Determinant Formula for the Solutions of the Quantum Knizhnik-Zamolodchikov Equation with $|q| = 1$

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Dedicated to Professor Kazuhiko Aomoto on his sixtieth birthday.

Abstract

The fundamental matrix solution of the quantum Knizhnik-Zamolodchikov equation associated with $U_q(\widehat{sl}_2)$ is constructed for $|q| = 1$. The formula for its determinant is given in terms of the double sine function.

1 Introduction

The aim of this paper is to compute the determinant of the fundamental matrix solution of the quantum Knizhnik-Zamolodchikov equation associated with $U_q(\widehat{sl}_2)$ for $|q| = 1$.

In [6] and [7], Tarasov and Varchenko studied the rational qKZ equation associated with the sl_2 Yangian and the trigonometric qKZ equation associated with $U_q(\widehat{sl}_2)$, respectively. The case studied in [6] is the limit $q \to 1$ of the case studied in [7]. However, there is a difference in these two cases. In the former, the unknown function $\psi(\beta_1, \dots, \beta_n)$ is defined on $\mathbf{C}^n$, while in the latter the unknown function $\tilde{\psi}(z_1, \dots, z_n)$ is defined on $(\mathbf{C}\backslash\{0\})^n$. If we set

$$q = e^{-\frac{\pi^2 i}{\rho}}, \quad z_m = e^{\frac{2\pi\beta_m}{\rho}} (1 \leq m \leq n),$$

the limit $q \to 1$ corresponds to $\rho \to \infty$. Therefore, the period ρi in β_m is lost in this limit.

We will study the qKZ equation on $\mathbf{C}^n$ with finite ρ. In [7], the solutions are single-valued on $(\mathbf{C}\backslash\{0\})^n$. The price for this restriction is that the multiplicative step p in z_m of the difference equation is restricted by the condition $0 < |p| < 1$ (or $|p| > 1$ in a different convention). We consider the case $p = e^{-\frac{2\pi i\lambda}{\rho}}$ with real ρ and λ. Namely, the additive step in β_m is $-\lambda i$. Then, the solutions are *not* single-valued in z_m.

In the applications to physics, this difference corresponds to the difference of the models. In the application to the XXZ spin chain in the massive regime [3], the deformation parameter

q satisfies $-1 < q < 0$, and the multiplicative step is given by $p = q^{-4}$ for the correlation functions and by $p = q^4$ for the form factors. On the other hand, in the application to the sine-Gordon model [5], the deformation parameter q satisfies $|q| = 1$ and the additive step is given by $\lambda = 2\pi$ (i.e., $p = q^4$).

We follow [4, 8, 7] for the construction of solutions of the qKZ equations. We consider the evaluation modules of the $U_q(sl_2)$ Verma modules with the spectral parameter z_m and the highest weight $2\Lambda_m$. The unknown function $\psi(\beta_1, \dots, \beta_n)$ takes a value in their tensor product. We will fix l and consider the subspace of weight $2(\sum_{m=1}^n \Lambda_m - l)$, which has dimensions $d = \binom{n+l-1}{n-1}$. The qKZ equation is defined on this subspace. We will choose a basis of this space. The qKZ equation is then written as a system of equations for a d-dimensional unknown function.

We will construct a non-degenerate $d \times d$ matrix such that each column solves the qKZ equation following a similar construction in [6]. We call it the fundamental solution. The fundamental solution is constructed symmetrically with respect to the parameters ρ and λ. Namely, each row solves another qKZ equation with

$$q = e^{-\frac{\pi^2 i}{\lambda}}, \quad z_m = e^{\frac{2\pi\beta_m}{\lambda}} (1 \le m \le n),$$

and $-\rho i$ being the additive step in β_m. We remark that in [1] the co-existence of two systems of equations has already been discussed in the setting of elliptic equations.

We give the fundamental solution in terms of l-dimensional integrals. To prove the non-degeneracy of the matrix we compute its determinant. The integrands of these integrals are meromorphic functions given explicitly in terms of the double sine function. We express the determinant itself as a simple product of double sine functions. In [6, 7], similar determinant formulas are obtained. We follow their line of arguments in the evaluation of the determinant.

We introduce another parameter μ which couples to the sl_2 generator h_1 acting on the m-th component of the tensor product. Therefore, the determinant is a function of the variables $\beta_1, \dots, \beta_n, \Lambda_1, \dots, \Lambda_n$ and ρ, λ, μ. The dependence on $\beta_1, \dots, \beta_n$ is easily determined by using the two sets of qKZ equations. In order to determine the multiplicative factor which is independent of $\beta_1, \dots, \beta_n$, we compute the asymptotics of the fundamental solution in the region $\beta_1 << \cdots << \beta_n$. This asymptotics is triangular, and the diagonal extries are the same kind of integrals with $n = 1$.

The computation of the $n = 1$ integrals is done in two steps. The first step is to derive difference equations in μ with steps $\frac{2\pi}{\rho}$ and $\frac{2\pi}{\lambda}$. The dependence on μ is determined by these equations. Finally, the multiplicative factor which is independent of μ is determined by taking the asymptotics when $\mu \to i\infty$.

The plan of the paper is as follows. Section 2 contains preliminaries of the qKZ equation. In Section 3, we construct the fundamental matrix solution. In Section 4, the asymptotics of the determinant is computed in terms of the $n = 1$ integrals. These integrals are computed in Section 5. The formula for the determinant is given in Section 6. A short Appendix is

given on the double sine function for the convenience of the reader.

2 The qKZ equation

Let q be a nonzero complex number which is not a root of unity. Consider the quantum affine algebra $\widehat{U_q} = U_q'(\widehat{sl_2})$ with generators $e_i, f_i, q^{\pm h_i}$ $(i = 0, 1)$ and relations:

$$\begin{aligned}
q^{h_i}q^{h_j} &= q^{h_j}q^{h_i}, \quad q^{h_i}q^{-h_i} = q^{-h_i}q^{h_i} = 1,\\
q^{h_i}e_jq^{-h_i} &= q^{a_{ij}}e_j, \quad q^{h_i}f_jq^{-h_i} = q^{-a_{ij}}f_j,\\
[e_i, f_j] &= \delta_{ij}\frac{q^{h_i} - q^{-h_i}}{q - q^{-1}},\\
e_i^3e_j - [3]_qe_i^2e_je_i + [3]_qe_ie_je_i^2 - e_je_i^3 &= 0 \quad (i \neq j),\\
f_i^3f_j - [3]_qf_i^2f_jf_i + [3]_qf_if_jf_i^2 - f_jf_i^3 &= 0 \quad (i \neq j).
\end{aligned}$$

Here $a_{00} = a_{11} = 2, a_{01} = a_{10} = -2$, and $[n]_q$ is the q-integer defined by $[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}$. We also use $[n]_q! = \prod_{m=1}^{n}[m]_q$.

We use the coproduct

$$\Delta(q^{h_i}) = q^{h_i} \otimes q^{h_i}, \quad \Delta(e_i) = e_i \otimes 1 + q^{h_i} \otimes e_i, \quad \Delta(f_i) = f_i \otimes q^{-h_i} + 1 \otimes f_i.$$

We denote by U_q the subalgebra generated by $e_1, f_1, q^{\pm h_1}$. Let Λ be a complex number and V_Λ^q be the Verma module for U_q with the highest weight $q^{2\Lambda}$. We denote the highest weight vector of V_Λ^q by $v^{(0)}$ and use the following basis $\{v^{(k)}\}_{k\geq 0}$.

$$e_1v^{(k)} = [2\Lambda - k + 1]_qv^{(k-1)}, \quad q^{h_1}v^{(k)} = q^{2(\Lambda-k)}v^{(k)}, \quad f_1v^{(k)} = [k + 1]_qv^{(k+1)}.$$

For a nonzero complex number z, we can give V_Λ^q the $\widehat{U_q}$ module structure by setting

$$e_0v = zf_1v, \quad q^{h_0}v = q^{-h_1}v, \quad f_0v = z^{-1}e_1v, \quad (v \in V_\Lambda^q).$$

We denote by $V_\Lambda^q(z)$ the $\widehat{U_q}$ module defined in this way.

Let Λ_1, Λ_2 be complex numbers, and $v_1^{(k)}, v_2^{(k)}$ the base vectors of $V_{\Lambda_1}^q, V_{\Lambda_2}^q$, respectively. For generic z_1, z_2, there exists a map $R_{\Lambda_1\Lambda_2}^q(z_1/z_2) \in \mathrm{End}\left(V_{\Lambda_1}^q(z_1) \otimes V_{\Lambda_2}^q(z_2)\right)$ such that

$$R_{\Lambda_1\Lambda_2}^q(z_1/z_2)v_1^{(0)} \otimes v_2^{(0)} = v_1^{(0)} \otimes v_2^{(0)}, \tag{2.1}$$

$$R_{\Lambda_1\Lambda_2}^q(z_1/z_2) \circ \Delta(x) = \sigma(\Delta(x)) \circ R_{\Lambda_1\Lambda_2}^q(z_1/z_2) \quad \text{for all } x \in \widehat{U_q}, \tag{2.2}$$

where σ is the permutation map $\sigma(a \otimes b) = b \otimes a$. Such a map $R_{\Lambda_1\Lambda_2}^q(z_1/z_2)$ is uniquely determined by (2.1) and (2.2).

Let $V_{\Lambda_1}^q \otimes V_{\Lambda_2}^q = \oplus_{l=0}^{\infty}V_{\Lambda_1+\Lambda_2-2l}^q$ be the decomposition of the U_q module into the irreducible components. We denote by Π_l the projection to the component $V_{\Lambda_1+\Lambda_2-2l}^q$. Then we have (see, e.g., [6])

$$R_{\Lambda_1\Lambda_2}^q(z_1/z_2) = R_{\Lambda_1\Lambda_2}^q(0)\sum_{l=0}^{\infty}\Pi_l\prod_{j=0}^{l-1}\frac{1 - q^{-2(\Lambda_1+\Lambda_2-j)}z_1/z_2}{1 - q^{2(\Lambda_1+\Lambda_2-j)}z_1/z_2}, \tag{2.3}$$

where

$$R^q_{\Lambda_1\Lambda_2}(0) = q^{2\Lambda_1\Lambda_2 - \frac{h_1\otimes h_1}{2}} \sum_{k=0}^{\infty} \frac{q^k(1-q^2)^{2k}}{\prod_{j=1}^k (1-q^{2j})} (q^{-h_1}e_1 \otimes q^{h_1}f_1)^k.$$

Note that $R^q_{\Lambda_1\Lambda_2}(0)$ is triangular with respect to the basis $v_1^{(j)} \otimes v_2^{(k)}$.

Consider the tensor product $W^q = V^q_{\Lambda_1}(z_1)\otimes\cdots\otimes V^q_{\Lambda_n}(z_n)$. Let $R^q_{ij}(z_i/z_j) \in \mathrm{End}W^q$ be the operator acting on the i-th and j-th components as $R^q_{\Lambda_i,\Lambda_j}(z_i/z_j) \in \mathrm{End}(V^q_{\Lambda_i}(z_i) \otimes V^q_{\Lambda_j}(z_j))$. We denote by H_m the operator on W^q given by

$$H_m = 1 \otimes \cdots \otimes \overset{m-\mathrm{th}}{h_1} \otimes \cdots \otimes 1.$$

Let p, r be non-zero complex numbers. For $m = 1, \ldots, n$, set

$$\begin{aligned}\tilde{K}_m(z_1, \cdots, z_n; q, p, r) = R^q_{m,m-1}(pz_m/z_{m-1}) \cdots R^q_{m,1}(pz_m/z_1) r^{2\Lambda_m - H_m} \\ \times R^q_{m,n}(z_m/z_n) \cdots R^q_{m,m+1}(z_m/z_{m+1}).\end{aligned}$$

The quantum Knizhnik-Zamolodchikov equation is the following system of equations for unknown function $\tilde{\psi}(z_1, \cdots, z_n)$ that takes a value in W^q:

$$\tilde{\psi}(z_1, \cdots, pz_m, \cdots, z_n) = \tilde{K}_m(z_1, \cdots, z_n; q, p, r)\tilde{\psi}(z_1, \cdots, z_n), (1 \le m \le n) \qquad (2.4)$$

In this paper, we consider the qKZ equation where the step p satisfies $|p| = 1$. In this case, solutions to (2.4) are not necessarily single-valued with respect to the parameters $z_1, \ldots, z_n, q, p, r$. Therefore, we rewrite (2.4) as follows. We set

$$\begin{aligned}&z_k^{(\rho)} = e^{\frac{2\pi\beta_k}{\rho}}, q^{(\rho)} = e^{-\frac{\pi^2 i}{\rho}}, p^{(\rho,\lambda)} = e^{-\frac{2\pi i\lambda}{\rho}}, r^{(\lambda)} = e^{-\frac{\mu\lambda i}{2}}, \\ &\psi(\beta_1, \cdots, \beta_n) = \tilde{\psi}(z_1^{(\rho)}, \cdots, z_n^{(\rho)}), \\ &K_m(\beta_1, \cdots, \beta_n; \rho, \lambda, \mu) = \tilde{K}_m(z_1^{(\rho)}, \cdots, z_n^{(\rho)}; q^{(\rho)}, p^{(\rho,\lambda)}, r^{(\lambda)}). \end{aligned} \qquad (2.5)$$

Then (2.4) is equivalent to the following system of equations for ψ:

$$\psi(\beta_1, \cdots, \beta_m - \lambda i, \cdots, \beta_n) = K_m(\beta_1, \cdots, \beta_n; \rho, \lambda, \mu)\psi(\beta_1, \cdots, \beta_n), (1 \le m \le n). \qquad (2.6)$$

For a non-negative integer l, set

$$W^q_l = \{v \in W^q; h_1 v = 2(\sum_{m=1}^n \Lambda_m - l)v\}.$$

Since the matrix K_m acts on W^q_l for each l, the equation (2.6) for ψ splits into the equations for the weight-l component $\psi_l \in W^q_l$. We denote by $K_{m,l}$ the W^q_l block of K_m. Then, we have

$$\psi_l(\beta_1, \cdots, \beta_m - \lambda i, \cdots, \beta_n) = K_{m,l}(\beta_1, \cdots, \beta_n; \rho, \lambda, \mu)\psi_l(\beta_1, \cdots, \beta_n), (1 \le m \le n). \qquad (2.7)$$

Let W_l^{q*} be the dual space of W_l^q with the coupling $\langle v^*, v\rangle$ for $v^* \in W_l^{q*}$ and $v \in W_l^q$. In Section 3, we construct $\Psi_l(\beta, \dots, \beta_n; \rho, \lambda, \mu)$ taking a value in $W_l^{q^{(\rho)}} \otimes W_l^{q^{(\lambda)}}$ such that for any $v^* \in W_l^{q^{(\lambda)}*}$,

$$\psi_l(\beta_1, \dots, \beta_n) = \langle v^*, \Psi_l(\beta, \dots, \beta_n; \rho, \lambda, \mu)\rangle \in W_l^{q^{(\rho)}}$$

satisfies (2.7) and also, for any $w^* \in W_l^{q^{(\rho)*}}$,

$$\psi_l(\beta_1, \dots, \beta_n) = \langle w^*, \Psi_l(\beta, \dots, \beta_n; \rho, \lambda, \mu)\rangle \in W_l^{q^{(\lambda)}}$$

satisfies (2.7) with ρ and λ interchanged.

3 The hypergeometric solutions

For a non-negative integer l, we set

$$\mathcal{Z}_l^n = \left\{ (l_1, \cdots, l_n) \in \mathbf{Z}_{\geq 0}^n; \sum_{m=1}^n l_m = l \right\}.$$

For $L = (l_1, \cdots, l_n) \in \mathcal{Z}_l^n$, set

$$\Gamma_m^L = \{l_1 + \cdots + l_{m-1} + 1, \cdots, l_1 + \cdots + l_m\},$$

and define a map $\gamma^L : \{1, \cdots, l\} \to \{1, \cdots, n\}$ by

$$\gamma^L(j) = m \overset{\text{def}}{\Longleftrightarrow} j \in \Gamma_m^L.$$

We also define a partial order $\preceq$ on $\mathcal{Z}_l^n$ by

$$L \preceq L' \overset{\text{def}}{\Longleftrightarrow} l_n \leq l'_n, l_{n-1} + l_n \leq l'_{n-1} + l'_n, \dots, l_2 + \cdots + l_n \leq l'_2 + \cdots + l'_n.$$

We define a function $w_L^{(\rho)}$ as follows:

$$\begin{aligned} w_L^{(\rho)}(\alpha_1, \cdots, \alpha_l; \beta_1, \cdots, \beta_n) &= \mathrm{Skew}\left(g_L^{(\rho)}(\alpha_1, \cdots, \alpha_l; \beta_1, \cdots, \beta_n)\right), \qquad (3.8) \\ g_L^{(\rho)}(\alpha_1, \cdots, \alpha_l; \beta_1, \cdots, \beta_n) &= q^{(\rho)\sum_{m<m'} l_m l_{m'}} \prod_{1 \leq j < j' \leq l} \mathrm{sh}\,\frac{\pi}{\rho}(\alpha_{j'} - \alpha_j - \pi i) \\ \times \prod_{j=1}^{l} \left(e^{-\frac{\pi}{\rho}(\alpha_j - \beta_{\gamma^L(j)} + \Lambda_{\gamma^L(j)}\pi i)} \prod_{m<\gamma^L(j)} \mathrm{sh}\,\frac{\pi}{\rho}(\alpha_j - \beta_m + \Lambda_m \pi i) \prod_{m>\gamma^L(j)} \mathrm{sh}\,\frac{\pi}{\rho}(\alpha_j - \beta_m - \Lambda_m \pi i) \right), \end{aligned}$$

where Skew is the skew-symmetrization with respect to $(\alpha_1, \cdots, \alpha_l)$, i.e.,

$$\mathrm{Skew} f(\alpha_1, \cdots, \alpha_l) = \frac{1}{l!} \sum_{\sigma \in \mathbf{S}_l} (\mathrm{sgn}\sigma) f(\alpha_{\sigma_1}, \cdots, \alpha_{\sigma_l}).$$

We abbreviate $w_L^{(\rho)}(\alpha_1,\cdots,\alpha_l;\beta_1,\cdots,\beta_n)$ to $w_L^{(\rho)}(\beta_1,\cdots,\beta_n)$ (or $w_L^{(\rho)}(\alpha_1,\cdots,\alpha_l)$) when the dependence on the abbreviated variables is irrelevant.

We set

$$\mathcal{F}^{(\rho)} = \sum_{L\in\mathcal{Z}_l^n} \mathbf{C} w_L^{(\rho)}.$$

Now we define a pairing between $\mathcal{F}^{(\rho)}$ and $\mathcal{F}^{(\lambda)}$. We use

$$\varphi(x;\Lambda) = \frac{1}{S_2(ix-\Lambda\pi)S_2(-ix-\Lambda\pi)}, \quad \phi(x) = \prod_{m=1}^{n} \varphi(x-\beta_m;\Lambda_m), \quad \psi(x) = \varphi(x;-1), \tag{3.9}$$

where $S_2(x) = S_2(x|\rho,\lambda)$ is the double sine function with periods ρ and λ (see Appendix).

In this paper, we assume that the auxiliary parameters, ρ, λ, μ and $-\Lambda_m$ $(1\le m\le n)$, are in $\mathbf{R}_{>0}$.

Suppose that f,g are entire functions in the variables $\alpha_1,\dots,\alpha_l$. For fixed $\beta_1,\dots,\beta_n\in\mathbf{R}$ we set

$$\begin{aligned} I(f,g) = \int_C \cdots \int_C d\alpha_1\cdots d\alpha_l\, e^{\mu\sum_{j=1}^l \alpha_j} \prod_{j=1}^{l} \phi(\alpha_j) \prod_{1\le j<j'\le l} \psi(\alpha_j-\alpha_{j'}) \\ \times f(\alpha_1,\dots,\alpha_l) g(\alpha_1,\dots,\alpha_l). \end{aligned} \tag{3.10}$$

Here the contour C for the variable α_j is taken to be the real line $\mathbf{R}$. Note that the poles of the integrand at

$$\beta_m - \Lambda_m\pi i + \rho i\mathbf{Z}_{\ge 0} + \lambda i\mathbf{Z}_{\ge 0}, \quad \alpha_{j'} + \pi i + \rho i\mathbf{Z}_{\ge 0} + \lambda i\mathbf{Z}_{\ge 0}$$

are above C and the poles at

$$\beta_m + \Lambda_m\pi i - \rho i\mathbf{Z}_{\ge 0} - \lambda i\mathbf{Z}_{\ge 0}, \quad \alpha_{j'} - \pi i - \rho i\mathbf{Z}_{\ge 0} - \lambda i\mathbf{Z}_{\ge 0}$$

are below C. As we will see in Section 4, there is a region of the parameters where the integral (3.10) is absolutely convergent.

The integral, in particular, defines the pairing $I(w^{(\rho)},w^{(\lambda)})$ between $w^{(\rho)}\in\mathcal{F}^{(\rho)}$ and $w^{(\lambda)}\in\mathcal{F}^{(\lambda)}$.

Theorem 3.1. *For $w^{(\lambda)}\in\mathcal{F}^{(\lambda)}$, we set*

$$\psi_l = \sum_{L\in\mathcal{Z}_l^n} I(w_L^{(\rho)},w^{(\lambda)}) v_L^{(\rho)} \ \textit{where}\ v_L^{(\rho)} = v_1^{(l_1)}\otimes\cdots\otimes v_n^{(l_n)} \in V_{\Lambda_1}^{q^{(\rho)}}(z_1^{(\rho)})\otimes\cdots\otimes V_{\Lambda_n}^{q^{(\rho)}}(z_n^{(\rho)}).$$

Then, ψ_l is a solution to (2.7).

Proof. For $L=(l_1,\dots,l_n)$, set $L^{(m)}=(l_1,\cdots,l_{m+1},l_m,\cdots,l_n)$ and $\overline{L}=(l_n,l_1,\cdots,l_{n-1})$. It is easy to see that ψ_l is a solution to (2.7) if the following relations (3.11) and (3.12) hold.

$$\begin{aligned}&\sum_{L\in\mathcal{F}_l^n} w^{(\rho)}_{L^{(m)}}(\beta_1,\cdots,\beta_{m+1},\beta_m,\cdots,\beta_n)v_L^{(\rho)}\\&=\sum_{L\in\mathcal{F}_l^n} R^{q^{(\rho)}}_{\Lambda_m,\Lambda_{m+1}}(z_m^{(\rho)}/z_{m+1}^{(\rho)})w_L^{(\rho)}(\beta_1,\cdots,\beta_n)v_L^{(\rho)},\end{aligned}\tag{3.11}$$

$$I(w^{(\rho)}_{\overline{L}}(\beta_n-\lambda i,\beta_1,\cdots,\beta_{n-1}),w^{(\lambda)}(\beta_1,\cdots,\beta_n-\lambda i))=e^{-\mu l_n\lambda i}I(w_L^{(\rho)},w^{(\lambda)}).\tag{3.12}$$

The relation (3.11) follows from Lemma 5.2.2 in [4]. We prove (3.12) in the form

$$I(g_L^{(\rho)},w^{(\lambda)})(\beta_1,\cdots,\beta_n+\lambda i)=e^{\mu l_n\lambda i}I(g^{(\rho)}_{\overline{L}}(\beta_n,\beta_1,\cdots,\beta_{n-1}),w^{(\lambda)}(\beta_1,\cdots,\beta_n)).\tag{3.13}$$

First note that

$$w^{(\lambda)}(\alpha_1,\cdots,\alpha_j+\lambda i,\cdots,\alpha_l)=(-1)^{n+l-1}w^{(\lambda)}(\alpha_1,\cdots,\alpha_j,\cdots,\alpha_l),\tag{3.14}$$

$$w^{(\lambda)}(\beta_1,\cdots,\beta_{n-1},\beta_n+\lambda i)=(-1)^l w^{(\lambda)}(\beta_1,\cdots,\beta_{n-1},\beta_n).\tag{3.15}$$

From (6.59), we find

$$\frac{\varphi(x-\lambda i;\Lambda)}{\varphi(x;\Lambda)}=-\frac{\mathrm{sh}\,\frac{\pi}{\rho}(x+\Lambda\pi i)}{\mathrm{sh}\,\frac{\pi}{\rho}(x-\Lambda\pi i-\lambda i)}.\tag{3.16}$$

Set

$$\begin{aligned}f_L(\alpha_1,\cdots,\alpha_l;\beta_1,\cdots,\beta_n)=e^{\mu\sum_{j=1}^l\alpha_j}&\prod_{j=1}^l\phi(\alpha_j)\prod_{1\le j<j'\le l}\psi(\alpha_j-\alpha_{j'})\\&\times g_L^{(\rho)}(\alpha_1,\cdots,\alpha_l;\beta_1,\cdots,\beta_n).\end{aligned}$$

Then

$$\begin{aligned}&I(g_L^{(\rho)},w^{(\lambda)})(\beta_1,\cdots,\beta_n)\\&=\int_C\cdots\int_C d\alpha_1\cdots d\alpha_l f_L(\alpha_1,\cdots,\alpha_l;\beta_1,\cdots,\beta_n)w^{(\lambda)}(\alpha_1,\cdots,\alpha_l;\beta_1,\cdots,\beta_n).\end{aligned}$$

First, we consider the case $l_n=0$. In this case, f_L has no poles at

$$\alpha_j=\beta_n+\Lambda_n\pi i-\rho i\mathbf{Z}_{\ge 0}.$$

Therefore, when we make the analytic continuation $\beta_n\to\beta_n+\lambda i$, no poles of f_L cross the contour C. From (3.15) and (3.16), we find

$$f_L(\beta_1,\cdots,\beta_n+\lambda i)w^{(\lambda)}(\beta_1,\cdots,\beta_n+\lambda i)=f_{\overline{L}}(\beta_n,\beta_1,\cdots,\beta_{n-1})w^{(\lambda)}(\beta_1,\cdots,\beta_n).$$

Therefore, the equation (3.13) holds if $l_n = 0$.

In the case of $l_n > 0$, if $j \in \Gamma_n^L$, the poles of f_L at

$$\alpha_j = \beta_n + \Lambda_n \pi i - \rho i \mathbf{Z}_{\geq 0}$$

may cross the contour C. In order to avoid this crossing, we shift the contour C to $C + \lambda i$ for all $j \in \Gamma_n^L$. We note that f_L has no poles at

$$\begin{aligned} &\alpha_j = \beta_m - \Lambda_m \pi i + \rho i \mathbf{Z}_{\geq 0} \quad (m = 1, \cdots, n-1), \\ &\alpha_j = \alpha_{j'} + \pi i + \rho i \mathbf{Z}_{\geq 0} \quad (j' \in \Gamma_m^L, m = 1, \cdots n-1) \end{aligned}$$

for $j \in \Gamma_n^L$. Hence, this shift of the contours does not cause any crossing of poles.

By changing the variables $\alpha_j \to \alpha_j + \lambda i \, (j \in \Gamma_n^L)$ after this shift and using (3.14), (3.15), and (3.16), we get

$$\begin{aligned} I(g_L^{(\rho)}, w^{(\lambda)})(\beta_1, \cdots, \beta_n + \lambda i) &= (-1)^{l_n(l-l_n)} e^{\mu l_n \lambda i} \int_C \cdots \int_C d\alpha_1 \cdots d\alpha_l \\ \times f_{\overline{L}}(\alpha_{l-l_n+1}, \cdots, \alpha_l, \alpha_1, \cdots, \alpha_{l-l_n}; \beta_n, \beta_1, \cdots, \beta_{n-1}) &w^{(\lambda)}(\alpha_1, \cdots, \alpha_l; \beta_1, \cdots, \beta_n) \quad (3.17) \end{aligned}$$

Symmetrizing the integrand of (3.17), we obtain (3.13). □

We define the fundamental matrix solution by

$$\Psi_l(\beta_1, \ldots, \beta_n; \rho, \lambda, \mu) = \sum_{L, L' \in \mathcal{Z}_l^n} I(w_L^{(\rho)}, w_{L'}^{(\lambda)}) v_L^{(\rho)} \otimes v_{L'}^{(\lambda)} \in W_l^{(\rho)} \otimes W_l^{(\lambda)}. \quad (3.18)$$

Then, it has the property announced in Section 2.

4 Asymptotics of the Determinant in $\beta_1, \cdots, \beta_n$

In the following sections we calculate the determinant

$$D_l(\beta_1, \cdots, \beta_n) = \det \left[I(w_L^{(\rho)}, w_{L'}^{(\lambda)}) \right]_{L, L' \in \mathcal{Z}_l^n}.$$

¿From Theorem 3.1, we find

$$\frac{D_l(\beta_1, \cdots, \beta_m - \lambda i, \cdots, \beta_n)}{D_l(\beta_1, \cdots, \beta_m, \cdots, \beta_n)} = \det K_{m,l}(\beta_1, \cdots, \beta_n; \rho, \lambda, \mu), \quad (4.19)$$

$$\frac{D_l(\beta_1, \cdots, \beta_m - \rho i, \cdots, \beta_n)}{D_l(\beta_1, \cdots, \beta_m, \cdots, \beta_n)} = \det K_{m,l}(\beta_1, \cdots, \beta_n; \lambda, \rho, \mu). \quad (4.20)$$

Using the formula (2.3), we see that

$$\begin{aligned} &\det K_{m,l}(\beta_1, \cdots, \beta_n; \rho, \lambda, \mu) \\ &= \left(e^{-\mu \lambda i}\right)^{\binom{n+l-1}{n}} \prod_{j=0}^{l-1} \left(\prod_{k=1}^{m-1} \frac{\operatorname{sh} \frac{\pi}{\rho}(\beta_m - \beta_k - \lambda i + (\Lambda_m + \Lambda_k - j)\pi i)}{\operatorname{sh} \frac{\pi}{\rho}(\beta_m - \beta_k - \lambda i - (\Lambda_m + \Lambda_k - j)\pi i)} \right. \\ &\qquad \left. \times \prod_{k=m+1}^{n} \frac{\operatorname{sh} \frac{\pi}{\rho}(\beta_m - \beta_k + (\Lambda_m + \Lambda_k - j)\pi i)}{\operatorname{sh} \frac{\pi}{\rho}(\beta_m - \beta_k - (\Lambda_m + \Lambda_k - j)\pi i)} \right)^{\binom{n+l-j-2}{n-1}}, \end{aligned}$$

where $\binom{a}{b}$ is the usual binomial coefficient.

We consider the following function

$$E_l(\beta_1,\cdots,\beta_n) = \left(e^{\mu\sum_{m=1}^n \beta_m}\right)^{\binom{n+l-1}{n}} \prod_{j=0}^{l-1}\left(\prod_{1\le m<m'\le n}\frac{S_2(i(\beta_m-\beta_{m'})+(\Lambda_m+\Lambda_{m'}-j)\pi)}{S_2(i(\beta_m-\beta_{m'})-(\Lambda_m+\Lambda_{m'}-j)\pi)}\right)^{\binom{n+l-j-2}{n-1}} \tag{4.21}$$

By using (6.59), we can check that $E_l(\beta_1,\cdots,\beta_n)$ satisfies both (4.19) and (4.20). Therefore, we have

Proposition 4.1.

$$D_l(\beta_1,\cdots,\beta_n) = c_l(\rho,\lambda,\mu;\Lambda_1,\cdots,\Lambda_n)E_l(\beta_1,\cdots,\beta_n), \tag{4.22}$$

where $c_l(\rho,\lambda,\mu;\Lambda_1,\cdots,\Lambda_n)$ is a constant independent of $\beta_1,\ldots,\beta_n$.

In order to determine $c_l(\rho,\lambda,\mu;\Lambda_1,\cdots,\Lambda_n)$, we consider the asymptotics of D_l/E_l as

$$\beta_1,\cdots,\beta_n\in\mathbf{R},\quad \beta_1\ll\cdots\ll\beta_n. \tag{4.23}$$

Hereafter we use the notation $\sim$ as follows:

$$f(\beta_1,\cdots,\beta_n)\sim g(\beta_1,\cdots\beta_n) \quad\overset{\text{def}}{\Longleftrightarrow}\quad \frac{f(\beta_1,\cdots,\beta_n)}{g(\beta_1,\cdots,\beta_n)}\to 1 \quad \text{in the limit.}$$

We use the abbreviation $\beta_{mm'}=\beta_m-\beta_{m'}$. From (6.57), we find

$$E_l(\beta_1,\cdots,\beta_n)\sim\frac{d_l}{\prod_{L\in\mathcal{Z}_l^n}P_L(\beta_1,\ldots,\beta_n)} \tag{4.24}$$

where

$$d_l=\exp\left(\frac{\rho+\lambda}{\rho\lambda}\pi^2 i\left(\binom{n+l-1}{n}(n-1)\sum_{m=1}^n\Lambda_m-\binom{n}{2}\binom{n+l-1}{n+1}\right)\right), \tag{4.25}$$

$$P_L(\beta_1,\ldots,\beta_n)=\exp\left(\frac{2\pi^2}{\rho\lambda}\left(\sum_{m,m'=1}^n l_m\Lambda_{m'}|\beta_{mm'}|-\sum_{m<m'}l_m l_{m'}\beta_{m'm}\right)-\mu\sum_{m=1}^n l_m\beta_m\right).$$

Here we used the equalities

$$\sum_{L\in\mathcal{Z}_l^n} l_{m_1}\cdots l_{m_k}=\binom{n+l-1}{n+k-1}\quad(1\le m_1<\cdots<m_k\le n).$$

We have

$$\frac{D_l(\beta_1,\cdots,\beta_n)}{E_l(\beta_1,\cdots,\beta_n)}\sim\frac{1}{d_l}\det\left[P_L I(w_{L'}^{(\rho)},w_L^{(\lambda)})\right]_{L,L'\in\mathcal{Z}_l^n}. \tag{4.26}$$

In the following, we consider the asymptotics of $P_L I(w_{L'}^{(\rho)}, w_L^{(\lambda)})$. We note that

$$I(w_{L'}^{(\rho)}, w_L^{(\lambda)}) = \frac{1}{l!}\sum_{\sigma\in\mathbf{S}_l}(\operatorname{sgn}\sigma) I(g_{L'}^{(\rho)}(\alpha_{\sigma_1},\dots,\alpha_{\sigma_l}), g_L^{(\lambda)}(\alpha_1,\dots,\alpha_l)).$$

By changing the integral variables $\alpha_p \to \alpha_p + \beta_{\gamma^L(p)}$, we have

$$\begin{aligned} I(g_{L'}^{(\rho)}(\alpha_{\sigma_1},\dots,\alpha_{\sigma_l}), g_L^{(\lambda)}(\alpha_1,\dots,\alpha_l)) &= \int_C\cdots\int_C d\alpha_1\cdots d\alpha_l e^{\mu\sum_{j=1}^l \alpha_j + \mu\sum_{m=1}^n l_m\beta_m} \\ \times\prod_{j=1}^l \phi(\alpha_j+\beta_{\gamma^L(j)}) &\prod_{1\le j<j'\le l}\psi(\alpha_j-\alpha_{j'}+\beta_{\gamma^L(j)\gamma^L(j')}) \\ \times g_{L'}^{(\rho)}(\alpha_{\sigma_1}+\beta_{\gamma^L(\sigma_1)},\dots,\alpha_{\sigma_l}+\beta_{\gamma^L(\sigma_l)}) &g_L^{(\lambda)}(\alpha_1+\beta_{\gamma^L(1)},\dots,\alpha_l+\beta_{\gamma^L(l)}). \end{aligned} \tag{4.27}$$

We set

$$J_{L,L'}^{\sigma}(\alpha_1,\dots,\alpha_l;\beta_1,\dots,\beta_n) = P_L(\beta_1,\dots,\beta_n)\times(\text{the integrand of (4.27)}).$$

From (6.58), we have

$$\varphi(x;\Lambda)\sim\exp\left(\mp\pi\frac{\rho+\lambda+2\Lambda\pi}{\rho\lambda}x\right),\quad (x\to\infty,\pm\operatorname{Re}x>0) \tag{4.28}$$

Therefore, if $|\operatorname{Im}x|<K$, we have an estimate

$$|\varphi(x;\Lambda)|\le\gamma_K\exp\left(-\pi\frac{\rho+\lambda+2\Lambda\pi}{\rho\lambda}|x|\right) \tag{4.29}$$

where γ_K is a constant independent of x. Set $\xi(x)=x+|x|$. Using (3.9), and (4.29), we obtain the following uniform estimate in the asymptotic region (4.23).

$$\begin{aligned} |J_{L,L'}^{\sigma}(\alpha_1,\dots,\alpha_l;\beta_1,\dots,\beta_n)| &\le \gamma\exp\Big(\mu\sum_{j=1}^l\alpha_j+\frac{2\pi^2}{\rho\lambda}\sum_{j<j'}|\alpha_j-\alpha_{j'}| \\ -\frac{2\pi^2}{\rho\lambda}\sum_{j=1}^l\sum_{m=1}^n\Lambda_m|\alpha_j| &-\frac{\pi}{\lambda}\sum_{j=1}^l\xi(\alpha_j)-\frac{\pi}{\rho}\sum_{j=1}^l\xi(\alpha_{\sigma_j}+\beta_{\gamma^L(\sigma_j)\gamma^{L'}(j)})\Big). \end{aligned} \tag{4.30}$$

Here γ is a constant independent of $\alpha_1,\dots,\alpha_l,\beta_1,\dots,\beta_n$.

Now, it is easy to see that $J_{L,L'}^{\sigma}$ is uniformly integrable in (4.23) if

$$\frac{2\pi^2}{\rho\lambda}\left(l-1-\sum_{m=1}^n\Lambda_m\right)<\mu<\frac{2\pi}{\lambda}-\frac{2\pi^2}{\rho\lambda}\left(l-1-\sum_{m=1}^n\Lambda_m\right). \tag{4.31}$$

For simplicity, in the following calculation, we assume that ρ and λ are sufficiently large. We have, in particular, that the region of convergence (4.31) is not void.

If $L\not\preceq L'$, then for any $\sigma\in\mathbf{S}_l$ there exists j such that $\gamma^L(\sigma_j)>\gamma^{L'}(j)$. Therefore, by Lebesgue's convergence theorem, we have

$$L\not\preceq L'\Longrightarrow P_L I(w_{L'}^{(\rho)}, w_L^{(\lambda)})\to 0$$

in the limit (4.23). Therefore, we get

$$\frac{D_l(\beta_1,\cdots,\beta_n)}{E_l(\beta_1,\cdots,\beta_n)} \sim \frac{1}{d_l}\prod_{L\in\mathcal{Z}_l^n} P_L I(w_L^{(\rho)}, w_L^{(\lambda)}). \tag{4.32}$$

Let us calculate the asymptotics of $P_L I(w_L^{(\rho)}, w_L^{(\lambda)})$. Note that

$$\gamma^L(\sigma_p) \le \gamma^L(p) \text{ for all } p \quad \Longleftrightarrow \quad \sigma \in \mathbf{S}_L \stackrel{\text{def}}{=} \mathbf{S}_{l_1}\times\cdots\times\mathbf{S}_{l_n} \subset \mathbf{S}_l.$$

Therefore, we have

$$P_L I(w_L^{(\rho)}, w_L^{(\lambda)}) \sim \frac{1}{l!}\sum_{\sigma\in\mathbf{S}_L}(\operatorname{sgn}\sigma)\int\cdots\int d\alpha_1\cdots d\alpha_l\, J^{\sigma}_{L,L}(\alpha_1,\dots,\alpha_l;\beta_1,\dots,\beta_n).$$

Using (3.9), and (4.28), we can calculate the limit of $J^{\sigma}_{L,L}$ ($\sigma\in\mathbf{S}_{l_1}\times\cdots\times\mathbf{S}_{l_n}$) as (4.23):

$$\begin{aligned} P_L I(w_L^{(\rho)}, w_L^{(\lambda)}) \to{}& \frac{l_1!\cdots l_n!(q^{(\rho)}q^{(\lambda)})^{\sum_{m<m'} l_m l_{m'}}}{4^{l(n-1)+\sum_{1\le m<m'\le n} l_m l_{m'}}\, l!} \\ &\times \prod_{m=1}^{n} F_{l_m}^{\Lambda_m}\Big(\mu + \frac{2\pi^2}{\rho\lambda}\left(\sum_{j=1}^{m-1}(l_j-\Lambda_j) - \sum_{j=m+1}^{n}(l_j-\Lambda_j)\right) - \frac{\rho+\lambda}{\rho\lambda}\pi\Big). \end{aligned} \tag{4.33}$$

Here $F_k^{\Lambda}(x)$ is given by

$$\begin{aligned} F_k^{\Lambda}(x) = {}& \int_C\cdots\int_C d\alpha_1\cdots d\alpha_k\, e^{x\sum_{j=1}^k \alpha_j}\prod_{j=1}^{k}\varphi(\alpha_j;\Lambda)\prod_{1\le j<j'\le k}\psi(\alpha_j-\alpha_{j'}) \\ &\times \operatorname{Skew}\left(\prod_{1\le j<j'\le k}\operatorname{sh}\frac{\pi}{\rho}(\alpha_{j'}-\alpha_j-\pi i)\right)\prod_{1\le j<j'\le k}\operatorname{sh}\frac{\pi}{\lambda}(\alpha_{j'}-\alpha_j-\pi i) \end{aligned} \tag{4.34}$$

¿From (4.22), (4.32), and (4.33), we get

Proposition 4.2.

$$\begin{aligned} c_l(\rho,\lambda,\mu;\Lambda_1,\cdots,\Lambda_n) = {}& \frac{(q^{(\rho)}q^{(\lambda)})^{\binom{n}{2}\binom{n+l-1}{n+1}+\binom{n+l-1}{n}\sum_{m=1}^n\Lambda_m}}{4^{n(n-1)\binom{n+l-1}{n}+\binom{n}{2}\binom{n+l-1}{n+1}}(l!)^{\binom{n+l-1}{n-1}}}\prod_{j=1}^{l}(j!)^{n\binom{n+l-j-2}{n-2}} \\ &\times\prod_{L\in\mathcal{Z}_l^n}\prod_{m=1}^{n} F_{l_m}^{\Lambda_m}\Big(\mu+\frac{2\pi^2}{\rho\lambda}\left(\sum_{j=1}^{m-1}(l_j-\Lambda_j)-\sum_{j=m+1}^{n}(l_j-\Lambda_j)\right)-\frac{\rho+\lambda}{\rho\lambda}\pi\Big), \end{aligned} \tag{4.35}$$

where $F_{l_m}^{\Lambda_m}(x)$ is defined by (4.34).

5 Calculation of the integral

In this section, we find an explicit formula of $F_l^{\Lambda}(x)$.

The integral (4.34) is absolutely convergent if

$$|\mathrm{Re}\, x| < \frac{\pi}{\rho} + \frac{\pi}{\lambda} - \frac{2\pi^2}{\rho\lambda}(l - 1 - \Lambda). \tag{5.36}$$

In fact, $F_l^\Lambda(x)$ is analytically continued in x to the whole complex plane. To see this, we derive difference equations satisfied by $F_l^\Lambda(x)$. Namely, we relate the values of F_l^Λ at $x + \frac{2\pi}{\rho}$ and $x + \frac{2\pi}{\lambda}$ to $F_l^\Lambda(x)$.

Proposition 5.1. *The function $F_l^\Lambda(x)$ satisfies*

$$\frac{F_l^\Lambda(x + \frac{\pi}{\lambda})}{F_l^\Lambda(x - \frac{\pi}{\lambda})} = \prod_{k=0}^{l-1} \frac{\mathrm{ch}\left(\frac{\rho i}{2}x - \frac{\pi^2 i}{\lambda}(k - \Lambda)\right)}{\mathrm{ch}\left(\frac{\rho i}{2}x + \frac{\pi^2 i}{\lambda}(k - \Lambda)\right)}, \tag{5.37}$$

$$\frac{F_l^\Lambda(x + \frac{\pi}{\rho})}{F_l^\Lambda(x - \frac{\pi}{\rho})} = \prod_{k=0}^{l-1} \frac{\mathrm{ch}\left(\frac{\lambda i}{2}x - \frac{\pi^2 i}{\rho}(k - \Lambda)\right)}{\mathrm{ch}\left(\frac{\lambda i}{2}x + \frac{\pi^2 i}{\rho}(k - \Lambda)\right)}. \tag{5.38}$$

Proof. Because of the symmetry between ρ and λ, it is enough to show (5.37).

We set

$$\begin{aligned} f^\Lambda(x|\alpha_1, \cdots, \alpha_l) &= e^{x\sum_{j=1}^l \alpha_j} \prod_{j=1}^l \varphi(\alpha_j; \Lambda) \prod_{1 \le j < j' \le l} \psi(\alpha_j - \alpha_{j'}) \\ &\quad \times \mathrm{Skew}\left(\prod_{1 \le j < j' \le l} \mathrm{sh}\, \frac{\pi}{\rho}(\alpha_{j'} - \alpha_j - \pi i) \right), \\ h(\alpha_1, \cdots, \alpha_l) &= \prod_{1 \le j < j' \le l} \mathrm{sh}\, \frac{\pi}{\lambda}(\alpha_{j'} - \alpha_j - \pi i). \end{aligned} \tag{5.39}$$

The poles of f^Λ in the variable α_j that are lying above the contour C are at $-\Lambda\pi i + \rho i \mathbf{Z}_{\ge 0} + \lambda i \mathbf{Z}_{\ge 0}$ and $\alpha_{j'} + \pi i + \rho i \mathbf{Z}_{\ge 0} + \lambda i \mathbf{Z}_{\ge 0}$.

Consider

$$\begin{aligned} I_k = \int_C \cdots \int_C \left(\int_{C+\rho i} - \int_C \right) d\alpha_1 \cdots d\alpha_l f^\Lambda(x|\alpha_1, \cdots, \alpha_l) \\ \times h(\alpha_1, \cdots, \alpha_l)\, \mathrm{sh}\, \frac{\pi}{\lambda}(\alpha_l + \Lambda\pi i) e^{-\frac{\pi}{\lambda}(\sum_{j=1}^{k-1} \alpha_j - \sum_{j=k}^{l-1} \alpha_j)}. \end{aligned} \tag{5.40}$$

The integrand has no poles inside the strip $0 < \mathrm{Im}\, \alpha_l < \rho$, and the integral is absolutely convergent if $|\mathrm{Re}\, x|$ is sufficiently small. Therefore, we have

$$I_k = 0.$$

Now we transform the variable α_l to $\alpha_l + \rho i$ so that the contour $C + \rho i$ is modified to C. From (3.9) and (3.16), we have

$$\frac{f^\Lambda(x|\alpha_1, \cdots, \alpha_l + \rho i)}{f^\Lambda(x|\alpha_1, \cdots, \alpha_l)} = (-1)^l e^{x\rho i} \frac{\mathrm{sh}\, \frac{\pi}{\lambda}(\alpha_l - \Lambda\pi i)}{\mathrm{sh}\, \frac{\pi}{\lambda}(\alpha_l + \Lambda\pi i + \rho i)} \prod_{j=1}^{l-1} \frac{\mathrm{sh}\, \frac{\pi}{\lambda}(\alpha_j - \alpha_l - \pi i)}{\mathrm{sh}\, \frac{\pi}{\lambda}(\alpha_l - \alpha_j - \pi i + \rho i)} \tag{5.41}$$

Therefore, the equality $I_k = 0$ gives rise to

$$0 = \int_C \cdots \int_C d\alpha_1 \cdots d\alpha_l f^\Lambda(x|\alpha_1, \cdots, \alpha_l) e^{-\frac{\pi}{\lambda}(\sum_{j=1}^{k-1} \alpha_j - \sum_{j=k}^{l-1} \alpha_j)}$$
$$\times \left\{ (-1)^l e^{x\rho i} \operatorname{sh} \frac{\pi}{\lambda}(\alpha_l - \Lambda\pi i) h(\alpha_l, \alpha_1, \cdots, \alpha_{l-1}) - \operatorname{sh} \frac{\pi}{\lambda}(\alpha_l + \Lambda\pi i) h(\alpha_1, \cdots, \alpha_l) \right\} \quad (5.42)$$

We use the notation

$$a_j = e^{\frac{2\pi}{\lambda}\alpha_j}, \quad \tau = e^{\frac{\Lambda\pi^2 i}{\lambda}}, \quad q = q^{(\lambda)}.$$

Symmetrizing the integrand of (5.42), we get

$$0 = \int_C \cdots \int_C d\alpha_1 \cdots d\alpha_l f^\Lambda(x - \frac{\pi}{\lambda}|\alpha_1, \cdots, \alpha_l) A_k(x|\alpha_1, \cdots, \alpha_l), \quad (5.43)$$

where

$$A_k(x|\alpha_1, \cdots, \alpha_l)$$
$$= \operatorname{Skew} \left\{ \left(-e^{x\rho i}(\tau^{-1} a_1 - \tau) a_{k+1} \cdots a_l - (\tau a_l - \tau^{-1}) a_k \cdots a_{l-1} \right) h(\alpha_1, \cdots, \alpha_l) \right\}.$$

Since $(a_j - q^{-2} a_{j+1}) h(\alpha_1, \cdots, \alpha_l)$ is symmetric with respect to α_j and α_{j+1}, we have

$$\operatorname{Skew}(a_j h(\alpha_1, \cdots, \alpha_l)) = q^{-2} \operatorname{Skew}(a_{j+1} h(\alpha_1, \cdots, \alpha_l)). \quad (5.44)$$

Using (5.44) repeatedly, we get

$$\operatorname{Skew}(a_1 a_{k+1} a_{k+2} \cdots a_l h(\alpha_1, \cdots, \alpha_l)) = q^{-2(k-1)} \operatorname{Skew}(a_k a_{k+1} \cdots a_l h(\alpha_1, \cdots, \alpha_l)),$$
$$\operatorname{Skew}(a_k \cdots a_{l-1} h(\alpha_1, \cdots, \alpha_l)) = q^{-2(l-k)} \operatorname{Skew}(a_{k+1} \cdots a_l h(\alpha_1, \cdots, \alpha_l)),$$

and therefore

$$A_k(x|\alpha_1, \cdots, \alpha_l) = -(\tau^{-1} e^{x\rho i} q^{-2(k-1)} + \tau) \operatorname{Skew}(a_k a_{k+1} \cdots a_l h(\alpha_1, \cdots, \alpha_l))$$
$$+ (\tau e^{x\rho i} + \tau^{-1} q^{-2(l-k)}) \operatorname{Skew}(a_{k+1} \cdots a_l h(\alpha_1, \cdots, \alpha_l)). \quad (5.45)$$

Substituting (5.45) for (5.43), we get

$$(\tau^{-1} e^{x\rho i} q^{-2(k-1)} + \tau) \int_C \cdots \int_C d\alpha_1 \cdots d\alpha_l f^\Lambda(x - \frac{\pi}{\lambda}|\alpha) \operatorname{Skew}(a_k \cdots a_l h(\alpha_1, \cdots, \alpha_l))$$
$$= (\tau e^{x\rho i} + \tau^{-1} q^{-2(l-k)}) \int_C \cdots \int_C d\alpha_1 \cdots d\alpha_l f^\Lambda(x - \frac{\pi}{\lambda}|\alpha) \operatorname{Skew}(a_{k+1} \cdots a_l h(\alpha_1, \cdots, \alpha_l)), \quad (5.46)$$

and therefore,

$$\prod_{k=0}^{l-1} (\tau^{-1} e^{x\rho i} q^{-2k} + \tau) \int_C \cdots \int_C d\alpha_1 \cdots d\alpha_l f^\Lambda(x - \frac{\pi}{\lambda}|\alpha) \operatorname{Skew}(a_1 \cdots a_l h(\alpha_1, \cdots, \alpha_l))$$
$$= \prod_{k=0}^{l-1} (\tau e^{x\rho i} + \tau^{-1} q^{-2k}) \int_C \cdots \int_C d\alpha_1 \cdots d\alpha_l f^\Lambda(x - \frac{\pi}{\lambda}|\alpha) \operatorname{Skew}(h(\alpha_1, \cdots, \alpha_l)).$$

If $|\operatorname{Re} x|$ is small, both $x \pm \frac{\pi}{\lambda}$ lie in the region (5.36). Therefore, we obtain (5.37). □

Consider

$$G_l^{\Lambda}(x) = \prod_{k=0}^{l-1} \frac{1}{S_2\left(\frac{\rho+\lambda}{2} - \pi(k-\Lambda) - \frac{\rho\lambda}{2\pi}x\right) S_2\left(\frac{\rho+\lambda}{2} - \pi(k-\Lambda) + \frac{\rho\lambda}{2\pi}x\right)}, \tag{5.47}$$

where $S_2(x) = S_2(x|\rho,\lambda)$. We can check that G_l^{Λ} also satisfies (5.37) and (5.38). Therefore, we obtain

Proposition 5.2.

$$F_l^{\Lambda}(x) = \widetilde{c}_l(\rho,\lambda;\Lambda) G_l^{\Lambda}(x), \tag{5.48}$$

where $\widetilde{c}_l(\rho,\lambda;\Lambda)$ is independent of x.

In order to determine $\widetilde{c}_l(\rho,\lambda;\Lambda)$, we consider the asymptotics of F_l^{Λ} and G_l^{Λ} as $x \to +i\infty$. From (6.58), we have

$$G_l^{\Lambda}(x) \sim \exp\left(-x\pi i \sum_{k=0}^{l-1}(\Lambda - k)\right), \quad (x \to +i\infty).$$

Therefore,

$$\widetilde{c}_l(\rho,\lambda;\Lambda) = \lim_{x\to+i\infty} \exp\left(x\pi i \sum_{k=0}^{l-1}(\Lambda-k)\right) F_l^{\Lambda}(x). \tag{5.49}$$

We calculate the limit (5.49). We use the following equalities.

$$\mathrm{Skew}\left(\prod_{1\le j<j'\le l} \mathrm{sh}\,\frac{\pi}{\rho}(\alpha_{j'} - \alpha_j - \pi i)\right) = \frac{[l]_{q^{(\rho)}}!}{l!} \prod_{1\le j<j'\le l} \mathrm{sh}\,\frac{\pi}{\rho}(\alpha_{j'} - \alpha_j), \tag{5.50}$$

$$e^{\frac{\pi^2 i}{2\rho}(l-1)(l-2\Lambda-2)} \mathrm{Skew}\left(e^{-\frac{\pi}{\rho}\sum_{j=1}^{l-1}\alpha_j} \prod_{j=2}^{l} \mathrm{sh}\,\frac{\pi}{\rho}(\alpha_j + \Lambda\pi i) \prod_{j=1}^{l-2}\prod_{j'=j+2}^{l} \mathrm{sh}\,\frac{\pi}{\rho}(\alpha_{j'} - \alpha_j - \pi i)\right)$$
$$= \frac{1}{l!} \prod_{1\le j<j'\le l} \mathrm{sh}\,\frac{\pi}{\rho}(\alpha_{j'} - \alpha_j). \tag{5.51}$$

Combining (4.34), (5.50), and (5.51), we have

$$\exp\left(x\pi i \sum_{k=0}^{l-1}(\Lambda-k)\right) F_l^{\Lambda}(x) = \frac{[l]_{q^{(\rho)}}![l]_{q^{(\lambda)}}!}{l!} e^{\frac{\pi^2 i}{2\rho}(l-1)(l-2\Lambda-2)}$$
$$\times \int_C \cdots \int_C d\alpha_1 \cdots d\alpha_l e^{x(\sum_{j=1}^{l}\alpha_j + \pi i \sum_{k=0}^{l-1}(\Lambda-k))} \prod_{j=1}^{l} \varphi(\alpha_j;\Lambda) \prod_{1\le j<j'\le l} \psi(\alpha_j - \alpha_{j'})$$
$$\times e^{-\frac{\pi}{\rho}\sum_{j=1}^{l-1}\alpha_j} \prod_{j=2}^{l} \mathrm{sh}\,\frac{\pi}{\rho}(\alpha_j + \Lambda\pi i) \prod_{j=1}^{l-2}\prod_{j'=j+2}^{l} \mathrm{sh}\,\frac{\pi}{\rho}(\alpha_{j'} - \alpha_j - \pi i) \prod_{1\le j<j'\le l} \mathrm{sh}\,\frac{\pi}{\lambda}(\alpha_{j'} - \alpha_j). \tag{5.52}$$

We denote by I_l^Λ the integrand of (5.52). The integral is zero in the limit $x \to +i\infty$ if

$$\mathrm{Im}\left(\sum_{j=1}^{l}\alpha_j + \pi i\sum_{k=0}^{l-1}(\Lambda-k)\right) > 0. \tag{5.53}$$

Therefore, we can compute the integral by shifting the contour in the positive imaginary direction, and taking the residues at the poles which we are crossing. Note that we assume that ρ and λ are large. Therefore, the only pole we cross is the one at $\alpha_1 = \Lambda\pi i$. Thus we get

$$\lim_{x\to i\infty}\int_C\cdots\int_C d\alpha_1\cdots d\alpha_l I_l^\Lambda = 2\pi i\lim_{x\to i\infty}\int_{C-\Lambda\pi i+i0}\cdots\int_{C-\Lambda\pi i+i0} d\alpha_2\cdots d\alpha_l \mathrm{res}_{\alpha_1=-\Lambda\pi i} I_l^\Lambda. \tag{5.54}$$

Now the integrand has a pole at $\alpha_2 = (1-\Lambda)\pi i$. Repeating a similar argument, we obtain

$$\lim_{x\to i\infty}\int_C\cdots\int_C d\alpha_1\cdots d\alpha_l I_l^\Lambda = (2\pi i)^l \mathrm{res}_{\alpha_l=(l-1-\Lambda)\pi i}\cdots\mathrm{res}_{\alpha_1=-\Lambda\pi i} I_l^\Lambda. \tag{5.55}$$

From (5.49), (5.52), (5.55) (6.60) and (6.61), we get

Proposition 5.3.

$$\widetilde{c}_l(\rho,\lambda;\Lambda) = \frac{[l]_{q^{(\rho)}}![l]_{q^{(\lambda)}}!}{4^{\binom{l}{2}}l!}\prod_{k=1}^{l}\frac{S_2(\pi)\sqrt{\rho\lambda}}{S_2(k\pi)S_2((k-2\Lambda-1)\pi)},$$

where $S_2(x) = S_2(x|\rho,\lambda)$.

6 Formula for the determinant

Combining (4.21), (4.22), (4.35), (5.47), (5.48), and (5.56), we obtain the following formula for the determinant $D_l(\beta_1,\ldots,\beta_n)$.

Theorem 6.1.

$$\begin{aligned}
&\det\left[I(w_{L'}^{(\rho)}, w_L^{(\lambda)})\right]_{L,L'\in\mathcal{Z}_l^n}\\
&= \frac{\left(q^{(\rho)}q^{(\lambda)}\right)^{\binom{n}{2}\binom{n+l-1}{n+1}+\binom{n+l-1}{n}\sum_{m=1}^n\Lambda_m}\left(S_2(\pi)\sqrt{\rho\lambda}\right)^{n\binom{n+l-1}{n}}}{4^{n(n-1)\binom{n+l-1}{n}+\binom{n+1}{2}\binom{n+l-1}{n+1}}(l!)^{\binom{n+l-1}{n-1}}}\\
&\times\prod_{j=1}^{l}\prod_{m=1}^{n}\left(\frac{[j]_{q^{(\rho)}}![j]_{q^{(\lambda)}}!}{\prod_{k=1}^{j}S_2(k\pi)S_2((k-2\Lambda_m-1)\pi)}\right)^{\binom{n+l-j-2}{n-2}}\\
&\quad\times\prod_{j=0}^{l-1}\left(\frac{S_2(\frac{\rho\lambda}{2\pi}\mu-(\sum_{m=1}^n\Lambda_m-j)\pi)}{S_2(\frac{\rho\lambda}{2\pi}\mu+(\sum_{m=1}^n\Lambda_m-j)\pi)}\right)^{\binom{n+j-1}{n-1}} E_l(\beta_1,\ldots,\beta_n).
\end{aligned}$$

Proof. It remains only to calculate $c_l(\rho,\lambda,\mu;\Lambda_1,\cdots,\Lambda_m)$ by using (4.35), (4.21) and (5.56).

It is easy to see that

$$\prod_{L\in\mathcal{Z}_l^n}\prod_{m=1}^{n}\widetilde{c}_{l_m}(\rho,\lambda|\Lambda_m)$$
$$=\frac{\left(S_2(\pi)\sqrt{\rho\lambda}\right)^{n\binom{n+l-1}{n}}}{4^{n\binom{n+l-1}{n+1}}}\prod_{j=1}^{l}\prod_{m=1}^{n}\left(\frac{[j]_{q^{(\rho)}}![j]_{q^{(\lambda)}}!}{j!\prod_{l=1}^{j}S_2(l\pi)S_2((l-2\Lambda_m-1)\pi)}\right)^{\binom{n+l-j-2}{n-2}}.$$

¿From (5.47), (6.59), and (6.60), we have

$$G_{l_m}^{\Lambda_m}\Big(\mu+\frac{2\pi^2}{\rho\lambda}\left(\sum_{j=1}^{m-1}(l_j-\Lambda_j)-\sum_{j=m+1}^{n}(l_j-\Lambda_j)\right)-\frac{\rho+\lambda}{\rho\lambda}\pi\Big)$$
$$=\prod_{l=0}^{l_m-1}\frac{S_2(\frac{\rho\lambda}{2\pi}\mu-\pi(\Lambda^{(m)}-\widetilde{l}_m-l))}{S_2(\frac{\rho\lambda}{2\pi}\mu-\pi(\Lambda^{(m-1)}-\widetilde{l}_m+l))},$$

where $\Lambda^{(m)}=\sum_{j=1}^{m}\Lambda_j-\sum_{j=m+1}^{n}\Lambda_j$, and $\widetilde{l}_m=\sum_{j=1}^{m-1}l_j-\sum_{j=m+1}^{n}l_j$.

Now we rewrite

$$\prod_{L\in\mathcal{Z}_l^n}\prod_{m=1}^{n}\prod_{k=0}^{l_m-1}\frac{S_2(\frac{\rho\lambda}{2\pi}\mu-(\Lambda^{(m)}-\widetilde{l}_m-k)\pi)}{S_2(\frac{\rho\lambda}{2\pi}\mu-(\Lambda^{(m-1)}-\widetilde{l}_m+k)\pi)}. \tag{6.56}$$

Let us consider the following sets:

$$\mathcal{T}_{\pm}^{(m)}=\bigsqcup_{L\in\mathcal{Z}_l^n}\{\widetilde{l}_m\pm k|k=0,1,\cdots,l_m-1\}\quad(m=1,\cdots,n),$$

where $\bigsqcup$ means a disjoint union. For $a\in\mathbf{Z}$, we set

$$\mathrm{mult}_{\pm}^{(m)}(a)=\#\{t\in\mathcal{T}_{\pm}^{(m)}|t=a\}.$$

We set $\mathrm{mult}_{\pm}^{(0)}(j)=\mathrm{mult}_{\pm}^{(n+1)}(j)\equiv 0$. Then, we have

$$(6.56)=\prod_{m=0}^{n}\prod_{j\in\mathbf{Z}}S_2\Big(\frac{\rho\lambda}{2\pi}\mu-(\Lambda^{(m)}-j)\pi\Big)^{\mathrm{mult}_{+}^{(m)}(j)-\mathrm{mult}_{-}^{(m+1)}(j)}$$
$$=\prod_{j=0}^{l-1}\left(\frac{S_2\left(\frac{\rho\lambda}{2\pi}\mu-(\Lambda^{(n)}-j)\pi\right)}{S_2\left(\frac{\rho\lambda}{2\pi}\mu-(\Lambda^{(0)}+j)\pi\right)}\right)^{\binom{n+j-1}{n-1}},$$

by using

$$\mathrm{mult}_{+}^{(n)}(j)=\mathrm{mult}_{-}^{(1)}(-j)=\begin{cases}\binom{n+j-1}{n-1}, & \text{if}\quad j=0,1,\cdots,l-1;\\ 0, & \text{otherwise,}\end{cases}$$

$$\mathrm{mult}_{+}^{(m)}(j)-\mathrm{mult}_{-}^{(m+1)}(j)=0,\quad(m=1,\cdots,n-1),\quad\text{for all}\quad j\in\mathbf{Z}.$$

This completes the proof. □

Appendix Double sine function

Here we summarize the property of the double sine function $S_2(x) = S_2(x|\omega_1, \omega_2)$ following [2].

We assume $\mathrm{Re}\,\omega_1 > 0, \mathrm{Re}\,\omega_2 > 0$. $S_2(x|\omega_1, \omega_2)$ is a meromorpic function of x and symmetric with respect to ω_1,ω_2. Its zeros and poles are given by

$$\text{zeros at } x = \omega_1 \mathbf{Z}_{\leq 0} + \omega_2 \mathbf{Z}_{\leq 0}, \quad \text{poles at } x = \omega_1 \mathbf{Z}_{\geq 1} + \omega_2 \mathbf{Z}_{\geq 1}.$$

Its asymptotic behavior is as follows:

$$\begin{aligned}&\log S_2(x) = \\ &= \pm\pi i\left(\frac{x^2}{2\omega_1\omega_2} - \frac{\omega_1+\omega_2}{2\omega_1\omega_2}x - \frac{1}{12}\left(\frac{\omega_1}{\omega_2} + \frac{\omega_2}{\omega_1} + 3\right)\right) + o(1),\ (x \to \infty,\ \pm\mathrm{Im}\,x > 0). \end{aligned} \tag{6.57}$$

This implies

$$\log S_2(a+x)S_2(a-x) = \pm\pi i\frac{2a-\omega_1-\omega_2}{\omega_1\omega_2}x + o(1), \quad (x \to \infty,\ \pm\mathrm{Im}\,x > 0). \tag{6.58}$$

The following formulae hold:

$$\frac{S_2(x+\omega_1)}{S_2(x)} = \frac{1}{2\sin\frac{\pi x}{\omega_2}}, \tag{6.59}$$

$$S_2(x)S_2(-x) = -4\sin\frac{\pi x}{\omega_1}\sin\frac{\pi x}{\omega_2}, \tag{6.60}$$

$$S_2(x) = \frac{2\pi}{\sqrt{\omega_1\omega_2}}x + O(x^2) \quad (x \to 0). \tag{6.61}$$

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