Extended Vertex Operator Algebras and Monomial Bases

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Abstract

We present a vertex operator algebra which is an extension of the level k vertex operator algebra for the $\widehat{sl}_2$ conformal field theory. We construct monomial basis of its irreducible representations.

1 Introduction

Recall the following well-known construction of the level-1 representations of the Lie algebra $\widehat{gl}_2$. Let $\widehat{V}$ be the space of functions from S^1 to $V \simeq \mathbf{C}^2$. Then the irreducible representations $\widehat{gl}_2$ are realized in the space $\Lambda^{\frac{\infty}{2}}(\widehat{V})$ — the semi-infinite exterior power of $\widehat{V}$. There are many ways to define the space $\Lambda^{\frac{\infty}{2}}(\widehat{V})$. One approach is as follows. Consider the Clifford algebra generated by the space $\widehat{V} \oplus \widehat{V}^*$ with the natural quadratic form. The irreducible representation of the Clifford algebra is the direct sum

$$\bigoplus_{i \in \mathbf{Z}} \Lambda^{\frac{\infty}{2}+i}(\widehat{V}). \tag{1}$$

If we choose a basis in $\widehat{V}$ then the basis in (**??**) consists of the semi-infinite wedge products of the basis vectors of $\widehat{V}$.

An alternative construction goes as follows. Let $\widehat{V}_n := V \otimes z^n \mathbf{C}[z^{-1}]$. Then $\widehat{V} = V \otimes \mathbf{C}[z, z^{-1}]$ is equal to the inductive limit:

$$\cdots \to \widehat{V}_{-1} \to \widehat{V}_0 \to \widehat{V}_1 \to \cdots$$

Note that $\widehat{V}_n$ is a graded space. Let ω_n be an element of the highest degree in $\Lambda^2(\widehat{V}_n)$. We can consider the sequence of embeddings:

$$\mu: \ \Lambda^0(\widehat{V}_0) \to \Lambda^2(\widehat{V}_1) \to \Lambda^4(\widehat{V}_2) \to \cdots$$

Dedicated to James B. McGuire on the occasion of his 65th birthday.

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Here the map $\Lambda^s(\widehat{V}_n) \to \Lambda^{s+2}(\widehat{V}_{n+1})$ is the composition of the map $\Lambda^s(\widehat{V}_n) \to \Lambda^s(\widehat{V}_{n+1})$ and the product $\Lambda^s(\widehat{V}_{n+1}) \wedge \omega_{n+1} \to \Lambda^{s+2}(\widehat{V}_{n+1})$. The inductive limit of the sequence μ is $\Lambda^{\frac{\infty}{2}}(\widehat{V})$.

The dual sequence

$$\mu^* : \Lambda^0(\widehat{V}_0)^* \leftarrow \Lambda^2(\widehat{V}_1)^* \leftarrow \Lambda^4(\widehat{V}_2)^* \leftarrow \cdots$$

has a "functional" description. Namely, let us identify the space $\widehat{V}_n^*$ with the space $U_n = z^{-n-1}\mathbf{C}[z]dz \otimes (\mathbf{C}^2)^*$. Here the pairing is given by the residue. Then $\Lambda^s(\widehat{V}_n)^* \cong \Lambda^s(\widehat{V}_n^*) \cong \Lambda^s(U_n)$. The space $\Lambda^s(U_n)$ is the space of functions in $z_1, \ldots, z_s$ with values in $(\mathbf{C}^2)^* \otimes \cdots \otimes (\mathbf{C}^2)^*$ (times $dz_1 \cdots dz_s$) which is

(a) skew-symmetric with respect to the permutations of $\{z_j\}$ and components in the tensor product $(\mathbf{C}^2)^* \otimes \cdots \otimes (\mathbf{C}^2)^*$;

(b) of the form $(z_1 \cdots z_s)^{-n-1}P(z_1, \ldots, z_s)$ where P is a vector-valued polynomial.

Roughly speaking, the projective limit of the sequence μ^* is the space of skew-symmetric functions in infinitely many variables.

In this paper we present a generalization of these level-1 constructions. First of all, we construct a vertex operator algebra A_k which play the role of the Clifford algebra. It is generated by the spaces $V \otimes \mathbf{C}[z, z^{-1}]$ and $V^* \otimes \mathbf{C}[z, z^{-1}]$ where $V \simeq \mathbf{C}^{k+1}$. The idea of the construction is as follows. Consider the operator algebra of the conformal field theory consisting of both currents and intertwiners. The latter generate an algebra which is an extension of the vertex operator algebra generated by the former. However, it is not a vertex operator algebra because the relations among these operators are not "local". In a vertex operator algebra the operators placed at distinct points must commute (or skew-commute). In some cases it is possible to find the combinations of vertex operators which are "local" and generate a vertex operator algebra. It gives us an "algebraic" extension of the vertex operator algebra of currents.

For example let us start with the $\widehat{sl}_2$ conformal field theory of level 1. We have the vertex operators $\mathbf{C}^2(z)$ associated with 2-dimensional representation of sl_2. Consider the product of this theory and the free field theory. Fermions in the Clifford algebra are the products of $\mathbf{C}^2(z)$ and some primary fields (i.e., vertex operators) of the free field theory.

For higher level we apply exactly the same construction. We consider the product of the $\widehat{sl}_2$ conformal field theory of level k and the free field theory. The new vertex operator algebra is generated by the products of $\widehat{sl}_2$ vertex operators with values in $\mathbf{C}^{k+1}$ and the vertex operators of the free field theory.

We will study an analogue of the space $\Lambda^{\frac{\infty}{2}}\widehat{V}$ and a monomial basis there. Here we describe the "functional" version of the semi-infinite construction. Return to the level-1 case for the moment. Let us realize $(\mathbf{C}^2)^*$ in the space of polynomials in the variable t of degree ≤ 1. Then the space U_n can be identified with the space of functions $\{z^{-n-1}Q(t, z)\}$ where Q is a polynomial in t, z of degree ≤ 1 in t. Similarly, $\Lambda^s U_n$ can be identified with the space of functions

of the form $(z_1 \cdots z_s)^{-n-1} Q(t_1, z_1, \ldots, t_s, z_s)$ where Q is a polynomial skew-symmetric with respect to the permutations of pairs (t_j, z_j), and of degree ≤ 1 in t_j.

For $k > 1$, let us introduce the space $(\Lambda^s U_n)^k$. It consists of expressions of the form

$$(z_1 \cdots z_s)^{-k(n+1)} R(t_1, z_1, \ldots, t_s, z_s) \tag{2}$$

where R is a polynomial in $(t_1, z_1, \ldots, t_s, z_s)$

(a) of degree $\leq k$ in t_j, $j = 1, \ldots, s$;

(b) symmetric with respect to the permutations of the pairs (t_j, z_j) if k is even, and skew-symmetric otherwise;

(c) subject to the conditions

$$\frac{\partial^{j_1}}{\partial t_1^{j_1}} \frac{\partial^{j_2}}{\partial z_1^{j_2}} R|_{t_1 = t_2;\ z_1 = z_2} = 0 \quad \text{for } j_1 + j_2 < k.$$

In other words, R has a zero of order k if $t_1 = t_2$ and $z_1 = z_2$.

Thus, the space $\Lambda^s U_n$ is identified with some space of polynomials, and $(\Lambda^s U_n)^k$ is the linear span of $\Lambda^s U_n \times \cdots \times \Lambda^s U_n$ (k times) where $\times$ denotes just the product of polynomials. We have a projective system of spaces

$$(\Lambda^0 U_0)^k \leftarrow (\Lambda^2 U_1)^k \leftarrow (\Lambda^4 U_2)^k \leftarrow \cdots, \tag{3}$$

where the map sends the element (??) in $(\Lambda^{2n} U_n)^k$ to

$$\frac{1}{k!(z_1 \cdots z_{2n-2})^{k(n+1)}} \left(\frac{\partial}{\partial t_{2n-1}} \right)^k R(t_1, z_1, \ldots, t_{2n}, z_{2n})|_{t_{2n-1} = t_{2n} = z_{2n-1} = z_{2n} = 0}.$$

Roughly speaking, the projective limit is a space of polynomials in infinitely many variables with some conditions on diagonals. Its dual space is our analogue of the space $\Lambda^{\frac{\infty}{2}} \widehat{V}$. Let us denote it by $(\Lambda^{\frac{\infty}{2}} \widehat{V})^k$. Note that the space $\mathbf{C}^{k+1} \otimes \mathbf{C}[z, z^{-1}]$ can be identified with the dual to the projective limit

$$\cdots \leftarrow U_{-1}^k \leftarrow U_0^k \leftarrow U_1^k \leftarrow \cdots. \tag{4}$$

Therefore, our construction gives the semi-infinite "power" of the space $\mathbf{C}^{k+1} \otimes \mathbf{C}[z, z^{-1}]$.

In the second half of the paper, we construct a monomial basis of $\left(\Lambda^{\frac{\infty}{2}} \widehat{V}\right)^k$.

An irreducible representations of the vertex operator algebra A_k is a direct sum of irreducible representations of $\widehat{gl}_2$. The latter are of the form $\pi_j \otimes \mathcal{H}_p$ where π_j is the irreducible representation for $\widehat{sl}_2$ of level k and spin $\frac{j}{2}$, and $\mathcal{H}_p$ is a bosonic Fock space. The value p of the zero-mode is chosen suitably. We remark that the space $(\Lambda^{\frac{\infty}{2}} \widehat{V})^k$ discussed above is $\pi_0 \otimes \mathcal{H}_0$. The generators of A_k, which we denote by $\varphi_a(z)$ and $\varphi_a^*(z)$, act as follows:

$$\varphi_a(z) : \pi_j \otimes \mathcal{H}_p \rightleftharpoons \pi_{k-j} \otimes \mathcal{H}_{p + \sqrt{\frac{k}{2}}} : \varphi_a^*(z). \tag{5}$$

To be precise, the Fourier coefficients of the vertex operators act as above.

Let ω be the highest weight vector of one of the subspaces $\pi_j \otimes \mathcal{H}_p$. The Fourier coefficients of $\varphi_a(z)$, which we denote by $\varphi_{a,n}$, create a set of vectors in the total representation space. The vector ω itself is created from another highest weight vector, say ω', by a Fourier coefficient, and we can go further back to ω'', etc. The spaces generated from $\omega, \omega', \ldots$ by the $\varphi_{a,n}$, are increasing, and in fact, exhaust the whole space.

We prove this in three steps. In the first step, we write the quadratic relations satisfied by the vertex operators. We show that the space generated from ω is spanned by a certain set of vectors, "normal-ordered" monomials of $\varphi_{a,n}$ acting on ω. In the second step we show that the set of normal-ordered monomials is linearly independent by showing the non-degeneracy of the dual coupling. Finally, we show that the union of the subspaces generated from $\omega, \omega', \ldots$ is equal to the whole space by calculating the characters.

Before passing we mention briefly some references closely related to this work.

The construction of the level-1 vertex operators goes back to the papers [?],[?],[?],[?],[?]. The idea of semi-infinite construction used in this paper is originally developed in [?] for the current generators of $\widehat{sl}_2$.

Our normal-ordered monomials are labeled by the "paths" known in the solvable lattice models. In [?] paths are used to label a basis for the higher level representations of $\widehat{sl}_r$. The construction in that paper uses the Chevalley generators of $\widehat{sl}_r$ in order to create the basis vectors. This is the point of difference from the present paper. We construct a basis of the level k irreducible $\widehat{gl}_2$ modules by using the Fourier components of the vertex operators taking values in $\mathbf{C}^{k+1}$ modified with bosonic vertex operators.

In the q-deformed situation, similar constructions were given in [?],[?]. However, their construction does not recover the construction in this paper in the limit $q = 1$. The difference lie in the following point. The choice of the bosonic vertex operators in our construction is uniquely determined by the locality condition as explained. On the other hand, the choice in [?],[?] is such that the quadratic relations are of finite forms in terms of the Fourier coefficients. These two conditions are not compatible.

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2 Vertex operator algebra A_k

In this section we construct the vertex operator algebra A_k by using the vertex operator algebra of the $\widehat{sl}_2$ conformal field theory of level k and that of the free bosons.

2.1 Definition of A_k

Recall first some well-known facts about minimal conformal field theories associated with the affine Lie algebra

$$\widehat{sl}_2 = sl_2 \otimes \mathbf{C}[t, t^{-1}] \oplus \mathbf{C}c \oplus \mathbf{C}d. \tag{6}$$

We identify $sl_2 \otimes 1 \subset \widehat{sl}_2$ with sl_2. We use the basis of sl_2:

$$E = \begin{pmatrix} & 1 \\ 0 & \end{pmatrix}, H = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}, F = \begin{pmatrix} & 0 \\ 1 & \end{pmatrix}. \tag{7}$$

We set $X_i = X \otimes t^i$ for $X \in sl_2$. We also use the Chevalley generators

$$e_0 = F_1, h_0 = -H_0 + c, f_0 = E_{-1}, e_1 = E_0, h_1 = H_0, f_1 = F_0. \tag{8}$$

Let $P = \mathbf{C}\Lambda_0 \oplus \mathbf{C}\Lambda_1 \oplus \mathbf{C}\delta$ be the weight lattice. The dual lattice is $P^* = \mathbf{C}h_0 \oplus \mathbf{C}h_1 \oplus \mathbf{C}d$ where $(\Lambda_0, \Lambda_1, \delta)$ and (h_0, h_1, d) are dual to each other.

We consider level $k \in \mathbf{Z}_{\geq 0}$ representations, i.e., $c = k$. There are $k+1$ integrable highest weight representations $\pi_0, \pi_1, \ldots, \pi_k$ of level k. The representation π_j is generated by the highest weight vector $|j\rangle$ satisfying $E_i|j\rangle = 0$ $(i \geq 0)$, $H_i|j\rangle = F_i|j\rangle = 0$ $(i \geq 1)$ $H_0|j\rangle = j|j\rangle$ and $d|j\rangle = 0$. These representations constitute the Verlinde algebra. We will need the following relations in the Verlinde algebra.

$$\pi_k \cdot \pi_j = \pi_{k-j}. \tag{9}$$

To each representation π_j we can correspond the vertex operators. In this paper we consider the vertex operators corresponding to π_k. Let V be the $(k+1)$-dimensional representation of sl_2. Fix the standard basis $\psi_0, \psi_1, \ldots, \psi_k$ in V: $E\psi_a = (k-a+1)\psi_{a-1}$, $H\psi_a = (k-2a)\psi_a$, $F\psi_a = (a+1)\psi_{a+1}$. We have $(k+1)$ vertex operators $\psi_a(z) = \sum_{n \in \mathbf{Z}} \psi_{a,n} z^{-n}$. These are a collection of operators, and satisfy

$$[X_i, \psi_a(z)] = z^i (X\psi_a)(z) \text{ for } X \in \widehat{sl}_2, \tag{10}$$

$$[d, \psi_a(z)] = -z\frac{d}{dz}\psi_a(z). \tag{11}$$

From (??) it follows that they are acting from π_j to π_{k-j}. We fix the normalization of $\psi_a(z)$ by the condition

$$\langle k-j|\psi_j(z)|j\rangle = 1. \tag{12}$$

We use the currents $E(z) = \sum_i E_i z^{-i-1}$, $H(z) = \sum_i H_i z^{-i-1}$, $F(z) = \sum_i F_i z^{-i-1}$. We also use symbolic notations $\int H_{>0}(z) = -\sum_{i>0} \frac{H_i}{i} z^{-i}$ and $\int H_{<0}(z) = -\sum_{i<0} \frac{H_i}{i} z^{-i}$. The currents act on the integrable representations of level k and satisfy

$$E(z)^{k+1} = F(z)^{k+1} = 0. \tag{13}$$

The semi-infinite construction given in [?] is based on (??).

Consider the group element

$$D = e^{f_0} e^{-e_0} e^{f_0} e^{f_1} e^{-e_1} e^{f_1}. \tag{14}$$

It satisfies

$$\mathrm{Ad}D\, E(z) = z^{-2}E(z), \mathrm{Ad}D\, H(z) = H(z) - 2kz^{-1}, \mathrm{Ad}D\, F(z) = z^2 F(z). \tag{15}$$

Proposition 1 *Consider the actions of $E(z)$ and $F(z)$ on the level k representation π_j. We have*

$$E(z)^k = c_j e^{\int H_{<0}(z)} D z^{H_0} e^{\int H_{>0}(z)}, \tag{16}$$

$$F(z)^k = c_j e^{-\int H_{<0}(z)} D^{-1} z^{-H_0} e^{-\int H_{>0}(z)}. \tag{17}$$

where $c_j = (-1)^j k!$.

Proof Note that the action of D on π_j is determined (up to a constant multiple) by (??). For $k = 1$ (??) and (??) follows from the well-known result [?]. In the realization of [?] the mutiplication by e^{α_1}, where α_1 is the simple root of sl_2, is used instead of D. It is easy to check (??) for e^{α_1} in place of D. Therefore, we have $e^{\alpha_1} = c_j D$ for some constant c_j. By calculating $D|j\rangle \in \pi_j$ we obtain $c_j = (-1)^j$ for $k = 1$.

Suppose we know that (??) is true for k. Consider the tensor product of the representation of a level k module with the currents $E(z)_1, H(z)_1, F(z)_1$, and a level 1 module with the currents $E(z)_2, H(z)_2, F(z)_2$. In the tensor product we have a level $(k+1)$ action of $E(z) = E(z)_1 + E(z)_2$. Because of (??) we have $E(z)^{k+1} = (k+1)E(z)_1^k E(z)_2$. Therefore $E(z)^{k+1}$ has the same form as (??). The proof of (??) is similar. □

The operators $E(z)^k$ and $F(z)^k$ generate a vertex subalgebra. Their operator product expansion reads as

$$E(z)^k F(w)^k = (z-w)^{-2k} \sum_{i \geq 0} (z-w)^i : C_i^{(1)}(z) :, \tag{18}$$

where each term $C_i^{(1)}(z)$ is a differential polynomial of $H(z)$ and, in particular, $C_0^{(1)}(z)$ is a constant.

Consider now the square root of the automorphism (??):

$$U(E(z)) = -z^{-1}E(z), U(F(z)) = -zF(z), U(H(z)) = H(z) - kz^{-1}. \tag{19}$$

This is an outer automorphism.

Let ι be the outer automorphhim of $\widehat{sl}_2$ induced from the non-trivial Dynkin diagram automorphism. It is involutive and $\iota(e_0) = e_1, \iota(h_0) = h_1, \iota(f_0) = f_1$. It acts on the currents:

$$\iota\big(E(z)\big) = zF(z), \iota\big(H(z)\big) = -H(z) + kz^{-1}. \tag{20}$$

It also acts from π_j to π_{k-j} in such a way that $\iota X \iota^{-1} = \iota(X)$ $(X \in \widehat{sl}_2)$ in End $\left(\oplus_{j=0}^{k} \pi_j\right)$.

Set

$$D^{\frac{1}{2}} = e^{f_0} e^{-e_0} e^{f_0} \iota. \tag{21}$$

Then we have $\left(D^{\frac{1}{2}}\right)^2 = D$ and $D^{\frac{1}{2}} X D^{-\frac{1}{2}} = U(X)$ $(X \in \widehat{sl}_2)$ in End $\left(\oplus_{j=0}^{k} \pi_j\right)$.

Proposition 2 *In* $\mathrm{Hom}(\pi_j, \pi_{k-j})$ *we have*

$$\psi_0(z) = z^{\frac{j}{2}} e^{\frac{1}{2}\int H_{<0}(z)} D^{\frac{1}{2}} z^{\frac{H_0}{2}} e^{\frac{1}{2}\int H_{>0}(z)}, \tag{22}$$

$$\psi_k(z) = (-1)^{k-j} z^{\frac{j}{2}} e^{-\frac{1}{2}\int H_{<0}(z)} D^{-\frac{1}{2}} z^{-\frac{H_0}{2}} e^{-\frac{1}{2}\int H_{>0}(z)}. \tag{23}$$

Proof The operator $\psi_0(z) : \pi_j \to \pi_{k-j}$ is determined (up to a constant multiple) by (??) and the commutation relation with $H(z)$ and $E(z)$. It is easy to check that the right hand side of (??) commutes with $E(z)$ and has the correct commutation relations with $H(z)$. For $k = 1$ one can also check directly (??) and (??). The general case follows from the following coonsideration.

Suppose $k_l \in \mathbf{Z}_{>0}$ $(l = 1,2)$ and consider the representations $(\pi_{j_l})_l$ of level k_l. We put $(\quad)_l$ only to distinguish the different values of level for $l = 1, 2$. We use similar notations for the vertex operators $(\psi_a(z))_l$.

Consider the operator

$$(\psi_0(z))_1 \otimes (\psi_0(z))_2 : (\pi_{j_1})_1 \otimes (\pi_{j_2})_2 \to (\pi_{k_1-j_1})_1 \otimes (\pi_{k_2-j_2})_2 \,. \tag{24}$$

The algebra $\widehat{sl}_2$ of level $k_1 + k_2$ is acting on $(\pi_{j_1})_1 \otimes (\pi_{j_2})_2$ and $(\pi_{k_1-j_1})_1 \otimes (\pi_{k_2-j_2})_2$ diagonally. Let us decompose

$$(\pi_{j_1})_1 \otimes (\pi_{j_2})_2 = \oplus_{\gamma=0}^{k_1+k_2} \pi_\gamma \otimes S_\gamma. \tag{25}$$

The representation $(\pi_{k_1-j_1})_1 \otimes (\pi_{k_2-j_2})_2$ is canonically isomorphic to

$$\oplus_{\gamma=0}^{k_1+k_2} \pi_{k_1+k_2-\gamma} \otimes S_\gamma. \tag{26}$$

¿From the explicit formula (??) it follows that the operator $(\psi_0(z))_1 \otimes (\psi_0(z))_2$ is acting from $\pi_\gamma \otimes S_\gamma$ to $\pi_{k_1+k_2-\gamma} \otimes S_\gamma$ and equal to $\psi_0(z) \otimes 1$. □

Set $\phi_a(z) = z^{-j/2} \psi_a(z)$. From (??) we can deduce the operator product expansion:

$$\phi_0(z)\phi_0(w) = (z-w)^{\frac{k}{2}} \left(\sum_{i \geq 0} : E(z)^k C_i^{(2)}(z) : (z-w)^i \right), \tag{27}$$

where $C_i^{(2)}(z)$ is a differential polynomial of $H(z)$ and, in particular, that $C_0^{(2)}$ is a constant. A similar result holds if we replace ϕ_0, E with ϕ_k, F. We have

also

$$\phi_0(z)\phi_k(w) = (z-w)^{-\frac{k}{2}} \left(\sum_{i\geq 0} : C_i^{(3)}(z) : (z-w)^i \right), \tag{28}$$

where $C_i^{(3)}(z)$ is a differential polynomial of $H(z)$ and, in particular, $C_0^{(3)}(z)$ is a constant.

Let B be the Heisenberg algebra with the basis $\{a_j\}$ ($j \in \mathbf{Z}$) and the relations $[a_i, a_j] = i\delta_{i+j,0}$. Let $\mathcal{H}_q$ be the irreducible representation of B; $\mathcal{H}_q$ contains the vacuum vector $v(q)$ such that $a_j v(q) = 0$ ($j > 0$) and $a_0 v(q) = qv(q)$. Introduce the vertex operator $w(p,z)$:

$$w(p,z) = \exp\left(-p \sum_{i<0} \frac{a_i}{i} z^{-i} \right) T^p z^{pa_0} \exp\left(-p \sum_{i>0} \frac{a_i}{i} z^{-i} \right). \tag{29}$$

The operator T^p acts from $\mathcal{H}_q$ to $\mathcal{H}_{q+p}$. It commutes with a_i ($i \neq 0$) and satisfies $T^p v(q) = v(q+p)$. The operator product expansion of $w(p,z)$ has the form

$$w(p_1,z_1)w(p_2,z_2) = (z_1-z_2)^{p_1 p_2} \left(\sum_{i\geq 0} (z_1-z_2)^i S_i(z_1) \right) \tag{30}$$

where $S_0(z) = w(p_1+p_2, z)$. We have

$$[a_n, w(p,z)] = pz^n w(p,z). \tag{31}$$

Introduce now the operators acting from $\pi_j \otimes \mathcal{H}_q$ to $\pi_{k-j} \otimes \mathcal{H}_{q\pm\sqrt{k/2}}$

$$\varphi_a(z) = \phi_a(z) w\big(\sqrt{k/2}, z\big), \tag{32}$$

$$\varphi_a^*(z) = (-1)^{k-j} \phi_a(z) w\big(-\sqrt{k/2}, z\big). \tag{33}$$

Definition 1 *A_k is the vertex operator algebra generated by $\{\varphi_a(z)\}, \{\varphi_a^*(z)\}$.*

2.2 Properties of A_k

We now present some properties of the algebra A_k without giving proofs.

Let $W \simeq V$ be the $(k+1)$-dimensional irreducible representation of gl_2. The matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is acting on W by the scalar $\sqrt{\frac{k}{2}}$. We write the basis of W as φ_a instead of ψ_a, i.e., the linear map $\psi_a \mapsto \varphi_a$ is sl_2 linear. Let W^* be the dual space. We identify W^* with V by the invariant coupling determined by $(\psi_0, \psi_k) = 1$. We write the basis of W^* as φ_a^* instead of ψ_a. We denote the map $W \to W^*$, $\varphi_a \mapsto \varphi_a^*$ by $*$.

The algebra A_k is generated by the spaces $W(z)$ with basis $\varphi_0(z),\dots,\varphi_k(z)$ and $W^*(z)$ with basis $\varphi_0^*(z),\dots,\varphi_k^*(z)$. For a vector $w\in W$ we will denote by $w(z)$ and $w^*(z)$ the corresponding operators in A_k.

(a) Operators $w_1(z), w_2(z)$ ($w_i\in W$) are commuting if k is even and skew-commuting if k is odd. The same is true for $w_1^*(z), w_2^*(z)$.

We call a vector $w\in W$ the highest if it is annihilated by a nilpotent matrix u in gl_2: $u(w)=0$. The set of all highest vectors form a cone $K\subset W$. The set K is an orbit by the SL_2 action on W.

(b) Let $w\in K$, then $w(z)w^{(l)}(z)=0$ for $l<k$ and $w^*(z)w^{*(l)}(z)=0$ also for $l<k$. Here $w^{(l)}(z)$ is the l-th derivative of $w(z)$.

(c) If $w_1, w_2\in W$, then

$$w_1(z_1)w_2^*(z_2)(z_1-z_2)^k = w_2^*(z_2)w_1(z_1)(z_2-z_1)^k. \tag{34}$$

The operator product expansion of $w_1(z_1)$ and $w_2^*(z_2)$ has the form:

$$w_1(z_1)w_2^*(z_2) = (z_1-z_2)^{-k}\left(S_0+(z_1-z_2)S_1(z_1)+\cdots\right) \tag{35}$$

where S_0 is a scalar, $S_0=(w_1,w_2)$, and $S_1(z)$ is a linear combination of Heisenberg algebra $a(z)=\sum a_i z^{-i}$ and $\widehat{sl}_2=\{E(z),H(z),F(z)\}$. Altogether they constitute the algebra $\widehat{gl}_2$. Therefore, we see that A_k contains $\widehat{gl}_2$ as a Lie subalgebra. The algebra A_k is generated by $\varphi_0(z),\varphi_0^*(z)$ and $\widehat{gl}_2$.

(d) Let $w\in K$. Then, the operator product of $w(z_1)$ and $w^*(z_2)$ has no singular terms. In particular, it means that

$$w(z_1)w^*(z_2) = (-1)^k w^*(z_2)w(z_1). \tag{36}$$

Cases $k=1$ and $k=2$. It is easy to see that for $k=1$ the operators $\{\varphi_0(z),\ \varphi_1(z),\ \varphi_0^*(z),\ \varphi_1^*(z)\}$ generate the usual Clifford algebra. In the case $k=2$ we have 6 generators $\{\varphi_0(z),\ \varphi_1(z),\ \varphi_2(z),\ \varphi_0^*(z),\ \varphi_1^*(z),\ \varphi_2^*(z)\}$. The operator product of $\varphi_a(z_1)$ and $\varphi_b^*(z_2)$ starts from $(z_1-z_2)^{-2}$ with a constant coefficient. The next term is $\widehat{gl}_2$. Therefore, all $\varphi_a(z),\varphi_a^*(z)$ generate (with respect to the bracket) the central extension of some Lie algebra. It is $\widehat{sp}_4$ with level 1.

(e) Fix two non-negative integers k_1,k_2. There is a homomorphism of algebras,

$$\kappa : A_{k_1+k_2} \to A_{k_1}\otimes A_{k_2}. \tag{37}$$

The map κ can be characterized by the following way. First we have $\widehat{sl}_2$ in $A_{k_1+k_2}$. This $\widehat{sl}_2$ goes by a diagonal way in $\widehat{sl}_2\oplus\widehat{sl}_2$. The same is true for the Heisenberg algebra B in $A_{k_1+k_2}$. Finally, $\varphi_0(z)\in A_{k_1+k_2}$ goes to the $\big(\varphi_0(z)\big)_1\otimes\big(\varphi_0(z)\big)_2$, and $\varphi_0^*(z)$ goes to $\big(\varphi_0^*(z)\big)_1\otimes\big(\varphi_0^*(z)\big)_2$.

2.3 Representations of A_k

Let us form the following space:

$$\cdots \oplus \left(\pi_k \otimes \mathcal{H}_{-\sqrt{\frac{k}{2}}}\right) \oplus (\pi_0 \otimes \mathcal{H}_0) \oplus \left(\pi_k \otimes \mathcal{H}_{\sqrt{\frac{k}{2}}}\right) \oplus \left(\pi_0 \otimes \mathcal{H}_{2\sqrt{\frac{k}{2}}}\right) \oplus \cdots \quad (38)$$

It is clear that the operators $W(z)$ and $W^*(z)$ are acting on this space in such a way that they can be expanded in z^n ($n \in \mathbf{Z}$). We can generalize this construction by writing

$$\left(\bigoplus_{s\in 2\mathbf{Z}} \pi_j \otimes \mathcal{H}_{q+s\sqrt{\frac{k}{2}}}\right) \oplus \left(\bigoplus_{s\in 2\mathbf{Z}+1} \pi_{k-j} \otimes \mathcal{H}_{q+s\sqrt{\frac{k}{2}}}\right) \quad (39)$$

where $q = m\sqrt{\frac{2}{k}}$, $m = 0, 1, \ldots, k-1$ if j is even, and $q = (m+\frac{1}{2})\sqrt{\frac{2}{k}}$, $m = 0, 1, \ldots, k-1$ if j is odd.

Therefore, our space is labeled by two numbers j ($0 \le j \le k$) and $r = q\sqrt{2k}$, ($0 \le r \le 2k-1$) where $j+r$ is even. Denote this space by $R(j,r)$. It is evident that $R(j,r) = R(j',r')$ if $j + j' = k$ and $r + k = r' \pmod{2k}$.

Without giving a proof we state

Proposition 3 *Each irreducible representation of the algebra A_k has the form $R(j,r)$ for some j,r. Therefore, the algebra A_k has $k(k+1)/2$ irreducible representations.*

Representations $R(j,r)$ of A_k form the minimal models. It means in particular, that representations $R(j,r)$ form the Verlinde algebra. We describe it here. Denote by $\mathcal{V}_k$ the Verlinde algebra of $\widehat{sl}_2$ of level k. It is an algebra with basis $\pi_0, \pi_1, \ldots, \pi_k$ and the product

$$\pi_i \pi_j = \pi_{i-j} + \pi_{i-j+2} + \cdots + \pi_s \quad (40)$$

where $j \le i$ and $s = \min(i+j, 2k-i-j)$. Note that $\pi_0 = 1$. Let $\mathcal{E}_{2k}$ be the algebra with basis $1, \varepsilon, \varepsilon^2, \ldots, \varepsilon^{2k-1}$, and product $\varepsilon^r \varepsilon^s = \varepsilon^{r+s \pmod{2k}}$. Consider the tensor product $\mathcal{V}_k \otimes \mathcal{E}_{2k}$. Define on $\mathcal{V}_k \otimes \mathcal{E}_{2k}$ the following operator γ:

$$\gamma(\pi_j \otimes \varepsilon^r) = (-1)^{j+r} \pi_j \otimes \varepsilon^r. \quad (41)$$

Let $[\mathcal{V}_k \otimes \mathcal{E}_{2k}]^\gamma$ be the set of fixed elements in $\mathcal{V}_k \otimes \mathcal{E}_{2k}$ by γ. This is a subalgebra. The Verlinde algebra of A_k is the quotient $[\mathcal{V}_k \otimes \mathcal{E}_{2k}]^\gamma / J$ where J is the ideal generated by the relation $\pi_k \otimes \varepsilon^k = 1$. The element $\pi_j \otimes \varepsilon^r$ corresponds to the representation $R(j,r)$.

2.4 Quadratic relations

The operators $W(z)$ acting on the irreducible representations of A_k satisfy the following quadratic relations.

$$\text{(a)} \qquad w_1(z)w_2(w) = (-1)^k w_2(w)w_1(z), w_1, w_2 \in W, \tag{42}$$

$$\text{(b)} \qquad w(z)w^{(l)}(z) = 0 (l < k) \text{ if } w \in K. \tag{43}$$

We call them the relations (R).

Suppose that operators $W(z)$ are acting on some space Q with the grading $Q = Q_0 \oplus Q_{-1} \oplus Q_{-2} \oplus \cdots$. Choose vectors $\rho \in Q$ and $\rho^* \in Q^*$. The correlation function is by definition the following matrix element:

$$\langle \rho^*, w_1(z_1) \cdots w_n(z_n)\rho \rangle, w_a \in W. \tag{44}$$

Changing $w_1, \ldots, w_n$ we get the vector-valued function $F^*(\rho^*, \rho; z_1, \ldots, z_n)$ with values in $\underbrace{W^* \otimes W^* \otimes \cdots \otimes W^*}_{n}$, which we will also call the correlation function. It is evident that $F^*(\rho^*, \rho; z_1, \ldots, z_n)$ is symmetric if k is even, and skew-symmetric if k is odd. (We permute simultaneously coordinates z_i and components in $W^* \otimes W^* \otimes \cdots \otimes W^*$). $F^*(\rho^*, \rho; z_1, \ldots, z_n)$ is a Laurent polynomial. W^* is an irreducible sl_2-module, so we can realize W^* in the space of sections of the line bundle on $\mathbf{C}P^1$. If we choose the coordinate t on $\mathbf{C}P^1$ and trivialize the line bundle on $\mathbf{C}P^1$ then W^* will be identified with the space of polynomials in t of degree less than or equal to k. Therefore, the tensor product $W^* \otimes \cdots \otimes W^*$ can be identified with the space of polynomials in $t_1, \ldots, t_n$ of degree less than or equal to k in each variable t_j. The correlation function $F^*(\rho^*, \rho; z_1, \ldots, z_n)$ can be viewed as a scalar function in variable $X_1, \ldots, X_n$ where $X_j = (t_j, z_j)$; we denote it by $F(\rho^*, \rho; X_1, \ldots, X_n)$.

Proposition 4 *Suppose that operators $W(z)$ satisfy the quadratic relations (R). Then the corresponding function $F(\rho^*, \rho, X_1, \ldots, X_n)$ has a zero of order at least k if $X_i = X_j$ $1 \leq i < j \leq n$. In other words,*

$$\frac{\partial^{j_1}}{\partial t_i^{j_1}} \frac{\partial^{j_2}}{\partial z_i^{j_2}} F(\rho^*, \rho, X_1, \ldots, X_n)\Big|_{X_i = X_j} = 0 \text{ if } j_1 + j_2 < k - 1. \tag{45}$$

We prove this proposition in 2.3.

Consider the irreducible representation $R(j, r)$ of the algebra A_k. We will call a vector in $\pi_i \otimes \mathcal{H}_q \subset R(j, r)$ "extremal" if it is the product of an extremal vector in π_i and the highest weight vector $v(q)$.

Proposition 5 *Introduce the Fourier coefficients of $\varphi_a(z)$:*

$$\varphi_a(z) = \sum_n \varphi_{a,n} z^{-n}. \tag{46}$$

Let $\omega \in R(j,r)$ be an extremal vector. Then there exists a set of integers N_a depending on ω and a such that the following are valid.

$$(i) \quad \varphi_{a,n}\omega = 0 \text{ if } n > N_a. \tag{47}$$

(ii) *Let B_ω be the subspace of $R(j,r)$: $B_\omega = \mathbf{C}[\varphi_{a,n}; 0 \le a \le k, n \in \mathbf{Z}]\omega$ Then we have an isomorphism*

$$\mathbf{C}[\varphi_{a,n}; 0 \le a \le k, n \in \mathbf{Z}]/N \tilde{\to} B_\omega \tag{48}$$

where the ideal N is generated by the relations (R) and (??).

Proof of (i) Using the automorphism of A_k induced by $D^{1/2}$ we can reduce the proof to the case where $\omega = \omega_{j,j+2l}$:

$$\omega_{j,j+2l} = |j\rangle \otimes v\Big(\frac{j+2l}{\sqrt{2k}}\Big). \tag{49}$$

We define the local energy function $h : I_k \otimes I_k \to I_k$ where $I_k = \{0, 1, \ldots, k\}$:

$$h_{a,b} = \min(a, k-b). \tag{50}$$

For $\omega = \omega_{j,j+2l}$ we set

$$N_a = h_{a,k-j} - j - l. \tag{51}$$

Then we have

$$\varphi_{a,n}\omega_{j,j+2l} = 0 \text{ if } n > N_a, \tag{52}$$

$$\varphi_{j,-l}\omega_{j,j+2l} = \omega_{k-j,j+2l+k}. \tag{53}$$

These equalities follow by a direct calculation. Because of the automorphism of A_k induced by $T^{\sqrt{2/k}}$, it is enough to prove it for one value of l. □

The proof of (ii) is given in 2.3.

3 Monomial bases and correlation functions

In the previous section we have introduced the vertex operator algebra A_k that are generated by $\varphi_a(z), \varphi_a^*(z)$ $(0 \le a \le k)$. In this section we construct a basis of its irreducible representations. We interprete the quadratic relations of the vertex operators as normal-ordering rules. The vectors in the basis are normal-ordered monomials (see below for the precise definition).

3.1 Symmetric and skew-symmetric tensors

In this section we work on the Fourier coefficients of the vertex operators: $\varphi_a(z) = \sum_{n\in\mathbf{Z}} \varphi_{a,n} z^{-n}$. First we consider the $\varphi_{a,n}$ as abstract generators of an algebra B_k where we assume only the (skew-) commutativity,

$$[\varphi_{a,m}, \varphi_{b,n}]\mp = 0 \text{ if } k \text{ is even (odd).} \tag{54}$$

The $\widehat{sl}_2$-action is given by

$$e_0\varphi_a(z) = (a+1)z\varphi_{a+1}(z), e_1\varphi_a(z) = (k-a+1)\varphi_{a-1}(z), \tag{55}$$
$$f_0\varphi_a(z) = (k-a+1)z^{-1}\varphi_{a-1}(z), f_1\varphi_a(z) = (a+1)\varphi_{a+1}(z), \tag{56}$$
$$h_0\varphi_a(z) = -(n-2a)\varphi_a(z), h_1\varphi_a(z) = (n-2a)\varphi_a(z). \tag{57}$$

The following diagram shows the action for $k = 2$ schematically.

$$\begin{array}{ccccccc}
\cdots & \varphi_{0,-1} & & \varphi_{0,0} & & \varphi_{0,1} & \cdots \\
 & \downarrow f_1 & \nwarrow f_0 & \downarrow f_1 & \nwarrow f_0 & \downarrow f_1 & \\
\cdots & \varphi_{1,-1} & & \varphi_{1,0} & & \varphi_{1,1} & \cdots \\
 & \downarrow f_1 & \nwarrow f_0 & \downarrow f_1 & \nwarrow f_0 & \downarrow f_1 & \\
\cdots & \varphi_{2,-1} & & \varphi_{2,0} & & \varphi_{2,1} & \cdots
\end{array}$$

We introduce an ordering of the index set $\{0,1,\dots,k\}\times\mathbf{Z}$: $(a,m) < (b,n)$ if and only if $m < n$, or $m = n$ and $a > b$. Note that f_0 and f_1 are lowering operators in this ordering.

If n is even, we have $B_k = \oplus_{s=0}^{\infty} B_k^{(s)}$ where

$$B_k^{(s)} = \oplus_{(a_1,n_1)\le\cdots\le(a_s,n_s)} \mathbf{C}\varphi_{a_1,n_1}\cdots\varphi_{a_s,n_s}. \tag{58}$$

If n is odd, we have

$$B_k^{(s)} = \oplus_{(a_1,n_1)<\cdots<(a_s,n_s)} \mathbf{C}\varphi_{a_1,n_1}\cdots\varphi_{a_s,n_s}. \tag{59}$$

In order to handle the quadratic relations (??), we need a completion of the algebra. For $N\in\mathbf{Z}$, let $B_{k,N}$ be the ideal of B_k that is generated by the set of elements $\{\varphi_{a,m}; m > N\}$. We set $B_{k,N}^{(s)} = B_k^{(s)} \cap B_{k,N}$, and define the completion of the algebra B_k.

$$\bar{B}_k = \oplus_{s=0}^{\infty} \bar{B}_k^{(s)}, \tag{60}$$
$$\bar{B}_k^{(s)} = \lim_{\leftarrow n} B_k^{(s)}/B_{k,n}^{(s)}. \tag{61}$$

The $\widehat{sl}_2$-action extends to $\bar{B}_k$.

Let $U \subset W \otimes W$ be a subspace. We define $D^{(m)}U$ to be the subspace of $\bar{B}_k^{(2)}$ spanned by the Fourier coefficients of

$$D^{(m)}w(z) = \sum_{a,b} c_{a,b}\varphi_a(z)\varphi_b^{(m)}(z) \tag{62}$$

where $w = \sum_{a,b} c_{a,b}\varphi_a \otimes \varphi_b \in U$ and $\varphi_b^{(m)}(z)$ is the m-th derivative of $\varphi_b(z)$.

We have the decomposition

$$W \otimes W = S \oplus A \tag{63}$$

where S and A are the symmetric and the skew-symmetric tensors, respectively. They are invariant with respect to the sl_2 action. In fact, we have the irreducible decomposition of S and A as follows.

$$\begin{aligned} S &= \mathbf{C}^{2k+1} \oplus \mathbf{C}^{2k-3} \oplus \cdots \\ &= \begin{cases} \mathbf{C}^1 \oplus \mathbf{C}^5 \oplus \cdots \text{ if } k \text{ is even;} \\ \mathbf{C}^3 \oplus \mathbf{C}^7 \oplus \cdots \text{ if } k \text{ is odd,} \end{cases} \end{aligned} \tag{64}$$

$$\begin{aligned} A &= \mathbf{C}^{2k-1} \oplus \mathbf{C}^{2k-5} \oplus \cdots \\ &= \begin{cases} \mathbf{C}^3 \oplus \mathbf{C}^7 \oplus \cdots \text{ if } k \text{ is even;} \\ \mathbf{C}^1 \oplus \mathbf{C}^5 \oplus \cdots \text{ if } k \text{ is odd.} \end{cases} \end{aligned} \tag{65}$$

Set $\Omega = \sum_{m=0}^{\infty} D^{(m)}(W \otimes W)$. The space of the quadratic relations is an $\widehat{sl}_2$-invariant subspace of Ω. We now determine these spaces (see Propositions ?? and ??).

In abuse of notation we write $D^{(m)}\mathbf{C}^{2j+1}$ $(0 \le j \le k)$ for $D^{(m)}U$ where U is the unique $(2j+1)$-dimensional component of $W \otimes W$. We have

$$D^{(0)}(W \otimes W) = D^{(0)}\mathbf{C}^1 \oplus D^{(0)}\mathbf{C}^5 \oplus \cdots. \tag{66}$$

If $m \ge 1$ we have

$$D^{(m)}(W \otimes W) = D^{(m)}\mathbf{C}^1 \oplus D^{(m)}\mathbf{C}^3 \oplus \cdots. \tag{67}$$

These are direct sums. However, the sum of $D^{(m)}(W \otimes W)$ for $m = 0, 1, \ldots$ is not direct. In fact, we have

Proposition 6 *(i) We have the irreducible decomposition of the sl_2 module Ω:*

$$\begin{aligned} \Omega &= (D^{(0)}\mathbf{C}^1 \oplus D^{(0)}\mathbf{C}^5 \oplus \cdots) \\ &\quad \oplus (D^{(1)}\mathbf{C}^3 \oplus D^{(1)}\mathbf{C}^7 \oplus \cdots) \\ &\quad \oplus (D^{(2)}\mathbf{C}^1 \oplus D^{(2)}\mathbf{C}^5 \oplus \cdots) \\ &\quad \oplus (D^{(3)}\mathbf{C}^3 \oplus D^{(3)}\mathbf{C}^7 \oplus \cdots) \oplus \cdots. \end{aligned} \tag{68}$$

(ii) The space $D^{(m)}\mathbf{C}^{2j+1}$ which does not appear in the above decomposition is contained in $\sum_{l=0}^{m-1} D^{(l)}\mathbf{C}^{2j+1}$.

Proof The statement (ii) follows from the (skew-)commutativity of $\varphi_{a,n}$ and the following equality.

$$\begin{aligned}&\varphi_a^{(m)}(z)\otimes\varphi_b(z)-(-1)^m\varphi_a(z)\otimes\varphi_b^{(m)}(z)\\&=\sum_{j=0}^{m-1}(-1)^j\binom{m}{j}\frac{d^{m-j}}{dz^{m-j}}\left(\varphi_a(z)\otimes\frac{d^j}{dz^j}\varphi_b(z)\right).\end{aligned}\tag{69}$$

We now prove that the summands in (??) are linearly independent. Suppose $w=\sum c_{ab}\varphi_a\otimes\varphi_b\in S$ and $k+m$ is even, or $w\in A$ and $k+m$ is odd. Consider the degree d term of $D^{(m)}w$:

$$\begin{aligned}&\sum_{d_1+d_2=d}c_{ab}(-d_2)(-d_2-1)\cdots(-d_2-m+1)\varphi_{a,d_1}\varphi_{b,d_2}\\=&\sum_{d_1+d_2=d}\frac{1}{2}c_{ab}\{(-d_2)(-d_2-1)\cdots(-d_2-m+1)\\&+(-1)^m(-d_1)(-d_1-1)\cdots(-d_1-m+1)\}\varphi_{a,d_1}\varphi_{b,d_2}\\=&\sum_{2d_1\le d}c_{ab}(\epsilon(d_1)d_1^m+\cdots)\varphi_{a,d_1}\varphi_{b,d-d_1},\end{aligned}\tag{70}$$

where

$$\epsilon(d_1)=\begin{cases}1\text{ if }2d_1=d;\\2\text{ otherwise.}\end{cases}\tag{71}$$

Namely, the coefficients increase in the m-th power of d_1. The proof of (i) is over. □

The sl_2 structure of Ω is clear from (??). We now determine the $\widehat{sl}_2$ structure of Ω. This is not semi-simple. The following table illustrates the closure relations of the $\widehat{sl}_2$-action.

	$m=0$		$m=1$		$m=2$		$m=3$		$m=4$
$k=0$	1				1				1
		↖		↙		↖		↙	
$k=1$			3				3		
		↙		↖		↙		↖	
$k=2$	**5**				5				5
		↖		↙		↖		↙	
$k=3$			**7**				7		
		↙		↖		↙		↖	
$k=4$	**9**				**9**				9
		↖		↙		↖		↙	
$k=5$			**11**				**11**		
		↙		↖		↙		↖	
$k=6$	**13**				**13**				**13**

(72)

The integer $2j+1$ in the m-th column signifies $D^{(m)}\mathbf{C}^{2j+1}$ in (??). We have

Proposition 7 *(i) Let $\Omega_1 \subset \Omega$ be an $\widehat{sl}_2$-invariant subspace. If $D^{(m)}\mathbf{C}^{2j+1} \subset \Omega_1$, then $D^{(m-1)}\mathbf{C}^{2j-1} \oplus D^{(m-1)}\mathbf{C}^{2j+3} \subset \Omega_1$.*

(ii) Consider a subdiagram I of (??). Let $\Omega(I)$ be the union of the corresponding subspaces in Ω. $\Omega(I)$ is $\widehat{sl}_2$-invariant if and only if I is closed with respect to the arrows.

Proof The 'only if' part of *(ii)* follows from *(i)*. Let us show *(i)* and the 'if' part of *(ii)*. Suppose that $D^{(m)}\mathbf{C}^{2j+1} \subset \Omega(I)$. Let $v_0^{(j)} = \sum c_{ab}^{(j)} \varphi_a \otimes \varphi_b \in \mathbf{C}^{2j+1} \subset W \otimes W$ be the highest weight vector. We will show that the Fourier coefficients of $e_0 D^{(m)} v_0^{(j)}(z)$ (see (??)) belongs to the closure of $D^{(m)}\mathbf{C}^{2j+1}$ in the sense of the arrows. Because $[e_0, f_1] = 0$, the same statement for $e_0 \left(D^{(m)} f_1^l v_0^{(j)}(z) \right)$ then follows.

We have

$$e_0 D^{(m)} v_0^{(j)}(z) = z f_1 \left(D^{(m)} v_0^{(j)}(z) \right) + v_1(z),$$
$$v_1(z) = m \sum c_{ab}^{(j)} \varphi_a(z) (f_1 \varphi_b(z))^{(m-1)}. \tag{73}$$

It is easy to see that $(1 \otimes f_1) v_0^{(j)}$ is a linear combination of $f_1^2 v_0^{(j+1)}$, $f_1 v_0^{(j)}$ and $v_0^{(j-1)}$ with non-zero coefficients. The proposition follows from this. □

3.2 Quadratic relations and normal-ordering rules

We assume that $k \geq 2$. We denote by $Q_k^{(2)}$ the $\widehat{sl}_2$-invariant subspace of $\bar{B}_k^{(2)}$ that is generated by $\varphi_0(z)\varphi_0^{(k-2)}(z)$ (or equivalently by $\varphi_k(z)\varphi_k^{(k-2)}(z)$).

For example, if $k = 2$, the $\widehat{sl}_2$ action generates

$$\varphi_0(z)^2, \varphi_0(z)\varphi_1(z), 2\varphi_0(z)\varphi_1(z) + \varphi_1(z)^2, \varphi_1(z)\varphi_2(z), \varphi_2(z)^2. \tag{74}$$

The following proposition follows immediately from Propositions ?? and ??.

Proposition 8 *The sl_2 decomposition of $Q_k^{(2)}$ is given by*

$$Q_k^{(2)} = \oplus_{\substack{j \equiv m \,(\mathrm{mod}\, 2), \\ 2m+5 \leq 2j+1 \leq 2k+1}} D^{(m)}\mathbf{C}^{2j+1} \tag{75}$$

The summands in (??) are indicated in (??) by the boldface letters. For example, $Q_2^{(2)}$ consists of $D^{(0)}\mathbf{C}^5$. $Q_3^{(2)}$ of $D^{(0)}\mathbf{C}^5$ and $D^{(1)}\mathbf{C}^7$, $Q_4^{(2)}$ of $D^{(0)}\mathbf{C}^5$, $D^{(1)}\mathbf{C}^7$, $D^{(0)}\mathbf{C}^9$, $D^{(2)}\mathbf{C}^9$, etc. We define the algebra R_k to be the quotient of $\bar{B}_k$ by the ideal generated by $Q_k^{(2)}$.

Our next aim is to rewrite the quadratic relations as the normal-ordering rules. Consider first a special case, $k = 2$ and the quadratic relation $\varphi_0(z)^2 = 0$. In terms of the Fourier coefficients the relations read as

$$\sum_{j_1 + j_2 = j} \varphi_{0,j_1} \varphi_{0,j_2} = 0. \tag{76}$$

We can rewrite them as follows:

$$\varphi_{0,l}\varphi_{0,l} = -2\sum_{n=1}^{\infty}\varphi_{0,l-n}\varphi_{0,l+n}\,, \tag{77}$$

$$\varphi_{0,l}\varphi_{0,l+1} = -\sum_{n=1}^{\infty}\varphi_{0,l-n}\varphi_{0,l+1+n}\cdot \tag{78}$$

The product $\varphi_{0,j_1}\varphi_{0,j_2}$ where (j_1, j_2) satisfying $j_1 \geq j_2 - 1$ are written as linear combinations of products $\varphi_{0,l_1}\varphi_{0,l_2}$ where (l_1, l_2) satisfying $l_1 \leq l_2 - 2$. We call such relations normal-ordering rules.

We now consider the general situation. The way to rewrite the quadratic relations as normal-ordering rules is not unique. Set $J = \{0, 1, \ldots, k\} \times \mathbf{Z}$. Choose and fix a subset $O \subset J \otimes J$. A pair $((a,m),(b,n))$ is called "normal-ordered" if $((a,m),(b,n)) \in O$. A product $\varphi_{a_1,j1}\cdots\varphi_{a_s,j_s}$ is called normal-ordered if each pair $((a_l, j_l),(a_{l+1}, j_{l+1}))$ $(1 \leq l \leq s-1)$ is normal-ordered.

A set of normal-ordered pairs O is called good if the following are valid.

(a) Any product $\varphi_{a_1,j_1}\cdots\varphi_{a_s,j_s}$ can be rewritten as a linear combination of normal-ordered products by using the normal-ordering rules.

(b) Normal-ordered products are linearly independent.

Our aim is to find a good subset O.

Define the local energy function $h : J \times J \to \mathbf{Z}$ by

$$h((a,m),(b,n)) = h_{a,b} - m + n \tag{79}$$

where $h_{a,b}$ is given by (??). We set

$$O = \{((a,m),(b,n)); h((a,m),(b,n)) \geq k\}. \tag{80}$$

Our goal is to prove

Proposition 9 *The set of normal-ordered pairs O given above is good.*

We give a proof of (a) in this section, and (b) in 2.3. For the proof of (a) it is enough to show the following

Proposition 10 *If $\varphi_{a_0,m_0}\varphi_{b_0,n_0}$ is not normal-ordered, one can rewrite it as a linear combination of $\varphi_{a,m}\varphi_{b,n}$ where $(a,m) < (a_0,m_0)$.*

Proof If $(a_0,m_0) > (b_0,n_0)$, or if k is odd and $(a_0,m_0) = (b_0,n_0)$, then the assertion is obvious because of the (skew-)commutativity of $\varphi_{a,n}$. Therefore, we assume that

$$(a_0,m_0) \leq (b_0,n_0) \text{ if } k \text{ is even}, \quad (a_0,m_0) < (b_0,n_0) \text{ if } k \text{ is odd}. \tag{81}$$

For $m_0 \leq n_0$ we set

$$F_{m_0,n_0} = \{(a_0,b_0); \Big((a_0,m_0),(b_0,n_0)\Big) \notin O \text{ and } (??) \text{ is satisfied}\}, \tag{82}$$

and $(F_{m_0,n_0})_d = \{(a,b) \in F_{m_0,n_0}; a+b=d\}$.

Let $U^*_{m_0,n_0}$ (resp. $(U^*_{m_0,n_0})_d$) $\subset (W\otimes W)^*$ be the subspace spanned by the set of elements $\varphi^*_{a_0}\otimes\varphi^*_{b_0}$ where $(a_0,b_0)\in F_{m_0,n_0}$ (resp. $(F_{m_0,n_0})_d$). Here $\langle\varphi^*_{a_1}\otimes\varphi^*_{b_1}, \varphi_{a_2}\otimes\varphi_{b_2}\rangle = \delta_{a_1a_2}\delta_{b_1b_2}$. Note that there is an action of sl_2 on $(W\otimes W)^*$ such that $\langle Xw^*, w\rangle + \langle w^*, Xw\rangle = 0$ for $w^*\in(W\otimes W)^*$ and $w\in W\otimes W$.

For $w=\sum c_{ab}\varphi_a\otimes\varphi_b \in W\otimes W$ we set $w_{m,n}=\sum c_{ab}\varphi_{a,m}\varphi_{b,n}\in R_k$. We denote the map $w\to w_{m,n}$ by $\pi_{m,n}$.

Suppose that $w\in \mathbf{C}^{2j+1}\subset W\otimes W$. If $n_0-m_0\geq 0$ and $D^{(n_0-m_0)}\mathbf{C}^{2j+1}$ belongs to (??), then by using Proposition ?? (ii) we have

$$\sum c_{ab}\varphi_a(z)\left(z^{n_0-1}\varphi_b(z)\right)^{(n_0-m_0)} = 0 \tag{83}$$

in R_k. The degree m_0+n_0 part of this equation contains $(-1)^{n_0-m_0}(n_0-m_0)!w_{m_0,n_0}$, and the rest is a linear combination of w_{m_0-l,n_0+l} $(l\geq 1)$. Similarly, if $n_0-m_0\geq 1$ and $D^{(n_0-m_0-1)}\mathbf{C}^{2j+1}$ belongs to (??), we have

$$\sum c_{ab}\varphi_a(z)\left(z^{n_0-1}\varphi_b(z)\right)^{(n_0-m_0-1)} = 0, \tag{84}$$

in R_k. The degree m_0+n_0 part of this equation contains $(-1)^{n_0-m_0-1}2(n_0-m_0-1)!w_{m_0,n_0}$, and the rest is a linear combination of w_{m_0-l,n_0+l} $(l\geq 1)$. Therefore, we have

(i) if $m_0=n_0$ and $w\in\mathbf{C}^{4j+1}\subset W\otimes W$ $(1\leq j\leq[\frac{k}{2}])$ then w_{m_0,n_0} is equal to a linear combination of w_{m_0-l,n_0+l} $(l\geq 1)$;

(ii) if $n_0-m_0\geq 1$ and $w\in\mathbf{C}^{2j+1}\subset W\otimes W$ $(n_0-m_0+1\leq j\leq k)$ then w_{m_0,n_0} is equal to a linear combination of w_{m_0-l,n_0+l} $(l\geq 1)$.

Let $U_{m_0,n_0}\subset W\otimes W$ be the sum of the irreducible components appearing in (i) or (ii).

Now we prove the statement of the proposition for $(a_0,b_0)\in F_{m_0,n_0}$. We assume that $0\leq a_0+b_0\leq k$. The case $k\leq a_0+b_0\leq 2k$ is similar.

For $U\subset W\otimes W$ set $U_d=\{w\in U; h_1w=2(k-d)w\}$. We claim that the the coupling between $(U^*_{m_0,n_0})_d$ and $(U_{m_0,n_0})_d$ is non-degenerate. If $m_0=n_0$ and $0\leq d\leq k-1$, this is clear because

$$(U_{m_0,n_0})_d = \begin{cases} S_d & \text{if } k \text{ is even;} \\ A_d & \text{if } k \text{ is odd.}\end{cases} \tag{85}$$

It is also clear if $m_0<n_0$ and $0\leq d\leq k-n_0+m_0-1$ because $(U_{m_0,n_0})_d = (W\otimes W)_d$. We have, in particular, that if $0\leq a_0+b_0\leq k-n_0+m_0-1$ then $\varphi_{a_0,m_0}\varphi_{b_0,n_0}\in\pi_{m_0n_0}(U_{m_0,n_0})$, and therefore, $\varphi_{a_0,m_0}\varphi_{b_0,n_0}$ can be written as a linear combination of $\varphi_{a_0,m_0-l}\varphi_{b_0,n_0+l}$ $(l\geq 1)$.

The non-degeneracy of the coupling for $k-n_0+m_0\leq d\leq k$ reduces to the case $d=k-n_0+m_0-1$ because $f_1^{d-k+n_0-m_0+1}: (U_{m_0,n_0})_{k-n_0+m_0-1}\to (U_{m_0,n_0})_d$ and $f_1^{d-k+n_0-m_0+1}: (U^*_{m_0,n_0})_d\to (U^*_{m_0,n_0})_{k-n_0+m_0-1}$ are both isomorphisms of vector spaces.

Thus, we have shown that if $(a_0, b_0) \in F_{m_0,n_0}$ the product $\varphi_{a_0,m_0}\varphi_{b_0,n_0}$ can be written as a linear combination of $\varphi_{a,m}\varphi_{b,n}$ where, in addition to $a+b = a_0+b_0, m+n = m_0+n_0$, we have "$m < m_0, n > n_0$" or "$m = m_0, n = n_0$ and $(a,b) \notin F_{m_0,n_0}$".

Observe that in the latter case we have $(a, m_0) < (a_0, m_0)$. It will finish the proof. $\square$

3.3 Correlation functions

We use the realization of the dual space, $W^* = \oplus_{a=0}^{k} \mathbf{C}t^a$ where $\langle t^a, \varphi_b \rangle = \delta_{a,b}$. The dual sl_2 action is given by

$$e_1 = t^2\frac{d}{dt} - kt, h_1 = 2t\frac{d}{dt} - k, f_1 = -\frac{d}{dt}. \tag{86}$$

In this picture $(W^*)^{\otimes s}$ is nothing but the space of polynomials in $t_1, \ldots, t_s$ whose degree in each t_j is less than or equal to k.

Fix $0 \leq r \leq k$. We extend this realization to the dual space of $B_k^{(s)}/N_{k,r}^{(s)}$, where $N_{k,r}^{(s)}$ is the subspace of $B_k^{(s)}$ that is the union of $B_k^{(s-1)}\varphi_{a,n}$ for $n > \min(a-r, 0)$. The dual space $\left(B_k^{(s)}/N_{k,r}^{(s)}\right)^*$ is realized as the space of polynomials in $t_1, \ldots, t_s, z_1, \ldots, z_s$ satisfying the following conditions:

(i) $f(t_1, \ldots, t_s, z_1, \ldots, z_s)$ is symmetric if k is even, and skew-symmetric if k is odd, with respect to the permutation of $(t_1, z_1), \ldots, (t_s, z_s)$;

(ii) the degree of $f(t_1, \ldots, t_s, z_1, \ldots, z_s)$ in each t_j is less than or equal to k.

(iii)

$$\frac{\partial^{j_1}}{\partial t_1^{j_1}}\frac{\partial^{j_2}}{\partial z_1^{j_2}} f(t_1, \ldots, t_s, z_1, \ldots, z_s)|_{t_1=z_1=0} = 0 \text{ for } j_1 + j_2 < r. \tag{87}$$

The dual coupling is induced from

$$\langle \prod_{j=1}^{s} t_j^{a_j} \prod_{j=1}^{s} z_j^{m_j}, \varphi_{b_1,n_1} \otimes \cdots \otimes \varphi_{b_s,n_s} \rangle = \delta_{a_1,b_1} \cdots \delta_{a_s,b_s}\delta_{m_1,-n_1} \cdots \delta_{m_s,-n_s}. \tag{88}$$

Now we consider the quotient $\mathcal{H}_{k,r}^{(s)}$ of $B_k^{(s)}/N_{k,r}^{(s)}$ by the quadratic relations discussed in 2.2. The dual space $\mathcal{H}_{k,r}^{*(s)}$ is a subspace of $\left(B_k^{(s)}/N_{k,r}^{(s)}\right)^*$.

A polynomial $f(t_1, \ldots, t_s, z_1, \ldots, z_s) \in \left(B_k^{(s)}/N_{k,r}^{(s)}\right)^*$ belongs to $\mathcal{H}_{k,r}^{*(s)}$ if and only if the following condition is satisfied:

(iv) if $\mathbf{C}^{2j+1} \subset W \otimes W$ is such that $D^{(m)}\mathbf{C}^{2j+1}$ is a component of (??) then we have

$$\langle \frac{\partial^m}{\partial z_1^m} f(t_1, \ldots, t_s, z_1, \ldots, z_s), \mathbf{C}^{2j+1} \rangle|_{z_1=z_2} = 0. \tag{89}$$

We call the function $f(t_1,\dots,t_s,z_1,\dots,z_s)$ satisfying $(i),(ii),(iii),(iv)$ an s-particle correlation function.

Proposition 11 *The condition* (??) *is equivalent to*

$$\frac{\partial^{j_1}}{\partial t_1^{j_1}}\frac{\partial^{j_2}}{\partial z_1^{j_2}} f(t_1,\dots,t_s,z_1,\dots,z_s)|_{t_1=t_2,z_1=z_2} = 0 \text{ for } j_1+j_2<k. \tag{90}$$

Proof We write

$$\frac{\partial^m}{\partial z_1^m} f(t_1,\dots,t_s,z_1,\dots,z_s)|_{z_1=z_2} = \sum_i (u_i)_{12}\otimes(v_i)_{3\cdots s} \subset W^*\otimes\cdots\otimes W^* \tag{91}$$

where $u_i \in W^*\otimes W^*$. Then, the condition (**??**) is equivalent to

$$u_i \in \oplus_{j=0}^m \mathbf{C}^{2j+1} \subset W^*\otimes W^*.$$

Using (**??**) one can show that this is further equivalent to the condition that u_i has the factor $(t_1-t_2)^{k-m}$. The statement of the proposition follows from this. □

The proof of Proposition **??** is similar to this.

Proposition 12 *The set of the normal-ordered monomials in* $\mathcal{H}_{k,r}^{(s)}$*, i.e., the set of vectors of the form*

$$\varphi_{a_1,n_1}\cdots\varphi_{a_s,n_s} \tag{92}$$

where $h((a_j,n_j),(a_{j+1},n_{j+1}))\geq k$ *for all* $1\leq j\leq s-1$ *and* $n_s\leq \min(a_s-r,0)$*, is linearly independent.*

Proof Set

$$\begin{aligned} I_{k,r}^{(s)} \quad = \quad & \{((a_1,n_1),\dots,(a_s,n_s)); h((a_j,n_j),(a_{j+1},n_{j+1}))\geq k \\ & \text{for all } 1\leq j\leq s-1 \text{ and } n_s\leq \min(a_s-r,0)\}. \end{aligned} \tag{93}$$

We define an order in $I_{k,r}^{(s)}$ by saying that

$$\begin{aligned} &((a_1,m_1),\dots,(a_s,m_s)) < ((b_1,n_1),\dots,(b_s,n_s)) \\ &\text{if and only if for some } l \text{ we have} \\ &(a_j,m_j)=(b_j,n_j) \text{ for } 1\leq j\leq l-1 \text{ and } (a_l,m_l)<(b_l,n_l). \end{aligned} \tag{94}$$

For $\rho=((b_1,n_1)\dots,(b_s,n_s))\in I_{k,r}^{(s)}$ we set $v_\rho=\varphi_{b_1,n_1}\cdots\varphi_{b_s,n_s}\in\mathcal{H}_{k,r}^{(s)}$. We will construct a set of correlation functions $\{f_\kappa(t_1,\dots,t_s,z_1,\dots,z_s);\kappa\in I_{k,r}^{(s)}\}$ such that

$$\langle f_\kappa(t_1,\dots,t_s,z_1,\dots,z_s),v_\rho\rangle = \begin{cases} 1 & \text{if } \kappa=\rho; \\ 0 & \text{if } \kappa>\rho. \end{cases} \tag{95}$$

For $0 \leq a \leq k$ and $m \leq 0$ we define a k-component monomial in t and z:

$$(P^{(1)}_{a,m}, \dots, P^{(k)}_{a,m}) = (t^{a^{(1)}} z^{m^{(1)}}, \dots, t^{a^{(k)}} z^{m^{(k)}}) \tag{96}$$

where $a^{(1)}, \dots, a^{(k)}$ and $m^{(1)}, \dots, m^{(k)}$ are uniquely determined by the following conditions:

$$\sum_j a^{(j)} = a, \quad \sum_j m^{(j)} = -m, \tag{97}$$

$$m^{(1)} \geq m^{(2)} \geq \cdots \geq m^{(k)} \geq m^{(1)} - 1, \tag{98}$$

$$m^{(1)} + a^{(1)} \geq m^{(2)} + a^{(2)} \geq \cdots m^{(k)} + a^{(k)} \geq m^{(1)} + a^{(1)} - 1. \tag{99}$$

For $p_1, \dots, p_s \in \mathbf{C}[t, z]$ we define

$$p_1 \odot \cdots \odot p_s = \sum_{\sigma \in S_s} \operatorname{sgn} \sigma \; p_1(t_{\sigma_1}, z_{\sigma_1}) \cdots p_s(t_{\sigma_s}, z_{\sigma_s}). \tag{100}$$

For $\kappa = ((a_1, m_1) \dots, (a_s, m_s)) \in I^{(s)}_{k,r}$ we define

$$f_\kappa(t_1, \dots, t_s, z_1, \dots, z_s) = \prod_{j=1}^{k} P^{(j)}_{a_1,m_1} \odot \cdots \odot P^{(j)}_{a_s,m_s}. \tag{101}$$

One can easily check $(i), (ii), (iii)$,(??) and (??). $\square$

Finally we give a proof of Proposition ?? (ii). The proof is similar to the above proof of Proposition ??. Since the surjectivity is clear, it is enough to show the injectivity. Without loss of generality we assume that $w = \omega_{r,r}$ $(0 \leq r \leq k)$.

It is enough to show that the set of vectors

$$\{v_\rho \omega_{r,r}; \rho \in I^{(s)}_{k,r}\} \tag{102}$$

is linearly independent.

We reduce the proof to the level 1 case by using the algebra map $A_k \to \underbrace{A_1 \otimes \cdots \otimes A_1}_{k}$ (see (??)). Let us write $\omega^{(k)}_{a,r}$ and $\varphi^{(k)}_a(z)$ for $\omega_{a,r}$ and $\varphi_a(z)$ of level k. We realize $\omega^{(k)}_{r,r}$ as $\underbrace{\omega^{(1)}_{1,1} \otimes \cdots \otimes \omega^{(1)}_{1,1}}_{r} \otimes \underbrace{\omega^{(1)}_{0,0} \otimes \cdots \otimes \omega^{(1)}_{0,0}}_{k-r}$. The action of $\varphi^{(k)}_a(z)$ is realized as

$$a! \sum_{a_1 + \cdots + a_k = a} \varphi^{(1)}_{a_1}(z) \otimes \cdots \otimes \varphi^{(1)}_{a_k}(z). \tag{103}$$

We will show the linear independence of the vectors (??) in this realization.

Consider the vectors

$$\omega^{i}_{a_1,\dots,a_s;m_1,\dots,m_s} = \varphi^{(1)}_{a_1,m_1}\cdots\varphi^{(1)}_{a_s,m_s}\omega^{(1)}_{i,i} \tag{104}$$

such that $-2m_1+a_1 > \cdots > -2m_s+a_s \geq i$ for $i=0,1$, and their dual vectors $\omega^{i*}_{a_1,\dots,a_s;m_1,\dots,m_s}$.

For $0 \leq a \leq k$, $m \leq 0$ and $1 \leq j \leq k$, we define $A^{(j)}_{a,m} \in \{0,1\}$ and $M^{(j)}_{a,m} \in \mathbf{Z}_{\leq 0}$ by

$$A^{(j)}_{a,m} = a^{(j)}, \quad M^{(j)}_{a,m} = -m^{(j)}, \tag{105}$$

where $a^{(j)}, m^{(j)}$ are given in (??).

For $\kappa = ((a_1,m_1),\dots,(a_s,m_s)) \in I^{(s)}_{k,r}$ and $1 \leq j \leq k$ we set

$$\omega^*_j(\kappa) = \omega^{i*}_{A^{(j)}_{a_1,m_1},\dots,A^{(j)}_{a_s,m_s};M^{(j)}_{a_1,m_1},\dots,M^{(j)}_{a_s,m_s}}, \tag{106}$$

where $i = \begin{cases} 1 & \text{if } 1 \leq j \leq r; \\ 0 & \text{if } r+1 \leq j \leq k. \end{cases}$ Then, it is easy to see that

$$\langle \omega^*_1(\kappa) \otimes \cdots \otimes \omega^*_k(\kappa), v_\rho \omega^{(k)}_{r,r} \rangle = \begin{cases} c_\rho & \text{if } \kappa = \rho; \\ 0 & \text{if } \kappa > \rho, \end{cases} \tag{107}$$

where c_ρ is a non-zero constant. $\square$

¿From (??) the vector $\omega_{r,r+2l} \in \pi_r \otimes \mathcal{H}_{\frac{r+2l}{\sqrt{2k}}}$ can be formally written as

$$\omega_{r,r+2l} = \varphi_{k-r,k-r-l}\varphi_{r,k-l}\varphi_{k-r,2k-r-l}\varphi_{r,2k-l}\cdots. \tag{108}$$

For $j \geq 1$ we define

$$a^{(r,r+2l)}_j = \begin{cases} k-r & \text{if } j \text{ is odd}; \\ r & \text{if } j \text{ is even}, \end{cases} \tag{109}$$

$$n^{(r,r+2l)}_j = \begin{cases} \frac{(j+1)k}{2} - r - l & \text{if } j \text{ is odd}; \\ \frac{jk}{2} - l & \text{if } j \text{ is even}. \end{cases} \tag{110}$$

We call a sequence $p = (a_j, n_j)_{j \geq 1}$ a *path* which belongs to $\omega_{r,r+2l}$ if the following are satisfied.

$$(i) \quad 0 \leq a_j \leq k \quad n_j \in \mathbf{Z}, \tag{111}$$

$$(ii) \quad a_j = a^{(r,r+2l)}_j, \quad n_j = n^{(r,r+2l)}_j \text{ if } j \gg 0, \tag{112}$$

$$(iii) \quad ((a_j,n_j),(a_{j+1},n_{j+1})) \text{ is normal-ordered.} \tag{113}$$

We denote the set of the paths which belong to $\omega_{r,r+2l}$ by $\mathcal{P}_{r,r+2l}$. We can associate a vector $\omega_p \in \pi_r \otimes \mathcal{H}_{\frac{r+2l}{\sqrt{2k}}}$ with each path $p \in \mathcal{P}_{r,r+2l}$:

$$\omega_p = \varphi_{a_1,n_1}\varphi_{a_2,n_2}\cdots. \tag{114}$$

From what we have proved it follows that the vectors ω_p are linearly independent. The weight of ω_p is given by the formula.

$$\mathrm{wt}(\omega_p) - \mathrm{wt}(\omega_{r,r+2l}) = -\sum_{j=1}^{\infty}(a_j - a_j^{(r,r+2l)})\alpha_1 + \sum_{j=1}^{\infty}(n_j - n_j^{(r,r+2l)})\delta. \quad (115)$$

We call a path $p = (a_j, n_j)_{j\geq 1} \in \mathcal{P}_{r,r+2l}$ "reduced" if

$$h((a_j, n_j), (a_{j+1}, n_{j+1})) = k \quad (116)$$

for all $j \geq 1$. We denote the set of the reduced paths in $\mathcal{P}_{r,r+2l}$ by $\mathcal{P}^{\mathrm{red}}_{r,r+2l}$.

We conclude the paper by the following

Proposition 13 *The space $\pi_r \otimes \mathcal{H}_{\frac{r+2l}{\sqrt{2k}}}$ is spanned by the set of vectors $\omega_p = \varphi_{a_1,n_1}\varphi_{a_2,n_2}\cdots$ for $p \in \mathcal{P}_{r,r+2l}$.*

Proof We compare the characters of the space $\pi_r \otimes \mathcal{H}_{\frac{r+2l}{\sqrt{2k}}}$, and its subspace spanned by the above vectors. Let us denote the former by $\chi_{r,r+2l}$ and the latter $\chi(\mathcal{P}_{r,r+2l})$. We also denote the character of the space spanned by the vectors corresponding to the reduced paths by $\chi(\mathcal{P}^{\mathrm{red}}_{r,r+2l})$. Then we have

$$\chi(\mathcal{P}_{r,r+2l}) = \frac{\chi(\mathcal{P}^{\mathrm{red}}_{r,r+2l})}{\prod_{n=1}^{\infty}(1-e^{-n\delta})}. \quad (117)$$

For a reduced path $p = (a_j, n_j)_{j\geq 1}$ we have

$$\sum_{j=1}^{\infty}(n_j - n_j^{(r,r+2l)}) = \sum_{j=1}^{\infty} j(h_{a_j,a_{j+1}} - h_{a_j^{(r,r+2l)},a_j^{(r,r+2l)}}). \quad (118)$$

Therefore, the assertion $\chi_{r,r+2l} = \chi(\mathcal{P}_{r,r+2l})$ follows from the known fact, Theorem 1.2 of [?]. $\square$

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