \Rightarrow x^0$;
2-dimensional indices:	$a,\ b,\ \cdots \Rightarrow \mu,\ \nu,\ \cdots$;
string component indices:	$\mu,\ \nu \Rightarrow M,\ N$;
string-space metric:	$g^{\mu\nu} \Rightarrow -\eta^{MN}$;
field operators:	$\frac{1}{\sqrt{\kappa}}X_\mu \Rightarrow \phi_M, \quad c^a \Rightarrow c^\mu,$
	$\bar{c}_0 \Rightarrow -\frac{1}{\sqrt{2}}\bar{c}^1, \quad \bar{c}_1 \Rightarrow -\frac{1}{\sqrt{2}}\bar{c}^0,$
	$B_0 \Rightarrow -\frac{1}{\sqrt{2}}\tilde{b}^1, \quad B_1 \Rightarrow -\frac{1}{\sqrt{2}}\tilde{b}^0$, where B_a is the B_a before field redefinition[d] is made;
zero-mode operators:	$\sqrt{\frac{\pi}{\kappa}}q_0^\mu \Rightarrow \boldsymbol{q}_0^M, \quad \sqrt{\frac{\kappa}{\pi}}p_0^\mu \Rightarrow \boldsymbol{p}_0^M,$
	$c_0 \Rightarrow \boldsymbol{c}_0, \quad \bar{c}_0 \Rightarrow -\frac{1}{\sqrt{2}}\bar{\boldsymbol{c}}_0;$
nonzero-mode operators:	${a_n}^\mu \Rightarrow {\boldsymbol{a}_n}^M,$
	$c_n \Rightarrow \boldsymbol{c}_n, \quad \bar{c}_n \Rightarrow -\frac{1}{\sqrt{2}}\bar{\boldsymbol{c}}_n.$

In our notation, their mode expansion formulae [Kato-Ogawa's (2.18)] are translated into

$$\phi^M(x) = \frac{1}{\sqrt{\pi}}{\boldsymbol{q}_0}^M + \frac{1}{\sqrt{\pi}}{\boldsymbol{p}_0}^M x^0 + \frac{1}{\sqrt{\pi}}\sum_{n=1}^\infty \frac{1}{\sqrt{n}}({\boldsymbol{a}_n}^M e^{-inx^0} + {\boldsymbol{a}_n}^{M\dagger} e^{inx^0})\cos nx^1, \tag{4.1}$$

$$c^0(x) = \frac{1}{\sqrt{\pi}}\boldsymbol{c}_0 + \frac{1}{\sqrt{\pi}}\sum_{n=1}^\infty(\boldsymbol{c}_n e^{-inx^0} + {\boldsymbol{c}_n}^\dagger e^{inx^0})\cos nx^1, \tag{4.2}$$

$$c^1(x) = -\frac{i}{\sqrt{\pi}}\sum_{n=1}^\infty(\boldsymbol{c}_n e^{-inx^0} - {\boldsymbol{c}_n}^\dagger e^{inx^0})\sin nx^1, \tag{4.3}$$

$$\bar{c}^1(x) = -\frac{i}{\sqrt{\pi}}\sum_{n=1}^\infty(\bar{\boldsymbol{c}}_n e^{-inx^0} - {\bar{\boldsymbol{c}}_n}^\dagger e^{inx^0})\sin nx^1, \tag{4.4}$$

$$\bar{c}^0(x) = \frac{1}{\sqrt{\pi}}\bar{\boldsymbol{c}}_0 + \frac{1}{\sqrt{\pi}}\sum_{n=1}^\infty(\bar{\boldsymbol{c}}_n e^{-inx^0} + {\bar{\boldsymbol{c}}_n}^\dagger e^{inx^0})\cos nx^1 \tag{4.5}$$

[d] See first two formulae of Kato-Ogawa's (2.13).

with

$$[\boldsymbol{p}_0{}^M,\ \boldsymbol{q}_0{}^N] = -i\eta^{MN}, \qquad [\boldsymbol{a}_n{}^M,\ \boldsymbol{a}_m{}^{N\dagger}] = \eta^{MN}\delta_{nm}, \tag{4.6}$$

$$\{\boldsymbol{c}_0,\ \bar{\boldsymbol{c}}_0\} = -\sqrt{2}, \qquad \{\boldsymbol{c}_n,\ \bar{\boldsymbol{c}}_m{}^\dagger\} = \{\boldsymbol{c}_n{}^\dagger,\ \bar{\boldsymbol{c}}_m\} = -\sqrt{2}\delta_{nm}, \tag{4.7}$$

others being zero. We rewrite (4.2)∼(4.5) as

$$\begin{aligned} c^\pm(x) &= \frac{1}{\sqrt{2\pi}}\left[\boldsymbol{c}_0 + \sum_{n=1}^\infty (\boldsymbol{c}_n e^{-in\sqrt{2}x^\pm} + \boldsymbol{c}_n{}^\dagger e^{in\sqrt{2}x^\pm})\right] \\ &= \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^\infty \boldsymbol{c}_n e^{-in\sqrt{2}x^\pm}, \end{aligned} \tag{4.8}$$

$$\begin{aligned} \bar{c}^\pm(x) &= \frac{1}{\sqrt{2\pi}}\left[\bar{\boldsymbol{c}}_0 + \sum_{n=1}^\infty (\bar{\boldsymbol{c}}_n e^{-in\sqrt{2}x^\pm} + \bar{\boldsymbol{c}}_n{}^\dagger e^{in\sqrt{2}x^\pm})\right] \\ &= \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^\infty \bar{\boldsymbol{c}}_n e^{-in\sqrt{2}x^\pm}, \end{aligned} \tag{4.9}$$

where $\boldsymbol{c}_{-n} \equiv \boldsymbol{c}_n{}^\dagger$, $\bar{\boldsymbol{c}}_{-n} \equiv \bar{\boldsymbol{c}}_n{}^\dagger$, and $\{\boldsymbol{c}_n,\ \bar{\boldsymbol{c}}_m\} = -\sqrt{2}\delta_{n,\,-m}$. Compared with (3.12), we note that the mode operators for the open string has no $\pm$ index. From (4.8) and (4.9), we obtain

$$\{c^\pm(x),\ \bar{c}^\pm(y)\} = -\frac{1}{\sqrt{2}\pi}\left[1 + 2\sum_{n=1}^\infty \cos n\sqrt{2}(x^\pm - y^\pm)\right], \tag{4.10}$$

$$\{c^\pm(x),\ \bar{c}^\mp(y)\} = -\frac{1}{\sqrt{2}\pi}\left[1 + 2\sum_{n=1}^\infty \cos n\sqrt{2}(x^\pm - y^\mp)\right]. \tag{4.11}$$

The nonvanishing of (4.11) is due to translational noninvariance.

In the Fourier expansion formula

$$\frac{1-\lambda^2}{1-2\lambda\cos\alpha+\lambda^2} = 1 + 2\sum_{n=1}^\infty \lambda^n \cos n\alpha \qquad (|\lambda| < 1), \tag{4.12}$$

we take the limit $\lambda \to 1$; we then see that the left-hand side of (4.12) tends to

$$2\pi \sum_{m=-\infty}^\infty \delta(\alpha - 2\pi m). \tag{4.13}$$

Hence (4.10) and (4.11) are rewritten as

$$\{c^\pm(x),\ \bar{c}^\pm(y)\} = -\sum_{m=-\infty}^\infty \delta(x^\pm - y^\pm - \sqrt{2}\pi m), \tag{4.14}$$

$$\{c^\pm(x),\ \bar{c}^\mp(y)\} = -\sum_{m=-\infty}^\infty \delta(x^\pm - y^\mp - \sqrt{2}\pi m), \tag{4.15}$$

respectively.

Next, from (4.1) with (4.6), we obtain

$$[\phi_M(x),\ \phi^N(y)] = -\frac{i}{\pi}\delta_M{}^N\Big\{x^0 - y^0 + \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n}\Big(\sin n\sqrt{2}(x^+ - y^+)$$

$$+\sin n\sqrt{2}(x^- - y^-) + \sin n\sqrt{2}(x^+ - y^-) + \sin n\sqrt{2}(x^- - y^+)\Big)\Big\}. \quad (4.16)$$

Using the formula

$$\sum_{n=1}^{\infty}\frac{\sin n\alpha}{n} = \frac{1}{2}(\pi - \alpha) + \pi\Big[\frac{\alpha}{2\pi}\Big], \quad (4.17)$$

where $[\,r\,]$ denotes the largest integer not greater than r, we obtain

$$[\phi_M(x),\ \phi^N(y)] = -\frac{1}{2}i\delta_M{}^N\Big\{2 + \Big[\frac{x^+ - y^+}{\sqrt{2}\pi}\Big]$$

$$+\Big[\frac{x^- - y^-}{\sqrt{2}\pi}\Big] + \Big[\frac{x^+ - y^-}{\sqrt{2}\pi}\Big] + \Big[\frac{x^- - y^+}{\sqrt{2}\pi}\Big]\Big\}, \quad (4.18)$$

and, therefore,

$$[\partial_\pm\phi_M(x),\ \phi^N(y)] = -\frac{1}{2}i\delta_M{}^N\sum_{m=-\infty}^{\infty}[\delta(x^\pm - y^\pm - \sqrt{2}\pi m)$$

$$+\ \delta(x^\pm - y^\mp - \sqrt{2}\pi m)]. \quad (4.19)$$

Now, we consider the B field. Kato and Ogawa first made field redefinition [Kato-Ogawa's (2.13)] and then wrote down the field equations in terms of the redefined fields [Kato-Ogawa's (2.16)]. We should, therefore, restore the field equation for the original B field from those formulae. We then find

$$\tilde{b}^\pm = \partial_\pm\phi_M\cdot\partial_\pm\phi^M - i(2\bar{c}^\pm\partial_\pm c^\pm + \partial_\pm\bar{c}^\pm\cdot c^\pm) \quad (4.20)$$

in our notation. As expected, (4.20) is identical with (2.15).

We calculate the commutators involving the B field by using (4.10), (4.11), (4.14), (4.15) and (4.19). We find

$$[\tilde{b}^\pm(x),\ c^\pm(y)] = -i[c^\pm(x)\partial_\pm + 2\partial_\pm c^\pm(x)]\sum_{m=-\infty}^{\infty}\delta(x^\pm - y^\pm - \sqrt{2}\pi m), \quad (4.21)$$

$$[\tilde{b}^\pm(x),\ c^\mp(y)] = -i[c^\pm(x)\partial_\pm + 2\partial_\pm c^\pm(x)]\sum_{m=-\infty}^{\infty}\delta(x^\pm - y^\mp - \sqrt{2}\pi m); \quad (4.22)$$

$$[\tilde{b}^\pm(x),\ \bar{c}^\pm(y)] = i[2\bar{c}^\pm(x)\partial_\pm + \partial_\pm\bar{c}^\pm(x)]\sum_{m=-\infty}^{\infty}\delta(x^\pm - y^\pm - \sqrt{2}\pi m), \quad (4.23)$$

$$[\tilde{b}^{\pm}(x),\ \bar{c}^{\mp}(y)] = i[2\bar{c}^{\pm}(x)\partial_{\pm} + \partial_{\pm}\bar{c}^{\pm}(x)] \sum_{m=-\infty}^{\infty} \delta(x^{\pm} - y^{\mp} - \sqrt{2}\pi m); \tag{4.24}$$

$$[\tilde{b}^{\pm}(x),\ \phi_M(y)] = -i\partial_{\pm}\phi_M(x) \sum_{m=-\infty}^{\infty} [\delta(x^{\pm} - y^{\pm} - \sqrt{2}\pi m) + \delta(x^{\pm} - y^{\mp} - \sqrt{2}\pi m)]; \tag{4.25}$$

$$[\tilde{b}^{\pm}(x),\ \tilde{b}^{\pm}(y)] = i(\tilde{b}^{\pm}(x) + \tilde{b}^{\pm}(y)) \sum_{m=-\infty}^{\infty} \delta'(x^{\pm} - y^{\pm} - \sqrt{2}\pi m), \tag{4.26}$$

$$[\tilde{b}^{\pm}(x),\ \tilde{b}^{\mp}(y)] = i(\tilde{b}^{\pm}(x) + \tilde{b}^{\mp}(y)) \sum_{m=-\infty}^{\infty} \delta'(x^{\pm} - y^{\mp} - \sqrt{2}\pi m). \tag{4.27}$$

In terms of mode operators, we have

$$[\tilde{\boldsymbol{b}}_n,\ \boldsymbol{c}_m] = -\sqrt{\frac{2}{\pi}}\,(2n+m)\boldsymbol{c}_{n+m}, \tag{4.28}$$

$$[\tilde{\boldsymbol{b}}_n,\ \bar{\boldsymbol{c}}_m] = \sqrt{\frac{2}{\pi}}\,(n-m)\bar{\boldsymbol{c}}_{n+m}, \tag{4.29}$$

$$[\tilde{\boldsymbol{b}}_n,\ \tilde{\boldsymbol{b}}_m] = \sqrt{\frac{2}{\pi}}\,(n-m)\tilde{\boldsymbol{b}}_{n+m}, \tag{4.30}$$

where $\tilde{\boldsymbol{b}}_n$ is the mode operator of the B field defined through

$$\tilde{b}^{\pm}(x) = \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} \tilde{\boldsymbol{b}}_n e^{-in\sqrt{2}x^{\pm}}. \tag{4.31}$$

One should note the parallelism between the above formulae and those for the closed string.

5. FP-ghost vacuum structure of the Kato-Ogawa theory

Kato and Ogawa[1] introduced the vacuum structure in a rather artificial way, especially for the FP-ghost. They introduced two FP-ghost vacua $|+\rangle$ and $|-\rangle$ such that $c_0|+\rangle = 0$, $\bar{c}_0|+\rangle = |-\rangle$, $\bar{c}_0|-\rangle = 0$, $c_0|-\rangle = |+\rangle$, where $\{c_0,\ \bar{c}_0\} = 1$. Since ${c_0}^{\dagger} = c_0$ and ${\bar{c}_0}^{\dagger} = \bar{c}_0$, both vacua are of zero norm. To overcome this trouble, they assumed $\langle +|-\rangle = \langle -|+\rangle = 1$ and introduced an indefinite metric $\eta = c_0 + \bar{c}_0$ by hand so that $\eta|+\rangle = |-\rangle$ and $\eta|-\rangle = |+\rangle$. This procedure is *not admissible*, however, because the introduction of η violates the operator hermitian conjugation at the representation level. Indeed, in the presence of η, the original action is no longer hermitian in the sense of Kato-Ogawa's inner product.

More naturally, we should start with the unique vacuum $|\,0\,\rangle$ with positive norm,

$$\langle\,0\,|\,0\,\rangle = 1. \tag{5.1}$$

Then the trouble encountered is how to calculate $\langle\,0\,|\,\boldsymbol{c}_0\,\bar{\boldsymbol{c}}_0\,|\,0\,\rangle$. This problem can be resolved in the following way.

From the consideration made in the present paper, it is now straightforward to calculate all Wightman functions in the Kato-Ogawa theory. For example, we have

$$\langle\,0\,|\,c^{\pm}(x_1)\bar{c}^{\pm}(x_2)\,|\,0\,\rangle = \frac{i}{2\pi}\sum_{m=-\infty}^{\infty}\frac{1}{{x_1}^{\pm} - {x_2}^{\pm} - \sqrt{2}\pi m - i0}, \tag{5.2}$$

$$\langle\,0\,|\,c^{\pm}(x_1)\bar{c}^{\mp}(x_2)\,|\,0\,\rangle = \frac{i}{2\pi}\sum_{m=-\infty}^{\infty}\frac{1}{{x_1}^{\pm} - {x_2}^{\mp} - \sqrt{2}\pi m - i0}, \tag{5.3}$$

We note from (4.8) and (4.9) that

$$\int_0^{\pi} dx^1\,(c^{+}(x) + c^{-}(x)) = \sqrt{2\pi}\,\boldsymbol{c}_0, \tag{5.4}$$

$$\int_0^{\pi} dx^1\,(\bar{c}^{+}(x) + \bar{c}^{-}(x)) = \sqrt{2\pi}\,\bar{\boldsymbol{c}}_0. \tag{5.5}$$

From (5.2)∼(5.5), we obtain

$$\begin{aligned}
&\langle\,0\,|\,\boldsymbol{c}_0\,\bar{\boldsymbol{c}}_0\,|\,0\,\rangle \\
&= \frac{i}{4\pi^2}\int_0^{\pi} dx^1 \int_0^{\pi} dy^1 \sum_{m=-\infty}^{\infty}\left[\frac{1}{x^{+} - y^{+} - \sqrt{2}\pi m - i0} + \frac{1}{x^{-} - y^{-} - \sqrt{2}\pi m - i0}\right. \\
&\qquad\qquad \left. + \frac{1}{x^{+} - y^{-} - \sqrt{2}\pi m - i0} + \frac{1}{x^{-} - y^{+} - \sqrt{2}\pi m - i0}\right].
\end{aligned} \tag{5.6}$$

Because of the periodicity, we may set $x^0 - y^0 = 0$. Then the real part of the integrand is seen to vanish. Thus (5.6) reduces to

$$\begin{aligned}
\langle\,0\,|\,\boldsymbol{c}_0\,\bar{\boldsymbol{c}}_0\,|\,0\,\rangle &= -\frac{1}{\sqrt{2}\pi}\int_0^{\pi} dx^1 \int_0^{\pi} dy^1\,\delta(x^1 - y^1) \\
&= -\frac{1}{\sqrt{2}}.
\end{aligned} \tag{5.7}$$

Likewise, we have

$$\langle\,0\,|\,\bar{\boldsymbol{c}}_0\,\boldsymbol{c}_0\,|\,0\,\rangle = -\frac{1}{\sqrt{2}}. \tag{5.8}$$

Of course, the sum of (5.7) and (5.8) is consistent with (4.7) and (5.1).

From the above consideration, it is natural to conclude that $|\,0\,\rangle$, $\boldsymbol{c}_0|\,0\,\rangle$, $\bar{\boldsymbol{c}}_0|\,0\,\rangle$, and $(\boldsymbol{c}_0\,\bar{\boldsymbol{c}}_0 - \bar{\boldsymbol{c}}_0\,\boldsymbol{c}_0)|\,0\,\rangle$ are *four linearly independent states*. The Kato-Ogawa vacua $|+\rangle$ and $|-\rangle$ are interpreted as

$$|+\rangle = \boldsymbol{c}_0(\alpha\,\bar{\boldsymbol{c}}_0 + \beta)|\,0\,\rangle, \tag{5.9}$$

$$|-\rangle = \bar{\boldsymbol{c}}_0(\alpha'\,\boldsymbol{c}_0 + \beta')|\,0\,\rangle, \tag{5.10}$$

where α, β, α' β' are arbitrary c-numbers. Each of them is *not a single state*. Indeed, the relation $\eta|\pm\rangle = |\mp\rangle$ holds only in the sense of the *subspace*.

The FP-ghost number operator is given by

$$\begin{aligned} iQ_c &= \frac{1}{\sqrt{2}}\int_0^\pi dx^1\,[\bar{c}^+(x)c^+(x) + \bar{c}^-(x)c^-(x)] \\ &= \frac{1}{\sqrt{2}}\sum_{n=-\infty}^{\infty}\bar{\boldsymbol{c}}_{-n}\,\boldsymbol{c}_n. \end{aligned} \tag{5.11}$$

Adjusting the zero-point value, we redefine Q_c by

$$iQ_c = \frac{1}{\sqrt{2}}\left[\frac{1}{2}(\bar{\boldsymbol{c}}_0\,\boldsymbol{c}_0 - \boldsymbol{c}_0\,\bar{\boldsymbol{c}}_0) + \sum_{n=1}^{\infty}(\bar{\boldsymbol{c}}_n{}^\dagger\boldsymbol{c}_n - \boldsymbol{c}_n{}^\dagger\bar{\boldsymbol{c}}_n)\right]. \tag{5.12}$$

The Kato-Ogawa vacua satisfy

$$iQ_c|\pm\rangle = \pm\frac{1}{2}|\pm\rangle. \tag{5.13}$$

From this result, Kato and Ogawa concluded that the FP-ghost numbers would be half-integers.

From our standpoint, the genuine vacuum $|\,0\,\rangle$ is *not* an eigenstate of iQ_c.[e] We can, however, bypass the trouble caused by this fact in the following way.

Let $P\,(=P^\dagger)$ be the projection operator to the subspace defined by the totality of the states which can be constructed from $|\,0\,\rangle$ by using the mode operators other than $\boldsymbol{c}_0$ and $\bar{\boldsymbol{c}}_0$. Then we introduce PQ_cP instead of Q_c. We find that iPQ_cP is unbroken and has integral eigenvalues. There is no anomalous feature for it. We should also introduce PQ_bP for the BRS charge in order to keep the relation between the BRS charge and the FP-ghost number.

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[e] Freeman and Olive[7] also adopted a vacuum which is not an eigenstate of iQ_c, but theirs is a linear combination of $|+\rangle$ and $|-\rangle$. In their formalism, therefore, no physical states have a definite FP-ghost number.

6. Nilpotent BRS charge in the Kato-Ogawa theory

In this section, we establish our main claim that it is possible to construct a BRS charge nilpotent for any value of D in the Kato-Ogawa string theory.

First, we briefly review how Kato and Ogawa[1] obtained their crucial result[f]

$$Q_B{}^2 = \frac{2}{\pi}\left[\frac{D-26}{24}\sum_{n=1}^{\infty} n^3 \boldsymbol{c}_n{}^\dagger \boldsymbol{c}_n - \left(\frac{D-26}{24} - \alpha_0 + 1\right)\sum_{n=1}^{\infty} n \boldsymbol{c}_n{}^\dagger \boldsymbol{c}_n\right]. \quad (6.1)$$

They define their BRS charge Q_B in terms of the BRS *Noether* current. That is, with

$$Q_b \equiv \frac{1}{\sqrt{2}}\int_0^\pi dx^1\,[j_b{}^-(x) + j_b{}^+(x)], \quad (6.2)$$

where

$$j_b{}^\mp = -i\bar{c}^\pm c^\pm \partial_\pm c^\pm - c^\pm \partial_\pm \phi_M \cdot \partial_\pm \phi^M, \quad (6.3)$$

which is the same as (2.31), Q_B is defined by the normal-product form of Q_b (more precisely, see below). They express Q_B as

$$Q_B = L\boldsymbol{c}_0 + M\bar{\boldsymbol{c}}_0 + \tilde{Q}_B, \quad (6.4)$$

where L, M and $\tilde{Q}_B$ involve neither $\boldsymbol{c}_0$ nor $\bar{\boldsymbol{c}}_0$.[g] Since the normal-product forms of M and $\tilde{Q}_B$ are the same as themselves, the difference between Q_B and Q_b arises only from L. That is, we can formally write

$$Q_B = Q_b + K\boldsymbol{c}_0, \quad (6.5)$$

with

$$K \equiv \frac{D-2}{2\sqrt{\pi}}\sum_{n=1}^{\infty} n + \frac{1}{\sqrt{\pi}}\alpha_0, \quad (6.6)$$

where α_0 is a regularization parameter of the Hamiltonian (interpreted as the zero intercept of the leading Regge trajectory). From our standpoint of generalized normal product stated in Sec.2, however, we should set $\alpha_0 = 0$. Note that $PQ_BP = PQ_bP$.

At the operator level, we, of course, have

$$Q_b{}^2 = 0, \quad (6.7)$$

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[f] Unfortunately, in their paper, the denominator factor is incorrectly written as 12 instead of 24.

[g] Slight changes of notation should be understood.

as can be straightforwardly verified by explicit calculation. Hence naive operator calculation yields ${Q_B}^2 = -\sqrt{2}KM$.

Thus it is impossible to derive (6.1) unambiguously by operator calculation. We emphasize that the reasonable derivation of (6.1) can be done only on the basis of the Fock representation of nonzero-mode operators. That is, *(6.1) is a formula which holds not at the operator level but at the representation level.* Indeed, the expression (6.1) can be obtained by calculating matrix elements of ${Q_B}^2$ with respect to Fock states. Especially, it is easy to see

$$\frac{1}{2}\langle\, 0\,|\,\bar{\boldsymbol{c}}_n {Q_B}^2 \bar{\boldsymbol{c}}_m{}^\dagger\,|\,0\,\rangle = \frac{2}{\pi}\Big[\frac{D-26}{24}(n^3-n)+(\alpha_0-1)n\Big]\delta_{n\,m} \qquad (n,\ m>0). \tag{6.8}$$

Since (6.1) is a result not at the operator level but at the representation level, it can be changed by using the field-equation anomaly, as discussed at the end of Sec.2. Owing to (4.20), (6.2) *equals*

$$\hat{Q}_b = \frac{1}{\sqrt{2}}\int_0^\pi dx^1\,[\hat{j}_b{}^-(x)+\hat{j}_b{}^+(x)] \tag{6.9}$$

with

$$\hat{j}_b{}^\mp = -\tilde{b}^\pm c^\pm + i\bar{c}^\pm c^\pm \partial_\pm c^\pm \tag{6.10}$$

at the operator level. We now demonstrate that

$$\langle\, 0\,|\,\bar{\boldsymbol{c}}_n(:\hat{Q}_b:)^2\bar{\boldsymbol{c}}_m{}^\dagger\,|\,0\,\rangle = 0 \qquad (n,\ m>0) \tag{6.11}$$

independently of the value of D.

In terms of mode operators, $:\hat{Q}_b:$ is given by

$$\begin{aligned} :\hat{Q}_b: \;&=\; -\frac{1}{\sqrt{2}}\sum_{n=-\infty}^{\infty} :\tilde{\boldsymbol{b}}_{-n}\boldsymbol{c}_n: + \frac{1}{\sqrt{2}\pi}\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} m :\bar{\boldsymbol{c}}_{-n-m}\boldsymbol{c}_n\boldsymbol{c}_m: \\ &=\; -\frac{1}{\sqrt{2}}\Big(\tilde{\boldsymbol{b}}_0\boldsymbol{c}_0 + \sum_{n=1}^{\infty}\tilde{\boldsymbol{b}}_n{}^\dagger\boldsymbol{c}_n + \sum_{n=1}^{\infty}\boldsymbol{c}_n{}^\dagger\tilde{\boldsymbol{b}}_n\Big) \\ &\quad + \frac{1}{\sqrt{2}\pi}\Big[-\boldsymbol{c}_0\sum_{n=1}^{\infty} n(\bar{\boldsymbol{c}}_n{}^\dagger\boldsymbol{c}_n + \boldsymbol{c}_n{}^\dagger\bar{\boldsymbol{c}}_n) + 2\bar{\boldsymbol{c}}_0\sum_{n=1}^{\infty} n\,\boldsymbol{c}_n{}^\dagger\boldsymbol{c}_n \\ &\qquad + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} m(\bar{\boldsymbol{c}}_{n+m}{}^\dagger\boldsymbol{c}_n\boldsymbol{c}_m - \boldsymbol{c}_n{}^\dagger\boldsymbol{c}_m{}^\dagger\bar{\boldsymbol{c}}_{n+m}) \\ &\qquad + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(n+2m)(\bar{\boldsymbol{c}}_n{}^\dagger\boldsymbol{c}_m{}^\dagger\boldsymbol{c}_{n+m} + \boldsymbol{c}_{n+m}{}^\dagger\boldsymbol{c}_m\bar{\boldsymbol{c}}_n)\Big]. \end{aligned} \tag{6.12}$$

Accordingly, by using (4.29) together with

$$\tilde{\boldsymbol{b}}_n|\,0\,\rangle = \{:\hat{Q}_b:,\ \bar{\boldsymbol{c}}_n\}|\,0\,\rangle =:\hat{Q}_b:\ \bar{\boldsymbol{c}}_n|\,0\,\rangle + \bar{\boldsymbol{c}}_n\ :\hat{Q}_b:\,|\,0\,\rangle = 0 \qquad (n>0), \tag{6.13}$$

where $:\hat{Q}_b:\,|\,0\,\rangle = 0$ holds as is shown at the end of Sec.7, we have

$$:\hat{Q}_b:\ \bar{\boldsymbol{c}}_m{}^\dagger|\,0\,\rangle = \left(\tilde{\boldsymbol{b}}_m{}^\dagger - \frac{1}{\sqrt{2}}\tilde{\boldsymbol{b}}_0\,\boldsymbol{c}_0\,\bar{\boldsymbol{c}}_m{}^\dagger + \frac{m}{\sqrt{\pi}}\boldsymbol{c}_0\,\bar{\boldsymbol{c}}_m{}^\dagger\right)|\,0\,\rangle \qquad (m>0), \tag{6.14}$$

and hence

$$\begin{aligned}\langle\,0\,|\,\bar{\boldsymbol{c}}_n(:\hat{Q}_b:)^2\,\bar{\boldsymbol{c}}_m{}^\dagger\,|\,0\,\rangle \;&=\; \langle\,0\,|\,\tilde{\boldsymbol{b}}_n\,\tilde{\boldsymbol{b}}_m{}^\dagger\,|\,0\,\rangle - \langle\,0\,|\,\tilde{\boldsymbol{b}}_n\,\boldsymbol{c}_0\left(\frac{1}{\sqrt{2}}\tilde{\boldsymbol{b}}_0 - \frac{m}{\sqrt{\pi}}\right)\bar{\boldsymbol{c}}_m{}^\dagger\,|\,0\,\rangle \\ &\quad -\langle\,0\,|\,\bar{\boldsymbol{c}}_n\left(\frac{1}{\sqrt{2}}\tilde{\boldsymbol{b}}_0 - \frac{n}{\sqrt{\pi}}\right)\boldsymbol{c}_0\,\tilde{\boldsymbol{b}}_m{}^\dagger\,|\,0\,\rangle \qquad (n,\ m>0).\end{aligned} \tag{6.15}$$

With the aid of (4.29), (4.30) and (6.13), we calculate each term of (6.15), and then find

$$\begin{aligned}\langle\,0\,|\,\bar{\boldsymbol{c}}_n(:\hat{Q}_b:)^2\,\bar{\boldsymbol{c}}_m{}^\dagger\,|\,0\,\rangle \;&=\; \frac{2\sqrt{2}n}{\sqrt{\pi}}\delta_{n\,m}\langle\,0\,|\,\tilde{\boldsymbol{b}}_0\,|\,0\,\rangle \\ &\quad +\frac{2n}{\sqrt{\pi}}\delta_{n\,m}\langle\,0\,|\,\tilde{\boldsymbol{b}}_0\,\bar{\boldsymbol{c}}_0\,\boldsymbol{c}_0\,|\,0\,\rangle + \frac{2n}{\sqrt{\pi}}\delta_{n\,m}\langle\,0\,|\,\tilde{\boldsymbol{b}}_0\,\boldsymbol{c}_0\,\bar{\boldsymbol{c}}_0\,|\,0\,\rangle \\ &=\; 0.\end{aligned} \tag{6.16}$$

This complete the proof of (6.11).

7. Unified treatment of infinite and finite strings

In this section, we present the calculation of ${Q_b}^2$ and ${\hat{Q}_b}{}^2$ in the x-space.[h] This approach enables us to calculate both infinite and finite strings simultaneously. We calculate

$$A \equiv \langle\,0\,|\,j_b{}^-(x_1)j_b{}^-(x_2)\bar{c}^+(x_3)\bar{c}^+(x_4)\,|\,0\,\rangle, \tag{7.1}$$

$$B \equiv \langle\,0\,|\,\hat{j}_b{}^-(x_1)\hat{j}_b{}^-(x_2)\bar{c}^+(x_3)\bar{c}^+(x_4)\,|\,0\,\rangle. \tag{7.2}$$

For simplicity, we write

$$\langle\,1,\,2\,\rangle \;=\; \begin{cases} \dfrac{i}{2\pi}\dfrac{1}{x_1{}^+ - x_2{}^+ - i0} & \text{for infinite string,} \\[2ex] \dfrac{i}{2\sqrt{2}\,\pi}\cot\left(\dfrac{x_1{}^+ - x_2{}^+}{\sqrt{2}} - i0\right) & \text{for finite string.}\end{cases} \tag{7.3}$$

[h] Since our calculation is made under the generalized normal-product rule, we need not take normal products for Q_b and $\hat{Q}_b$ explicitly.

Then the following identity hold:

$$\langle 1, 2\rangle\partial_+{}^{x_1}\partial_+{}^{x_2}\langle 1, 2\rangle = 2\partial_+{}^{x_1}\langle 1, 2\rangle\cdot\partial_+{}^{x_2}\langle 1, 2\rangle + \frac{\gamma}{2\pi i}\partial_+{}^{x_1}\langle 1, 2\rangle, \tag{7.4}$$

$$\partial_+{}^{x_1}\langle 1, 2\rangle\cdot\partial_+{}^{x_2}\langle 1, 2\rangle = \frac{i}{12\pi}[(\partial_+{}^{x_1})^3\langle 1, 2\rangle + 2\gamma\partial_+{}^{x_1}\langle 1, 2\rangle], \tag{7.5}$$

where

$$\gamma = \begin{cases} 0 & \text{for infinite string,} \\ 1 & \text{for finite string.} \end{cases} \tag{7.6}$$

Substituting (2.31) into (7.1), we have

$$\begin{aligned} A = & -\langle 0\,|\,\bar{c}^+(x_1)c^+(x_1)\partial_+c^+(x_1)\cdot\bar{c}^+(x_2)c^+(x_2)\partial_+c^+(x_2)\cdot\bar{c}^+(x_3)\bar{c}^+(x_4)\,|\,0\rangle \\ & +\langle 0\,|\,c^+(x_1)\partial_+\phi_M(x_1)\partial_+\phi^M(x_1)\cdot c^+(x_2)\partial_+\phi_N(x_2)\partial_+\phi^N(x_2)\cdot\bar{c}^+(x_3)\bar{c}^+(x_4)\,|\,0\rangle. \end{aligned} \tag{7.7}$$

Since all fields involved in (7.7) are free fields, it is expressible in terms of the 2-point functions

$$\langle 0\,|\,c^+(x_1)\bar{c}^+(x_2)\,|\,0\rangle = \langle 0\,|\,\bar{c}^+(x_1)c^+(x_2)\,|\,0\rangle = \langle 1, 2\rangle, \tag{7.8}$$

$$\langle 0\,|\,\partial_+\phi_M(x_1)\partial_+\phi^N(x_2)\,|\,0\rangle = \frac{i}{2}\delta_M{}^N\partial_+{}^{x_1}\langle 1, 2\rangle \tag{7.9}$$

only. We find

$$\begin{aligned} A = & \Big[\frac{1}{2}(D-2)\partial_1\langle 1, 2\rangle\cdot\partial_2\langle 1, 2\rangle\cdot\langle 1, 4\rangle\langle 2, 3\rangle + \partial_1\langle 1, 2\rangle\cdot\langle 1, 2\rangle\langle 1, 4\rangle\partial_2\langle 2, 3\rangle \\ & +\langle 1, 2\rangle\partial_2\langle 1, 2\rangle\cdot\partial_1\langle 1,4\rangle\cdot\langle 2, 3\rangle - \langle 1, 2\rangle^2\partial_1\langle 1, 4\rangle\cdot\partial_2\langle 2, 3\rangle\Big] \\ & -(3\leftrightarrow 4), \end{aligned} \tag{7.10}$$

where $\partial_1 \equiv \partial_+{}^{x_1}$. After some manipulation, A can be rewritten as

$$\begin{aligned} A = & \Big[\frac{1}{2}(D-10)\partial_1\langle 1, 2\rangle\cdot\partial_2\langle 1, 2\rangle\cdot\langle 1, 4\rangle\langle 2, 3\rangle \\ & -4\langle 1, 2\rangle\partial_1\partial_2\langle 1, 2\rangle\cdot\langle 1, 4\rangle\langle 2, 3\rangle\Big] - (3\leftrightarrow 4) \\ & +\Delta, \end{aligned} \tag{7.11}$$

where Δ is a total-divergence part given by

$$\begin{aligned} \Delta \equiv & \Big\{2\partial_1\partial_2[\langle 1, 2\rangle^2\langle 1, 4\rangle\langle 2, 3\rangle] \\ & -\frac{3}{2}\partial_1[\langle 1, 2\rangle^2\langle 1, 4\rangle\partial_2\langle 2, 3\rangle] - \frac{3}{2}\partial_2[\langle 1, 2\rangle^2\partial_1\langle 1, 4\rangle\cdot\langle 2, 3\rangle]\Big\} \\ & -(3\leftrightarrow 4). \end{aligned} \tag{7.12}$$

Then, by the help of (7.4) and (7.5), (7.1) finally becomes

$$A = \left\{ \frac{i}{24\pi}(D-26)(\partial_1)^3\langle\, 1,\, 2\,\rangle + \gamma\frac{i}{12\pi}(D-2)\partial_1\langle\, 1,\, 2\,\rangle \right\}[\langle\, 1,\, 4\,\rangle\langle\, 2,\, 3\,\rangle - \langle\, 1,\, 3\,\rangle\langle\, 2,\, 4\,\rangle] + \Delta. \tag{7.13}$$

The calculation of (7.2) is straightforward but more lengthly because we encounter not only 2-point functions but also 3-point and 4-point functions. We omit the details of the calculation. The result is simply

$$B = \Delta. \tag{7.14}$$

We return to (7.13). First, we consider the infinite string. From (2.32) (${j_b}^+$ does not contribute) and (7.13), we have

$$\begin{aligned} &\langle\, 0\,|\, {Q_b}^2 \bar{c}^+(x_3)\bar{c}^+(x_4)\,|\, 0\,\rangle \\ &\quad = \frac{i}{24\pi}(D-26)\frac{1}{2}\left(\frac{i}{2\pi}\right)^3 \int_{-\infty}^{\infty} d{x_1}^1 \int_{-\infty}^{\infty} d{x_2}^1 ({\partial_+}^{x_1})^3 \frac{1}{{x_1}^+ - {x_2}^+ - i0} \\ &\qquad \left(\frac{1}{{x_1}^+ - {x_4}^+ - i0}\frac{1}{{x_2}^+ - {x_3}^+ - i0} - \frac{1}{{x_1}^+ - {x_3}^+ - i0}\frac{1}{{x_2}^+ - {x_4}^+ - i0} \right). \end{aligned} \tag{7.15}$$

But (7.15) turns out to vanish, as is seen by carrying out the integration over ${x_1}^1$ as a contour integral in the lower half-plane. This kind of reasoning applies to any Wightman function which has Q_b at the left or right end.[i] That is, we may infer that

$$\langle\, 0\,| Q_b = 0, \qquad Q_b |\, 0\,\rangle = 0 \tag{7.16}$$

hold at the representation level.

In order to have anomaly, therefore, we should consider $\langle\, 0\,|\, \bar{c}^+(x_3){Q_b}^2\bar{c}^+(x_4)\,|\, 0\,\rangle$, for which $-i0$ is replaced by $+i0$ in all denominator factors involving ${x_3}^+$ in (7.15). We then find

$$\langle\, 0\,|\, \bar{c}^+(x_3){Q_b}^2\bar{c}^+(x_4)\,|\, 0\,\rangle = \frac{D-26}{8\pi^2}\frac{1}{({x_3}^+ - {x_4}^+ - i0)^4}. \tag{7.17}$$

We note that this result can be reproduced also if we use (7.16) and (2.30):

$$\begin{aligned} \langle\, 0\,|\, \bar{c}^+(x_3){Q_b}^2\bar{c}^+(x_4)\,|\, 0\,\rangle &= \langle\, 0\,|\{\bar{c}^+(x_3),\, Q_b\}\{Q_b,\, \bar{c}^+(x_4)\}|\, 0\,\rangle \\ &= \langle\, 0\,|\, \tilde{\mathcal{T}}^+(x_3)\tilde{\mathcal{T}}^+(x_4)\,|\, 0\,\rangle \\ &= \frac{D-26}{8\pi^2}\frac{1}{({x_3}^+ - {x_4}^+ - i0)^4}. \end{aligned} \tag{7.18}$$

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[i] Of course, sufficient damping of the integrand as $|x^1| \to \infty$ is needed.

Next, we consider the finite strings. We calculate the integrations of (7.13) by substituting the formula [cf.(3.10)]

$$\langle\, 1,\, 2\,\rangle = -\frac{1}{\sqrt{2}\,\pi}\Big[\frac{1}{2}+\sum_{n=1}^{\infty}e^{-in\sqrt{2}({x_1}^+-{x_2}^+)}\Big]. \tag{7.19}$$

For the closed string, the integration ranges of ${x_1}^1$ and ${x_2}^1$ are $[0,\ 2\pi]$, while, for the open string, those are $[0,\ \pi]$ but ${j_b}^+$ also contributes. We obtain the same result for both strings.

We can again infer (7.16) by using the fact that the summation over n in (7.19) is restricted to $n>0$ because the integral

$$\int_0^\pi d{x_1}^1\ \cos(n{x_1}^1+m{x_1}^1) = \pi\delta_{n+m,\,0} \tag{7.20}$$

vanishes for $n,\ m>0$, as long as the purely zero-mode term of Q_b does not contribute. A simple manipulation yields

$$\langle\, 0\,|\,\bar{c}^+(x_3){Q_b}^2\bar{c}^+(x_4)\,|\,0\,\rangle = \frac{2}{\pi^2}\sum_{n=1}^{\infty}\Big[\frac{D-26}{24}n^3-\frac{D-2}{24}n\Big]e^{-in\sqrt{2}({x_3}^+-{x_4}^+)}. \tag{7.21}$$

If we calculate $\langle\, 0\,|\,\bar{c}^+(x_3){Q_B}^2\bar{c}^+(x_4)\,|\,0\,\rangle$ by using (6.1) with $\alpha_0=0$, we find that it precisely equals the right-hand side of (7.21). Thus our approach is consistent with Kato-Ogawa's one apart from the introduction of α_0.

Carrying out the summation over n in (7.21), we obtain

$$\begin{aligned}&\langle\, 0\,|\,\bar{c}^+(x_3){Q_b}^2\bar{c}^+(x_4)\,|\,0\,\rangle\\ &\quad=\frac{D-26}{32\pi^2}\Big(\sin\frac{{x_3}^+-{x_4}^+}{\sqrt{2}}\Big)^{-4}+\frac{1}{2\pi^2}\Big(\sin\frac{{x_3}^+-{x_4}^+}{\sqrt{2}}\Big)^{-2}.\end{aligned} \tag{7.22}$$

This result is precisely equal to $\langle\, 0\,|\,\tilde{\mathcal{T}}^+(x_3)\tilde{\mathcal{T}}^+(x_4)\,|\,0\,\rangle$, as is verified by direct calculation.

Finally, we consider the case of $\hat{Q}_b$. By the same reasoning as that of (7.16), we see

$$\langle\, 0\,|\hat{Q}_b = 0, \qquad \hat{Q}_b|\,0\,\rangle = 0. \tag{7.23}$$

From (7.14), we have

$$\langle\, 0\,|\,\bar{c}^+(x_3){\hat{Q}_b}^2\bar{c}^+(x_4)\,|\,0\,\rangle = 0. \tag{7.24}$$

Corresponding to (7.18), (7.24) can be reproduced also by considering

$$\begin{aligned}\langle\, 0\,|\,\bar{c}^+(x_3){\hat{Q}_b}^2\bar{c}^+(x_4)\,|\,0\,\rangle &= \langle\, 0\,|\,\{\bar{c}^+(x_3),\,\hat{Q}_b\}\{\hat{Q}_b,\,\bar{c}^+(x_4)\}\,|\,0\,\rangle\\ &= \langle\, 0\,|\,\tilde{b}^+(x_3)\tilde{b}^+(x_4)\,|\,0\,\rangle = 0.\end{aligned} \tag{7.25}$$

8. Discussion

In the present paper, we have clarified how the Kato-Ogawa string theory can be understood in terms of our approach to the conformal-gauge two-dimensional quantum gravity. Our way of constructing Wightman functions reproduces the formulation of the Kato-Ogawa theory except for the introduction of a regularization parameter α_0.

We have shown that Kato-Ogawa's claim, ${Q_B}^2 \neq 0$ for $D \neq 26$, is not a result intrinsic to the BRS quantization of the string theory. It is possible to construct explicitly a BRS charge nilpotent for any value of D. *The BRS invariance itself is not anomalous.* What is anomalous is the B field equation, which is anomaly-free only at the critical dimension $D = 26$. It should be noted, however, that the absence of the field-equation anomaly at $D = 26$ is *not* a general property; in the de Donder gauge,the field-equation anomaly does not disappear for any value of D, as was shown elsewhere.[2] Even in that case, the BRS invariance is not broken. As was clarified already,[8] the appearance of the critical dimension $D = 26$ itself is *not* an intrinsic result in the de Donder-gauge two-dimensional quantum gravity.

Our next problem is to reformulate the no-ghost theorem. Since Kato and Ogawa eliminated the B field from the outset, their treatment of the no-ghost theorem is rather different from the original form of the Kugo-Ojima quartet mechanism.[9] We think that the B field should be adopted as a member of the Kugo-Ojima quartet. It will be possible to construct the physical subspace explicitly as the proper framework of the two-dimensional quantum gravity rather than as the string theory.

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