ABUNDANCE THEOREM FOR SEMI LOG CANONICAL THREEFOLDS

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0. Introduction

The main purpose of this paper is to prove the abundance theorem for semi log canonical threefolds. The abundance conjecture is a very important problem in the birational classification of algebraic varieties. The abundance theorem for semi log canonical surfaces was proved in [12, Chapter 8, 12] by L.-Y. Fong, S. Keel, J. Kollár, and J. McKernan. Their proof uses semiresolution, etc. and has some combinatorial complexities. So we simplify their proof and strengthen the theorem for semi divisorial log terminal surfaces a little (See Corollary (4.8)). By our method we can reduce the problem to the irreducible case and the finiteness of some groups. This shows that if the log Minimal Model Program (log MMP, for short), the log abundance conjecture, and the finiteness of B-pluricanonical representations (See Section 3) hold for n-folds, then the log abundance conjecture for semi log canonical n-folds is true almost automatically (See Theorem (5.1)). But unfortunately the log MMP and the log abundance conjecture are only conjectures for n-folds with $n \geq 4$. So we prove

Theorem 0.1 (Abundance theorem for slc threefolds). *Let (X, Δ) be a proper slc threefold with $K_X + \Delta$ nef. Then $K_X + \Delta$ is semi-ample.*

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This theorem is a generalization of the log abundance theorem for threefolds proved by S. Keel, K. Matsuki and J. McKernan (See [10]). According to them, the log abundance theorem for threefolds is considered to be the first step towards a proof of the abundance conjecture in dimension four. We believe that the abundance theorem for slc threefolds is the second step.

The notion of semi log canonical singularity was first introduced in [14] for the problem of compactifying the moduli of surfaces. For the further development of this direction, we recommend the readers to see [5].

We sketch the contents of this paper. Section 1 sets up some basic definitions and facts. In Section 2, we treat the reduced boundaries of dlt n-folds. This is a reformulation of [12, 12.3.2]. Section 3 deals with B-pluricanonical representations (precise definitions are given in Definition (3.1)) and we prove the finiteness of them for curves and surfaces. It plays an important role in our proof of abundance theorem for slc n-folds. Section 4 is the main section of this paper. We prove the abundance theorem for slc threefolds in this section. In Section 5, we reformulate the main theorem under some assumptions such as log MMP for n-folds and collect some known results for readers' convenience.

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Notation. (1) The word *scheme* is used for schemes which are separated and of finite type over $\mathbb{C}$ and the word *variety* stands for a reduced and irreducible scheme. When we say a normal scheme, we mean the disjoint union of irreducible normal schemes.

(2) We use the notation that (X, Δ) being *Kawamata log terminal, log terminal, divisorial log terminal, log canonical* (frequently abbreviated as klt, lt, dlt, and lc), and *terminal* as in [13] (See also [18]). In the definition in [13, Section 2.3] Δ is not necessarily effective, but in this paper we assume Δ is an effective $\mathbb{Q}$-divisor.

(3) Let $f : X \dashrightarrow Y$ be a rational map. We say that a $\mathbb{Q}$-divisor D is horizontal (resp. vertical) if D is dominating (resp. not dominating) over Y.

(4) The log MMP means the log MMP for $\mathbb{Q}$-factorial dlt pairs.

1. Definitions and Preliminaries

In this section, we make the basic notation and collect the necessary results.

Definition 1.1. Let X be a reduced S_2 scheme. We assume that it is pure n-dimensional and normal crossing in codimension 1. Let Δ be an effective $\mathbb{Q}$-Weil divisor on X (cf. [12, 16.2]) such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier.

Let $X = \cup X_i$ be a decomposition into irreducible components, and $\mu : X' := \coprod X'_i \to X = \cup X_i$ the normalization. A $\mathbb{Q}$-divisor Θ on X' is defined by $K_{X'} + \Theta := \mu^*(K_X + \Delta)$ and a $\mathbb{Q}$-divisor Θ_i on X'_i by $\Theta_i := \Theta|_{X'_i}$.

We say that (X, Δ) is a *semi log canonical n-fold* (an slc n-fold, for short) if (X'_i, Θ_i) is lc for every i.

We say that (X, Δ) is a *semi divisorial log terminal n-fold* (an sdlt n-fold, for short) if X_i is normal, that is, X'_i is isomorphic to X_i, and (X_i, Θ_i) is dlt for every i.

Remark 1.2. (1) The slc in Definition (1.1) is equivalent to the slc in [12, Chapter 12] (See [11, 4.2]).

(2) If (X, Δ) is an slc n-fold, then X is seminormal (See [12, 12.2.1.(8)] and [1, Remark 4.7]).

(3) If (X, Δ) is a dlt n-fold, then $(\llcorner \Delta \lrcorner, \mathrm{Diff}(\Delta - \llcorner \Delta \lrcorner))$ is an sdlt $(n-1)$-fold (See [12, 17.5] and [13, 5.52]).

(4) Let (X, Δ) be lc. Then $(\llcorner \Delta \lrcorner, \mathrm{Diff}(\Delta - \llcorner \Delta \lrcorner))$ is not necessarily slc (See [12, 17.5.2 Example]). Note that [12, (16.9.1)] is not true.

The next lemma plays an important role in Section 2.

Lemma 1.3 (Connectedness Lemma). ([17, 5.7], [12, 17.4], [8, 1.4]) *Let X and Y be normal varieties and let $f : X \to Y$ be a proper surjective morphism with connected fibers. Let $\Delta = \sum d_i \Delta_i$ be a $\mathbb{Q}$-divisor on X. Let $g : Z \to X$ be a log resolution* (cf. [13, Notation 0.4 (10)]) *such that $h := f \circ g$. Let*

$$K_Z = g^*(K_X + \Delta) + \sum e_i E_i, \text{ and } F := -\sum_{i : e_i \leq -1} e_i E_i.$$

Assume that

(1) *if $d_i < 0$, then $f(\Delta_i)$ has codimension at least two in Y, and*
(2) *$-(K_X + \Delta)$ is f-nef and f-big.*

Then $\mathrm{Supp} F = \mathrm{Supp} \llcorner F \lrcorner$ is connected in a neighborhood of any fiber of h. In particular, if (X, Δ) is lc and $(X, \Delta - \llcorner \Delta \lrcorner)$ is klt, then $\llcorner \Delta \lrcorner \cap f^{-1}(y)$ is connected for every point $y \in Y$.

Lemma-Definition 1.4 ($\mathbb{Q}$-factorial dlt model)**.** (cf. [12, 8.2.2]) *Let (X, Δ) be an lc n-fold with $n \leq 3$. Then there is a projective birational morphism $f : (Y, \Theta) \to (X, \Delta)$ such that (Y, Θ) is $\mathbb{Q}$-factorial dlt and $K_Y + \Theta = f^*(K_X + \Delta)$. Furthermore, if (X, Δ) is dlt, then we can make f a small projective morphism. We say that (Y, Θ) is a $\mathbb{Q}$-factorial dlt model of (X, Δ).*

Definition 1.5. Let $(X, \Delta) = \coprod_{i=1}^{n}(X_i, \Delta_i)$, $(X', \Delta') = \coprod_{i=1}^{n}(X'_i, \Delta'_i)$ be normal schemes with $\mathbb{Q}$-divisor such that $K_X + \Delta$ and $K_{X'} + \Delta'$ are $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisors.

We say that $f : (X, \Delta) \dashrightarrow (X', \Delta')$ is a B-birational map (resp. morphism) if $f : X \dashrightarrow X'$ is a proper birational map (resp. morphism) and there exists a common resolution $\alpha : T \to X$, $\beta : T \to X'$ of $f : X \dashrightarrow X'$ such that $\alpha^*(K_X + \Delta) = \beta^*(K_{X'} + \Delta')$. That is, there exists a permutation σ such that $f_i : X_i \dashrightarrow X'_{\sigma(i)}$ is a proper birational map (resp. morphism) and there exists a common resolution $\alpha_i : T_i \to X_i$, $\beta_i : T_i \to X'_{\sigma(i)}$ of f_i such that $\alpha_i^*(K_{X_i} + \Delta_i) = \beta_i^*(K_{X'_{\sigma(i)}} + \Delta'_{\sigma(i)})$ on T_i for every i. The last condition means that if we write

$$\begin{aligned} K_{T_i} &= \alpha_i^*(K_{X_i} + \Delta_i) + F, \\ K_{T_i} &= \beta_i^*(K_{X'_{\sigma(i)}} + \Delta'_{\sigma(i)}) + E, \end{aligned}$$

then $F = E$.

If there is a B-birational map from (X, Δ) to (X', Δ'), we say that (X, Δ) is B-birational equivalent to (X', Δ') and write $(X, \Delta) \sim_B (X', \Delta')$. Here the symbol B is the initial of *boundary*.

Lemma 1.6. *Let (X, Δ) and (Z, Δ') be normal varieties with $\mathbb{Q}$-divisors such that $K_X + \Delta$ and $K_Z + \Delta'$ are $\mathbb{Q}$-Cartier. Let $f : X \to R$ and $h : Z \to R$ be proper surjective morphisms onto a normal variety R and $p : X \dashrightarrow Z$ a birational map such that $f = h \circ p$. Assume that*

(1) *p^{-1} has no exceptional divisors,*
(2) *$\Delta' = p_*\Delta$, and*
(3) *$K_X + \Delta \equiv_f 0$, $K_Z + \Delta' \equiv_h 0$.*

Then p is a B-birational map.

Proof. Let $\beta : W \to X$ be a resolution such that the induced rational map $\alpha = p \circ \beta : W \to Z$ is a morphism. Let $m > 0$ be a sufficiently divisible integer. We have linear equivalences

$$\begin{aligned} -mK_W &\sim -\alpha^*(m(K_Z + \Delta')) - F, \\ mK_W &\sim \beta^*(m(K_X + \Delta)) + E. \end{aligned}$$

Adding the two we obtain that

$$\beta^*(K_X + \Delta) - \alpha^*(K_Z + \Delta') = F - E.$$

By assumption, $F-E$ is α-exceptional and numerically α-trivial. Then $F = E$, that is, $\alpha^*(K_Z + \Delta') = \beta^*(K_X + \Delta)$. □

2. Reduced boundaries of dlt n-folds

The following is a reformulation of [12, 12.3.2], which fits better in our arguments.

Proposition 2.1. (cf. [17, 6.9], [12, 12.3.2]) *Let (X, Θ) be a $\mathbb{Q}$-factorial dlt n-fold with $n \leq 3$. Let $f : X \to R$ be a proper surjective morphism onto a normal variety R with connected fibers. Assume that $K_X + \Theta$ is numerically f-trivial. Then one of the following holds:*

(0) $\dim R = 0$.
  (0.1) $\llcorner \Theta \lrcorner$ *is connected.*
  (0.2) $\llcorner \Theta \lrcorner$ *has two connected components Δ_1 and Δ_2 and there exists a rational map $v : X \dashrightarrow (V, P)$ onto a $\mathbb{Q}$-factorial lc $(n-1)$-fold (V, P) with general fiber $\mathbb{P}^1$. Furthermore there exists an irreducible component $D_i' \subset \Delta_i$ such that $v|_{D_i'} : (D_i', \mathrm{Diff}(\Theta - D_i')) \dashrightarrow (V, P)$ is a B-birational map for $i = 1, 2$.*

(1) $\dim R \geq 1$.
  (1.1) $\llcorner \Theta \lrcorner \cap f^{-1}(r)$ *is connected for every $r \in R$.*
  (1.2) *The number of connected components of $\llcorner \Theta \lrcorner \cap f^{-1}(r)$ is at most two for every $r \in R$. There exists a rational map $v : X \dashrightarrow (V, P)$ onto a $\mathbb{Q}$-factorial lc $(n-1)$-fold (V, P) with general fiber $\mathbb{P}^1$. The horizontal part Θ^h of $\llcorner \Theta \lrcorner$ is one of the following:*
    (i) $\Theta^h = D_1'$ *which is irreducible and the mapping degree $\deg[D_1' : V] = 2$ and there is a B-birational involution on $(D_1', \mathrm{Diff}(\Theta - D_1'))$ over V.*
    (ii) $\Theta^h = D_1' + D_2'$ *such that D_i' is irreducible and $v|_{D_i'} : (D_i', \mathrm{Diff}(\Theta - D_i')) \dashrightarrow (V, P)$ is a B-birational map for $i = 1, 2$.*

Remark 2.2. (1) In Proposition (2.1) the assumption $K_X + \Theta \equiv_f 0$ is equivalent to $K_X + \Theta \sim_{\mathbb{Q}, f} 0$. It is because the relative log abundance theorem holds when $\dim X \leq 3$ (See Theorem (5.2)).

(2) If the log MMP holds for n-folds, then Proposition (2.1) is also true for n-folds.

First we prove the next lemma.

Lemma 2.3. *Let (Z, Δ) be a $\mathbb{Q}$-factorial lc n-fold with $n \geq 2$ and $\llcorner \Delta \lrcorner \neq 0$. Let $h : Z \to R$ be a proper surjective morphism onto a normal variety R with connected fibers. Assume the following conditions:*

(1) $(Z, 0)$ *is lt.*
(2) $K_Z + \Delta \equiv_h 0$.
(3) *There is a* $(K_Z + \Delta - \varepsilon \llcorner \Delta \lrcorner)$*-extremal Fano contraction* $u : Z \to V$ *over* R *for* $0 < \varepsilon \ll 1$, *such that* $\dim V = n - 1$.

Then the horizontal part Δ^h *of* $\llcorner \Delta \lrcorner$ *is one of the following:*

(a) $\Delta^h = D_1$ *which is irreducible and* $\deg[D_1 : V] = 2$.
(b) $\Delta^h = D_1$ *which is irreducible and* $\deg[D_1 : V] = 1$.
(c) $\Delta^h = D_1 + D_2$ *such that* D_i *is irreducible and* $\deg[D_i : V] = 1$ *for* $i = 1, 2$.

In the cases (a) and (c), the number of connected components of $\llcorner \Delta \lrcorner \cap h^{-1}(r)$ *is at most two for every* $r \in R$.
In the case (b), $\llcorner \Delta \lrcorner \cap h^{-1}(r)$ *is connected for every* $r \in R$.
Furthermore, there is a $\mathbb{Q}$*-divisor* P *on* V *such that* (V, P) *is a* $\mathbb{Q}$*-factorial lc* $(n-1)$*-fold and* $K_{D_i} + \mathrm{Diff}(\Delta - D_i) = u|_{D_i}{}^*(K_V + P)$ *for* $i = 1, 2$.

In the case (a), there is a B-birational involution ι *over* V, *that is,* $\iota : (D_1, \mathrm{Diff}(\Delta - D_1)) \dashrightarrow (D_1, \mathrm{Diff}(\Delta - D_1))$ *over* V *is a B-birational map and* $\iota^2 = id$.

In the case (c), $u|_{D_i} : D_i \to V$ *is a B-birational morphism for* $i = 1, 2$. *In particular,* $(D_1, \mathrm{Diff}(\Delta - D_1)) \sim_B (D_2, \mathrm{Diff}(\Delta - D_2))$.

Proof. Since $\llcorner \Delta \lrcorner$ is u-ample by the assumptions (2) and (3), we have $\Delta^h \neq 0$. So the general fiber of $Z \to V$ is $\mathbb{P}^1$ and $\deg[\Delta^h : V] \leq 2$ because $K_Z + \Delta$ is numerically h-trivial. Since $u : Z \to V$ is extremal, the vertical component Δ^v of $\llcorner \Delta \lrcorner$ is a pull-back of a $\mathbb{Q}$-divisor on V. Then the first part is proved. Let $H_1, H_2, \cdots, H_{n-2}$ be general hypersurfaces on V. Consider

$$u^{-1}(H_1 \cap H_2 \cap \cdots \cap H_{n-2}) \to H_1 \cap H_2 \cap \cdots \cap H_{n-2}.$$

By using [7, 3.5.1, 3.5.2] and [12, 12.3.4] to the above morphism, we have a $\mathbb{Q}$-divisor P on V satisfying $K_{D_i} + \mathrm{Diff}(\Delta - D_i) = u|_{D_i}{}^*(K_V + P)$. The normalization of $(D_1, \mathrm{Diff}(\Delta - D_1))$ is lc and the normalization of (V, P) in the function field $\mathbb{C}(D_1)$ is lc, since $K_{D_1} + \mathrm{Diff}(\Delta - D_1) = u|_{D_1}{}^*(K_V + P)$. Then (V, P) is lc, and $\mathbb{Q}$-factorial since Z is $\mathbb{Q}$-factorial and u is extremal. □

Proof of Proposition (2.1). If f is birational then the connectedness lemma (1.3) implies that we are in Case (0.1) or (1.1). Thus we may assume that $\dim R < \dim X$.

We run the $(K_X + \Theta - \varepsilon \llcorner \Theta \lrcorner)$-MMP on X over R for $0 < \varepsilon \ll 1$. The end result is a birational map $p : X \dashrightarrow Z$ over R. Let $h : Z \to R$ be the induced morphism. Since $K_X + \Theta \equiv_f 0$, we obtain that

$K_Z + p_*\Theta \equiv_h 0$. Then $(Z, p_*\Theta)$ is a $\mathbb{Q}$-factorial lc pair (See Lemma (1.6)) and $(Z, p_*\Theta - \varepsilon\llcorner p_*\Theta\lrcorner)$ is klt.

Step 1. If $(Z, p_*\Theta - \varepsilon\llcorner p_*\Theta\lrcorner)$ is a minimal model, then $K_Z + p_*\Theta - \varepsilon\llcorner p_*\Theta\lrcorner$ is h-nef and $K_Z + p_*\Theta \equiv_h 0$. So $-\llcorner p_*\Theta\lrcorner$ is h-nef. If $\dim R = 0$, then $\llcorner p_*\Theta\lrcorner = 0$. By Lemma (2.4) below, we have $\llcorner\Theta\lrcorner = 0$. If $0 < \dim R < \dim X$, then $\llcorner p_*\Theta\lrcorner \cap h^{-1}(r)$ is connected for every $r \in R$. By Lemma (2.4) we have Case (1.1).

Step 2. If there exists an extremal Fano contraction $u : Z \to V$ over R, then $-(K_Z + p_*\Theta - \varepsilon\llcorner p_*\Theta\lrcorner)$ is u-ample. Let $g : V \to R$ be the induced morphism. We note $h = g \circ u$.

First assume that $\dim R = 0$. If $\llcorner p_*\Theta\lrcorner$ is connected, we have Case (0.1) by Lemma (2.4). Thus we can assume that $\llcorner p_*\Theta\lrcorner$ is not connected. Since $\llcorner p_*\Theta\lrcorner$ is relatively ample, we have $\dim V = \dim X - 1$. Then by Lemma (2.3) we have case (2.3.c) and $u|_{D_i} : D_i \to V$ are finite where D_i are as in (2.3.c) for $i = 1, 2$. It is because D_1 intersects D_2 by the relative ampleness of D_1 and D_2 if $u|_{D_1}$ or $u|_{D_2}$ is not finite. We have $D_i \simeq V$ by Zariski's Main Theorem since V is normal. By Lemma (1.6), the adjunction, and Lemma (2.4), we have Case (0.2).

Thus we may assume $\dim R \geq 1$.

If $\dim V = \dim X - 1$, we get Case (1.2) by Lemmas (1.6), (2.3), (2.4) below, and the adjunction.

If $\dim R = 1$, $\dim V = 1$, $\dim X = 3$, then $V \simeq R$. Since u is extremal, $\llcorner p_*\Theta\lrcorner$ is h-ample and $\rho(Z/R) = 1$. Then every horizontal irreducible component of $\llcorner p_*\Theta\lrcorner$ is h-ample, and every vertical irreducible component of $\llcorner p_*\Theta\lrcorner$ is a pull-back of a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on R. Then $\llcorner p_*\Theta\lrcorner \cap h^{-1}(r)$ is connected for every $r \in R$. We have Case (1.1). □

The next lemma was used in the proof of Proposition (2.1).

Lemma 2.4. *If $f : X \to R$, $h : Z \to R$, $p : X \dashrightarrow Z$ are as in the proof of Proposition (2.1), then the number of connected components of $\llcorner\Theta\lrcorner \cap f^{-1}(r)$ is equal to the number of connected components of $\llcorner p_*\Theta\lrcorner \cap h^{-1}(r)$ for every $r \in R$.*

Proof. $p : X \dashrightarrow Z$ is a composition of flips and divisorial contractions. $X \dashrightarrow X^1 \dashrightarrow X^2 \dashrightarrow \cdots \dashrightarrow X^i \dashrightarrow \cdots \dashrightarrow Z$. Use the connectedness lemma (1.3) in each step. Let Θ^i be the proper transform of Θ on X^i. Note that $\llcorner\Theta^i\lrcorner$ is relatively ample for each flipping or divisorial contraction and $K_{X^i} + \Theta^i$ is numerically trivial over R. □

We need the following lemma in the case where there are no horizontal parts.

Lemma 2.5. *Let $f : X \to R$ be a proper surjective morphism between normal varieties with connected fibers. Assume that*

(1) *(X, Θ) is a $\mathbb{Q}$-factorial dlt n-fold,*
(2) *$f(\llcorner \Theta \lrcorner) \subsetneq R$,*
(3) *$K_X + \Theta \sim_{\mathbb{Q},f} 0$.*

Then there is an exact sequence

$$0 \to f_*\mathcal{O}_X(-\llcorner \Theta \lrcorner) \to \mathcal{O}_R \to f_*\mathcal{O}_{\llcorner \Theta \lrcorner} \to 0.$$

Proof. There is a positive integer m and a Cartier divisor D on R such that $m(K_X + \Theta) = f^*D$. Observe that

$$m(K_X + \Theta) - \llcorner \Theta \lrcorner - (K_X + \{\Theta\}) = (m-1)(K_X + \Theta).$$

Since $K_X + \{\Theta\}$ is klt and $(m-1)(K_X + \Theta)$ is f-semi-ample by the assumptions (1) and (3), $R^1 f_*\mathcal{O}_X(m(K_X + \Theta) - \llcorner \Theta \lrcorner)$ is torsion free (See [9, 1-2-7]). By the assumption (2), $f_*\mathcal{O}_{\llcorner \Theta \lrcorner}(m(K_X + \Theta))$ is a torsion sheaf. Then we have an exact sequence

$$\begin{aligned} 0 \to f_*\mathcal{O}_X(m(K_X + \Theta) - \llcorner \Theta \lrcorner) &\to f_*\mathcal{O}_X(m(K_X + \Theta)) \\ &\to f_*\mathcal{O}_{\llcorner \Theta \lrcorner}(m(K_X + \Theta)) \to 0. \end{aligned}$$

Tensoring $\mathcal{O}_R(-D)$ gives the result. □

3. Finiteness of B-pluricanonical representations

We consider the birational automorphism groups of pairs.

Definition 3.1. Let (X, Δ) be a pair of a normal scheme and a $\mathbb{Q}$-divisor such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier. We define

$$\begin{aligned} \mathrm{Bir}(X, \Delta) &:= \{\sigma : X \dashrightarrow X \mid \sigma \text{ is a B-birational map}\}, \\ \mathrm{Aut}(X, \Delta) &:= \{\sigma : X \to X \mid \sigma \text{ is an automorphism and } \sigma^*\Delta = \Delta\}. \end{aligned}$$

Since $\mathrm{Bir}(X, \Delta)$ acts on $H^0(X, \mathcal{O}_X(m(K_X + \Delta)))$ for every integer m such that $m(K_X + \Delta)$ is a Cartier divisor, we can define B-pluricanonical representation $\rho_m : \mathrm{Bir}(X, \Delta) \to \mathrm{Aut}\, H^0(X, m(K_X + \Delta))$.

The following conjecture plays an important role when we reduce the problem to the irreducible case.

Conjecture 3.2 (Finiteness of B-pluricanonical representations)**.** *Let (X, Δ) be an n-dimensional (not necessarily connected) proper lc pair. Assume that $K_X + \Delta$ is nef. Then there is a positive integer m_0 such that $\rho_{m_1 m_0}(\mathrm{Bir}(X, \Delta))$ is finite for every $m_1 \in \mathbb{N}$.*

For Conjecture (3.2), it is obviously sufficient to prove it under the assumption that X is irreducible. In Theorems (3.3), (3.4), and (3.5), we will prove the conjecture for curves and surfaces.

Theorem 3.3. (cf. [12, 12.2.11]) *Let (C, Δ) be a proper lc curve. Then there is a positive integer m_0 such that $\rho_{m_1 m_0}(\mathrm{Aut}(C, \Delta))$ is finite for every $m_1 \in \mathbb{N}$.*

Proof. If the genus $g(C) \geq 2$, then it is trivial. If $g(C) = 1$ and $\Delta \neq 0$, then it is also true by [15, p.60 Application1]. If $g(C) = 1$ and $\Delta = 0$, then we put $m_0 = 12$ and theorem holds by [12, 12.2.9.1]. So we assume that C is a rational curve. If $\deg(K_C + \Delta) < 0$, then there is nothing to be proved. If $|\mathrm{Supp}\Delta| \geq 3$, then $\mathrm{Aut}(C, \Delta)$ is a finite group. So we can reduce to the case where $\Delta = \llcorner \Delta \lrcorner = \{\text{two points}\}$. In this case we can prove easily that $\rho_m(\mathrm{Aut}(C, \Delta))$ is trivial if m is even. □

Theorem 3.4. *Let (S, Δ) be a proper klt surface. Then there is a positive integer m_0 such that $\rho_{m_1 m_0}(\mathrm{Bir}(S, \Delta))$ is finite for every $m_1 \in \mathbb{N}$.*

Proof. Let $h : S' \to S$ be a terminal model, that is, (S', Δ') is terminal and $K_{S'} + \Delta' = h^*(K_S + \Delta)$ (See [12, 6.9.4]). Note that Δ' is effective and $\llcorner \Delta' \lrcorner = 0$ by the construction. The discrepancy of every exceptional divisor over (S', Δ') is positive and that of another divisor is nonpositive. The composite $\sigma' := h^{-1} \circ \sigma \circ h$ does not change discrepancies because σ' is a B-birational map. Thus σ' is a biregular morphism such that $\sigma'^* \Delta' = \Delta'$. We may assume (S, Δ) is terminal such that $\llcorner \Delta \lrcorner = 0$ and replace $\mathrm{Bir}(S, \Delta)$ with $\mathrm{Aut}(S, \Delta)$. Let $f : S' \to S$ be a finite sequence of blowing-ups whose centers are over Δ such that $K_{S'} + f_*^{-1}\Delta = f^*(K_S + \Delta) + \sum a_i E_i$ and $\mathrm{Supp}(f_*^{-1}\Delta \cup \sum E_i)$ is a simple normal crossing divisor. Let $m_0 \geq 2$ be an integer such that $m_0(K_S + \Delta)$ is a Cartier divisor. We put $D := f_*^{-1}\Delta + \sum (1/m_0) E_i$. Then we obtain

$$K_{S'} + D = f^*(K_S + \Delta) + \sum (a_i + \frac{1}{m_0}) E_i.$$

Note that D is effective and $\llcorner D \lrcorner = 0$ and $\mathrm{Supp} D$ is a simple normal crossing divisor. Since $S' - D \simeq_f S - \Delta$, $\sigma' := f^{-1} \circ \sigma \circ f$ acts on $H^0(S', \mathcal{O}_{S'}(mK_{S'} + (m-1)\ulcorner D \urcorner))$. (cf. [16, Theorem 2.1, Proposition 1.4], [4, §11.1]) Then the image of $\mathrm{Aut}(S, \Delta) \to \mathrm{Aut} H^0(S', \mathcal{O}_{S'}(mK_{S'} + (m-1)\ulcorner D \urcorner))$ is a finite group for every m by Nakamura, Ueno, Deligne, and Sakai. (See [16, Theorem 5.1] and [19, §14].) Since

$$\begin{aligned} H^0(S, \mathcal{O}_S(m_1 m_0 (K_S + \Delta))) &= H^0(S', \mathcal{O}_{S'}(m_1 m_0 (K_{S'} + D))) \\ &\subset H^0(S', \mathcal{O}_{S'}(mK_{S'} + (m-1)\ulcorner D \urcorner)) \end{aligned}$$

for every positive integer m_1 with $m = m_1 m_0$, the image of $\rho_{m_1 m_0} : \mathrm{Aut}(S, \Delta) \to \mathrm{Aut} H^0(S, m_1 m_0 (K_S + \Delta))$ is a finite group. □

Theorem 3.5. *Let (S, Δ) be a proper lc surface. Assume that $K_S + \Delta$ is nef. Then there is a positive integer m_0 such that $\rho_{m_1 m_0}(\mathrm{Bir}(S, \Delta))$ is finite for every $m_1 \in \mathbb{N}$.*

Proof. Let $f := \Phi_{|k(K_S+\Delta)|} : S \to R$ be a morphism with connected fibers for a sufficiently large and divisible integer k by the log abundance theorem.

Case (1). $\nu(S, K_S + \Delta) = 2$.
Then f is a birational morphism and $\mathrm{Bir}(S, \Delta)$ acts on R biregularly. We put $\Xi := f_*\Delta$. Then $K_S + \Delta = f^*(K_R + \Xi)$ and Ξ is $\mathrm{Bir}(S, \Delta)$-invariant. Let $h : S' \to R$ be the unique minimal resolution, so we have $K_{S'} + \Delta' = h^*(K_R + \Xi)$ and $\mathrm{Bir}(S, \Delta)$ acts on S' biregularly and Δ' is $\mathrm{Bir}(S, \Delta)$-invariant. Thus we may reduce to the case where (S, Δ) is lc, S is smooth and $\mathrm{Aut}(S, \Delta)$ as Theorem (3.4). Since $K_S + \Delta$ is big, we obtain an effective Cartier divisor D such that $am_0(K_S+\Delta) \sim \llcorner \Delta \lrcorner + D$ where a sufficiently large integer a and a sufficiently divisible integer m_0, that is, $m_0(K_S + \Delta)$ is a Cartier divisor. Observing

$$(m_1 + a)(m_0(K_S + \Delta) - \frac{1}{m_1 + a}\llcorner \Delta \lrcorner) = m_1 m_0(K_S + \Delta) + D,$$

we have

$$H^0(S, \mathcal{O}_S(m_1 m_0(K_S + \Delta))) \subset H^0(S, \mathcal{O}_S((m_1 + a)m_0(K_S + \Delta) - \llcorner \Delta \lrcorner)).$$

By using Theorem (3.4) for right hand side, we obtain the result.

Case (2). $\nu(S, K_S + \Delta) = 1$.
Let $g := S' \to S$ be a minimal resolution and $\Delta' := f^*(K_S+\Delta)-K_{S'} > 0$. We may replace (S, Δ) with (S', Δ'). By contracting (-1)-curves in the fibers we may reduce to the case where $f : S \to R$ is a $\mathbb{P}^1$-bundle or a minimal elliptic surface. When the horizontal part $\Delta^h \neq 0$, we take an irreducible component D of Δ^h. By elementary transformations we may assume D is smooth. Then

$$H^0(S, \mathcal{O}_S(m_1 m_2(K_S + \Delta))) \subset H^0(D, \mathcal{O}_D(m_1 m_2(K_D + \mathrm{Diff}(\Delta - D)))).$$

So we have the result by Theorem (3.3). Next we may assume $\Delta^h = 0$. When $f : S \to R$ is a $\mathbb{P}^1$-bundle, $\llcorner \Delta \lrcorner = \sum f^* p_i$ for some $p_i \in R$. When $f : S \to R$ is a minimal elliptic surface, we have $K_S \sim_{\mathbb{Q},f} 0$ and $\Delta \equiv_f 0$. Then $\Delta = \sum a_i f^* p_i$ for some $p_i \in R$ and positive rational numbers a_i (See [2, VIII.3, VIII.4]). $\mathrm{Bir}(S, \Delta)$ acts on $f(\llcorner \Delta \lrcorner)$. We define $B := \sum_{p_i \in f(\llcorner \Delta \lrcorner)} p_i$. Let H be an ample Cartier divisor on R such that $m_0(K_S + \Delta) \sim f^*H$. Then we have $bH \sim B + D$ for some effective divisor D and some sufficiently large integer b. Then

$bm_0(K_S+\Delta) \sim f^*(B+D)$. We put $A := f^*B$. Observing

$$(m_1+b)(m_0(K_S+\Delta) - \frac{1}{m_1+b}A) = m_1m_0(K_S+\Delta) + f^*D,$$

we have

$$H^0(S, \mathcal{O}_S(m_1m_0(K_S+\Delta))) \subset H^0(S, \mathcal{O}_S((m_1+b)m_0(K_S+\Delta) - A)).$$

By using Theorem (3.4) for right hand side we obtain a result.

Case (3). $\nu(S, K_S+\Delta) = 0$.
$\mathrm{Bir}(S,\Delta)$ acts on $H^0(S, m_1m_2(K_S+\Delta))$ trivially. We can prove it by using Lemma (4.7) below. First we replace (S,Δ) with its $\mathbb{Q}$-factorial dlt model (See Lemma-Definition (1.4)). So we may assume that (S,Δ) is dlt. By Proposition (4.3), we can take a preadmissible section s (cf. Definition (4.1)). Then $g^*s = s$ for every $g \in \mathrm{Bir}(S,\Delta)$ by Lemma (4.7). Since $\nu(S, K_S+\Delta) = 0$, we have the result. □

4. The abundance theorem for slc threefolds

We introduce the notion of preadmissible and admissible sections for the inductive proof of the abundance conjecture for slc n-folds.

Definition 4.1. Let (X,Δ) be a proper sdlt n-fold and m a sufficiently large and divisible integer. We define admissible and preadmissible sections inductively on dimension.

- $s \in H^0(X, \mathcal{O}_X(m(K_X+\Delta)))$ is *preadmissible* if the restriction $s|_{(\coprod_i \llcorner \Theta_i \lrcorner)} \in H^0(\coprod_i \llcorner \Theta_i \lrcorner, \mathcal{O}_{(\coprod_i \llcorner \Theta_i \lrcorner)}(m(K_{X'}+\Theta)|_{(\coprod_i \llcorner \Theta_i \lrcorner)}))$ is admissible.
- $s \in H^0(X, \mathcal{O}_X(m(K_X+\Delta)))$ is *admissible* if s is preadmissible and $g^*(s|_{X_j}) = s|_{X_i}$ for every B-birational map $g : (X_i,\Theta_i) \dashrightarrow (X_j,\Theta_j)$ for every i,j.

Note that if $s \in H^0(X, \mathcal{O}_X(m(K_X+\Delta)))$ is admissible, then $s|_{X_i}$ is $\mathrm{Bir}(X_i,\Theta_i)$-invariant for every i.

We define linear subspaces of $H^0(X, \mathcal{O}_X(m(K_X+\Delta)))$ as follows;

$$\mathrm{PA}(X, m(K_X+\Delta)) := \{s \text{ is preadmissible}\},$$

and

$$\mathrm{A}(X, m(K_X+\Delta)) := \{s \text{ is admissible}\}.$$

When $\dim X = 1$, the preadmissible section is a slight generalization of the *normalized* section (See [12, 12.2.9]). But in the higher dimensional case, the admissible and preadmissible sections behave much better in the inductive proof of the abundance conjecture for slc n-folds.

Lemma 4.2. *Let (X, Δ) be a proper $\mathbb{Q}$-factorial dlt n-fold, and $K_X+\Delta$ nef and $S = \llcorner \Delta \lrcorner \neq 0$. Assume that $f = \Phi_{|k(K_X+\Delta)|} : X \to R$ is a proper surjective morphism onto a normal variety R with connected fibers for a sufficiently large and divisible integer k and $f(\llcorner \Delta \lrcorner) \subsetneq R$. If there exist sections $\{s_i\}_{i=1}^p \subset H^0(S, \mathcal{O}_S(m(K_X + \Delta)|_S))$ without common zeros, then there exist sections $\{u_i\}_{i=1}^l \subset H^0(X, \mathcal{O}_X(rm(K_X + \Delta)))$ for some integer r such that*

(1) $u_i|_S = \begin{cases} s_i^r & \text{for } 1 \leq i \leq p \\ 0 & \text{for } p < i \leq l \end{cases}$

(2) $\{u_i\}_{i=1}^l$ *have no common zeros.*

Proof. There is an ample $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor H on R such that $K_X + \Delta \sim_{\mathbb{Q}} f^*H$. We consider the following commutative diagram

$$\begin{array}{ccc} H^0(X, \mathcal{O}_X(rm(K_X + \Delta))) & \longrightarrow & H^0(S, \mathcal{O}_S(rm(K_X + \Delta)|_S)) \\ \uparrow \cong & & \uparrow \cong \\ H^0(R, \mathcal{O}_R(rmH)) & \longrightarrow & H^0(T, \mathcal{O}_T(rmH|_T)). \end{array}$$

Let $T := f(S)$ and I_T defining ideal of T. By Lemma (2.5), $(f|_S)_*\mathcal{O}_S = \mathcal{O}_T$. So the vertical arrows are isomorphisms. Since H is ample, the horizontal arrows are surjective and $\mathcal{O}_R(rmH) \otimes I_T$ is generated by global sections for a sufficiently large integer r. So we can get sections $\{u_i\}_{i=1}^l$ with required properties. □

The next proposition is the main part of the proof of the abundance theorem for slc n-folds.

Proposition 4.3. *Let (X, Δ) be a pure n-dimensional (not necessarily connected) proper $\mathbb{Q}$-factorial dlt pair with $n \leq 3$. Let m be a sufficiently large and divisible integer. Assume that*

(1) *$K_X + \Delta$ is nef,*

(2) *$\mathrm{A}(\llcorner \Delta \lrcorner, m(K_X + \Delta)|_{\llcorner \Delta \lrcorner})$ generates $\mathcal{O}_{\llcorner \Delta \lrcorner}(m(K_{\llcorner \Delta \lrcorner} + \mathrm{Diff}(\Delta - \llcorner \Delta \lrcorner)))$.*

Then

$$\mathrm{PA}(X, m(K_X + \Delta)) \to \mathrm{A}(\llcorner \Delta \lrcorner, m(K_X + \Delta)|_{\llcorner \Delta \lrcorner})$$

is surjective, and $\mathrm{PA}(X, m(K_X + \Delta))$ generates $\mathcal{O}_X(m(K_X + \Delta))$.

Proof. It is sufficient to prove this proposition for each connected component. So we can assume that (X, Δ) is a proper $\mathbb{Q}$-factorial irreducible dlt n-fold.

Apply the log abundance theorem. We get $f := \Phi_{|k(K_X+\Delta)|} : X \to R$ for a sufficiently large and divisible integer k. If $\llcorner \Delta \lrcorner = 0$, then the

proposition is trivial. Thus we may assume $\llcorner\Delta\lrcorner \neq 0$. We have the following possibilities by Proposition (2.1).

(1) The numerical Kodaira dimension $\nu(X, K_X + \Delta) = 0$ and $\llcorner\Delta\lrcorner$ is connected.
(2) $\nu(X, K_X+\Delta) \geq 1$ and $f^{-1}(r)\cap\llcorner\Delta\lrcorner$ is connected for every $r \in R$ and $f(\llcorner\Delta\lrcorner) = R$.
(3) $\nu(X, K_X+\Delta) \geq 1$ and $f^{-1}(r)\cap\llcorner\Delta\lrcorner$ is connected for every $r \in R$ and $f(\llcorner\Delta\lrcorner) \subsetneq R$.
(4) $f^{-1}(r) \cap \llcorner\Delta\lrcorner$ is not connected for some $r \in R$.

Case (1). Consider the following exact sequence.

$$\begin{aligned} 0 \to H^0(X, \mathcal{O}_X(m(K_X + \Delta) - \llcorner\Delta\lrcorner)) &\to H^0(X, \mathcal{O}_X(m(K_X + \Delta))) \\ \to H^0(\llcorner\Delta\lrcorner, \mathcal{O}_{\llcorner\Delta\lrcorner}(m(K_X + \Delta)|_{\llcorner\Delta\lrcorner})) &\to \cdots \end{aligned}$$

Since $\nu(X, K_X + \Delta) = 0$, $H^0(X, \mathcal{O}_X(m(K_X + \Delta) - \llcorner\Delta\lrcorner)) = 0$ and second and third terms are one dimensional. Thus we get the result.

Case (2). We construct a morphism $\varphi : \llcorner\Delta\lrcorner \to C$ by the linear system corresponding to $\mathrm{A}(\llcorner\Delta\lrcorner, m(K_X + \Delta)|_{\llcorner\Delta\lrcorner})$. Since every curve in any fiber of $f|_{\llcorner\Delta\lrcorner}$ goes to a point by φ, then there exists a morphism $\psi : R \to C$ such that $\psi \circ (f|_{\llcorner\Delta\lrcorner}) = \varphi$. For $s \in \mathrm{A}(\llcorner\Delta\lrcorner, m(K_X + \Delta)|_{\llcorner\Delta\lrcorner})$, there exists a section t on C such that $s = \varphi^* t$. Thus we obtain $u := f^*\psi^* t \in \mathrm{PA}(X, m(K_X + \Delta))$ such that $u|_{\llcorner\Delta\lrcorner} = s$. Note that if we write $K_X+\Delta \sim_{\mathbb{Q}} f^*H$ as in Lemma (4.2), we have $H^0(X, \mathcal{O}_X(m(K_X + \Delta))) \simeq H^0(R, \mathcal{O}_R(mH)) \simeq H^0(\llcorner\Delta\lrcorner, \mathcal{O}_{\llcorner\Delta\lrcorner}(m(K_X + \Delta)|_{\llcorner\Delta\lrcorner}))$. Since $\mathrm{A}(\llcorner\Delta\lrcorner, m(K_X + \Delta)|_{\llcorner\Delta\lrcorner})$ generates $\mathcal{O}_{\llcorner\Delta\lrcorner}(m(K_{\llcorner\Delta\lrcorner} + \mathrm{Diff}(\Delta - \llcorner\Delta\lrcorner)))$, $\mathrm{PA}(X, m(K_X + \Delta))$ generates $\mathcal{O}_X(m(K_X + \Delta))$.

Case (3). We put $T := f(\llcorner\Delta\lrcorner) \subsetneq R$. By Lemma (2.5) we obtain $\mathcal{O}_T = f_*\mathcal{O}_{\llcorner\Delta\lrcorner}$. Then T is seminormal (See [1, Proposition 4.5]). As in Case (2), we construct $\varphi : \llcorner\Delta\lrcorner \to C$ by $\mathrm{A}(\llcorner\Delta\lrcorner, m(K_X+\Delta)|_{\llcorner\Delta\lrcorner})$ and get $\psi : T \to C$. By Lemma (4.2), we can pull back $s \in \mathrm{A}(\llcorner\Delta\lrcorner, m(K_X+\Delta)|_{\llcorner\Delta\lrcorner})$ to $u \in \mathrm{PA}(X, m(K_X + \Delta))$ if m is a sufficiently large and divisible integer. By Lemma (4.2) we can also check that $\mathrm{PA}(X, m(K_X + \Delta))$ generates $\mathcal{O}_X(m(K_X + \Delta))$ if m is a sufficiently large and divisible integer.

Case (4). In this case, X is generically a $\mathbb{P}^1$-bundle over V by Proposition (2.1). Let Δ^h be the horizontal part and Δ^v the vertical part of $\llcorner\Delta\lrcorner$. If $s \in \mathrm{A}(\llcorner\Delta\lrcorner, m(K_X + \Delta)|_{\llcorner\Delta\lrcorner})$, then $s|_{\Delta^h}$ is admissible. So it descends to a section on (V, P). Then there exists a section $u \in H^0(X, \mathcal{O}_X(m(K_X+\Delta)))$ given by the isomorphism $H^0(X, \mathcal{O}_X(m(K_X+\Delta))) \simeq H^0(V, \mathcal{O}_V(m(K_V + P)))$. If m is even, then $u|_{\Delta^h} = s|_{\Delta^h}$. It is because there is a small analytic open set U in V such that $v^{-1}(U)$ is

biholomorphic to $\mathbb{P}^1 \times U$ and $v|_{v^{-1}(U)} : \mathbb{P}^1 \times U \to U$ is a second projection (cf. [12, 12.3.4]). Since $\Delta^h \cap \Delta^v \neq \emptyset$ by Proposition (2.1) and $u|_{\Delta^h \cap \Delta^v}$ is admissible, $u|_{\llcorner \Delta \lrcorner} = s$. Since there are admissible sections on (V, P) without common zeros, we can prove the latter part. □

Remark 4.4. In Proposition (4.3) the assumption $n \leq 3$ is used for the log MMP and the log abundance theorem which are used in Proposition (2.1) too.

Lemma 4.5. *If* $\dim X \leq 2$ *in Proposition (4.3), then we can replace* $\mathrm{PA}(X, m(K_X + \Delta))$ *with* $\mathrm{A}(X, m(K_X + \Delta))$.

Proof. For every $s \in \mathrm{A}(\llcorner \Delta \lrcorner, m(K_X + \Delta)|_{\llcorner \Delta \lrcorner})$, we can take a section $s' \in \mathrm{PA}(X, m(K_X + \Delta))$ such that $s'|_{\llcorner \Delta \lrcorner} = s$ by Proposition (4.3). If we put $t := 1/|G| \sum_{g \in G} g^* s' \in \mathrm{A}(X, m(K_X + \Delta))$, then $t|_{\llcorner \Delta \lrcorner} = s$ by Lemma (4.7) below. Therefore

$$\mathrm{A}(X, m(K_X + \Delta)) \to \mathrm{A}(\llcorner \Delta \lrcorner, m(K_X + \Delta)|_{\llcorner \Delta \lrcorner})$$

is surjective. Let k be a sufficiently large and divisible integer. Then $\mathrm{Bir}(X, \Delta)$ acts on

$$\bigoplus_{l \geq 0} \mathrm{PA}(X, lk(K_X + \Delta)).$$

Let $N := |\rho_k(\mathrm{Bir}(X, \Delta))| < \infty$ by Section 3 and let σ_i be the i-th elementary symmetric polynomial. We obtain:

$$\{s = 0\} \supset \bigcap_{j=1}^{N} \{g_j^* s = 0\} = \bigcap_{i=1}^{N} \{\sigma_i(g_1^* s, \cdots, g_N^* s) = 0\}.$$

If $s \in \mathrm{PA}(X, k(K_X + \Delta))$, then $\sigma_i^{N-i}(g_1^* s, \cdots, g_N^* s) \in \mathrm{A}(X, Nk(K_X + \Delta))$. $\mathrm{PA}(X, k(K_X + \Delta))$ generates $\mathcal{O}_X(X, k(K_X + \Delta))$ by the Proposition (4.3). Thus we can prove that $\mathrm{A}(X, Nk(K_X + \Delta))$ generates $\mathcal{O}_X(Nk(K_X + \Delta))$ by using Proposition (4.3) and Lemma (4.7). □

In order to prove Lemma (4.7) we need the next definition.

Definition 4.6. Assume that X is nonsingular and $\mathrm{Supp}\Delta$ is a simple normal crossing divisor and $\Delta = \sum_i d_i \Delta_i$ is a $\mathbb{Q}$-divisor such that $d_i \leq 1$ for every i. In this case we say that (X, Δ) is *B-smooth*.

Let (X, Δ) be dlt or B-smooth. A subvariety W of X is said to be a *center of log canonical singularities* if there is a proper birational morphism from a normal variety $\mu : Y \to X$ and a prime divisor E on Y with the discrepancy $a(E, X, \Delta) = -1$ such that $\mu(E) = W$ (cf. [8, Definition 1.3]).

Let (X, Δ) be dlt or B-smooth. We write $\Delta = \sum_i d_i \Delta_i$ such that Δ_i are distinct prime divisors. Then the *B-part* of Δ is defined by $\Delta^B := \sum_{d_i=1} \Delta_i$.

If (X, Δ) is dlt or B-smooth, then a center of log canonical singularities is an irreducible component of an intersection of some B-part divisors. (See the Divisorial Log Terminal Theorem of [18] and [13, Section 2.3].) When we consider a center of log canonical singularities W, we always consider the pair (W, Ξ) such that $K_W + \Xi = (K_X + \Delta)|_W$, where Ξ is defined by using the adjunction repeatedly. Note that if (X, Δ) is dlt (resp. B-smooth), then (W, Ξ) is dlt (resp. B-smooth) by the adjunction.

If (X, Δ) is dlt pair or B-smooth and W is a center of log canonical singularities, then we write $W \Subset X$.

Lemma 4.7. *Let (X, Δ) be a pure n-dimensional (not necessarily connected) proper dlt pair with $K_X + \Delta$ nef and let m be a sufficiently large and divisible integer. We write $G = \rho_m(\mathrm{Bir}(X, \Delta))$. If $s \in \mathrm{PA}(X, m(K_X + \Delta))$, then $g^*s \in \mathrm{PA}(X, m(K_X + \Delta))$ for every $g \in G$ and $g^*s|_{\llcorner \Delta \lrcorner} = s|_{\llcorner \Delta \lrcorner}$.*

In particular if $|G|$ is finite, then

$$\sum_{g \in G} g^*s \in \mathrm{A}(X, m(K_X + \Delta)), \ \prod_{g \in G} g^*s \in \mathrm{A}(S, m|G|(K_X + \Delta)),$$

and

$$\prod_{g \in G} g^*s|_{\llcorner \Delta \lrcorner} = (s|_{\llcorner \Delta \lrcorner})^{|G|}.$$

Proof. In this proof we will omit the restriction symbols like $|_{\Theta^B}$ when there is no confusion. Let $\alpha, \beta : (Y, \Theta) \to (X, \Delta)$ be a common log resolution of a B-birational map $\sigma : X \dashrightarrow X$ such that $\alpha = \sigma \circ \beta$ and $\rho_m(\sigma) = g$. Since $\llcorner \Delta \lrcorner$ is seminormal and $\Theta^B \to \llcorner \Delta \lrcorner$ has connected fibers by connectedness lemma (1.3), we have $\alpha_* \mathcal{O}_{\Theta^B} = \mathcal{O}_{\llcorner \Delta \lrcorner}$ and $\beta_* \mathcal{O}_{\Theta^B} = \mathcal{O}_{\llcorner \Delta \lrcorner}$. Then α^* and β^* induce the isomorphisms between $H^0(\llcorner \Delta \lrcorner, \mathcal{O}_{\llcorner \Delta \lrcorner}(m(K_X + \Delta)|_{\llcorner \Delta \lrcorner}))$ and $H^0(\Theta^B, \mathcal{O}_{\Theta^B}(m(K_Y + \Theta)|_{\Theta^B}))$. So in order to prove $g^*s|_{\llcorner \Delta \lrcorner} = s|_{\llcorner \Delta \lrcorner}$ it is sufficient to check that $\alpha^*(s|_{\llcorner \Delta \lrcorner}) = \beta^*(s|_{\llcorner \Delta \lrcorner})$ in $H^0(\Theta^B, \mathcal{O}_{\Theta^B}(m(K_Y + \Theta)|_{\Theta^B}))$ for some common log resolution (Y, Θ).

For this purpose we prove the next two claims.

Claim (A_n). Let (T, Θ) and (S, Ξ) be n-dimensional B-smooth pairs and $h : S \to T$ a B-birational morphism. If $W \Subset T$, then there is a $W' \Subset S$ such that $W' \to W$ is a B-birational morphism.

Claim (B_n). Let (T, Θ) and (S, Ξ) be n-dimensional B-smooth pairs and $h : S \to T$ a B-birational morphism. Assume that $W \Subset S$.

If $W \to h(W)$ is not B-birational, then there is a $W' \Subset W$ such that $W' \to h(W)$ is a B-birational morphism and that the inclusion $W' \to W$ induces the isomorphism $H^0(W, \mathcal{O}_W(m(K_S + \Xi)|_W)) \simeq H^0(W', \mathcal{O}_{W'}(m(K_S + \Xi)|_{W'}))$.

Proof of Claims. We prove Claim (A_n) and Claim (B_n) by induction on n. If $n = 1$, then (A_1), (B_1) are trivial.

First we treat (A_n). If W is a divisor, then we can take the proper transform of W as W'. Otherwise, take a divisor $V \Subset T$ such that $W \Subset V$. We define $U \Subset S$ as the proper transform of V. By using the (A_{n-1}) to the B-birational morphism $U \to V$, we obtain $W' \Subset U$ such that $W' \to W$ is a B-birational morphism. Thus we have (A_n).

Next we treat (B_n) by using (A_l) with $l < n$. Let $u : (S', \Xi') \to S$ be the blowing-up of S with center W. If we prove (B_n) for the pair S' and the exceptional divisor $E \Subset S'$, then we can prove (B_n) for S. It is because $G \Subset E$ implies $u(G)$ is the required center of log canonical singularities. So we may assume that W is a divisor. By [13, 2.45] we have a sequence of blowing-ups $T_0 \to T_1 \to \cdots \to T_k = T$ whose centers are the centers associated to the valuation W such that the rational map $S \dashrightarrow T_0$ is an isomorphism at the generic point of W. We can replace (S, Ξ) with the elimination of indeterminacy (S', Ξ') since W and its proper transform are B-birational. So we can assume that $f_0 : S \to T_0$ is a B-birational morphism. We can apply (A_{n-1}) to the B-birational morphism $f_0 : W \to f_0(W)$. So we only have to prove (B_n) to the B-birational morphism $T_0 \to T$. So we may assume that $(S, \Xi) = (T_0, \Theta_0)$. We use the induction on the number k of the blowing-ups. When $k = 1$, we take a divisor $D \Subset T_1$ such that $h(W) \Subset D$ and its proper transform $D' \Subset T_0$. By applying (B_{n-1}) to $D' \cap W \Subset D'$ and $h(D' \cap W) = h(W) \Subset D$, we have the result. Next we apply the induction hypothesis to $a : T_0 \to T_{k-1}$. We have $U \Subset W$ such that $a : U \to a(W)$ is a B-birational morphism. Since $b : T_{k-1} \to T_k$ is one blowing-up whose center is the center associated to the valuation W, we can take a divisor V on T_{k-1} such that $a(U) = a(W) \Subset V$ and $b : V \to b(V)$ is B-birational. By using (B_{n-1}) to $b : V \to b(V)$ we have a $V' \Subset a(U)$ such that $V' \to (b \circ a)(W) = h(W)$ is B-birational. By using (A_l) with $l < n$ to $U \to a(U)$ we obtain a $W' \Subset U \Subset W$ such that $W' \to V'$ is B-birational. So $W' \to h(W)$ is B-birational. By the construction of W', the inclusion $W' \to W$ induces the isomorphism $H^0(W, \mathcal{O}_W(m(K_S + \Xi)|_W)) \simeq H^0(W', \mathcal{O}_{W'}(m(K_S + \Xi)|_{W'}))$. This proves (B_n). □

Proof of Lemma (4.7) continued. We return to the proof of the lemma. Let $f : (X', \Delta') \to (X, \Delta)$ be Szabó's resolution such that $K_{X'} + \Delta' =$

$f^*(K_X+\Delta)$, that is, a log resolution whose discrepancy is greater than -1 (See the Divisorial Log Terminal Theorem of [18]). Let $\alpha = \coprod \alpha_i$, $\beta = \coprod \beta_i : (Y,\Theta) = \coprod (Y_i,\Theta_i) \to (X,\Delta)$ be a common log resolution of a B-birational map $\sigma : X \dashrightarrow X$ passing through (X',Δ') such that $\alpha = \sigma \circ \beta$ and $\rho_m(\sigma) = g$. We write $\alpha', \beta' : Y \to X'$ such that $\alpha = f \circ \alpha', \beta = f \circ \beta'$. We take an irreducible divisor $E \subset \Theta^B$. Apply (B_n) to the $\alpha_i' : Y_i \to \alpha_i'(Y_i)$ such that $E \Subset Y_i$. Then we obtain an $E' \Subset E$ or $E' = E$ such that $E' \to \alpha'(E)$ is B-birational. Apply (B_n) to the $\beta_i' : Y_i \to \beta_i'(Y_i)$. Then we obtain an $E'' \Subset E'$ or $E'' = E'$ such that $E'' \to \beta'(E')$ is B-birational. By repeating the above construction we obtain an $F \Subset E$ or $F = E$ such that $\alpha' : F \to \alpha'(F)$ and $\beta' : F \to \beta'(F)$ are B-birational. The morphism $f : \alpha'(F) \to \alpha(F)$ and $f : \beta'(F) \to \beta(F)$ are B-birational since f is Szabó's resolution. Then $\alpha : F \to \alpha(F)$ and $\beta : F \to \beta(F)$ are B-birational. Since $s \in \mathrm{PA}(X, m(K_X+\Delta))$, $\alpha^*(s|_{\alpha(F)}) = \beta^*(s|_{\beta(F)})$ on F. Since $H^0(E, m(K_Y+\Theta)|_E) \simeq H^0(F, m(K_Y+\Theta)|_F)$ by (B_n), we have $\alpha^* s = \beta^* s$ on E. Since E is an arbitrary irreducible component of Θ^B, we have $\alpha^*(s|_{\llcorner\Delta\lrcorner}) = \beta^*(s|_{\llcorner\Delta\lrcorner})$. Thus $g^* s|_{\llcorner\Delta\lrcorner} = s|_{\llcorner\Delta\lrcorner}$.

The latter part is trivial. □

Corollary 4.8. (cf. [12, 12.1.1, 8.5]) *Let (X,Δ) be a proper slc n-fold such that $K_X+\Delta$ is nef, where $n \le 2$. Then $|m(K_X+\Delta)|$ is free for a sufficiently large and divisible integer m. In particular, if (X,Δ) is proper sdlt, then $\mathrm{A}(X, m(K_X+\Delta))$ generates $\mathcal{O}_X(m(K_X+\Delta))$.*

Proof. We prove this corollary inductively on $\dim X$ (See Proposition (4.3)). We take the normalization of (X,Δ) and take a $\mathbb{Q}$-factorial dlt model for each irreducible component by Lemma-Definition (1.4). We write it $(Y,\Theta) = \coprod (Y_i,\Theta_i)$. By Proposition (4.3) and Lemma (4.5), $\mathrm{A}(Y, m(K_Y+\Theta))$ generates $\mathcal{O}_Y(m(K_Y+\Theta))$. Since X is S_2 and normal crossing in codimension 1, sections in $\mathrm{A}(Y, m(K_Y+\Theta))$ descend to sections on (X,Δ). Thus we have the result. When (X,Δ) is sdlt, the above construction shows that $\mathrm{A}(X, m(K_X+\Delta))$ generates $\mathcal{O}_X(m(K_X+\Delta))$. □

The next corollary is the main theorem of this paper (See Theorem (0.1)), which can be proved by the same way used in the proof of Corollary (4.8) (See also Theorem (5.1)).

Corollary 4.9. *Let (X,Δ) be a proper slc threefold with $K_X+\Delta$ nef. Then $K_X+\Delta$ is semi-ample.*

5. Appendix

We can reformulate Corollaries (4.8) and (4.9) in the following form if we list all the necessary results (e.g. log MMP) as the assumption.

Theorem 5.1. *Assume the log MMP for dimension n.*

(1) *If the abundance conjecture holds for lc n-folds and if the finiteness of* B*-pluricanonical representations (See Conjecture (3.2)) is true for dimension n, then the abundance conjecture is true for sdlt and slc n-folds, that is, Corollary (4.8) holds in dimension n.*

(2) *If the abundance conjecture holds for klt n-folds and slc $(n-1)$-folds, then the abundance conjecture is true for lc n-folds.*

Proof. First for (1). See Remarks (2.2), (4.4) . Next for (2). By using the same proof in [10, Section7], we can prove (2). □

We list the following two results for the readers' convenience.

Theorem 5.2 (Relative log abundance theorem)**.** *Let (X, Δ) be lc and $\dim X \leq 3$. Let $f : X \to S$ be a proper surjective morphism onto a variety S. Assume that $K_X + \Delta$ is f-nef. Then $K_X + \Delta$ is f-semi-ample.*

Proof. If $\dim S = 0$, then this is nothing but the log abundance theorem (See [3], [12, Chapter 11] and [10]). So we may assume $\dim S \geq 1$. If (X, Δ) is klt, then it is true (See [9, 6-1-11] and [6]). When (X, Δ) is lc, we can use the arguments in [10, Section7] in the relative setting. (See also [12, 8.5].) □

Corollary 5.3 (Threefold log canonical flips)**.** ([12, 8.1]) *Threefold log canonical flips exist.*

Proof. Let (X, Δ) be an lc pair and $f : (X, \Delta) \to S$ a flipping contraction. We take a $\mathbb{Q}$-factorial dlt model (X', Δ') (See Lemma-Definition (1.4)) and run the log MMP over S. Then we obtain a relative minimal model (X'', Δ'') over S. By using Theorem (5.2) we have a relative canonical model (X^+, Δ^+). It is easy to check that (X^+, Δ^+) is the flip of f. □

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