NORMALIZED TAUTOLOGICAL DIVISORS OF SEMI-STABLE VECTOR BUNDLES

NOBORU NAKAYAMA

1. Introduction

Let $\mathcal{E}$ be a vector bundle, i.e., a locally free sheaf of finite rank, over a non-singular complex projective variety X. Let $\pi\colon \mathbb{P} = \mathbb{P}_X(\mathcal{E}) \to X$ be the associated $\mathbb{P}^{r-1}$-bundle, where $r = \operatorname{rank}\mathcal{E}$, and let $H = H_{\mathcal{E}}$ be the *tautological divisor*, that is a divisor of $\mathbb{P}$ with the property $\pi_*\mathcal{O}_{\mathbb{P}}(H) \simeq \mathcal{E}$. We define the *normalized tautological divisor* $\Lambda = \Lambda_{\mathcal{E}}$ by

$$\Lambda_{\mathcal{E}} := H_{\mathcal{E}} - \frac{1}{r}\pi^* \det \mathcal{E}.$$

Note that Λ is only a $\mathbb{Q}$-divisor and $r\Lambda$ is linearly equivalent to the relative anti-canonical divisor $-K_{\mathbb{P}/X}$. The *averaged first Chern class* $\mu(\mathcal{F})$ of a non-zero torsion free coherent sheaf $\mathcal{F}$ of X is defined by

$$\mu(\mathcal{F}) = \frac{1}{\operatorname{rank}\mathcal{F}} c_1(\mathcal{F}).$$

For an ample divisor A, we set $\mu_A(\mathcal{F}) = \mu(\mathcal{F}) \cdot A^{d-1}$, where $d = \dim X$. A *reflexive sheaf* $\mathcal{F}$ is a torsion free coherent sheaf isomorphic to its double dual $\mathcal{F}^{\vee\vee} = (\mathcal{F}^{\vee})^{\vee}$, where $\mathcal{F}^{\vee} = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$. A reflexive sheaf $\mathcal{F}$ is called *A-stable* or *A-semi-stable* according as the inequality $\mu_A(\mathcal{G}) < \mu_A(\mathcal{F})$ or $\mu_A(\mathcal{G}) \leq \mu_A(\mathcal{F})$ holds, for any coherent subsheaf $0 \subsetneq \mathcal{G} \subsetneq \mathcal{F}$.

Suppose that $d = \dim X = 1$. Then $\mathcal{E}$ is semi-stable if and only if Λ is *nef*, i.e., $\Lambda \cdot C \geq 0$ for any irreducible curve C of $\mathbb{P}$ (cf. [NS], [M2, 3.1]). Further in this case, every effective divisor of $\mathbb{P}$ is nef. If $\mathcal{E}$ is not semi-stable, then Λ is a *big* divisor, i.e., $\kappa(\Lambda, \mathbb{P}) = \dim \mathbb{P}$. Therefore, if $\dim X = 1$, Λ is *pseudo-effective*, i.e., the numerical equivalence class of Λ in the Néron–Severi group $N^1(\mathbb{P}) = \mathrm{NS}(\mathbb{P}) \otimes \mathbb{R}$ is a limit of those of effective $\mathbb{Q}$-divisors.

Next, we consider higher dimensional case. The normalized tautological divisor is not pseudo-effective in general.

Example. Let T_X be the tangent bundle of the projective plane $X = \mathbb{P}^2$. Then $\mathbb{P} = \mathbb{P}_X(T_X)$ is a hypersurface of $\mathbb{P}^2 \times \mathbb{P}^2$ and there are two $\mathbb{P}^1$-bundle structures $p_1, p_2 \colon \mathbb{P} \to \mathbb{P}^2$. We consider p_1 as the associated $\mathbb{P}^1$-bundle structure of T_X. The tautological divisor H of T_X is linearly equivalent to $p_1^*\ell + p_2^*\ell$, where $\ell \subset \mathbb{P}^2$ is a line. Thus the normalized tautological divisor is written by $\Lambda = p_2^*\ell - (1/2)p_1^*\ell$. Since $\Lambda \cdot (p_2^*\ell)^2 = -1/2 < 0$, it is not pseudo-effective.

The following result may be well-known. It is derived from the study of stable vector bundles and Hermitian–Einstein metrics by Narasimhan and Seshadri [NS], Mehta and Ramanathan [MR1], [MR2], Donaldson [D], Uhlenbeck and Yau [UY1], [UY2], and Bando and Siu [BS].

Theorem A. *Let $\mathcal{E}$ be a reflexive sheaf of rank r on a non-singular complex projective variety X of dimension d. Then the following three conditions are equivalent*:

(1) *$\mathcal{E}$ is locally free and the normalized tautological divisor $\Lambda_{\mathcal{E}}$ is nef.*

(2) *$\mathcal{E}$ is A-semi-stable and*

$$(c_2(\mathcal{E}) - \frac{r-1}{2r}c_1^2(\mathcal{E})) \cdot A^{d-2} = 0$$

for an ample divisor A.

(3) *$\mathcal{E}$ is locally free and there is a filtration of subbundles*

$$0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_l = \mathcal{E},$$

where $\mathcal{E}_i/\mathcal{E}_{i-1}$ are projectively flat and the averaged first Chern classes $\mu(\mathcal{E}_i/\mathcal{E}_{i-1})$ are numerically equivalent to $\mu(\mathcal{E})$ for any i.

Here, a vector bundle $\mathcal{E}$ is called *projectively flat* if it admits a projectively flat Hermitian metric h, i.e., its curvature tensor Θ_h is written by

$$\Theta_h = \omega \cdot \mathrm{id}_{\mathcal{E}}$$

for a 2-form ω, as an $\mathcal{E}nd(\mathcal{E})$-valued 2-form.

We next consider a weaker condition: Λ is pseudo-effective. We have the following result in the case $\operatorname{rank}\mathcal{E} = 2$.

Theorem B. *Let $\mathcal{E}$ be an A-semi-stable vector bundle of rank two on a non-singular complex projective variety X of dimension $d \geq 2$, for an ample divisor A. Suppose that the normalized tautological divisor $\Lambda_{\mathcal{E}}$ is pseudo-effective. Then $\Lambda_{\mathcal{E}}$ is nef except for the following three cases*:

(A) *There exist divisors M_1, M_2 such that $M_1 \cdot A^{d-1} = M_2 \cdot A^{d-1}$ and*

$$\mathcal{E} \simeq \mathcal{O}_X(M_1) \oplus \mathcal{O}_X(M_2).$$

(B) *There exist an unramified double covering $\tau : Y \to X$ and a divisor M of Y such that*

$$\mathcal{E} \simeq \tau_* \mathcal{O}_Y(M).$$

(C) *There is an exact sequence*

$$0 \to \mathcal{O}_X(L_1) \to \mathcal{E} \to I\mathcal{O}_X(L_2) \to 0,$$

where I is an ideal sheaf of $\operatorname{codim} \operatorname{Supp} \mathcal{O}_X/I = 2$, *and the divisor L_1 is numerically equivalent to L_2.*

Here $\Lambda = \Lambda_{\mathcal{E}}$ is pseudo-effective in these exceptional cases. Further Λ is nef if and only if M_1 is numerically equivalent to M_2 in the case (A), and M is numerically equivalent to $\sigma^* M$, where $\sigma : Y \to Y$ is the non-trivial involution over X in the case (B), respectively. In the case (C), Λ is not nef. The idea of our proof of Theorem B is to consider the σ-decomposition of a pseudo-effective divisor developed in [N], which is a generalization of the Zariski-decomposition over a surface.

In section 2, we recall the relation between projectively flatness and semi-stability and give a proof of Theorem A. After discussing three exceptional cases above, we shall prove Theorem B in section 3. As an application, in section 4, we shall show some results on vector bundles of rank two over surfaces. For example, we can show: the tautological divisor of the tangent bundle of any K3 surface is not pseudo-effective.

2. Projectively flatness and semi-stability

Let $\mathrm{CH}^i(X)$ be the Chow group of algebraic cycles of codimension i on a non-singular projective variety X, modulo rational equivalence. The Chern classes $c_i(\mathcal{E})$ of a vector bundle $\mathcal{E}$ of X are usually defined as elements of $\mathrm{CH}^i(X)$. However, here we consider

$c_i(\mathcal{E})$ only as an element of $N^i(X) \subset H^{2i}(X, \mathbb{R})$, that is the vector subspace generated by algebraic cycles of codimension i. By the Poincaré duality, the vector subspace $N_i(X) \subset H_{2i}(X, \mathbb{R})$ generated by algebraic cycles of dimension i is isomorphic to $N^{d-i}(X)$, where $d = \dim X$. The cup product of $H^\bullet(X, \mathbb{R})$ induces the intersection homomorphism

$$N^i(X) \times N^j(X) \to N^{i+j}(X),$$

which is compatible with the ring structure of the Chow ring $\mathrm{CH}^\bullet(X)$. By the natural identification $N^d(X) \simeq \mathbb{R}$, two vector spaces $N^i(X)$ and $N^{d-i}(X)$ are dual to each other by the intersection pairing. For a morphism $f\colon Y \to X$ of non-singular projective varieties with $d = \dim X$ and $e = \dim Y$, we have the natural homomorphisms

$$f^*\colon N^i(X) \to N^i(Y), \quad f_*\colon N_i(Y) \simeq N^{e-i}(Y) \to N_i(X) \simeq N^{d-i}(X),$$

where the following projection formula holds for $u \in N^i(X)$ and $v \in N^j(Y)$.

$$f_*(f^*u \cdot v) = u \cdot f_*v \in N^{d-e+(i+j)}(X).$$

For a vector bundle $\mathcal{E}$ of rank r on X, we define the following two polynomials:

$$P_{\mathcal{E}}(t) := \sum_{i=0}^{r} (-1)^i c_i(\mathcal{E}) t^{r-i},$$

$$Q_{\mathcal{E}}(t) := P_{\mathcal{E}}(t + \frac{1}{r} c_1(\mathcal{E})) = \sum_{i=0}^{r} (-1)^i \Delta_i(\mathcal{E}) t^{r-i}.$$

Here the coefficients $\Delta_k(\mathcal{E}) \in N^k(X)$ are written by Chern classes $c_i(\mathcal{E})$ as follows:

$$\Delta_k(\mathcal{E}) = \sum_{j=0}^{k} \frac{(-1)^j}{r^j} \binom{r-k+j}{j} c_1(\mathcal{E})^j \cdot c_{k-j}(\mathcal{E}).$$

For example,

$$\Delta_0(\mathcal{E}) = c_0(\mathcal{E}) = 1, \quad \Delta_1(\mathcal{E}) = 0,$$

$$\Delta_2(\mathcal{E}) = c_2(\mathcal{E}) - \frac{r-1}{2r} c_1(\mathcal{E})^2.$$

Let $\pi\colon \mathbb{P} = \mathbb{P}_X(\mathcal{E}) \to X$ be the associated $\mathbb{P}^{r-1}$-bundle and let $H = H_{\mathcal{E}}$ be the tautological divisor. Then

$$P_{\mathcal{E}}(H) = \sum_{i=0}^{r} (-1)^i \pi^* c_i(\mathcal{E}) H^{r-i} = 0$$

in $N^r(\mathbb{P})$. Thus for the normalized tautological divisor

$$\Lambda = \Lambda_{\mathcal{E}} = H_{\mathcal{E}} - \frac{1}{r}\pi^* c_1(\mathcal{E}),$$

we have

$$Q_{\mathcal{E}}(\Lambda) = \sum_{i=0}^{r} (-1)^i \pi^* \Delta_i(\mathcal{E}) \Lambda^{r-i} = 0.$$

Since $\pi_* \Lambda^j = 0$ for $j < r-1$ and $\pi_* \Lambda^{r-1} = 1 \in N^0(X)$, we have

$$\pi_* \Lambda^r = \Delta_1(\mathcal{E}) = 0, \quad \pi_* \Lambda^{r+1} = -\Delta_2(\mathcal{E}), \quad \pi_* \Lambda^{r+2} = \Delta_3(\mathcal{E}),$$
$$\pi_* \Lambda^{r+3} = \Delta_2(\mathcal{E})^2 - \Delta_4(\mathcal{E}), \text{ etc.}$$

Let $\mathcal{F}$ be a coherent sheaf of X. By using a resolution by locally free sheaves, we can define the Chern classes $c_i(\mathcal{F})$. Suppose that $\operatorname{codim} \operatorname{Supp} \mathcal{F} = k > 0$. Then by Grothendieck–Hirzebruch–Riemann–Roch formula, we have $c_i(\mathcal{F}) = 0$ for $0 < i < k$ and

$$c_k(\mathcal{F}) = (-1)^{k-1}(k-1)! \sum_{Z \subset \operatorname{Supp} \mathcal{F}} l_Z(\mathcal{F}) Z,$$

where $Z \subset \operatorname{Supp} \mathcal{F}$ is an irreducible component of codimension k and the length $l_Z(\mathcal{F})$ is defined as follows: There is a filtration

$$\mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_l = \mathcal{F}$$

of coherent sheaves such that $\mathcal{F}_i / \mathcal{F}_{i-1}$ is a non-zero torsion free $\mathcal{O}_Z$-module for $i > 0$ and Z is not contained in $\operatorname{Supp} \mathcal{F}_0$. Here

$$l_Z(\mathcal{F}) := \sum_{i>0} \operatorname{rank} \mathcal{F}_i / \mathcal{F}_{i-1}$$

does not depend on the choice of such filtration. For example, if $\mathcal{F}$ is a skyscraper sheaf, then

$$\dim H^0(X, \mathcal{F}) = \chi(X, \mathcal{F}) = (-1)^{d-1} \frac{1}{(d-1)!} c_d(\mathcal{F}).$$

Therefore if $\operatorname{codim} \operatorname{Supp} \mathcal{F} \geq k$, then

$$(-1)^{k-1} c_k(\mathcal{F}) \cdot A^{d-k} \geq 0$$

for an ample divisor A and the equality holds only when $\operatorname{codim}\operatorname{Supp}\mathcal{F} \geq k+1$. For a non-zero torsion free coherent sheaf $\mathcal{F}$, we define the averaged first Chern class

$$\mu(\mathcal{F}) := \frac{1}{\operatorname{rank}\mathcal{F}} c_1(\mathcal{F}) \in N^1(X).$$

For an ample divisor A, we also define $\mu_A(\mathcal{F}) := \mu(\mathcal{F}) \cdot A^{d-1}$. Let

$$0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0$$

be a short exact sequence of non-zero torsion free coherent sheaves. Then we have the following formula:

$$(\triangle) \qquad \Delta_2(\mathcal{F}) = \Delta_2(\mathcal{E}) + \Delta_2(\mathcal{G}) - \frac{(\operatorname{rank}\mathcal{E})(\operatorname{rank}\mathcal{G})}{2(\operatorname{rank}\mathcal{F})}(\mu(\mathcal{E}) - \mu(\mathcal{G}))^2.$$

Thus if $\mu_A(\mathcal{E}) = \mu_A(\mathcal{F}) = \mu_A(\mathcal{G})$ for an ample divisor A, then

$$\Delta_2(\mathcal{F}) \cdot A^{d-2} \geq \Delta_2(\mathcal{E}) \cdot A^{d-2} + \Delta_2(\mathcal{G}) \cdot A^{d-2}$$

by Hodge index theorem. Here the equality holds if and only if $\mu(\mathcal{F}) = \mu(\mathcal{E}) = \mu(\mathcal{G})$.

Let $\mathrm{U}(r)$ be the unitary group of degree r and let $\mathrm{PU}(r)$ be the quotient group $\mathrm{U}(r)/\mathrm{U}(1)$ by the center $\mathrm{U}(1) \simeq \mathrm{S}^1$. Let $\mathcal{O}_X^\star \times \mathrm{U}(r)$ be the direct product of the sheaf $\mathcal{O}_X^\star$ of germs of holomorphic unit functions and the constant sheaf $\mathrm{U}(r)$. Let $\mathrm{GL}(r, \mathcal{O}_X)$ be the sheaf of germs of $r \times r$ holomorphic regular matrices and let $\mathcal{O}_X^\star\,\mathrm{U}(r)$ be the image of the natural homomorphism

$$\mathcal{O}_X^\star \times \mathrm{U}(r) \to \mathrm{GL}(r, \mathcal{O}_X).$$

Then we have an exact sequence:

$$\begin{array}{ccccccccc} 1 & \to & \mathrm{S}^1 & \to & \mathcal{O}_X^\star \times \mathrm{U}(r) & \to & \mathcal{O}_X^\star\,\mathrm{U}(r) & \to & 1. \\ & & s & \mapsto & (s^{-1}, s) & & & & \end{array}$$

Lemma 2.1. *The image of the homomorphism $H^1(X, \mathcal{O}_X^\star\,\mathrm{U}(r)) \to H^1(X, \mathrm{GL}(r, \mathcal{O}_X))$ is the set of all the isomorphism classes of vector bundles $\mathcal{E}$ of X of rank r which admit projectively flat Hermitian metrics.*

Proof. Let $(\mathcal{E}, h)$ be a Hermitian vector bundle of rank r, where h is projectively flat, i.e., there is a d-closed $(1,1)$-form ω such that the curvature tensor Θ_h is written by

$$\Theta_h = \omega \cdot \mathrm{id}_{\mathcal{E}}.$$

Then there is an open covering $\{U_\lambda\}$ of X and positive valued C^∞-functions a_λ on U_λ such that $a_\lambda^{-1}h$ is a flat metric on U_λ. Thus we may assume that there exist holomorphic sections

$$e_1^\lambda, e_2^\lambda, \dots, e_r^\lambda \in H^0(U_\lambda, \mathcal{E}),$$

such that for any $1 \leq i, j \leq r$,

$$h(e_i^\lambda, e_j^\lambda) = a_\lambda \delta_{i,j},$$

where $\delta_{i,j}$ denotes Kronecker's δ. Let $T_{\lambda,\mu}$ be the transition matrix of $\mathcal{E}$ with respect to the frame $\{(U_\lambda, e_i^\lambda)\}$, i.e.,

$$(e_1^\lambda, e_2^\lambda, \dots, e_r^\lambda) \cdot T_{\lambda,\mu} = (e_1^\mu, e_2^\mu, \dots, e_r^\mu).$$

Then $T_{\lambda,\mu}$ are holomorphic $r \times r$ regular matrices and satisfy

$${}^{\mathrm{t}}T_{\lambda,\mu}\overline{T_{\lambda,\mu}} = a_\mu a_\lambda^{-1} \cdot \mathrm{id}.$$

Locally on $U_\lambda \cap U_\mu$, there is a holomorphic function u such that $a_\mu a_\lambda^{-1} = |u|^2$. Thus $u^{-1}T_{\lambda,\mu}$ is unitary. Hence $T_{\lambda,\mu} \in H^0(U_\lambda \cap U_\mu, \mathcal{O}_X^\star \mathrm{U}(r))$. Therefore $\mathcal{E} \in H^1(X, \mathrm{GL}(r, \mathcal{O}_X))$ is coming from $H^1(X, \mathcal{O}_X^\star \mathrm{U}(r))$.

Next suppose that $\mathcal{E}$ is contained in the image of $H^1(X, \mathcal{O}_X^\star \mathrm{U}(r))$. Then for a suitable frame $\{(U_\lambda, e_i^\lambda)\}$, the corresponding transition matrix $T_{\lambda,\mu}$ is contained in $H^0(U_\lambda \cap U_\mu, \mathcal{O}_X^\star \mathrm{U}(r))$. Thus

$${}^{\mathrm{t}}T_{\lambda,\mu}\overline{T_{\lambda,\mu}} = v_{\lambda,\mu} \cdot \mathrm{id},$$

for a positive valued C^∞-function $v_{\lambda,\mu}$ on $U_\lambda \cap U_\mu$. By replacing the open covering $\{U_\lambda\}$ by a finer one, we may assume that there is a positive valued C^∞-function a_λ on U_λ such that $v_{\lambda,\mu} = a_\mu a_\lambda^{-1}$. Let h_λ be the Hermitian metric of $\mathcal{E}_{|U_\lambda}$ defined by

$$h_\lambda(e_i^\lambda, e_j^\lambda) = a_\lambda \delta_{i,j}.$$

Then we see that $h_\lambda = h_\mu$ on $U_\lambda \cap U_\mu$. Hence we have a projectively flat metric on $\mathcal{E}$. □

Corollary 2.2. *A vector bundle $\mathcal{E}$ of rank r is projectively flat, if and only if the associated $\mathbb{P}^{r-1}$-bundle $\pi : \mathbb{P}_X(\mathcal{E}) \to X$ is induced from a projective unitary representation $\pi_1(X) \to \mathrm{PU}(r)$.*

Proof. There is a commutative diagram of exact sequences:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathcal{O}_X^\star & \longrightarrow & \mathcal{O}_X^\star \,\mathrm{U}(r) & \longrightarrow & \mathrm{PU}(r) & \longrightarrow & 1 \\ & & \| & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mathcal{O}_X^\star & \longrightarrow & \mathrm{GL}(r,\mathcal{O}_X) & \longrightarrow & \mathrm{PGL}(r,\mathcal{O}_X) & \longrightarrow & 1. \end{array}$$

Here note that $\mathcal{O}_X^\star$ is the center of $\mathcal{O}_X^\star \,\mathrm{U}(r)$ and $\mathrm{GL}(r,\mathcal{O}_X)$, respectively. Let $\mathcal{E}$ be an element of $H^1(X,\mathrm{GL}(r,\mathcal{O}_X))$ whose image in $H^1(X,\mathrm{PGL}(r,\mathcal{O}_X))$ is contained in the image of $H^1(X,\mathrm{PU}(r))$. Then we can check the $\mathcal{E}$ is coming from $H^1(X,\mathcal{O}_X^\star \,\mathrm{U}(r))$ by diagram chasing. □

Lemma 2.3. *Let $Y \subset X$ be a non-singular ample divisor of a non-singular projective variety X of dimension $d \geq 3$. Let $\mathcal{E}_Y$ be a vector bundle of Y and let $\mathcal{L}$ be a line bundle of X such that $\mathcal{E}_Y$ is projectively flat and $\det \mathcal{E}_Y \simeq \mathcal{L} \otimes \mathcal{O}_Y$. Then there is a projectively flat vector bundle $\mathcal{E}$ of X satisfying $\det \mathcal{E} \simeq \mathcal{L}$ and $\mathcal{E} \otimes \mathcal{O}_Y \simeq \mathcal{E}_Y$.*

Proof. We shall consider the following two homomorphisms

$$\det\colon \mathcal{O}_X^\star \,\mathrm{U}(r) \to \mathcal{O}_X^\star, \quad p\colon \mathcal{O}_X^\star \,\mathrm{U}(r) \to \mathrm{PU}(r).$$

Let $\boldsymbol{\mu}_r \subset \mathbb{C}^\star$ be the group of r-th roots of unity. Then we have an exact sequence

$$1 \to \boldsymbol{\mu}_r \to \mathcal{O}_X^\star \,\mathrm{U}(r) \stackrel{(\det,p)}{\to} \mathcal{O}_X^\star \times \mathrm{PU}(r) \to 1.$$

Thus the following exact sequence is derived:

$$H^1(X,\boldsymbol{\mu}_r) \to H^1(X,\mathcal{O}_X^\star \,\mathrm{U}(r)) \to H^1(X,\mathcal{O}_X^\star) \times H^1(X,\mathrm{PU}(r)) \to H^2(X,\boldsymbol{\mu}_r).$$

By the weak Lefschetz theorem, we have isomorphisms

$$H^1(X,\boldsymbol{\mu}_r) \simeq H^1(Y,\boldsymbol{\mu}_r), \quad H^1(X,\mathrm{PU}(r)) \simeq H^1(Y,\mathrm{PU}(r))$$

and injective homomorphisms

$$H^1(X,\mathcal{O}_X^\star) \hookrightarrow H^1(Y,\mathcal{O}_Y^\star), \quad H^2(X,\boldsymbol{\mu}_r) \hookrightarrow H^2(Y,\boldsymbol{\mu}_r).$$

Thus we can find the $\mathcal{E}$ by diagram chasing. □

The following result is proved in [BS, Corollary 3] in Kähler situation. But here, we give another proof by using the argument of [MR2, 5.1] only in projective situation.

Lemma 2.4. *Let $\mathcal{E}$ be an A-stable sheaf with $\Delta_2(\mathcal{E}) \cdot A^{d-2} = 0$ for an ample divisor A. Then $\mathcal{E}$ is a projectively flat vector bundle.*

Proof. If $\mathcal{E}$ is locally free, then it follows from works of Donaldson [D], Mehta–Ramanathan [MR1], [MR2] as well as Uhlenbeck–Yau [UY1], [UY2]. Thus we have only to prove $\mathcal{E}$ is locally free in the case $d \geq 3$. Let S be the complete intersection of smooth divisors $A_1, A_2, \ldots, A_{d-2}$ contained in the linear system $|mA|$ for sufficiently large $m > 0$. Then $\mathcal{E}_{|S} = \mathcal{E} \otimes \mathcal{O}_S$ is a locally free sheaf and it is stable by [MR2]. Hence $\mathcal{E}_{|S}$ is a projectively flat vector bundle. By Lemma 2.3, there is a projectively flat vector bundle $\mathcal{E}'$ such that

$$\det \mathcal{E}' \simeq \det \mathcal{E}, \quad \mathcal{E}' \otimes \mathcal{O}_S \simeq \mathcal{E} \otimes \mathcal{O}_S.$$

By the same argument of [MR2, 5.1], we have an isomorphism $\mathcal{E} \simeq \mathcal{E}'$. □

Proposition 2.5. *Let $\mathcal{E}$ be an A-semi-stable sheaf with $\Delta_2(\mathcal{E}) \cdot A^{d-2} = 0$ for an ample divisor A. Then $\mathcal{E}$ is locally free.*

Proof. We shall prove by induction on $\operatorname{rank} \mathcal{E}$. We may assume $\mathcal{E}$ is not A-stable by Lemma 2.4. Then there is an exact sequence

$$0 \to \mathcal{F} \to \mathcal{E} \to \mathcal{G} \to 0,$$

where $\mathcal{F}$ and $\mathcal{G}$ are non-zero torsion free sheaves satisfying $\mu_A(\mathcal{F}) = \mu_A(\mathcal{E}) = \mu_A(\mathcal{G})$. Thus $\mathcal{F}$ and the double dual $\mathcal{G}^\wedge$ are also A-semi-stable sheaves. Especially, we have Bogomolov's inequalities:

$$\Delta_2(\mathcal{F}) \cdot A^{d-2} \geq 0, \quad \Delta_2(\mathcal{G}^\wedge) \cdot A^{d-2} \geq 0.$$

Note that $\Delta_2(\mathcal{G}) - \Delta_2(\mathcal{G}^\wedge)$ is represented by an effective algebraic cycle of codimension two supported in $\operatorname{Supp} \mathcal{G}^\wedge/\mathcal{G}$. By the formula $(\triangle)$, we see that $\Delta_2(\mathcal{G}) = \Delta_2(\mathcal{G}^\wedge)$,

$$\Delta_2(\mathcal{F}) \cdot A^{d-2} = \Delta_2(\mathcal{G}^\wedge) \cdot A^{d-2} = 0,$$

and $\mu(\mathcal{F}) = \mu(\mathcal{G}) = \mu(\mathcal{E})$. By the induction, $\mathcal{F}$ and $\mathcal{G}^\wedge$ are locally free. Suppose that $\mathcal{G} \neq \mathcal{G}^\wedge$. Then the $\mathcal{E}$ defines a non-zero element in the cohomology group $H^0(X, \mathcal{E}xt^1(\mathcal{G}, \mathcal{F}))$. On the other hand, we have $\mathcal{E}xt^2(\mathcal{G}^\wedge/\mathcal{G}, \mathcal{F}) = 0$, since $\operatorname{codim} \operatorname{Supp} \mathcal{G}^\wedge/\mathcal{G} \geq 3$. It implies $\mathcal{E}xt^1(\mathcal{G}, \mathcal{F}) = 0$. This is a contradiction. Hence $\mathcal{G} = \mathcal{G}^\wedge$ and $\mathcal{E}$ is also locally free. □

Proof of Theorem A. (1) $\Longrightarrow$ (2). Let $C \subset X$ be a smooth projective curve. Then the normalized tautological divisor of the restriction $\mathcal{E}_{|C}$ is also nef. Thus $\mathcal{E}_{|C}$ is semi-stable. Hence $\mathcal{E}$ is A-semi-stable and Bogomolov's inequality $\Delta_2(\mathcal{E}) \cdot A^{d-2} \geq 0$ holds for any ample divisor A. On the other hand,

$$0 \leq \Lambda^{r+1} \cdot \pi^* A^{d-2} = -\Delta_2(\mathcal{E}) \cdot A^{d-2}.$$

Thus $\Delta_2(\mathcal{E}) = 0$ in $N^2(X)$.

(2) $\Longrightarrow$ (3). If $\mathcal{E}$ is A-stable, then $\mathcal{E}$ is a projectively flat vector bundle by Lemma 2.4. Otherwise, there is an exact sequence:

$$0 \to \mathcal{F} \to \mathcal{E} \to \mathcal{G} \to 0$$

such that $\mathcal{F}$ and $\mathcal{G}$ are non-zero torsion free sheaf and $\mu_A(\mathcal{E}) = \mu_A(\mathcal{F}) = \mu_A(\mathcal{G})$. As in the same argument in the proof of Proposition 2.5, we see that $\mathcal{F}$ and $\mathcal{G}$ are also A-semi-stable vector bundles with $\Delta_2(\mathcal{F}) \cdot A^{d-2} = \Delta_2(\mathcal{G}) \cdot A^{d-2} = 0$. Thus we have a filtration satisfying the condition (3).

(3) $\Longrightarrow$ (1). If $\mathcal{E}$ is projectively flat, then $f^*\mathcal{E}$ is semi-stable for any morphism $f : C \to X$ from a non-singular projective curve. Thus if $\mathcal{E}$ has a filtration satisfying the condition (3), then $f^*\mathcal{E}$ is also semi-stable. This is equivalent to $\Lambda_{\mathcal{E}}$ is nef. □

For a nef $\mathbb{Q}$-divisor D of a non-singular projective variety Z, the numerical D-dimension $\nu(D) = \nu(D, Z)$ is defined by:

$$\nu(D) := \max\{k \in \mathbb{Z}_{\geq 0} \,|\, D^k \neq 0 \text{ in } N^k(Z)\}.$$

Corollary 2.6. *If Λ is nef, then $\nu(\Lambda) = r - 1$.*

3. Vector bundles of rank two with pseudo-effective normalized tautological divisors

Lemma 3.1. *Let M_1, M_2 be divisors of a d-dimensional non-singular projective variety X such that $M_1 \cdot A^{d-1} = M_2 \cdot A^{d-1}$ for an ample divisor A. Then the vector bundle $\mathcal{E} := \mathcal{O}_X(M_1) \oplus \mathcal{O}_X(M_2)$ is A-semi-stable and the linear system $|2\Lambda_{\mathcal{E}}|$ is not empty. The $\Lambda_{\mathcal{E}}$ is nef if and only if M_1 and M_2 are numerically equivalent.*

Proof. If $\mathcal{L} \subset \mathcal{E}$ is an invertible subsheaf, then it is a subsheaf of $\mathcal{O}_X(M_1)$ or $\mathcal{O}_X(M_2)$. Thus $\mathcal{L} \cdot A^{d-1} \leq (1/2)c_1(\mathcal{E}) \cdot A^{d-1}$. The symmetric tensor $S^2\mathcal{E}$ contains $\mathcal{O}_X(M_1 + M_2) \simeq \det \mathcal{E}$

as a direct summand. Hence $|2\Lambda_{\mathcal{E}}| \neq \emptyset$. If M_1 and M_2 are numerically equivalent, then $\Lambda_{\mathcal{E}}$ is nef. Conversely if $\Lambda_{\mathcal{E}}$ is nef, then $M_1 - M_2$ is numerically trivial, since

$$\Delta_2(\mathcal{E}) = -\frac{1}{4}(M_1 - M_2)^2 = 0. \qquad \square$$

Lemma 3.2. *Let $\tau\colon Y \to X$ be an unramified double covering of non-singular varieties of dimension d and let M be a divisor of Y. Then for the vector bundle $\mathcal{E} := \tau_*\mathcal{O}_Y(M)$, we have an isomorphism*

$$\tau^*\mathcal{E} \simeq \mathcal{O}_Y(M) \oplus \mathcal{O}_Y(\sigma^*M),$$

where $\sigma\colon Y \to Y$ is the non-trivial involution over X. In particular, $\mathcal{E}$ is semi-stable with respect to any ample divisor of X and $\Lambda_{\mathcal{E}}$ is pseudo-effective. Further $\Lambda_{\mathcal{E}}$ is nef if and only if M is numerically equivalent to σ^*M.

Proof. Let us consider the natural homomorphism $\phi\colon \tau^*\tau_*\mathcal{O}_Y \to \mathcal{O}_Y$. Then $\phi + \sigma^*\phi$ gives an isomorphism

$$\tau^*\tau_*\mathcal{O}_Y \simeq \mathcal{O}_Y \oplus \mathcal{O}_Y.$$

Similarly from the natural homomorphism $\varphi\colon \tau^*\tau_*\mathcal{O}_Y(M) \to \mathcal{O}_Y(M)$, we have the homomorphism

$$\varphi + \sigma^*\varphi\colon \tau^*\mathcal{E} = \tau^*\tau_*\mathcal{O}_Y(M) \to \mathcal{O}_Y(M) \oplus \mathcal{O}_Y(\sigma^*M).$$

Since $\mathcal{O}_Y(M)$ is an invertible sheaf, we see that the homomorphism also is an isomorphism by considering it locally over X. $\square$

Lemma 3.3. *Let Z be a closed subscheme locally of complete intersection of a scheme X with* $\operatorname{codim} Z = 2$ *and let $\mathcal{L}$ be an invertible sheaf of X. If there exists a locally free sheaf $\mathcal{E}$ with an exact sequence*

$$(\dagger) \qquad 0 \to \mathcal{O}_X \to \mathcal{E} \to I_Z\mathcal{L} \to 0,$$

where I_Z is the defining ideal sheaf of Z, then

$$(\ddagger) \qquad \mathcal{E}xt^2(\mathcal{O}_Z, \mathcal{L}^{-1}) \simeq \mathcal{O}_Z.$$

Conversely, if the isomorphism (‡) *exists, then there is a naturally defined cohomology class* $\delta(Z,\mathcal{L}) \in H^2(X,\mathcal{L}^{-1})$ *such that* $\delta(Z,\mathcal{L}) = 0$ *if and only if there is a locally free sheaf* $\mathcal{E}$ *with the exact sequence* (†).

Proof. Suppose that the locally free sheaf $\mathcal{E}$ exists. Then from the exact sequence (†), we have a long exact sequence

$$0 \to \mathcal{H}om(I_Z\mathcal{L},\mathcal{O}_X) \to \mathcal{H}om(\mathcal{E},\mathcal{O}_X) \to \mathcal{H}om(\mathcal{O}_X,\mathcal{O}_X) \to \mathcal{E}xt^1(I_Z\mathcal{L},\mathcal{O}_X) \to 0.$$

Therefore

$$\mathcal{O}_Z \simeq \mathcal{E}xt^1(I_Z\mathcal{L},\mathcal{O}_X) \simeq \mathcal{E}xt^2(\mathcal{O}_Z,\mathcal{L}^{-1}).$$

Next suppose the isomorphism (‡) exists. From the spectral sequence

$$E_2^{p,q} = H^p(X,\mathcal{E}xt^q(I_Z\mathcal{L},\mathcal{O}_X)) \Longrightarrow E^{p+q} = \mathrm{Ext}^{p+q}(I_Z\mathcal{L},\mathcal{O}_X),$$

we have an exact sequence

$$0 \to H^1(X,\mathcal{L}^{-1}) \to \mathrm{Ext}^1(I_Z\mathcal{L},\mathcal{O}_X) \to H^0(Z,\mathcal{O}_Z) \to H^2(X,\mathcal{L}^{-1}).$$

Let $\delta = \delta(Z,L)$ be the image of $1 \in H^0(Z,\mathcal{O}_Z)$ under the last homomorphism. Then $\delta = 0$ if and only if there is an extension of sheaves

$$0 \to \mathcal{O}_X \to \mathcal{E} \to I_Z\mathcal{L} \to 0$$

such that $\mathcal{E}xt^1(\mathcal{E},\mathcal{O}_X) = 0$. □

Example. If X is a non-singular surface with $p_g(X) = \dim H^2(X,\mathcal{O}_X) = 0$, then for any point $x \in X$, there is a locally free sheaf $\mathcal{E}$ with an exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{E} \to \mathfrak{m}_x \to 0,$$

where $\mathfrak{m}_x$ is the maximal ideal at x.

Let $\tau\colon Y \to X$ be a generically finite proper surjective morphism over a non-singular variety X. Suppose that Y is a variety with only Gorenstein singularities and let $\nu\colon V \to Y$ be the normalization. By duality, there are trace maps $\nu_*\omega_V \to \omega_Y$ and $\tau_*\omega_Y \to \omega_X$. The first homomorphism induces an effective divisor C of V, which is called the conductor of Y, such that $K_V \sim \nu^*K_Y - C$. If $C = 0$, then ν is an isomorphism. The pullback of differential forms induces a homomorphism $\nu^*\tau^*\omega_X \to \omega_V$, which gives a splitting of the

composition of trace maps above. Thus there exist an effective divisor $R_{V/X}$ of V and an effective Cartier divisor $R_{Y/X}$ of Y such that

$$K_V \sim \nu^*\tau^*K_X + R_{V/X}, \quad K_Y \sim \tau^*K_X + R_{Y/X}, \quad R_{V/X} = \tau^*R_{Y/X} - C.$$

The $R_{Y/X}$ and $R_{V/X}$ are called the ramification divisors.

Lemma 3.4. *If the ramification divisor $R_{Y/X}$ is zero, then τ is a finite étale morphism.*

Proof. Since the ramification divisor $R_{V/X}$ is effective, the conductor C must be zero. Hence Y is normal. Let

$$Y \stackrel{\mu}{\to} W \stackrel{p}{\to} X$$

be the Stein factorization of τ. Then the dualizing sheaf ω_W is the double dual of $\mu_*\omega_Y$. Since $R_{Y/X} = 0$, we have isomorphisms $\omega_W \simeq p^*\omega_X$ and $\omega_Y \simeq \mu^*\omega_W$. Thus $W \to X$ is étale, since p is a finite morphism. In particular, W is non-singular. Consequently, the birational morphism $Y \to W$ is an isomorphism. □

Proof of Theorem B. We have Bogomolov's inequality

$$\Delta_2(\mathcal{E}) \cdot A^{d-2} \geq 0,$$

where the equality holds if and only if $\Lambda = \Lambda_{\mathcal{E}}$ is nef by Theorem A. We have only to show the equality $\Delta_2(\mathcal{E}) \cdot A^{d-2} = 0$ except for the three exceptional cases. Let $\Lambda = P + N$ be the σ-decomposition of the pseudo-effective divisor Λ defined in [N]. Then there exist an $\mathbb{R}$-divisor D of X and a real number b such that

$$N \sim_{\text{num}} b\Lambda + \pi^*D, \quad P \sim_{\text{num}} (1-b)\Lambda - \pi^*D,$$

where $\sim_{\text{num}}$ denotes the numerical equivalence relation. For a fiber F of the $\mathbb{P}^1$-bundle $\pi\colon \mathbb{P} \to X$, we have $P \cdot F \geq 0$ and $N \cdot F \geq 0$. Thus $0 \leq b \leq 1$. Let $A_1, A_2, \ldots, A_{d-1}$ be general members of the linear system $|mA|$ for sufficiently large $m > 0$. Then for the smooth curve

$$C = A_1 \cap A_2 \cap \cdots \cap A_{d-1},$$

$\mathcal{E}_{|C}$ is semi-stable by [MR1]. In particular, if

$$(\Lambda + \pi^*E)_{|\pi^{-1}(C)}$$

is pseudo-effective on $\pi^{-1}(C)$ for an $\mathbb{R}$-divisor E of X, then $E \cdot A^{d-1} \geq 0$. Note that $N_{|\pi^{-1}(C)}$ and $P_{|\pi^{-1}(C)}$ are pseudo-effective. Thus $D \cdot A^{d-1} \geq 0$ in the case $b > 0$ and $D \cdot A^{d-1} \leq 0$ in the case $b < 1$.

First suppose that $b < 1$. Since P is movable, P^2 is a pseudo-effective cycle, i.e., the cohomology class of P^2 in $N^2(\mathbb{P})$ is a limit of the cohomology classes of effective algebraic cycles of codimension two. Therefore

$$\pi_*(P^2) = -2(1-b)D$$

is a pseudo-effective $\mathbb{R}$-divisor. Thus $-D$ is pseudo-effective. If $b > 0$, then D is numerically trivial since $D \cdot A^{d-1} = 0$. Hence $N \sim_{\text{num}} b\Lambda$ and $P \sim_{\text{num}} (1-b)\Lambda$. This is a contradiction. Therefore $b = 0$. Thus $-N \sim_{\text{num}} -\pi^*D$ is pseudo-effective. Hence $N = 0$ and Λ is movable. Since $\Lambda^2 = -\pi^*\Delta_2(\mathcal{E})$, we have

$$-\Delta_2(\mathcal{E}) = \pi_*(H \cdot \Lambda^2) = \pi_*((H + m\pi^*A) \cdot \Lambda^2),$$

for any integer m. If $m > 0$ is large, then $H + m\pi^*A$ is ample and thus $(H + m\pi^*A) \cdot \Lambda^2$ is pseudo-effective. Hence $-\Delta_2(\mathcal{E})$ is pseudo-effective. By Bogomolov's inequality, we have $\Delta_2(\mathcal{E}) \cdot A^{d-2} = 0$.

Next suppose that $b = 1$. Since $P \sim_{\text{num}} -\pi^*D$ is movable, so is $-D$. On the other hand, $b > 0$ implies $D \cdot A^{d-1} \geq 0$. Hence D and P are numerically trivial. Let

$$N = \sum \sigma_i \Gamma_i$$

be the prime decomposition. For each i, there are non-negative integers b_i and $\mathbb{Q}$-divisors D_i such that

$$\Gamma_i \sim_{\mathbb{Q}} b_i\Lambda + \pi^*D_i,$$

where $\sim_{\mathbb{Q}}$ is the $\mathbb{Q}$-linear equivalence relation. Since $\Lambda - \sigma_i\pi^*D_i$ is pseudo-effective and since $\mathcal{E}_{|C}$ is semi-stable, we have $D_i \cdot A^{d-1} \leq 0$. Hence $b_i > 0$. Further $D_i \cdot A^{d-1} = 0$, since

$$D \sim_{\mathbb{Q}} \sum \sigma_i D_i$$

is numerically trivial. We consider the following three cases.

(I) $b_i \geq 2$ for some i.

(II) N has at least two irreducible components and $b_i = 1$ for every components Γ_i.

(III) N has only one irreducible component Γ_1 and $b_1 = 1$.

Let Y be an irreducible component Γ_1. Then $\pi : Y \to X$ is a generically finite surjective morphism of degree b_1. From adjunction, we have

$$K_Y \sim \pi^* K_X + ((b_1 - 2)\Lambda + \pi^* D_1)_{|Y}.$$

Therefore the ramification divisor $R_{Y/X}$ is linearly equivalent to $((b_1 - 2)\Lambda + \pi^* D_1)_{|Y}$. Since $R_{Y/X}$ is effective,

$$\pi_*(((b_1 - 2)\Lambda + \pi^* D_1)_{|Y}) = \pi_*(((b_1 - 2)\Lambda + \pi^* D_1) \cdot (b_1\Lambda + \pi^* D_1)) = 2(b_1 - 1)D_1$$

is an effective divisor of X.

We consider the case (I). We may assume that $b_1 \geq 2$. Then $D_1 \sim_{\mathbb{Q}} 0$, since $D_1 \cdot A^{d-1} = 0$. Hence $Y \sim_{\mathbb{Q}} b_1\Lambda$. By the definition of σ-decomposition, we have

$$\sigma_i = \sigma_{\Gamma_i}(\Lambda) = \frac{1}{b_1}\sigma_{\Gamma_i}(Y).$$

Thus N has only one irreducible component Y and $N = (1/b_1)Y$. The divisor $(b_1 - 2)\Lambda_{|Y}$ is $\mathbb{Q}$-linearly equivalent to the effective divisor $R_{Y/X}$. Let $m > 0$ be a positive integer with $H + m\pi^* A$ is ample. Then

$$(*) \qquad \pi_*((H + m\pi^* A) \cdot ((b_1 - 2)\Lambda) \cdot Y) = b_1(b_1 - 2)\pi_*(H \cdot \Lambda^2) = -b_1(b_1 - 2)\Delta_2(\mathcal{E})$$

is also a pseudo-effective cycle. Hence by Bogomolov's inequality, if $b_1 \geq 3$, then $\Delta_2(\mathcal{E}) = 0$ and hence Λ is nef by Theorem A, which contradicts to $P \sim_{\text{num}} 0$. Therefore $b_1 = 2$. Since every positive dimensional fibers of $\pi : Y \to X$ are $\mathbb{P}^1$, the ramification divisor $R_{Y/X}$ is zero by the formula $(*)$. By Lemma 3.4, $\pi : Y \to X$ is an étale double covering. From the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}}(H - Y) \to \mathcal{O}_{\mathbb{P}}(H) \to \mathcal{O}_Y(H) \to 0,$$

we see that $\mathcal{E} \simeq \pi_* \mathcal{O}_Y(H)$. Thus $\mathcal{E}$ is of type (B).

Next we consider the case (II). Let Γ_1, Γ_2 be two irreducible components of N. Since $b_1 = b_2 = 1$,

$$\pi_*(\Gamma_1 \cdot \Gamma_2) = D_1 + D_2$$

is an effective divisor of X. Further $D_1 + D_2 \sim 0$, since $(D_1 + D_2) \cdot A^{d-1} = 0$. Therefore

$$\Gamma_1 + \Gamma_2 \sim 2\Lambda.$$

Hence N has only two components and $\sigma_1 = \sigma_2 = 1/2$. By the vanishing $\pi_*(\Gamma_1 \cdot \Gamma_2) = 0$, every component of $\Gamma_1 \cap \Gamma_2$ is contracted by π. Therefore

$$\pi_*(H \cdot \Gamma_1 \cdot \Gamma_2) = -\Delta_2(\mathcal{E}) + D_1 \cdot D_2 = -\Delta_2(\mathcal{E}) - D_1^2$$

is an effective cycle. On the other hand, the ramification divisor $R_{\Gamma_1/X}$ is linearly equivalent to

$$(-\Lambda + \pi^* D_1)_{|\Gamma_1}.$$

Thus we have also an effective cycle

$$\pi_*(H \cdot R_{\Gamma_1/X}) = \pi_*(H \cdot (-\Lambda + \pi^* D_1) \cdot (\Lambda + \pi^* D_1)) = D_1^2 + \Delta_2(\mathcal{E}).$$

Hence $-\Delta_2(\mathcal{E}) = D_1^2$ in $N^2(X)$ and $\Gamma_1 \cap \Gamma_2 = \emptyset$. In particular, Γ_1 and Γ_2 are disjoint sections of the $\mathbb{P}^1$-bundle. Therefore

$$\mathcal{E} \simeq \pi_* \mathcal{O}_{\Gamma_1}(H) \oplus \pi_* \mathcal{O}_{\Gamma_2}(H).$$

Thus it is of type (A).

Finally, we treat the case (III). For the unique component $Y = \Gamma_1$, there is a divisor L_1 such that $Y \sim H - \pi^* L_1$. Since $N = \sigma_1 Y \sim_{\text{num}} \Lambda$, $\sigma_1 = 1$ and $\det \mathcal{E}$ is numerically equivalent to $2L_1$. The ramification divisor $R = R_{Y/X}$ is linearly equivalent to

$$(-H + \pi^*(-L_1 + \det \mathcal{E}))_{|Y}.$$

By applying π_* to the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}}(H - Y) \to \mathcal{O}_{\mathbb{P}}(H) \to \mathcal{O}_Y(H) \to 0,$$

we have another exact sequence

$$0 \to \mathcal{O}_X(L_1) \to \mathcal{E} \to I\mathcal{O}_X(L_2) \to 0,$$

where L_2 is a divisor linearly equivalent to $\det \mathcal{E} - L_1$ and I is the ideal sheaf defined by

$$I := \pi_* \mathcal{O}_Y(-R).$$

Therefore $\mathcal{E}$ is of type (C). This completes the proof. □

Corollary 3.5. *If $\mathcal{E}$ is an A-stable vector bundle of rank two for an ample divisor A such that the normalized tautological divisor $\Lambda_{\mathcal{E}}$ is pseudo-effective. Then $\Lambda_{\mathcal{E}}$ is nef except for the case (B) in Theorem B.*

For a $\mathbb{Q}$-divisor D of a non-singular projective variety Z, the numerical D-dimension $\kappa_\sigma(D) = \kappa_\sigma(D, Z)$ is defined as follows (cf. [N]):

$$\kappa_\sigma(D) := \max\{\sigma(D; A) \,|\, A \text{ is an ample divisor of } Z\},$$

where

$$\sigma(D; A) := \max\{k \in \mathbb{Z}_{\geq 0} \cup \{-\infty\} \,|\, \overline{\lim}_{m\to\infty} m^{-k} \dim H^0(Z, \mathcal{O}_Z(\llcorner mD \lrcorner + A)) > 0\}.$$

Here $\llcorner mD \lrcorner$ denotes the round-down of the $\mathbb{Q}$-divisor mD. If D is nef, then $\kappa_\sigma(D) = \nu(D)$. If D is pseudo-effective, then $\kappa_\sigma(D) \geq 0$.

Corollary 3.6. *If $\mathcal{E}$ is an A-semi-stable vector bundle of rank two, then*

$$\kappa_\sigma(\Lambda_{\mathcal{E}}) \leq 1.$$

Proof. We may assume that $\Lambda = \Lambda_{\mathcal{E}}$ is pseudo-effective. By Corollary 2.6, we may assume further that Λ is not nef. By the proof of Theorem B, the positive part P of the σ-decomposition is numerically trivial and hence $\Lambda \sim_{\text{num}} N$. Thus $\kappa_\sigma(\Lambda) = 0$. □

4. Vector bundles of rank two over surfaces

Theorem 4.1. *The tautological divisor of the tangent bundle of a K3 surface is not pseudo-effective.*

Proof. For the tangent bundle $\mathcal{E} = T_X$ of a K3 surface X, $\det(\mathcal{E}) = \mathcal{O}_X$ and $c_2(\mathcal{E}) = 24$. By [Y], $\mathcal{E}$ is A-stable for any ample divisors A. Since X is simply connected, $\Lambda_{\mathcal{E}} = H_{\mathcal{E}}$ is not pseudo-effective by Corollary 3.5. □

Remark. Kobayashi has shown $\kappa(\Lambda) = -\infty$ in [K, Theorem C]. On the other hand, the tangent bundle is *generically semi-positive* in the sense of Miyaoka [M1].

Problem. For a K3 surface X, are there infinitely many prime divisors $\Gamma \subset \mathbb{P}_X(T_X)$ such that $H_{|\Gamma}$ are not pseudo-effective?

Actually, for some K3 surface X, there is a nef divisor L of $\mathbb{P}_X(T_X)$ with $H \cdot L^2 < 0$ (cf. [O]). For example, if X is a smooth quadric surface, then $L = H + 2\pi^*A$ is free, i.e., $|L|$ is base point free, where A is a hyperplane section. In this case, $H \cdot L^2 = -8 < 0$. A general member $\Gamma \in |L|$ is a non-singular surface birational to X, with $K_\Gamma^2 = -40$. Here $H_{|\Gamma}$ is not pseudo-effective. In particular, the pullback of T_X in Γ is not A'-semi-stable for an ample divisor A' of Γ.

Problem. Let $\mathcal{E}$ be a vector bundle of rank two on a non-singular projective surface X. Suppose that for any generically finite morphism $f : Y \to X$ from any non-singular projective surface Y and for any ample divisor A of Y, $f^*\mathcal{E}$ is A-semi-stable. Then is $\Lambda_\mathcal{E}$ nef?

If the $\Lambda_\mathcal{E}$ is not nef, then it is not pseudo-effective by Theorem B.

Proposition 4.2. *If $\mathcal{E}$ is a vector bundle of rank two on a non-singular projective surface whose normalized tautological divisor is not pseudo-effective, then $\mathcal{E}$ is A-semi-stable for an ample divisor A.*

Proof. Assume the contrary. Then there is an exact sequence

$$0 \to \mathcal{L} \to \mathcal{E} \to I_Z\mathcal{M} \to 0$$

such that I_Z is the ideal sheaf of subscheme Z of $\dim Z \leq 0$ and $(\mathcal{L} - \mathcal{M}) \cdot A > 0$ for any ample divisors A. Therefore $\mathcal{L} - \mathcal{M}$ is pseudo-effective. By the formula,

$$\Lambda_\mathcal{E} = H_\mathcal{E} - \frac{1}{2}\pi^*(\mathcal{L} + \mathcal{M}) = H_\mathcal{E} - \pi^*\mathcal{L} + \frac{1}{2}\pi^*(\mathcal{L} - \mathcal{M}),$$

$\Lambda_\mathcal{E}$ is pseudo-effective. □

Corollary 4.3. *Let $\mathcal{E}$ be a vector bundle of rank two of a non-singular projective surface X. If D is a pseudo-effective $\mathbb{R}$-divisor of X with $3D^2 \geq \Delta_2(\mathcal{E})$, then $\Lambda_\mathcal{E} + \pi^*D$ is pseudo-effective.*

Proof. We may assume that $\Lambda = \Lambda_\mathcal{E}$ is not pseudo-effective. By Proposition 4.2, $\mathcal{E}$ is A-semi-stable for an ample divisor A. Thus we have Bogomolov's inequality $\Delta_2(\mathcal{E}) \geq 0$. Let D be a $\mathbb{Q}$-divisor with $3D^2 > \Delta_2(\mathcal{E})$ and let m be a positive integer such that $m\Lambda$

and mD are Cartier divisors. Then D is big, i.e., $\kappa(D) = 2$, by Hodge index theorem and

$$\pi_*\mathcal{O}_{\mathbb{P}}(m(\Lambda + \pi^*D)) \simeq \pi_*\mathcal{O}_{\mathbb{P}}(m\Lambda) \otimes \mathcal{O}_X(mD),$$

where $\pi_*\mathcal{O}_{\mathbb{P}}(m\Lambda)$ is an A-semi-stable vector bundle with trivial first Chern class. Therefore,

$$H^2(X, \pi_*\mathcal{O}_{\mathbb{P}}(m(\Lambda + \pi^*D)))^\vee \simeq H^0(X, \pi_*\mathcal{O}_{\mathbb{P}}(m\Lambda)^\vee \otimes \mathcal{O}_X(K_X - mD)) = 0$$

for $m \gg 0$. Now we have

$$H^p(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(m(\Lambda + \pi^*D))) \simeq H^p(X, \pi_*\mathcal{O}_{\mathbb{P}}(m(\Lambda + \pi^*D)))$$

for any $p \geq 0$. Since $(\Lambda + \pi^*D)^3 = -\Delta_2(\mathcal{E}) + 3D^2 > 0$, we have

$$\overline{\lim}_{m\to\infty} m^{-3}\chi(\mathbb{P}, m(\Lambda + \pi^*D)) > 0.$$

Therefore $\Lambda + \pi^*D$ is big, i.e., $\kappa(\Lambda + \pi^*D) = 3$. □

Problem. Let $\mathcal{E}$ be a vector bundle of rank two on a non-singular projective variety X. Suppose that the normalized tautological divisor $\Lambda = \Lambda_{\mathcal{E}}$ is not pseudo-effective. Describe the set

$$V(X, \mathcal{E}) := \{D \in N^1(X) \,|\, \Lambda + \pi^*D \text{ is pseudo-effective}\}.$$

For example, if $X = \mathbb{P}^2$ and $\mathcal{E} = T_X$, then $V(X, \mathcal{E}) = \{a\ell | a \geq 1/2\}$, where $\ell \subset \mathbb{P}^2$ is a line.

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Research Institute for Mathematical Sciences

Kyoto University, Kyoto 606-8502 Japan

E-mail address: nakayama@kurims.kyoto-u.ac.jp