Symmetry Breaking Patterns

–Spontaneous Collapse of SUSY and Others–

Dedicated to Professor Takeyuki Hida on the occasion of his 70th birthday

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1 Introduction

The motivation for classifying possible symmetry breaking patterns in quantum field theory (QFT) arises from the observation of *universal* breakdowns of Lorentz invariance [1] and of supersymmetry [2] in thermal situations:

i) Spontaneous breakdown of Lorentz invariance at $T \neq 0°$ K:
Since a thermal equilibrium state is characterized by the KMS condition in reference to a *fixed time axis* which is necessarily changed by any Lorentz boost, Lorentz symmetry is always broken spontaneously in any thermal equilibrium [1]. Even if we shift to any mixed phases consisting of KMS states in different frames, there is *no* Lorentz-invariant equilibrium states. However, a unitary representation of Lorentz group can be realized in suitable mixed phases as an induced representation from the unbroken spatial rotation group.

Reciprocally, this means the thermal equilibrium is broken by the action of Lorentz boosts, which gives us some useful hints for the attempts of formulating local temperature states in the general framework of QFT (D.Buchholz, I.O., H.Roos, work in progress).

ii) Spontaneous collapse of supersymmetry outside of vacua:
Here "collapse" means that there is *no* chance to restore the symmetry going over to any *mixed phases*. This makes a sharp contrast with the familiar picture of spontaneous symmetry breakdown (SSB) with a compact group which is always associated with invariant states in mixed phases.

To place these symmetry breakings in the general context in contrast with other known patterns of symmetry realizations or breakings, it is necessary to generalize and make precise the usual criterion of SSB. For this purpose, we will see that the notion of "(thermodynamic) phases" and the distinction between pure and mixed phases play a crucial role.

2 Basic notions and classified patterns of symmetry breaking

2.1 Spontaneous symmetry breakdown

In the usual discussions of SSB, its criterion is given in the following way. First we suppose that our physical system is described as a C*-dynamical system consisting of a C*-algebra $\mathfrak{F}$ of physical quantities (including also *non*observables) with a spacetime symmetry group (such as Poincaré group $\mathcal{P}_+^\uparrow = \mathbb{R}^4 \times SL(2,\mathbb{C})$, including a one-parameter group of dynamics) as well as some internal symmetry described by a Lie group G, both of which act on $\mathfrak{F}$ through automorphisms. This means, for instance in the latter case, that the mapping $\tau : G \ni g \mapsto \tau_g \in Aut(\mathfrak{F})$ (=automorphism group of $\mathfrak{F}$) is a group homomorphism (and strongly continuous in the sense of $g \mapsto \tau_g(F)$ being continuous for each $F \in \mathfrak{F}$). In its infinitesimal version, this can also be formulated by a Lie algebra homomorphism $\delta : \mathfrak{g} \ni X \mapsto \delta_X \in Der(\mathfrak{F})$ (=Lie algebra of derivations on $\mathfrak{F}$) which is related to τ through

$$\delta_X(F) \doteq \frac{d}{dt}\tau_{e^{tX}}(F)\Big|_{t=0}, \tag{1}$$

or

$$\tau_{e^{tX}}(F) = e^{t\delta_X}(F) \tag{2}$$

for $X \in \mathfrak{g}$ and for F (in Eq.(1), in a dense subalgebra of τ-differentiable elements) in $\mathfrak{F}$.

Let ω be a state on $\mathfrak{F}$ formulated as a positive linear functional (expectation functional), $\omega : \mathfrak{F} \ni F \mapsto \omega(F) \in \mathbb{C}$. Corresponding to this state ω, we can always construct a representation π_ω of the algebra $\mathfrak{F}$ in a Hilbert space $\mathfrak{H}_\omega$ with a cyclic vector Ω_ω, $\mathfrak{H}_\omega = \overline{\pi_\omega(\mathfrak{F})\Omega_\omega}$, satisfying the relation, $\omega(F) = \langle \Omega_\omega \,|\, \pi_\omega(F)\ \Omega_\omega \rangle$. This is called a GNS representation and denoted by $(\mathfrak{H}_\omega, \pi_\omega, \Omega_\omega)$ (see, e.g., [3]).

If ω is invariant, $\omega = \omega \circ \tau_g \quad (\forall g \in G)$, under the action of (G, τ), then $U_\omega(g)$ defined in the GNS space $\mathfrak{H}_\omega$ by

$$U_\omega(g)\pi_\omega(F)\Omega_\omega \doteq \pi_\omega(\tau_g(F))\Omega_\omega \tag{3}$$

is easily seen to give a unitary representation of G satisfying $U_\omega(g)\Omega_\omega = \Omega_\omega$ and

$$\pi_\omega(\tau_g(F)) = U_\omega(g)\pi_\omega(F)U_\omega(g)^*. \tag{4}$$

When G-action τ has a unitary representation $(U_\omega, \mathfrak{H}_\omega)$ satisfying (4), it is said to be *unitarily implemented* (or *implementable*) in $(\pi_\omega, \mathfrak{H}_\omega)$. Thus the G-invariance of a state ω gives a sufficient condition for the symmetry G to be unitarily implementable in the GNS representation space of ω.

For a translation invariant vacuum state ω of $\mathfrak{F}$ characterized by the spectrum condition in its GNS representation, the converse statement holds if it is a *pure* state (with its GNS representation irreducible). Namely, the famous Goldstone theorem [4, 5] asserts that the symmetry is *not* unitarily implementable in the corresponding GNS representation space $\mathfrak{H}_\omega$ if ω is *not invariant* under the symmetry transformation τ:

$$\exists g \in G \text{ s.t. } \omega \circ \tau_g \neq \omega, \tag{5}$$

or infinitesimally,

$$\exists X \in \mathfrak{g}\ \exists F \in \mathfrak{F} \text{ s.t. } \omega(\delta_X(F)) \neq 0, \tag{6}$$

and that the operator $F \in \mathfrak{F}$ responsible for this non-invariance (6) necessarily contains *massless* spectrum called Goldstone bosons. In this case, the symmetry (G, τ) is said to be *spontaneously broken* in ω, or, ω is said to suffer from an SSB with respect to this symmetry.

2.2 How to judge SSB in general context: pure and mixed phases

In many of the discussions in QFT concentrating upon translation invariant pure vacuum situations, these two notions, non-implementability and non-invariance (5, 6) (with the important restriction to the vacuum being implicitly understood), are treated without explicit distinction. However, when we start to extend our discussion to other states than a pure vacuum, such as a thermal equilibrium state, as in the case of SSB of Lorentz symmetry mentioned at the beginning, it becomes necessary to be careful about this distincion as well as such restrictions imposed on the state ω as its translation invariance and purity. From the general viewpoint what is most essential for characterizing SSB is not the invariance property of a state but whether the symmetry is unitarily implementable or not.

Since the invariance property can be checked more conveniently, however, it is desirable to clarify when and how we can reduce the general criterion for implementability or SSB to such a convenient form as (6) or (5). For this purpose, we need to examine more closely the necessary restrictions imposed upon the states to be examined there. For instance, if one forgets about the purity of ω, one encounters the following situation: Let the group G of symmetry be compact equipped with a normalized Haar measure μ. Even in the case that ω suffers from an SSB, one can easily recover the invariance by averaging over the G-orbit $\{\omega \circ \tau_g;\ g \in G\}$ through ω in the state space in the form of a G-invariant *mixed* state given by

$$\widehat{\omega} \doteq \int_G d\mu(g)\omega \circ \tau_g, \tag{7}$$

$$\widehat{\omega} = \widehat{\omega} \circ \tau_g \quad (\forall g \in G). \tag{8}$$

Eq.(8) implies the unitary implementability of G as shown above. In this situation, we find in $\mathfrak{H}_{\widehat{\omega}}$ many vacua orthogonal to each other, because of which SSB has been understood historically as the situation with *degenerate vacua*.

If one concluded from this example that the criterion (5) could not be applied to *mixed states*, however, one would exclude away from the present context all thermal equilibrium states which are necessarily mixed states with reducible GNS representations. Instead of sticking to the superficial difference between pure and mixed states, we must understand the more general meaning of this difference in a form applicable also to thermal states. For this purpose, we note the following equivalence of properties valid for vacua [6]:

i) ω is a pure state,

ii) all the translation invariant vectors belong to a one-dimensional subspace of the GNS space $\mathfrak{H}_\omega$ spanned by the GNS cyclic vector Ω_ω,

iii) ω satisfies the *cluster property* (i.e., absence of long range order), $\omega(A\alpha_{\mathbf{x}}(B)) \underset{\mathbf{x}\to\infty}{\to} \omega(A)\omega(B)$, with $\alpha_{\mathbf{x}}$ denoting a spatial translation by $\mathbf{x} \in \mathbb{R}^3$.

While the first property i) is meaningless in the thermal situations, the equivalence between the latter two remains true there. To incorporate the essence of i) in our discussion, we need only to replace the word pure *state* with pure *phase*: What we should look for is the condition to characterize an *indecomposable unit* of thermal states, just corresponding to a *pure* vacuum state which is *indecomposable among vacua*.

The key observation for this task consists in the simplicial structure of the set of KMS states [3] describing thermal equilibria at a fixed temperature. Denoting K_β the set of all KMS states at an inverse temperature β, we find its desirable property in that K_β is not only a convex closed subset in the set $E_{\mathfrak{F}}$ of all states on $\mathfrak{F}$ but also that it is a "simplex"; this means that any state belonging to K_β can be *decomposed uniquely* into a convex combination of extremal KMS states. On the basis of this fact, a thermodynamic *pure phase* (at a temperature $1/k_B\beta$) is defined as an *extremal* KMS state. Then, the condition for a KMS state ω_β to be a pure phase is known [3] to be equivalent to the condition that its corresponding GNS representation is a "*factor*", i.e., $\mathfrak{M} := \pi_\beta(\mathfrak{F})''$ has a trivial centre: $\mathfrak{Z} := \mathfrak{M} \cap \mathfrak{M}' = \mathbb{C}\mathbf{1}_{\tilde{\mathfrak{H}}_\beta}$. Any pair of pure phases belonging to K_β are shown to be either identical or *disjoint* [3]. (Disjointness, much stronger than unitary inequivalence, means that any pair of nontrivial subrepresentations taken from each of GNS representations corresponding to these two states are always unitarily inequivalent.)

In the case of a quantum system with finite degrees of freedom, a factor representation is reduced to a direct sum of an irreducible one (repeated many times, in general). In the infinite cases, however, we cannot attach an unambiguous meaning to such a decomposition, and hence, it is appropriate to treat a factor representation as an entity in itself whose further decomposition is meaningless. A state with a factor representation as its GNS representation is called a factor state. Pure states are the special case of factor states. While a factor state is a mixture in general, it shares many features of pure states. For instance, under the assumption of local commutativity (or in a more general form, of asymptotic abelianness with respect to spatial translations), any factor state satisfies the cluster property (see [3]): By denoting by $A(\mathbf{x}) = \alpha_{\mathbf{x}}(A)$ the spatial translate of a physical quantitiy $A \in \mathfrak{F}$ by $\mathbf{x} \in \mathbb{R}^3$, the following relation holds for a factor state ω,

$$|\omega(A(\mathbf{x})B) - \omega(A(\mathbf{x}))\omega(B)| \underset{|\mathbf{x}|\to\infty}{\longrightarrow} 0. \tag{9}$$

Note here that spatial-translation invariance of ω need not be assumed for this cluster property. These nice properties shared by thermodynamic pure phases are actually not restricted to thermal equilibrium, but essentially follow from their factoriality (under the assumption of local commutativity or asymptotic abelianness). For instance, any pair of factor states are shown to be either disjoint or quasi-equivalent (the latter of which means the unitary equivalence up to multiplicities, or equivalently, isomorphism of von Neumann algebras arising from GNS representations). Therefore, it is legitimate to call *pure phase* any factor state in general which may not be a KMS state.

A state which is not a pure phase is a *mixed phase* to which corresponds

a non-trivial centre $\mathfrak{Z} = \mathfrak{M} \cap \mathfrak{M}' \neq \mathbb{C}\mathbf{1}$. By definition, $\mathfrak{Z}$ is a commutative algebra, all the element of which can be "diagonalized simultaneously". Thus the centre $\mathfrak{Z}$ becomes isomorphic to a function algebra $L^\infty(Spec\mathfrak{Z})$ (Gel'fand theorem [3]) on a (compact Hausdorff) space $Spec\mathfrak{Z}$ consisting of the spectrum (="generalized eigenvalues") of $\mathfrak{Z}$ which can be interpreted as the (realized values of) macroscopic *order parameters* distinguishing among different phases. A pure phase is characterized by sharp c-number values of order parameters, which are statistically fluctuating in the case of mixed phases. Here the notion of pure and mixed phases should not be confused with that of pure and mixed states: Any thermodynamic phase, pure or mixed, is described by a mixed state; in the case of vacuum states the notions of pure phase and pure state coincide owing to the speciality of the vacuum situation arising from the spectrum condition. In this sense, pure phases give a natural extension of pure vacuum states to the general situations, and mixed phases correspond to degenerate vacua.

On the basis of these observations, we now adopt a scheme to classify symmetry breaking patterns which consists of two steps, first to examine the criterion of SSB restricted to *pure phases* and, if symmetry is broken, then to examine the possibility of its *restoration* in the mixed phase situation. In this scheme the notion of SSB can be extended from the restricted situation of vacuum at $T = 0°$ K to general situations, including not only temperature states but also more general ones even without translation invariance such as local-temperature states.

For the sake of simplicity, however, we consider here only spatial-translation invariant states under the assumption of local commutativity (or, more generally, asymptotic abelianness with respect to spatial translations). As explained above, a spatial-translation invariant state ω in this situation is a pure phase if and only if the set of spatial-translation invariant vectors in its GNS space is the one-dimensional subspace spanned by Ω_ω. If G commutes with spatial translations, then its unitary implementability implies the G-invariance of ω, because $U_\omega(g)\Omega_\omega$ should belong to the above subspace as another spatial-translation invariant vector. In this way, the criterion of SSB for a spatial-translation invariant pure phase can be reduced to examining its G-invariance just in the same way as the case of vacuum states. In Sec.3, we will see that this applies also to the case of supersymmetry, even irrespective of pure or mixed phases.

2.3 Classification of symmetry breaking patterns

Now we obtain the following classification of symmetries and their breaking patterns.

i) Unbroken symmetry in a pure phase [Wigner mode of symmetry]:

A pure phase ω belongs to this case if there exists a unitary representation U_ω of G in the GNS representation space $(\pi_\omega, \mathfrak{H}_\omega)$ of ω satisfying the relation $\pi_\omega(\tau_g(F)) = U_\omega(g)\pi_\omega(F)U_\omega(g)^*$.

ii) Symmetry spontaneously broken in a pure phase and restored in a mixed phase [Nambu-Goldstone mode of symmetry]:

A pure phase ω belongs to this case if there exist *no* unitary representation U_ω of G in $\mathfrak{H}_\omega$ satisfying $\pi_\omega(\tau_g(F)) = U_\omega(g)\pi_\omega(F)U_\omega(g)^*$. However, when the group G of symmetry is amenable having an *invariant mean* μ (which is the case for G compact or abelian), then we can always restore the broken symmetry in the averaged mixed phase $\widehat{\omega}$ constructed as Eq.(7), which is G-invariant. This covers all the familiar cases of SSB.

iii) Symmetry spontaneously broken in a pure phase with *partial* restoration in a mixed phase:

The known typical example is just the breakdown of Lorentz symmetry at $T \neq 0°\mathrm{K}$ mentioned above. Similarly to the above ii), we can construct a unitary representation of broken symmetry G in a mixed phase, but, for lack of invariant means in such a non-amenable group as Lorentz group, G-invariant states cannot be found in any mixed phases related to a KMS state.

iv) "Spontaneous collapse"= symmetry breaking without any restoration in any mixed phases:

The only known example is just the case of breakdown of supersymmetry outside of the vacuum regimes [2]. As mentioned at the beginning, there is no possibility of symmetry restoration.

Remark: a) The most familiar example of ii) is provided by a ferromagnet. If $\omega_{\boldsymbol{\theta}}$ describes such a state with sharp direction $\boldsymbol{\theta}(= (\vartheta, \varphi))$ of magnetization, spatial rotations cannot be defined on the corresponding Hilbert space (except for the rotations around the axis $\boldsymbol{\theta}$). But the mixed phase corresponding to the spherical average of these states, $\omega = \int d\boldsymbol{\theta}\ \omega_{\boldsymbol{\theta}}$, passes the test $\omega \circ \delta = 0$ with respect to infinitesimal rotations δ. Hence, there exist generators for rotations on the corresponding "enlarged" Hilbert space. In other words, while the result of a rotation cannot be described in the Hilbert space of each pure phase since it cannot be accomplished by the action of polynomials in the local fields on the corresponding vectors, it is still meaningful to speak about the action of rotations on these states. Generators

inducing this action can be defined in the Hilbert space of the averaged state ω since it comprises states with arbitrary directions of magnetization.

b) In contrast to the case of pure phases, the equivalence between the invariance property and the implementability cannot be assured in mixed phases even for translation invariant states. Exhibiting this situation, the case iii) represents just the intermediate situation between ii) and iv): In view of the absence of *an invariant state*, it is close to iv), while it is similar to ii) in the sense of "restoration of symmetry" interpreted as the existence of a *unitary implementation.* As for massless Goldstone bosons, it is shown in [1] that this symmetry breakdown does not imply its existence. If other examples belonging to iii) are found, it would be interesting to examine this problem there from more systematic viewpoints.

c) Since the above i)-iii) cover symmetries described by *locally compact* groups, such cases as local gauge symmetries described by infinite-dimensional groups are excluded here. The case iv) will be explained in the next section.

3 The case of SUSY: spontaneous collapse

Here we explain how the above notions for classifying symmetry breaking patterns are used, discussing the status of supersymmetry(SUSY) in a general setting including thermal states. We will establish the following result: If supercharges can be defined in a given spatial-translation invariant state, then this state describes the vacuum. In this way it is shown that supersymmetry is inevitably broken spontaneously in thermal states.

Before entering the detailed discussion, we note that the analysis of symmetry breaking patterns can be done in a clear-cut way only by putting our system in an infinite-volume limit (i.e., *thermodynamic limit*), where such naive formulae as symmetry generators expressed as the volume integral of current densities become very touchy because of the problems of non-implementability of symmetry and so on. When such mixed states as thermal equilibria enter in our discussion, we need also treat *reducible* representations of physical variables, which implies the existence of operators commuting with the basic fields. This introduces certain arbitrariness or ambiguities in the definition of symmetry generators. These situations require us to adopt the *algebraic* formulation of quantum (field) theory starting from basic notions as independent of specific representationas as possible. To formulate the essence of SUSY in this spirit, we start by introducing the $\mathbb{Z}_2$-grading $\mathcal{F} = \mathcal{F}_+ + \mathcal{F}_-$ in the C*-algebra $\mathcal{F}$ of quantum fields, corresponding to the distinction between bosonic ($\mathcal{F}_+$) and fermionic ($\mathcal{F}_-$) fields which satisfies

$\mathcal{F}_+\mathcal{F}_- = \mathcal{F}_-\mathcal{F}_+ \subseteq \mathcal{F}_-$ and $\mathcal{F}_\pm\mathcal{F}_\pm \subseteq \mathcal{F}_+$. Then the supersymmetry transformations are formulated in terms of (densely defined) $\mathbb{Z}_2$-graded derivations $(\delta_\alpha)_{\alpha=1,2}$ and $(\bar{\delta}_{\dot{\beta}})_{\dot{\beta}=\dot{1},\dot{2}}$ (with α, $\dot{\beta}$ undotted and dotted spinor indices under the Lorentz group $SL(2,\mathbb{C})$, i.e., ${\delta_\alpha}' = {A_\alpha}^\beta \delta_\beta$, $\quad {\bar{\delta}_{\dot{\alpha}}}' = \overline{{A_\alpha}^\beta}\, \bar{\delta}_{\dot{\beta}} = {\bar{A}_{\dot{\alpha}}}^{\dot{\beta}} \bar{\delta}_{\dot{\beta}}$), commuting with the spatial translations $\delta_\alpha(F(\mathbf{x})) = (\delta_\alpha(F))(\mathbf{x})$, $\bar{\delta}_{\dot{\beta}}(F(\mathbf{x})) = (\bar{\delta}_{\dot{\beta}}(F))(\mathbf{x})$. They are assumed to satisfy $\delta_\alpha(\mathcal{F}_\pm) \subset \mathcal{F}_\mp$, $\bar{\delta}_{\dot{\beta}}(\mathcal{F}_\pm) \subset \mathcal{F}_\mp$ and

$$\delta_\alpha(F_\pm)^* = \mp\, \bar{\delta}_{\dot{\alpha}}(F_\pm^*). \tag{10}$$

The meaning of "$\mathbb{Z}_2$-graded derivations" is made precise by the relation

$$\delta_\alpha(F_\pm F) = \delta_\alpha(F_\pm)F \pm F_\pm \delta_\alpha(F), \tag{11}$$

for $F_\pm \in \mathcal{F}_\pm, F \in \mathcal{F}$, and the similar one for $\bar{\delta}_{\dot{\beta}}$. The fundamental relation characterizing supersymmetry can now be formulated as

$$\bar{\delta}_{\dot{1}} \circ \delta_1 + \delta_1 \circ \bar{\delta}_{\dot{1}} + \bar{\delta}_{\dot{2}} \circ \delta_2 + \delta_2 \circ \bar{\delta}_{\dot{2}} = 4\delta_0, \tag{12}$$

where δ_0 is the derivation arising from the time translations,

$$\delta_0(F) = -i\frac{d}{dx_0}F(x_0)\Big|_{x_0=0}. \tag{13}$$

The above relation among maps δ_α, $\bar{\delta}_{\dot{\beta}}$ and δ_0 is meaningful independently of the specific choice of a state, in sharp contrast with such a familiar expression as $\sum_{\alpha=1}^2 \{Q_\alpha,\ Q_{\dot{\alpha}}^*\} = 4H$ in terms of supercharges which *may not exist* in the case of spontaneously broken SUSY.

It is our aim to show that a spatial-translation invariant state ω must be a vacuum if supersymmetry is not broken in it. The argument will be given as follows.

(a) First, it is shown by standard arguments [5, 7] that the Bose-Fermi superselection rule is not broken spontaneously in any spatial-translation invariant state.

Let $F_- \in \mathcal{F}_-$ be a fermionic operator. From the assumption of the invariance of ω under spatial translations it follows that $\omega(F_-) = \omega(F_-(\mathbf{x})) = (1/|V|)\int_V d^3\mathbf{x}\ \omega(F_-(\mathbf{x}))$ for any bounded spatial region V in $\mathbb{R}^3$ with $|V|$ being its volume. In the limit of $V \nearrow \mathbb{R}^3$ the right-hand side of this equality is shown to vanish because of the following bound on the norm of the spatial

mean of fermionic vectors,

$$\begin{aligned}
&\|\frac{1}{|V|}\int_V d^3\mathbf{x} F_-(\mathbf{x})\,\Omega_\omega\|^2 + \|\frac{1}{|V|}\int_V d^3\mathbf{x} F_-(\mathbf{x})^*\,\Omega_\omega\|^2 \\
&= \frac{1}{|V|}\int_V d^3\mathbf{x}\,\frac{1}{|V|}\int_V d^3\mathbf{x}'\;\omega(\,\{\,F_-(\mathbf{x})^*,\,F_-(\mathbf{x}')\,\}\,) \\
&\leq \frac{1}{|V|}\int d^3\mathbf{z}\;|\omega(\,\{\,F_-(\mathbf{z})^*, F_-\,\}\,)|. \qquad (14)
\end{aligned}$$

Note that the latter integral exists since the anti-commutator vanishes for large spatial translations $\mathbf{z}$. From this, we conclude that

$$\omega(F_-) = 0 \quad \text{for } F_- \in \mathcal{F}_- \qquad (15)$$

which asserts the validity of the Bose-Fermi superselection rule.

(b) Next we show that the SUSY invariance of ω

$$\omega \circ \delta_\alpha = \omega \circ \bar{\delta}_{\dot\beta} = 0 \qquad (16)$$

follows from the assumption of *implementability* of SUSY in ω in the sense that there exist supercharges Q_α and $Q^*_{\dot\beta}$ (hermitian conjugate of Q_β) having vectors $F\,\Omega_\omega$ ($F \in \mathcal{F}$) in their domains of definition and satisfying

$$\begin{aligned}
Q_\alpha F_\pm\,\Omega_\omega &= \delta_\alpha(F_\pm)\,\Omega_\omega \pm F_\pm Q_\alpha\,\Omega_\omega, \qquad (17)\\
Q^*_{\dot\beta} F_\pm\,\Omega_\omega &= \bar{\delta}_{\dot\beta}(F_\pm)\,\Omega_\omega \pm F_\pm Q^*_{\dot\beta}\,\Omega_\omega. \qquad (18)
\end{aligned}$$

Remark: We do not assume that $Q_\alpha\,\Omega_\omega = Q^*_{\dot\beta}\,\Omega_\omega = 0$ or that Q_α and $Q^*_{\dot\beta}$ commute with translations, because of the ambiguities involved in the definition of generators in such mixed states as thermal ones.

From the Bose-Fermi superselection rule we have, for any $F_+ \in \mathcal{F}_+$,

$$\omega(\delta_\alpha(F_+)) = \omega(\bar{\delta}_{\dot\beta}(F_+)) = 0, \qquad (19)$$

because of $\delta_\alpha(F_+), \bar{\delta}_{\dot\beta}(F_+) \in \mathcal{F}_-$. Making use of Eqs.(17), (18), we show that these expressions vanish also for fermionic operators $F_- \in \mathcal{F}_-$,

$$\omega(\delta_\alpha(F_-)) = \omega(\bar{\delta}_{\dot\beta}(F_-)) = 0. \qquad (20)$$

Combining Eqs.(17), (18) and the commutativity of the anti-derivation δ_α with spatial translations $\mathbf{x}$, we obtain, as in step (a),

$$\begin{aligned}
\omega(\delta_\alpha(F_-)) &= \frac{1}{|V|}\int_V d^3\mathbf{x}\;\langle\Omega_\omega|\,\delta_\alpha(F_-(\mathbf{x}))\,\Omega_\omega\rangle \\
&= \frac{1}{|V|}\int_V d^3\mathbf{x}\;\langle\Omega_\omega|\,(Q_\alpha F_-(\mathbf{x}) + F_-(\mathbf{x})Q_\alpha)\,\Omega_\omega\rangle. \qquad (21)
\end{aligned}$$

Using the Cauchy-Schwarz inequality, we arrive at the estimate

$$\begin{aligned} |\omega(\delta_\alpha(F_-))| \;\leq\; & \|Q_\alpha^*\,\Omega_\omega\| \cdot \|\frac{1}{|V|}\int_V d^3\mathbf{x}\; F_-(\mathbf{x})\,\Omega_\omega\| \\ & + \|\frac{1}{|V|}\int_V d^3\mathbf{x}\; F_-(\mathbf{x})^*\,\Omega_\omega\| \cdot \|Q_\alpha\,\Omega_\omega\| \end{aligned} \tag{22}$$

for arbitrary V. The right-hand side of this inequality vanishes for $V \nearrow \mathbb{R}^3$ according to relation (14), which shows that $\omega(\delta_\alpha(F_-)) = 0$ for $F_- \in \mathcal{F}_-$. By the same token we obtain also $\omega(\bar{\delta}_{\dot\beta}(F_-)) = 0$, $F_- \in \mathcal{F}_-$, which completes the proof of relation (20). As a consequence of the fundamental relation Eq.(12) characterizing supersymmetry, the invariance of ω under time translations automatically follows: $\omega \circ \delta_0 = 0$. It is important to note that we did not assume in the above discussion the cluster property with respect to spatial translations, and hence, the result is valid even if the state ω describes an arbitrary mixed thermodynamic phase. The only condition on ω is that supercharges can be defined.

(c) Now we show that a state ω is a vacuum, if it satisfies Eq.(16) and complies with the Bose-Fermi superselection rule (15). More precisely, it is invariant under spacetime translations with their generators satisfying the relativistic spectrum condition (positivity of energy in all Lorentz frames). To prove this, we consider the expectation values $\omega(F^*\delta_0(F))$ for $F \in \mathcal{F}$. Since $F = F_+ + F_-$ with $F_\pm \in \mathcal{F}_\pm$ and $\omega(F_\pm^*\delta_0(F_\mp)) = 0$ by (15), we may restrict our attention to the expectation values $\omega(F_\pm^*\delta_0(F_\pm))$. From the fundamental relation of SUSY Eq.(12) we have

$$4\omega(F_+^*\,\delta_0(F_+)) = \omega(F_+^*\,(\bar{\delta}_{\dot 1}\circ\delta_1 + \delta_1\circ\bar{\delta}_{\dot 1} + \bar{\delta}_{\dot 2}\circ\delta_2 + \delta_2\circ\bar{\delta}_{\dot 2})(F_+)), \tag{23}$$

in which each term on the right-hand side will be shown to be positive. From the $\mathbb{Z}_2$-graded derivation properties of $\delta_\alpha, \bar{\delta}_{\dot\beta}$ and of Eq.(10), we obtain

$$\begin{aligned} \bar{\delta}_{\dot 1}(F_+^*\,\delta_1(F_+)) &= \bar{\delta}_{\dot 1}(F_+^*)\,\delta_1(F_+) + F_+^*\,\bar{\delta}_{\dot 1}(\delta_1(F_+)) \\ &= -\delta_1(F_+)^*\,\delta_1(F_+) + F_+^*\,\bar{\delta}_{\dot 1}\circ\delta_1(F_+). \end{aligned} \tag{24}$$

Since $\omega \circ \bar{\delta}_{\dot 1} = 0$, this implies

$$\omega(F_+^*\,\bar{\delta}_{\dot 1}\circ\delta_1(F_+)) = \omega(\delta_1(F_+)^*\,\delta_1(F_+)) \geq 0 \tag{25}$$

and a similar argument applies to the remaining terms. So $\omega(F_+^*\,\delta_0(F_+)) \geq 0$ and the same result holds with F_+ replaced by $F_- \in \mathcal{F}_-$. Thus we arrive at

$$\omega(F^*\,\delta_0(F)) \geq 0 \qquad \text{for } F \in \mathcal{F}. \tag{26}$$

Since δ_0 is a derivation satisfying $\delta_0(F)^* = -\delta_0(F^*)$ and $\omega \circ \delta_0 = 0$, the equality

$$P_0 F\,\Omega_\omega = \delta_0(F)\,\Omega_\omega \qquad \text{for } F \in \mathcal{F} \tag{27}$$

gives a well-defined operator P_0 which is essentially self-adjoint and satisfies $P_0\Omega_\omega = \delta_0(\mathbf{1})\Omega_\omega = 0$. Because of

$$\langle \Omega_\omega |\, F^*\, P_0\, F\, \Omega_\omega \rangle = \omega(F^*\delta_0(F))) \geq 0, \tag{28}$$

P_0 is a positive operator, and hence, in view of $P_0\,\Omega_\omega = 0$, we conclude that Ω_ω is a ground state for P_0.

From the spinorial properties of δ_α and $\bar{\delta}_{\dot\beta}$ under the Lorentz transformations, it follows that ω is a ground state in any Lorentz frame. They imply that the fundamental relation (12) holds for the transformed maps ${\delta_\alpha}', {\bar\delta_{\dot\beta}}'$ and ${\delta_0}'$ in any Lorentz frame. Moreover, since a change of Lorentz frame amounts to a linear transformation of these maps, i.e.,

$${\delta_\alpha}' = A_\alpha{}^\beta \delta_\beta, \quad {\bar\delta_{\dot\alpha}}' = \overline{A_\alpha{}^\beta}\,\bar\delta_{\dot\beta} = \bar{A}_{\dot\alpha}{}^{\dot\beta}\bar\delta_{\dot\beta} \tag{29}$$

with $A \in SL(2,\mathbb{C})$, it follows that, if supersymmetry is unbroken in some Lorentz frame in the sense of Eq.(16), this holds true in any other frame. Applying the preceding arguments to the primed maps, we arrive at the conclusion that the corresponding generators ${P_0}'$ of time translations are positive in all Lorentz frames and satisfy ${P_0}'\,\Omega_\omega = 0$. Hence ω is a vacuum state, as claimed.

One can also demonstrate that this state definitely does not describe a thermal equilibrium situation, which we omit here.

In summary, the validity of Bose-Fermi superselection rule follows from the spatial-translation invariance of a state ω and the existence of supercharges in the corresponding GNS representation space implies the strict invariance (16) of ω under supersymmetry transformations. Any state with these two properties is necessarily a vacuum state and does not satisfy the KMS condition for finite β. Thus supersymmetry is broken in all thermal equilibrium states, irrespective of whether they describe pure or mixed phases. Moreover, generators of supersymmetry cannot be defined in such states. Hence, thermal effects induce an inevitable collapse of supersymmetry. If one wishes to repeat in this case the process of restoring a broken symmetry by considering mixed phases, one could do it *formally*, which, however, results in such unphysical objects as "states" taking Grassmann-number values owing to the necessity of some Brezin-type integral. Since the order parameters in Grassmann numbers cannot be measured physically, this gives only a formal analogy without proper physical relevance.

4 Discussions: Microscopic and macroscopic fluctuations

We discuss here the physical implications of symmetry restoration in mixed phases with non-trivial centre. Since the elements of the centre have the physical meaning as order parameters to discriminate among different phases, the restoration problem concerns directly the symmetry properties at the level of the macroscopic phenomena. This means that spontaneously broken symmetries belonging to the case (ii) (and also (iii)) in Sec.2 restorable in some mixed phases involves two different levels of symmetric worlds, the microscopic abstract level of G-invariant dynamics and the macroscopic statistical level consisting of order parameters, between which we encounter asymmetric individual outcomes of data arising from G-non-invariant states in pure phases. Actually, this is just the pattern frequently encountered in the discoveries of fundamental symmetries such as Lorentz invariance: What we actually measure in the experiments, e.g., the life time of accelerated muons, are quantities statistically fluctuating in a manner depending upon (the relation between each muon's rest frame and) a specific choice of Lorentz frame associated with the experimental apparatus, and only through the comparison of data arising from different Lorentz frames (due to different velocities of individual muons) the Lorentz invariance mediated by Lorentz transformations emerges out of fluctuating sample data. If the physical world were totally symmetric without this intermediate level of fluctuating asymmetric individual outcomes, it would be impossible for us to recognize the presence of symmetries in physical laws.

What is more important in this context is the mutual relationship between *quantum fluctuations* inside factor states in pure phases and *statistical* ones in the direction of centre over different phases, which is mediated by "order fields" or "central sequences" [8]. The prototype of such a relationship is already supplied by the familiar Born's statistical formula: It describes statistical fluctuations as (a classical notion of) probability distributions of measured values in terms of wave functions controlling quantum fluctuations. This raises an important problem to investigate the relations between macroscopic classical (=commutative) probabilities and microscopic quantum (=non-commutative) ones from some unified viewpoints. I expect that important clues to this direction will be given by different versions of low energy theorems corresponding to the classified patterns of symmetry breaking discussed in Sec.2 which control behaviours of order parameters in reference to the broken symmetries.

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References

[1] I.Ojima, Lecture Notes in Physics **176** (K. Kikkawa, et al. eds.) (Springer, Berlin, 1983) pp.161-165; Lett. Math. Phys. **11,** 73 (1986).

[2] D.Buchholz and I.Ojima, Nucl. Phys. **B498**, 228 (1997).

[3] O.Bratteli and D.W.Robinson, *Operator Algebras and Quantum Statistical Mechanics*, Vols. 1 and 2 (Springer, Berlin, 1979 and 1981).

[4] D.Kastler, D.W.Robinson and J.A.Swieca, Comm. Math. Phys. **2**, 108 (1966).

[5] D.Buchholz, S.Doplicher, R.Longo and J.E.Roberts, Rev. Math. Phys. Special Issue, 49 (1992).

[6] R.F.Streater and A.S.Wightman, *PCT, Spin and Statistics and All That* (Benjamin, New York, 1964); R.Haag, *Local Quantum Physics* (Springer, Berlin, 1992 [1st ed.], 1996 [2nd ed.]).

[7] H.Narnhofer and W.Thirring, Lett.Math.Phys. **30,** 307 (1994).

[8] G.L.Sewell, *Quantum Theory of Collective Phenomena*, Oxford University Press (1986); D.Buchholz, *Foundation and Recent Results in Algebraic Quantum Field Theory*, Series Lectures given at RIMS, Kyoto University (1996).