COMPLETENESS OF THE AUTOMORPHISM GROUPS OF FREE META-ABELIAN GROUPS

KATSUTOSHI YAMANOI

Research Institute for Mathematical Sciences, Kyoto University

ABSTRACT. For a given group G, one of the basic questions related to G is to determine its automorphism group $Aut(G)$, and in particular its outer automorphism group $Out(G)$, the quotient of $Aut(G)$ modulo the inner automorphism group $Int(G)$. When F_r is a free group of finite rank r, $Aut(F_r)$ has been one of the main object in Combinatorial group theory. In this case, $Aut(F_r)$ is known to be generated by 4 kinds of *elementary automorphisms*, and the fundamental relations have also been determined by Nielsen [12]. Moreover, $Aut(F_r)$ is complete, i.e. $Aut(AutF_r) = AutF_r$. On the other hand, the automorphism group of the abelianization $\mathbb{Z}^{\oplus r} = F_r/F_r'$ of F_r is the general linear group $GL_r(\mathbb{Z})$, and its outer automorphism group is known to be finite of order at most 4 [7,Theorem4]. In this paper, we shall study $Out(AutG)$ when G is a finer quotient of F_r than F_r/F_r', namely the free meta-abelian group of rank r. It is, by definition, the quotient $M_r = F_r/F_r''$ where $F_r' = (F_r, F_r)$ and $F_r'' = (F_r', F_r')$, $(\ ,\ )$ being the commutator subgroup. We shall prove that also in this case, $AutM_r$ is complete, basically following the method of [5] used in the case of F_r, but we need some new devices to prove the meta-abelian versions of various lemmas on free groups used there.

§1 Introduction.

The main purpose of this paper is to prove the following.

Theorem A. *Let M_r be a free meta-abelian group of finite rank $r \geq 4$ and $AutM_r$ be its automorphism group. Then $AutM_r$ has trivial center,and every automorphism of $AutM_r$ is inner. In other words, the canonical homomorphism $AutM_r \to Aut(AutM_r)$ is bijective.*

Since M_r has trivial center (see.eg.p9 Remark),and since it is known that for any group G which has trivial center, the two conditions

(a)the canonical homomorphism $AutG \longrightarrow Aut(AutG)$ is bijective,

(a')$G(\simeq IntG)$ is a characteristic subgroup of $AutG$,

are equivalent [4,p95], the above Theorem A is equivalent to the following Theorem A'. Here and in what follows, for any group G with trivial center, we shall always identify G with its inner automorphism group and regard it as a normal subgroup of AutG.

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Theorem A'.
When $r \geq 4$, M_r is a characteristic subgroup of $AutM_r$

In the following, we shall prove Theorem A'. Our proof basically follows the method of John L. Dyer and Edward Fomanek [5] used to prove the F_r version of Theorem A'. The content of this paper is as follows. In §2 ,we shall review some basic facts of Magnus representations which are used in later sections. In §3 ,we shall show that $M_r' = (M_r, M_r)$ is a characteristic subgroup of $AutM_r$, assuming some results proved in §4,5 . In §4, we shall show that certain two subgroups of $AutM_r$ are characteristic. In §5, we shall show that

$$K_r = ker(AutM_r \to Aut(M_r/M_r'))$$

is residually nilpotent, i.e. , the intersection of the members of its lower central series reduces to the identity. In §6, by using the result of §3, we shall deduce that M_r itself is characteristic in $AutM_r$. Finally, in the Appendix, we shall study some question related to §6.

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Notations. In this paper,we shall denote for free generators of F_r by $\widetilde{x}_1, \cdot\cdot, \widetilde{x}_r$, and the images of $\widetilde{x}_1, \cdots, \widetilde{x}_r$ by the projection $F_r \to F_r/F_r''$ (resp.$F_r \to F_r/F_r'$) are denoted by $x_1, \cdots, x_r$(resp.$\overline{x_1}, \cdots, \overline{x_r}$).
These generators of F_r, F_r/F_r', F_r/F_r'' will be fixed throughout, and provide a fixed isomorphism of $Aut(F_r/F_r')$ with $GL_r(\mathbb{Z})$ under which $a \in Aut(F_r/F_r')$ corresponds to $(a_{ij}) \in GL_r(\mathbb{Z})$ where for $i = 1, \cdots, r$, $a(\overline{x_i}) = \sum_l a_{li}\overline{x_l}$

§2. To study the structure of the group $M_r = F_r/F_r''$, its automorphism group $AutM_r$, and the kernel $K_r = ker(AutM_r \to Aut(M_r/M_r'))$, it is useful to consider the group ring

$$A_r = \mathbb{Z}[M_r/M_r'] = \mathbb{Z}[\overline{x_1}^{\pm 1}, \cdots, \overline{x_r}^{\pm 1}]$$

of $M_r/M_r' = \mathbb{Z}^r$. Indeed, by the theory of W.Magnus [9],S.Bachmuth [2] (see also [11]), and for pro-ℓ case Y.Ihara [8], we can embed M_r into $A_r^{\oplus r}$, and K_r into $GL_r(A_r)$ in the following natural manner. Call $\pi_r : M_r \to M_r/M_r' \hookrightarrow A_r$ the canonical projection.
(1) There exists a unique injective mapping

$$f : M_r \to A_r^{\oplus r} = A_r t_1 \oplus \cdots \oplus A_r t_r$$

satisfy,

· $f(ab) = f(a) + \pi_r(a)f(b)$

· $f(x_i) = t_i \qquad (1 \leq i \leq r)$

· if $f(y) = \sum y_i t_i$ then $\sum(\overline{x_i} - 1)y_i = \pi_r(y) - 1$

· an element $\sum y_i t_i$ of $A_r^{\oplus r}$ is in the image of f if and only if $\sum(\overline{x_i} - 1)y_i = \overline{x_1}^{\alpha_1} \cdots \overline{x_r}^{\alpha_r} - 1$ for some integer α_i $(1 \leq i \leq r)$.

· the restriction of f to the $M_r' = F_r'/F_r''$ gives an A_r module isomorphism of M_r' with submodule of $A_r^{\oplus r}$, $\{(y_1, \cdots, y_r) \in A_r^{\oplus r} | \sum(\overline{x_i} - 1)y_i = 0\}$.Here, the action of A_r on M_r' is defined by the linear extension of the action of $F_r/F_r' = M_r/M_r'$ by conjugation.
Thus $\overline{x_i}$ acts on M_r' as $u \mapsto x_i u x_i^{-1} (mod M_r'')$ $(u \in M_r')$.

(2) There exists an injective group homomorphism $g : K_r \to GL_r(A_r)$ satisfy,

· an element $\tau \in GL_r(A_r)$ is in the image of g if and only if $(\overline{x_1} - 1, \cdots, \overline{x_r} - 1)\tau = (\overline{x_1} - 1, \cdots, \overline{x_r} - 1)$

· following diagram is commutative for every $k \in K$

$$\begin{array}{ccc} M_r & \xrightarrow{f} & A_r^{\oplus r} \\ {\scriptstyle k}\downarrow & & \downarrow{\scriptstyle g(k)} \\ M_r & \xrightarrow[f]{} & A_r^{\oplus r} \end{array}$$

where an element of $A_r^{\oplus r}$ is considered as column vector and $GL_r(A_r)$ acts by left multiplication.

(3) The inclusion $M \subset K$ and (1),(2) are related as follows.
For every $y \in M$

$$g(y) = \pi_r(y)^{-1} f(y)(\overline{x_1} - 1, \cdots, \overline{x_r} - 1) + \pi_r(y)^{-1} I_r$$

where $f(y)$ is considered as column vector and I_r is an identity matrix.

(4) For each prime number ℓ, let $\mathbb{Z}_\ell$ be the ring of ℓ adic integers, and call $\hat{A}_r = \mathbb{Z}_\ell[[\overline{x_1} - 1, \cdots, \overline{x_r} - 1]]$ the completed group algebra of M_r/M_r' over $\mathbb{Z}_\ell$, which is isomorphic to the ring of formal power series in $\overline{x_i} - 1$ $(1 \leq i \leq r)$ over $\mathbb{Z}_\ell$. Call $\pi_r^{(\ell)} : M_r^{(\ell)} \to M_r^{(\ell)}/(M_r^\ell)' \hookrightarrow \hat{A}_r$ the canonical projection, where $M_r^{(\ell)}$ denotes pro-ℓ completion of M_r. Then $M_r^{(\ell)}$can be embedded into $\hat{A}_r t_1 \oplus \cdots \oplus \hat{A}_r t_r$ by a same way as (1). There exists a unique injective mapping.

$$f^{(\ell)} : M_r^{(\ell)} \to \hat{A}_r{}^{\oplus r} = \hat{A}_r t_1 \oplus \cdots \oplus \hat{A}_r t_r$$

satisfy,

· $f^{(\ell)}(ab) = f^{(\ell)}(a) + \pi_r^{(\ell)}(a) f^{(\ell)}(b)$
· $f^{(\ell)}(x_i) = t_i \qquad (1 \le i \le r)$
· if $f^{(\ell)}(y) = \sum y_i t_i$ then $\sum(\overline{x_i} - 1)y_i = \pi_r^{(\ell)}(y) - 1$
· an element $\sum y_i t_i$ of $\hat{A}_r^{\oplus r}$ is in the image of $f^{(\ell)}$ if and only if $\sum(\overline{x_i} - 1)y_i = \overline{x_1}^{\alpha_1} \cdots \overline{x_r}^{\alpha_r} - 1$ for some ℓ adic integer α_i $(1 \le i \le r)$.

§3. In what follows, we shall basically fix $r \ge 4$ and abbreviate as $M = M_r$ e.t.c. As the center of M is trivial (see.e.g.p9 Remark), we shall identify M with the inner automorphism group of M itself and regard M as a normal subgroup of $AutM$. we shall prove in §3 that M' is a characteristic subgroup of $AutM$, assuming Proposition B,C below which will be proved in §4,5 respectively.

Proposition B.
Let $K = ker(AutM \to Aut(M/M'))$. Then K and MK' are characteristic subgroups of $AutM$.

Proposition C.
$\bigcap_j (M\gamma_j(K)) = M$ *in* $AutM$.
where in general,for group G, we shall denote by $\{\gamma_j(G)\}_{j=1}^{\infty}$ its lower central series defined by $\gamma_1(G) = G$,$\gamma_{j+1}(G) = (G, \gamma_j(G))$ $(j \ge 1)$.

The basic two known facts that are key to our arguments in §3 are the following.

Lemma 1 [11,TheoremA]**.**
The projection $Aut(F_r) \to Aut(M_r)$ is surjective for $r \ge 4$

Lemma 2 [1,p254] [3,lemma3 and 5]**.**
Put $\widetilde{K} = ker(AutF_r \to Aut(F_r/F_r'))$. Then the sequence

$$1 \to \widetilde{K}' \to AutF_r \to Aut(F_r/\gamma_3(F_r)) \to 1$$

is exact.

We shall also need the following two lemmas

Lemma 3.
The sequence $1 \to K' \to AutM_r \to Aut(M_r/\gamma_3(M_r)) \to 1$ is exact.

Lemma 4.
$K' \cap M = M'$

Proof of lemma 3.

We shall reduce lemma3 to lemma 2. By lemma 2, $\widetilde{K}'$ coincides with the kernel of $AutF_r \to Aut(F_r/\gamma_3(F_r))$, but since $F'' \subset \gamma_3(F)$, this homomorphism

factors through $u : AutF \to Aut(F/F'')$. therefore, $keru \subset \widetilde{K}'$. So lemma 2 induces an exact sequence

$$1 \to \widetilde{K}'/keru \to AutF_r/keru \to Aut(F_r/\gamma_3(F_r)) \to 1$$

But this is exactly what we want. $\square$

Proof of lemma 4.

By lemma 3, K' is the kernel of $v : AutM \to Aut(M/\gamma_3(F_r))$ hence $M \cap K'$ is the kernel of its restriction v_M to M. But this is equal to M', because the center of $M/\gamma_3(F_r)$ is $M'/\gamma_3(F_r)$. $\square$

We shall also make frequent use of the following two equality:

(i) $(N_1, N_2N_3) = (N_1, N_2)(N_1, N_3)$

for the commutators of normal subgroup N_1, N_2, N_3 of any group G;

(ii) $(M, K') = \gamma_3(F_r)$

for the groups $M = M_r$, $K = K_r$ $(r \geq 4)$.

The second equality (ii) follows directly from the exact sequence of lemma 3, $1 \to K' \to AutM_r \to Aut(M_r/\gamma_3(M_r)) \to 1$. In fact, since K' acts trivially on $M_r/\gamma_3(M_r)$, we have $(M, K') \subset \gamma_3(M)$. But since $(M, K') \supset (M, M') = \gamma_3(M)$, we conclude that $(M, K') = \gamma_3(M)$.

Now we are going to prove that M' is a characteristic subgroup of $Aut(M)$. To prove this, let ϕ be any element of $Aut(AutM)$. Then ϕ acts on $AutM$, and M is regarded as a subgroup of $AutM$. So, $\phi(M) \subset AutM$. We first prove by induction on $j \geq 3$ that

$$M' = (M, \phi(M))\gamma_j(M) \qquad (*)$$

The case $j = 3$

$$\begin{aligned} M' = (M,M) = (M,M)\gamma_3(M) &\underset{\text{(ii)}}{=} (M,M)(M,K') \underset{\text{(i)}}{=} (M,MK') \\ &\underset{\text{PropB}}{=} (M,\phi(MK')) \underset{\text{PropB}}{=} (M,\phi(M)K') \underset{\text{(i)}}{=} (M,\phi(M))(M,K') \\ &\underset{\text{(ii)}}{=} (M,\phi(M))\gamma_3(M) \end{aligned}$$

The induction step $j \to j+1$

$$\begin{aligned} M' = (M,\phi(M))\gamma_j(M) &\supseteq (M,\phi(M))\gamma_{j+1}(M) = (M,\phi(M))(M,\gamma_j(M)) \\ &\underset{\text{(i)}}{=} (M,\phi(M))(M,\phi(M)\gamma_j(M)) \supseteq (M,\phi(M))(M,(M,\phi(M))\gamma_j(M)) \\ &= (M,\phi(M))(M,M') = (M,\phi(M))\gamma_3(M) = M' \end{aligned}$$

Therefore $M' = (M, \phi(M))\gamma_{j+1}(M)$. This proves (*).

Hence,

$M' = (M, \phi(M))\gamma_j(M) \subset \phi(M)\gamma_j(M) \subset \phi(M)\gamma_j(K) \underset{\text{PropB}}{=} \phi(M\gamma_j(K))$.

Therefore $\phi^{-1}(M') \subset M\gamma_j(K)$ for all $j \geq 3$. But by Prop.C, $\bigcap_j M\gamma_j(K) = M$. Thus,we have $\phi^{-1}(M') \subset M$. On the other hand, since $M' = M \cap K' \subset K'$, and K' is a characteristic subgroup of $AutM$ (Prop.B), we have $\phi^{-1}(M') \subset \phi^{-1}(K') = K'$.Hence,$\phi^{-1}(M') = \phi^{-1}(M') \cap K' \subseteq M \cap K' = M'$. Therefore $M' \subset \phi(M')$.Apply this to ϕ^{-1} to conclude the opposite inclusion $M' \supset \phi(M')$. Therefore $M' = \phi(M')$.

§**4.** In this section, we shall prove Prop.B

Proof of Proposition B.

(i)That K is a characteristic subgroup of $AutM$.

We need the following lemma.

Lemma 5 [5,Theorem B]**.**

The kernel $\widetilde{K}$ of $AutF_r \to Aut(F_r/F_r') \simeq GL_r(\mathbb{Z})$ is contained in the kernel of any matrix representation $\rho : AutF_r \to GL_m(R)$ of size $m \leqslant r$, where R is any integral domain of characteristic $\neq 2$.

Take $\phi \in Aut(AutM)$, the composite homomorphism

$$AutF_r \xrightarrow{u} AutM_r \xrightarrow{\phi} AutM_r \to AutM_r/K \simeq GL_r(\mathbb{Z})$$

has kernel $u^{-1}(\phi^{-1}(K))$.

By lemma 5 (applied to $m = r$, $R = \mathbb{Z}$),$u^{-1}(\phi^{-1}(K)) \supseteq \widetilde{K}$, and by definition, $\widetilde{K} = u^{-1}(K)$. By lemma 1, u is surjective so we obtain $\phi^{-1}(K) \supseteq K$ $\square$

(ii) That MK' is a characteristic subgroup of $AutM$.

Since K is a normal subgroup of $AutM$, each $\sigma \in AutM$ induces an automorphism of K by conjugation $x \mapsto \sigma x \sigma^{-1}$, and also an automorphism of K^{ab}. And by this action each $k \in K \subset AutM$ acts on K^{ab} trivially, so $AutM/K$ acts on K^{ab}. But since $AutM/K \simeq Aut(M/M') \simeq GL_r(\mathbb{Z})$, K^{ab} is a left $GL_r(\mathbb{Z})$ module. By same reason $\widetilde{K}^{ab}$ is a left $GL_r(\mathbb{Z})$ module.

Let $u : AutF \to AutM$ be the natural surjection(c.f.lemma 1), and let $u_{\widetilde{K}}$ be the restriction of u to $\widetilde{K}$. Then $u_{\widetilde{K}}$ is a surjection $u_{\widetilde{K}} : \widetilde{K} \to K$ and $keru = keru_{\widetilde{K}}$, so we get $\widetilde{K}/keru \simeq K$. By lemma 2, since $\widetilde{K}'$ is a kernel of composition of homomorphisms $AutF_r \to Aut(F_r/F_r'') \to Aut(F_r/\gamma_3(F_r))$, we have $keru \subseteq \widetilde{K}'$, so $u_{\widetilde{K}} : \widetilde{K} \to K$ induces isomorphism, $\widetilde{K}^{ab} \to K^{ab}$ and by definition of the $GL_r(\mathbb{Z})$ action on $\widetilde{K}^{ab}$ and K^{ab}, this isomorphism is an isomorphism of

$GL_r(\mathbb{Z})$ modules. By lemma 4, $K' \cap M = M'$; hence $M^{ab} \subseteq K^{ab}$. Since x_r generate M as an $AutM$ group, it follows that $\overline{x_r}(x_r mod M')$ generate M^{ab} as an $GL_r(\mathbb{Z})$ module. By the first part of Prop.B, K is a characteristic subgroup of $AutM$, so $Aut(AutM)$ acts on $AutM/K \simeq GL_r(\mathbb{Z})$ and on $K/K' \simeq K^{ab}$ and as a $GL_r(\mathbb{Z})$ module K^{ab}, $Aut(AutM)$ acts semilinearly, i.e. if $g \in GL_r(\mathbb{Z})$, $\sigma \in Aut(AutM)$ then $g(k)^\sigma = g^\sigma(k^\sigma)$. From the fact $K' \subset MK' \subset K$, to show that MK' is a characteristic subgroup of $AutM$, we have to show that $MK'/K' \simeq M^{ab}$ is stable under the $Aut(AutM)$ action in $K/K' \simeq K^{ab}$. Since $\overline{x_r}$ generate M^{ab} as $GL_r(\mathbb{Z})$ module, it is sufficient to show $\overline{x_r}^\sigma \subset M^{ab}$ for every $\sigma \in Aut(AutM)$. We now quote two lemmas.

Lemma 6 [7,Theorem 4]**.**

Let J be the subgroup of $Aut(GL_r(\mathbb{Z}))$ where an element of J is an inner automorphism of $GL_r(\mathbb{Z})$. Then $[Aut(GL_r(\mathbb{Z})) : J] = 4$ (r:even ≥ 4) $[Aut(GL_r(\mathbb{Z})) : J] = 2$ (r :odd ≥ 3). Moreover for each $X \in GL_r(\mathbb{Z})$ let X^t denote the transpose of X,

(1) if r is even ≥ 4, the four automorphism of $GL_r(\mathbb{Z})$ $R_1 = identity$ $R_2 : X \mapsto (X^t)^{-1}$, $R_3 : X \mapsto (detX)X$, $R_4 = R_2R_3$ represent the 4 cosets of J in $Aut(GL_r(\mathbb{Z}))$.

(2) if r is odd ≥ 3, the 2 cosets of J in $Aut(GL_r(\mathbb{Z}))$ are represented by R_1 and R_2.

Lemma 7 [5,Proposition 15]**.**

Let $N \subseteq GL_r(\mathbb{Z})$ be the image of $\mathcal{N} \subseteq AutF_r$ defined by $\mathcal{N} = \{\alpha \in AutF_r | \widetilde{x_r}^\alpha = \widetilde{x_r}\}$

(1) If $r \geq 3$, $\widetilde{K}^{ab}_N =< \overline{x_r} >$ and $\widetilde{K}^{ab}_{R_2(N)} = \{1\}$

(2) If $r \geq 4$ is even, $\widetilde{K}^{ab}_{R_3(N)} = \widetilde{K}^{ab}_{R_4(N)} = \{1\}$

where if L is any subgroup of $GL_r(\mathbb{Z})$, a subgroup $\widetilde{K}^{ab}_L$ of $\widetilde{K}^{ab}$ is defined by

$$\widetilde{K}^{ab}_L = \{k \in \widetilde{K}^{ab} | l(k) = k \quad for\ all \quad l \in L\}$$

Although lemma 7 is stated for $\widetilde{K}^{ab}$, as $GL_r(\mathbb{Z})$module $\widetilde{K}^{ab}$ and K^{ab} is isomorphic, the same statement holds for K^{ab} .

Now we conclude our proof of Prop.B. Take $\sigma \in Aut(AutM)$. Since the action of an element of $AutM \subset Aut(AutM)$ on M^{ab} is just the $GL_r(\mathbb{Z})$ action, we can multiply a suitable element of $AutM$ on σ , and assume without loss of generality that σ acts on $GL_r(\mathbb{Z})$ as one of the R_i's in lemma 6. We have

$1 \neq \sigma(< \overline{x_r} >) = \sigma(K^{ab}_N) = K^{ab}_{N^\sigma} = K^{ab}_{R_i(N)}$ for some i, so, by lemma 7 we conclude that $i = 1$ and that $\sigma(< \overline{x_1} >) =< \overline{x_1} > \subset M^{ab}$. This proves Prop.B. $\square$

§**5.** Now we are going to prove Prop.C. We start with some lemma. An element x of M will be called primitive if it is an $AutM$ transform of x_1.

Lemma 8.

Suppose $\alpha \in AutM$ is such that x is conjugate to $\alpha(x)$ in M for every primitive $x \in M$. Then $\alpha \in M$.

Proof of lemma 8.

We can take $\beta \in AutM$ such that $\beta = \tau \circ \alpha$ where $\tau \in M$, and $\beta(x_1) = x_1$, and $\beta(x)$ is conjugate to x in M for every primitive $x \in M$. Since $\beta \in K$, we can regard β as an element of $GL_r(A)$ (see §2) and write $\beta = (b_{ij})$ $(1 \leqslant i, j \leqslant r)$, $b_{ij} \in A$. Since $\beta(x_1) = x_1$, and $f(x_1) = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$, we obtain

$$(b_{ij}) \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \qquad \text{so} \quad b_{i1} = \begin{cases} 1 & i = 1 \\ 0 & i \neq 1 \end{cases}$$

Next, since $\beta(x_j x_1^m)$ is conjugate to $x_j x_1^m$ in M (because $x_j x_1^m$ is primitive) and $f(x_j x_1^m) = \begin{pmatrix} * \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} < j$, we obtain

$$(b_{ij}) \begin{pmatrix} * \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} < j = \begin{pmatrix} * \\ b_{2j} \\ \vdots \\ \vdots \\ \vdots \\ b_{rj} \end{pmatrix} = (\overline{x}_1^m \overline{x}_j - 1) \begin{pmatrix} * \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ * \end{pmatrix} + \begin{pmatrix} * \\ 0 \\ \vdots \\ 0 \\ * \\ 0 \\ \vdots \\ 0 \end{pmatrix} < j$$

so if $i \neq 1, j$, b_{ij} is divisible by $(\overline{x}_1^m \overline{x}_j - 1)$ for every integer $m \geq 1$, and we obtain $b_{ij} = 0$ $(i \neq 1, j)$. Next we can choose $u \in M (\subset AutM)$ such that

$u\beta(x_2) = x_2$ and $u\beta(x)$ is conjugate to x in M for every primitive $x \in M$, and by the same as before, we obtain

$$(\pi(u)^{-1} f(u)(\overline{x}_1 - 1, \cdots, \overline{x}_r - 1) + \pi(u)^{-1} I_r)(b_{ij}) = \begin{pmatrix} * & 0 & & \dots & 0 \\ * & 1 & * & \dots & * \\ 0 & 0 & * & 0 & 0 \\ & & & \ddots & 0 \\ 0 & 0 & & 0 & * \end{pmatrix}$$

Now write $f(u) = \begin{pmatrix} u_1 \\ \vdots \\ u_r \end{pmatrix}$ and compute the $(1, j)$ component of this matrix. Then we obtain

$$u_1(\overline{x}_j - 1) + b_{1j} = 0$$

On the other hand, since an element of K fixes $(\overline{x}_1 - 1, \cdots, \overline{x}_r - 1)$, we have $(\overline{x}_1 - 1)b_{1j} + (\overline{x}_j - 1)b_{jj} = (\overline{x}_j - 1)$, so we obtain $b_{jj} = 1 + (\overline{x}_1 - 1)u_1$. Since $(b_{ij}) \in GL_r(A)$ is upper triangular, $b_{jj} \in A$ is a unit; hence $b_{jj} = \overline{x}_1^N$ for some $N \in \mathbb{Z}$, and $u_1 = \frac{\overline{x}_1^N - 1}{\overline{x}_1 - 1}$. Therefore we obtain

$$(b_{ij}) = \begin{pmatrix} 1 & -(\overline{x}_2 - 1)\frac{\overline{x}_1^N - 1}{\overline{x}_1 - 1} & \dots & -(\overline{x}_r - 1)\frac{\overline{x}_1^N - 1}{\overline{x}_1 - 1} \\ 0 & \overline{x}_1^N & 0 & 0 \\ & 0 & \ddots & 0 \\ 0 & & 0 & \overline{x}_1^N \end{pmatrix}$$

But this is just the conjugation by x_1^{-N}; hence $\beta \in M$; hence $\alpha \in M$ $\square$

Remark. At the same time, this proof of lemma 8 states that the centralizer of x_1 is $\{x_1^N | N \in \mathbb{Z}\}$, so the center of M is trivial.

Lemma 9.

Suppose $x, y \in M$ are primitive elements. If x, y are conjugate in the group $M^{(\ell)}$ for some prime number ℓ, then they are conjugate in M.

Proof of lemma 9.

We can assume without loss of generality that $x = x_1$. Since x_1 and y are conjugate in $M^{(\ell)}$, we can write,

$$(\overline{x}_1 - 1)\begin{pmatrix} u_1 \\ \vdots \\ u_r \end{pmatrix} + \overline{x}_1^{\alpha_1} \cdots \overline{x}_r^{\alpha_r} \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} y_1 \\ \vdots \\ y_r \end{pmatrix} \qquad (*)$$

$$f(y) = \begin{pmatrix} y_1 \\ \vdots \\ y_r \end{pmatrix}, \quad y_i \in A, \quad u_i \in \hat{A}, \quad \alpha_i \in \mathbb{Z}_\ell$$

By considering (*) mod($\overline{x}_1 - 1$) in $\hat{A}/(\overline{x}_1 - 1) \simeq \mathbb{Z}_\ell[[\overline{x}_2 - 1, \cdots, \overline{x}_r - 1]]$ we obtain $\overline{x}_2^{\alpha_2} \cdots \overline{x}_r^{\alpha_r} \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} \overline{y_1} \\ \vdots \\ \overline{y_r} \end{pmatrix}$, $\overline{y_i} = y_i (mod \overline{x}_1 - 1) \in \mathbb{Z}_\ell[\overline{x}_2^{\pm 1}, \cdots, \overline{x}_r^{\pm 1}]$

But since $\{\overline{x}_2^{\mathbb{Z}_\ell} \cdots \overline{x}_r^{\mathbb{Z}_\ell}\} \cap \mathbb{Z}[\overline{x}_2^{\pm 1}, \cdots, \overline{x}_r^{\pm 1}] = \{\overline{x}_2^{\mathbb{Z}} \cdots \overline{x}_r^{\mathbb{Z}}\}$
in $\mathbb{Z}_\ell[[\overline{x}_2 - 1, \cdots, \overline{x}_r - 1]]$, we have $\alpha_2, ..., \alpha_r \in \mathbb{Z}$, so (*) becomes

$$(\overline{x}_1 - 1) \begin{pmatrix} u_1' \\ \vdots \\ u_r' \end{pmatrix} + \overline{x}_2^{\alpha_2} \cdots \overline{x}_r^{\alpha_r} \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} y_1 \\ \vdots \\ y_r \end{pmatrix} \qquad (*)'$$

here $u_i' \in \hat{A}$. It remains to show that $u_i' \in A$ $(1 \leqslant i \leqslant r)$. But this follows directly from the following lemma 10. Thus lemma 9 is reduced to lemma 10.

Lemma 10.

Let $a \in A$, $b \in \hat{A}$ be such that $(\overline{x}_1 - 1)b = a$. Then $b \in A$.

Proof of lemma 10.

Let $\mathfrak{m} = (\ell, \overline{x}_1 - 1, ..., \overline{x}_r - 1)$ be the maximal ideal of A generated by $\ell, \overline{x}_1 - 1, ..., \overline{x}_r - 1$. Let $A_\mathfrak{m}$ be the localization of A at $\mathfrak{m}$. Then $A_\mathfrak{m}$ is a local ring, and the $\mathfrak{m}$ - adic completion of $A_\mathfrak{m}$ is $\hat{A}$. We first show that $b \in A_\mathfrak{m}$. For all $i \geq 0$, we can choose $b_i \in A_\mathfrak{m}$ such that $b - b_i \in \mathfrak{m}^i \hat{A}$. Then we have $a - (\overline{x}_1 - 1)b_i \in A_\mathfrak{m} \cap \mathfrak{m}^i \hat{A} = \mathfrak{m}^i A_\mathfrak{m}$, so $a \in \mathfrak{m}^i A_\mathfrak{m} + (\overline{x}_1 - 1)A_\mathfrak{m}$ and $a \in \bigcap_i (\mathfrak{m}^i A_\mathfrak{m} + (\overline{x}_1 - 1)A_\mathfrak{m})$. By the Krull's theorem, $\bigcap_i (\mathfrak{m}^i A_\mathfrak{m} + (\overline{x}_1 - 1)A_\mathfrak{m}) = (\overline{x}_1 - 1)A_\mathfrak{m}$ so $a \in (\overline{x}_1 - 1)A_\mathfrak{m}$ and $b \in A_\mathfrak{m}$. Since A is a unique factorization domain, we can write $b = b'/b''$ where b' and b'' has no common factor, and $b'' \notin \mathfrak{m}$. From $(\overline{x}_1 - 1)b'/b'' = a$, we have $(\overline{x}_1 - 1)b' = b''a$, so b'' is a unit and this proves $b \in A$. $\square$

Now we prove Prop.C. Define for $j = 1, 2, ..$

$$K_j = ker(AutM \to Aut(M/\gamma_j(M)))$$

Then since $K_j \supseteq \gamma_j(K)$, to prove Prop.C, it suffices to prove Prop.C'. (The statement $K_j \supseteq \gamma_j(K)$ follows from the fact $\tilde{K}_j \supseteq \gamma_j(\tilde{K})$ ([1,Theorem 1.1]) and

commutative diagram

$$\begin{array}{ccc} AutF & \longrightarrow & Aut(F/\gamma_j(F)) \\ \downarrow & & \downarrow \\ AutM & \longrightarrow & Aut(M/\gamma_j(M)) \end{array}$$

where $\widetilde{K_j}$ is $ker(AutF \to Aut(F/\gamma_j(F)))$

Proposition C'.
$\cap M \cdot K_j = M$

Proof of Proposition C'.
$\cap M \cdot K_j \supset M$ is obvious, so we shall prove $\cap M \cdot K_j \subset M$. Suppose $\alpha \in \cap M \cdot K_j$. For each j choose $k_j \in K_j$, $u_j \in M$ such that $\alpha = u_j k_j$. Then for all primitive $x \in M$, $\alpha(x) = u_j k_j(x) u_j^{-1} \equiv u_j x u_j^{-1} (mod \gamma_{j+1}(M))$. Since $\alpha(x)$ and x are conjugate modulo $\gamma_{j+1}(M)$ for all j, they are conjugate at all finite quotients of ℓ power order, so they are conjugate in $M^{(\ell)}$. So they are conjugate in M (From lemma 7), and therefore $\alpha \in M$ (From lemma 6), so $\cap M \cdot K_j \subset M$. $\square$

§6. We shall consider $M' = F'/F''$ as an A submodule of $A^{\oplus r}$. Let Q be the quotient field of A, and define Q module $M'_Q = M' \bigotimes_A Q$, $K_Q = \{k \in GL_r(Q) | (\overline{x}_1 - 1, \cdots, \overline{x}_r - 1)k = (\overline{x}_1 - 1, \cdots, \overline{x}_r - 1)\}$. First we prove the following lemma. Here, for any group G,$Z(G)$ denote its center.

Lemma 11.
(1) $1 \to M' \to K \to Aut_A M'$ *is exact.*
(2) $Z(K/M') \subset Z(Aut_A M') = A^\times$

Proof of lemma 11(1).
The only nontrivial part is the exactness of $M' \to K \to Aut_A M'$. To prove this, consider the commutative diagram of group homomorphisms

$$\begin{array}{ccccccc} 1 & \longrightarrow & M' & \longrightarrow & K & \longrightarrow & Aut_A M' \\ & & \bigcap & & \bigcap & & \bigcap \\ 1 & \longrightarrow & M'_Q & \longrightarrow & K_Q & \longrightarrow & Aut_Q M'_Q \longrightarrow 1 \end{array}$$

Since the lower sequence is exact (which can be easily checked by changing the coordinate system so that the vector $(\overline{x}_1 - 1, \cdots, \overline{x}_r - 1)$ has the coordinate $(1, 0, \cdots, 0)$), and $K \cap M'_Q = M'$, the upper sequence is also exact. $\square$

Proof of lemma 11 (2).

We first show that K is Zariski dense in K_Q. Let σ be the matrix

$$\begin{pmatrix} \overline{x}_1 - 1 & \overline{x}_2 - 1 & \dots & \overline{x}_r - 1 \\ 0 & 1 & 0 & 0 \\ & 0 & \ddots & 0 \\ 0 & & 0 & 1 \end{pmatrix} \in GL_r(Q)$$

Then we have

$$\sigma K_Q \sigma^{-1} = \{ \begin{pmatrix} 1 & 0 & \dots & 0 \\ & & * & \end{pmatrix} \in GL_r(Q) \}$$

and $\sigma K \sigma^{-1} \supset \Gamma$ where the group Γ is defined by

$$\Gamma = \{ \gamma \in GL_r(A) \cap \sigma K_Q \sigma^{-1} | \gamma \equiv \begin{pmatrix} 1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & 1 \end{pmatrix} \mod(\overline{x}_1 - 1) \}$$

Let $U_{ij} (i \neq j, i \neq 1)$,$T_i (i \neq 1)$ be the subgroup of $\sigma K_Q \sigma^{-1}$ defined by

$$U_{ij} = \{ \begin{pmatrix} 1 & 0 & & 0 \\ 0 & 1 & c_{ij} & \\ & & \ddots & 0 \\ 0 & 0 & & 1 \end{pmatrix} \mid \quad c_{ij} \in Q \}$$

$$T_i = \{ \begin{pmatrix} 1 & 0 & & & 0 \\ 0 & \ddots & 0 & & \\ & 0 & c_i & 0 & \\ & & 0 & \ddots & 0 \\ 0 & & & 0 & 1 \end{pmatrix} \mid \quad c_i \in Q^\times \}$$

To show that Γ is Zariski dense in $\sigma K_Q \sigma^{-1}$, it is sufficient to show that $\Gamma \cap U_{ij}$ (resp. $\Gamma \cap T_i$) are Zariski dense in U_{ij} (resp. T_i). But this can be proved easily by using the facts $U_{ij} \simeq \mathbb{G}_a$ (the additive group of Q), $T_i \simeq \mathbb{G}_m$(the multiplicative group of $Q^\times$), and $\Gamma \cap \mathbb{G}_a \supset \{(\overline{x}_1 - 1)\overline{x}_1^N\}$, $\Gamma \cap \mathbb{G}_m \supset \{\overline{x}_1^N\}$. Therefore Γ is zariski dense in $\sigma K_Q \sigma^{-1}$, so by $\sigma K \sigma^{-1} \supset \Gamma$, K is zariski dense in K_Q.

Now look at the diagram (**). Since K/M' and $Aut_A M'$ are Zariski dense in
$Aut_Q(M'_Q) \simeq GL_{r-1}(Q)$, we have $Z(K/M') \subset Z(Aut_A(M')) \subset Z(Aut_Q M'_Q = GL_{r-1}(Q)) = Q^\times$(the scalar matrices). But since $Aut_A M' \cap Q^\times = A^\times$, this proves lemma 11(2). $\square$

Now we shall complete the proof of Theorem A'.

Theorem A'.
$M \triangleleft AutM$ is characteristic.

Proof of Theorem A'.
Let H be the subgroup of K defined by

$$H = \{\alpha \in K | (\alpha, K) \subseteq M'\}$$

Then M is contained in H. By definition, H/M' is the center of K/M', and so by lemma 9, H/M' is a subgroup of $\{\pm\overline{x}_1^{\alpha_1} \cdots \overline{x}_r^{\alpha_r} | \alpha_i \in \mathbb{Z}\}$. But since $M/M' = \{\overline{x}_1^{\alpha_1} \cdots \overline{x}_r^{\alpha_r} | \alpha_i \in \mathbb{Z}\}$, H/M' is equal to M/M' or $\{\pm\overline{x}_1^{\alpha_1} \cdots \overline{x}_r^{\alpha_r} | \alpha_i \in \mathbb{Z}\}$. Now we shall show that $H/M' = M/M'$. In the second case, $-1 \in H/M'$ and -1 acts on $M'/\gamma_3(M)$ non-trivially. But any element of K acts on $M'/\gamma_3(M)$ trivially.(This last statement is equivalent to the statement $(K, M') \subset \gamma_3(M)$ which follows directly from the equation $\gamma_3(K) \cap M = \gamma_3(M)$. But this equation is a direct consequence of the known result $\gamma_3(\widetilde{K}) \cap F = \gamma_3(F)$ ([10,Exer.5,p347] and lemma 2) and the fact $\gamma_3(F) \supseteq F''$) Therefore $-1 \notin K/M'$, a contradiction. Hence $H = M$. But since K and M' are characteristic subgroups of $AutM$, so is H. Therefore, M is a characteristic subgroup of $AutM$. This completes proof of Theorem A'. $\square$

APPENDIX. In this appendix, we shall study an exact sequence ($\diamondsuit$) bellow which is related to lemma11(1).
Let $\mathfrak{A} = (\overline{x}_1 - 1, \cdots, \overline{x}_r - 1)$ be the ideal of A. Then since A submodule $\mathfrak{A}M'$ of M' is stable under $Aut_A M'$ action, there is a homomorphism $Aut_A M' \to Aut_A M'/\mathfrak{A}M'$. Then we have that the sequence

$$1 \to M' \to K \to Aut_A M' \to Aut_A M'/\mathfrak{A}M' \qquad (\diamondsuit)$$

is exact. And an image of morphism $Aut_A M' \to Aut_A M'/\mathfrak{A}M'$ is $\{\pm 1\}$.

Proof We shall prove an exactness of $K \to Aut_A M' \to Aut_a M'/\mathfrak{A}M'$, the other parts are proved in lemma11. By definition, for an element $m \in M'$, we have $x_i m x_i^{-1} m^{-1} = (\overline{x_i} - 1)m$ for $1 \leq i \leq r$ so $\gamma_3(M) = \mathfrak{A}M'$. Since K acts on $M'/\gamma_3(M)$ trivially (c.f. p12), we have $Im(K \to Aut_A M') \subset ker(Aut_A M' \to Aut_A(M'/\mathfrak{A}M'))$, and since $-1 \in Aut_A M'$ acts on $Aut_A(M'/\mathfrak{A}M')$ nontrivially, we have $ker(Aut_A M' \to Aut_A(M'/\mathfrak{A}M')) \underset{\neq}{\subset} Aut_A M'$. Hence to prove ($\diamondsuit$), it suffices to prove that $Im(K \to Aut_A M')$ contains a subgroup of $Aut_A M'$ whose index in $Aut_A M'$ is at most 2.

Let Y be a closed subscheme of $SpecA$ defined by the ideal $\mathfrak{A}$ of A, and put $X = SpecA - Y$. And cover X with a finite number of open affines $X_i = SpecA[(\overline{x_i} - 1)^{-1}] \quad (1 \leq i \leq r)$. Now we shall consider a sheafication of a sequence

$$1 \to M' \to K \to Aut_A M' \to 1 \qquad (*)$$

on $SpecA$. Namely, let $\widetilde{M'}$ be an associated coherent sheaf of A module M', let $\mathcal{K}$ be a sheaf of groups represented by a closed subscheme of GL_r over $SpecA$ which fixes $(\overline{x}_1 - 1, \cdots, \overline{x}_r - 1)$ by right multiplication and let $\mathcal{A}ut\widetilde{M'}$ be a sheaf of groups of unit elements of $\mathcal{H}om(\widetilde{M'}, \widetilde{M'})$. And consider a sequence $\widetilde{(*)}$ of sheaves on $SpecA$;

$$1 \to \widetilde{M'} \to \mathcal{K} \to \mathcal{A}ut\widetilde{M'} \to 1 \qquad \widetilde{(*)}$$

and put $\widetilde{(*)}_X$ to be a restriction of $\widetilde{(*)}$ to X. Then it follows,

(i) $\Gamma(X, \widetilde{(*)}_X) = (*)$

(ii) $\widetilde{(*)}_X$ is an exact sequence of sheaves.

(i) can be checked from the fact $\Gamma(X, \mathcal{O}_X) = A$ e.t.c. (ii) follows from the fact that for any affine scheme V contained in X_i for some i, the sequence $\Gamma(V, \widetilde{(*)}_X)$ is exact. Now we shall take a cohomology exact sequence of $\widetilde{(*)}_X$, and using (i) above, we have

$$1 \to M' \to K \to Aut_A M' \underset{\phi}{\to} H^1(X, \widetilde{M'}) \qquad (1)$$

where ϕ is not a homomorphism of groups, but a 1-cocycle map, i.e. $\phi(ab) = \phi(a) + a(\phi(b))$ for $a, b \in Aut_A M'$, here $H^1(X, \widetilde{M'})$ is canonically regarded as an $Aut_A M'$-module. Next we shall compute a cohomology group $H^1(X, \widetilde{M'})$. There is an exact sequence of A-module,

$$0 \to M' \to A^{\oplus r} \to \mathfrak{A} \to 0$$

where a homomorphism $A^{\oplus r} \to \mathfrak{A}$ is defined by $(a_1, \cdots, a_r) \mapsto \sum(\overline{x_i} - 1)a_i$. And take an associated exact sequence of sheaves on X,

$$0 \to \widetilde{M'}|_X \to \widetilde{A^{\oplus r}}|_X \to \widetilde{\mathfrak{A}}|_X \to 0$$

and consider an associated cohomology exact sequence,

$$\cdots \to H^0(X, \widetilde{A^{\oplus r}}|_X) \to H^0(X, \widetilde{\mathfrak{A}}|_X) \to H^1(X, \widetilde{M'}|_X) \to H^1(X, \widetilde{A^{\oplus r}}|_X) \to \cdot\cdot$$

Since $\widetilde{\mathfrak{A}}|_X \simeq \widetilde{A}|_X$, we obtain $coker(H^0(X, \widetilde{A^{\oplus r}}|_X) \to H^0(X, \widetilde{\mathfrak{A}}|_X)) = \mathbb{Z}$, hence above cohomology exact sequence becomes

$$0 \to \mathbb{Z} \to H^1(X, \widetilde{M'}|_X) \to H^1(X, \widetilde{A^{\oplus r}}|_X) = H^1(X, \widetilde{A}|_X)^{\oplus r} \to \cdots \qquad (2)$$

Next since $X = SpecA - Y$, we obtain a long exact sequence of cohomology groups,

$$\cdots \to H^1(SpecA, \widetilde{A}) \to H^1(X, \widetilde{A}) \to H^2_Y(SpecA, \widetilde{A}) \to \cdots \quad (3)$$

where $H_Y(SpecA, \widetilde{A})$ is cohomology groups with supports in Y.Since $SpecA$ is affine and $\widetilde{A}$ is coherent, we obtain $H^1(SpecA, \widetilde{A}) = 0$. And since we are assuming $r \geq 4$, it follows $depth_{\mathfrak{A}} A \geq 4$, which induces $H^2_Y(SpecA, \widetilde{A}) = 0$ (c.f. [6, Ex. 3.4]). Here from (2),(3), we obtain $H^1(X, \widetilde{M'}|_X) \simeq \mathbb{Z}$ and (1) becomes

$$1 \to M' \to K \to Aut_A M' \underset{\phi}{\to} \mathbb{Z}$$

Finally we shall show that for an element $a \in ker(Aut_A M' \to \mathbb{Z}^\times)$,we have $\phi(a) = 0$ i.e. $a \in Im(K \to Aut_A M')$.Since $ker(Aut_A M' \to \mathbb{Z}^\times)$ is a subgroup of $Aut_A M'$ of index 2, this will complete the proof of ($\diamondsuit$). Take $a \in ker(Aut_A M' \to \mathbb{Z}^\times)$, since $-1 \in Aut_A M'$ commutes with a,and acts on $\mathbb{Z}$ as -1 multiple, we have

$$\begin{aligned} \phi(-1 \cdot a) &= \phi(-1) - \phi(a) \\ \| & \\ \phi(a \cdot (-1)) &= \phi(a) + \phi(-1) \end{aligned}$$

hence we have $\phi(a) = 0$.

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Oiwake-cho, Sakyo-ku, Kyoto, 606-8502, Japan
E-mail address: ya@kurims.kyoto-u.ac.jp