On beta and gamma functions associated with the Grothendieck-Teichmüller group. II

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(*To the memory of Jürgen Neukirch*)

In the previous paper under the same title, cited as [BΓ], we studied the beta function $B_\sigma(s,t)$ ($s,t \in \mathbb{Q}/\mathbb{Z}$; which takes values in a certain arithmetic ring) associated with each element σ of the Grothendieck-Teichmüller group GT. We have shown there, by using the 5-cycle relation defining GT, that B_σ splits into the product of the "hyperadelic gamma functions" $\Gamma_\sigma(s)$, as

$$B_\sigma(s,t) = \frac{\Gamma_\sigma(s)\Gamma_\sigma(t)}{\Gamma_\sigma(s+t)},$$

and also that the logarithmic derivative

$$D\log\Gamma_\sigma(s) - D\log\Gamma_\sigma(0)$$

of $\Gamma_\sigma(s)$ can be expressed explicitly in terms of the generalized logarithmic Kummer quasi 1-cocycles

$$\Psi_n^{(j)}(\sigma) \quad (n \in \mathbb{N},\ j \in \mathbb{Z}/n,\ \sigma \in GT).$$

These are generalizations of results of G. Anderson [A] (who refined and generalized [I_1] from a new point of view) from the case $\sigma \in G_{\mathbb{Q}} = \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ to $\sigma \in GT$, with essentially different methods of proofs. We still do not know whether $G_{\mathbb{Q}} \subsetneq GT$, but we suspect so,

and in particular that Anderson's multiplication formula [A] Cor 8.6.3 for Γ_σ ($\sigma \in G_{\mathbb{Q}}$):

$$\text{(A)} \qquad \left(\prod_{nc=0} \Gamma_\sigma(s+c)\Gamma_\sigma(c)^{-1} \right) \Gamma_\sigma(ns)^{-1} = 1 \otimes \exp(2\pi ns \Psi_n^{(0)}(s))$$

($s \in \mathbb{Q}/\mathbb{Z}, n \in \mathbb{N}$) could be a key test condition for an element σ of GT to belong to $G_{\mathbb{Q}}$.

The main purpose of this article is to present two other necessary conditions (K) and (A$'$) for $\sigma \in GT$ to belong to $G_{\mathbb{Q}}$, and discuss logical relations among (K) (A) (A$'$). One of them, (K), arises from the compatibility of the $G_{\mathbb{Q}}$-actions on the algebraic fundamental groups over $\bar{\mathbb{Q}}$ associated with the following morphisms of curves:

$$\begin{array}{ccccc} \mathbb{P}^1 - \{0,1,\infty\} & \leftarrow & \mathbb{P}^1 - \{0, \boldsymbol{\mu}_n, \infty\} & \hookrightarrow & \mathbb{P}^1 - \{0,1,\infty\} \\ t^n & \leftarrow\!\shortmid & t & \mapsto & t \end{array}$$

($n \in \mathbb{N}$). This compatibility gives rise to defining a certain subgroup, which we call GTK, of GT containing $G_{\mathbb{Q}}$. Recall that GT arose from the compatibility associated with certain morphisms $M_{0,4} \xrightarrow{\partial_i} \partial M_{0,5}$ ($M_{0,n}$: the moduli space of ordered n distinct points on $\mathbb{P}^1$; $M_{0,4} = \mathbb{P}^1 - \{0,1,\infty\}$; $i \in \mathbb{Z}/5$). The new condition (K) appears to be a quite different (independent) sort of restrictions. So, although we could not prove $GTK \subsetneq GT$ — a construction of non-trivial elements of GT seems out of reach by our present techniques — the author thinks it worthwhile to study this natural condition (K) and its consequences. Another reason for this study, which is also a part of our motivation, is a recent (yet unpublished) work of F. Pop who has succeeded in characterizing $G_{\mathbb{Q}}$ as the group of compatible automorphisms of fundamental groups of all varieties over $\bar{\mathbb{Q}}$. In contrast, here, we use only the simplest possible systems of curves and make conditions explicit, but do not know whether $G_{\mathbb{Q}} = GTK$. Thus, our first task is to define GTK explicitly (§1). For $\pi_1(\mathbb{P}^1 - \{0,1,\infty\})$, with a tangential base point $\vec{01}$, our previous field-theoretic terminology is more convenient. Then in §2, we discuss its relations with the above (A), and with another natural condition on our quasi 1-cocycles $\Psi_n^{(j)}(\sigma)$'s, called (A$'$), each defining some subgroups GTA, GTA' of GT containing $G_{\mathbb{Q}}$. Our main purpose of §2 is to show that $GTK \subseteq GTA'$ and also that (A$'$) is equivalent to "$D \log A$" by using Theorem 2 of [BΓ], which implies in particular that $GTA \subseteq GTA'$. We note here that the kernel of the derivation D contains more than the constants, and accordingly, to derive (A) conversely

from $(D \log A)$ is, if possible, not merely the determination of a multiplicative constant (which itself is trivial).

In a sense, we are discussing relations among various intermediate subgroups of $G_{\mathbb{Q}} \subseteq GT$ without knowing whether $G_{\mathbb{Q}} \neq GT$, but we hope that it is of some worth, as a step towards understanding $G_{\mathbb{Q}}$ as a subgroup of the automorphism group of $\pi_1(\mathbb{P}^1 - \{0, 1, \infty\}, \vec{01})$, and also in order to understand each geometric reason for each basic arithmetic property of $B_\sigma(s_1, s_2)$. We might add that this provides, for example, a proof for $(D \log A)$ for $\sigma \in G_{\mathbb{Q}}$, which is essentially different from the Anderson's proofs in [A].

We have also found some unfortunate sign errors in our earlier paper [PGC] during the process of this work. In view of their close relations with the present paper, the indications for corrections will be listed in the Supplement.

Throughout the following, we shall use the symbol $e[z]$ for $\exp(2\pi i z) \quad (z \in \mathbb{C})$.

§1

As in [BΓ], let $\hat{F}_2$ be a free profinite group of rank 2 given together with its generators x and y, and GT be the group of all automorphisms σ of $\hat{F}_2$ of the form $\sigma(x) = x^\lambda, \sigma(y) = f^{-1} y^\lambda f$, where $\lambda = \lambda_\sigma \in \hat{\mathbb{Z}}^\times$ and $f = f_\sigma \in \hat{F}_2'$ are subject to the 2-cycle, 3-cycle and 5-cycle relations (cf. also [I$_2$]). The absolute Galois group $G_{\mathbb{Q}} = \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ over $\mathbb{Q}$ is regarded as a subgroup of GT, via its action on the algebraic fundamental group of $\mathbb{P}^1 - \{0, 1, \infty\}$ over $\bar{\mathbb{Q}}$ w.r.t. the tangential base point $\vec{01}$.

Now, for each $n \in \mathbb{N}$, denote by H_n the kernel of the homomorphism $\hat{F}_2 \to \boldsymbol{\mu}_n$ defined by $x \to e[1/n], y \to 1$. It is free profinite of rank $n+1$ generated by x^n and $x^{-j} y x^j$ $(0 \leq j < n)$. Note that $f_\sigma \in \hat{F}_2' \subset H_n$ for each $\sigma \in GT$, $n \in \mathbb{N}$. Call

$$\varphi_n : H_n \to \hat{F}_2$$

the unique (surjective) homomorphism defined by

$$\varphi_n(x^n) = x, \ \varphi_n(y) = y, \ \varphi_n(x^{-j} y x^j) = 1 \ (1 \leq j < n).$$

Theorem 1. *If $\sigma \in G_{\mathbb{Q}}$, then $f_\sigma \in \hat{F}_2'$ satisfies the relation*

$$(K)_n \qquad \varphi_n(f_\sigma) = y^{\Psi_n^{(0)}(\sigma)} f_\sigma$$

for all $n \in \mathbb{N}$. Here, $\Psi_n^{(0)}(\sigma)$ is the unique element of $\hat{\mathbb{Z}}$ satisfying

$$(n^{-r})^{\sigma-1} = e[r \cdot \Psi_n^{(0)}(\sigma)], \text{ all } r \in \mathbb{Q}.$$

By using a general definition of $\Psi_n^{(0)}(\sigma)$ ($\sigma \in GT$) given in [BΓ], we shall regard $(K)_n$ ($n \in \mathbb{N}$) as necessary conditions for $\sigma \in GT$ to belong to $G_{\mathbb{Q}}$. We shall first show the following

Proposition 1. *Let σ be any element of GT, and $n \in \mathbb{N}$. Then the action of σ on $\hat{F}_2$ leaves $H_n \subset \hat{F}_2$ stable. Moreover, the condition $(K)_n$ for a given $\sigma \in GT$ is equivalent with*

$$(K)_n' \qquad \varphi_n \circ \sigma|_{H_n} = \sigma \circ \varphi_n.$$

Proof of Proposition 1. As in [BΓ], let M be the maximal Galois extension of the rational function field $K = \bar{\mathbb{Q}}(t)$ unramified outside the places $t = 0, 1, \infty$ of K over $\bar{\mathbb{Q}}$, embedded in the field $\bar{\mathbb{Q}}\{\{t\}\}$ of all formal Puiseux series in t. Fix any positive integer n, put $K_n = \bar{\mathbb{Q}}(t^{1/n})$, and let M_n be the maximal Galois extension of K_n unramified outside $t^{1/n} = 0, 1, \infty$. Then $K \subset K_n \subset M_n \subset M$, and the automorphism $\sum_{\nu \in \mathbb{Q}} a_\nu t^\nu \to \sum_{\nu \in \mathbb{Q}} a_\nu t^{n\nu}$ of $\bar{\mathbb{Q}}\{\{t\}\}$ induces a field isomorphism $\Phi_n : M_n \xrightarrow{\sim} M$ such that $\Phi_n(K_n) = K$. We identify $\hat{F}_2$ with $\mathrm{Gal}(M/K)$ in the way described in [I$_2$][BΓ]. Note that H_n is the (open normal) subgroup of $\hat{F}_2$ corresponding to K_n ([BΓ] §1.8). Thus, for each $h \in H_n = \mathrm{Gal}(M/K_n)$, if h_{M_n} denotes its restriction to (an automorphism of) M_n, then

$$H_n \ni h \to \Phi_n \circ h_{M_n} \circ \Phi_n^{-1} \in \mathrm{Gal}(M/K) = \hat{F}_2$$

gives a group homomorphism $H_n \to \hat{F}_2$. We claim that this is nothing but our φ_n; in other words,

[Claim 1] $\qquad \Phi_n(h(F)) = \varphi_n(h)\Phi_n(F) \quad (F \in M_n, h \in H_n).$

(Proof) We may assume that h is one of the generators $x^n, x^{-j}yx^j$ $(0 \leq j < n)$ of H_n. First, let $h = x^n$, and put $F = \sum_{\nu \in \mathbb{Q}} a_\nu t^\nu$. Then $h(F) = x^n(F) = \sum_\nu a_\nu e[n\nu]t^\nu$; hence $\Phi_n(h(F)) = \sum_\nu a_\nu e[n\nu]t^{n\nu}$. On the other hand, $\Phi_n(F) = \sum_\nu a_\nu t^{n\nu}$; hence $\varphi_n(h)\Phi_n(F) = x(\Phi_n(F)) = \sum_\nu a_\nu e[n\nu]t^{n\nu} = \Phi_n(h(F))$. Secondly, let $h = x^{-j}yx^j$ with $0 < j < n$. Then, since M_n/K_n is unramified above the place $t^{1/n} = e[j/n]$ of K_n, h acts trivially on M_n. Since $\varphi_n(h) = 1$, our claim is valid also in this case. Finally, let $h = y$. Consider the place of M over $t \to 1$ defined as follows. Regard each element of M as a univalent function on the open interval (0,1) as described in [I$_2$]§1.1, and take the limit value at 1 from inside (0,1). Then this gives a place of M above the place $t \to 1$ of $\bar{\mathbb{Q}}(t)$, which is invariant by $y \in \hat{F}_2 = \mathrm{Gal}(M/K)$. Since Φ_n transforms this place of M to its restriction to M_n, and since the inertia group of this place is generated by y, we see that $\Phi_n y_{M_n} \Phi_n^{-1} = y^\alpha$ with some $\alpha \in \hat{\mathbb{Z}}^\times$. By letting this element of $\mathrm{Gal}(M/K)$ act on $(1-t)^r = \Phi_n((1-t^{1/n})^r)$, where $r \in \mathbb{Q}$, we see directly that $\alpha = 1$. Therefore, our claim holds also when $h = y$, and hence for all cases.

[Claim 2] The action of GT on $\hat{F}_2$ leaves H_n and the kernel of φ_n stable.

(Proof) First, $\sigma \in GT$ leaves H_n stable because H_n is normally generated by x^n in $\hat{F}_2$, and $\sigma(x^n) = (x^n)^{\lambda_\sigma}$ with $\lambda_\sigma \in \hat{\mathbb{Z}}^\times$. Since $\ker(\varphi_n)$ is normally generated by $x^{-j}yx^j$ $(j \not\equiv 0 \pmod{n})$ in H_n, and σ maps $x^{-j}yx^j$ to

$$(x^{-j\lambda_\sigma} f_\sigma^{-1} x^{j\lambda_\sigma})(x^{-j\lambda_\sigma} y x^{j\lambda_\sigma})^{\lambda_\sigma}(x^{-j\lambda_\sigma} f_\sigma x^{j\lambda_\sigma}),$$

with $x^{-j\lambda_\sigma} f_\sigma x^{j\lambda_\sigma} \in \hat{F}_2' \subset H_n$, $\ker(\varphi_n)$ is also invariant under σ.

[Claim 3] The two conditions $(K)_n$ and $(K)_n'$ are equivalent.

(Proof) The condition $(K)_n'$, i.e.,

$$\varphi_n(\sigma(h)) = \sigma\varphi_n(h) \quad (h \in H_n),$$

when h is specified to one of the generators $x^n, x^{-j}yx^j$ $(0 \leq j < n)$ of H_n, reads more explicitly as follows. When $h = x^n, \varphi_n(\sigma(h)) = \varphi_n(x^{n\lambda_\sigma}) = x^{\lambda_\sigma} = \sigma(x) = \sigma\varphi_n(h)$;

hence the equation holds automatically. When $h = x^{-j}yx^j$ with $j \not\equiv 0 \ (mod\ n)$, one has $\varphi_n(h) = 1$, and $\sigma(h)$ is also in $\ker(\varphi_n)$, as we have seen above. Now let $h = y$. Then $\sigma(y) = f_\sigma^{-1} y^{\lambda_\sigma} f_\sigma$ and $\varphi_n(y) = y$; hence $\varphi_n(\sigma(y)) = \varphi_n(f_\sigma)^{-1} y^{\lambda_\sigma} \varphi_n(f_\sigma)$. On the other hand, $\sigma\varphi_n(y) = \sigma(y) = f_\sigma^{-1} y^{\lambda_\sigma} f_\sigma$. Since the centralizer of y^{λ_σ} $(\lambda_\sigma \in \hat{\mathbb{Z}}^\times)$ in $\hat{F}_2$ consists only of power of y, the condition is equivalent with that

$$(K)_n'' \qquad\qquad \varphi_n(f_\sigma) f_\sigma^{-1} = y^\alpha$$

with some $\alpha \in \hat{\mathbb{Z}}^\times$. But in this case, α can be computed explicitly as follows. Take any $r \in \mathbb{Q}$, and let $\theta_r : \hat{F}_2 \to \boldsymbol{\mu}_\infty$ be the homomorphism defined by $\theta_r(x) = 1, \theta_r(y) = e[r]$. Recall ([BΓ] §1.8) that $\kappa_{n,r}^{(0)}(\sigma) = e[r\Psi_n^{(0)}(\sigma)]$ is the image of f_σ under the homomorphism $H_n \to \boldsymbol{\mu}_\infty$ defined by $x^n \to 1, y \to e[r], x^{-j}yx^j \to 1 \ (0 < j < n)$. Note that this homomorphism is nothing but $\theta_r \circ \varphi_n$. Now apply θ_r to both sides of $(K)_n''$ to get

$$e[r\Psi_n^{(0)}(\sigma)] = e[\alpha r]$$

(Note that $\theta_r(f_\sigma) = 1$, as $f_\sigma \in \hat{F}_2'$.) Since $r \in \mathbb{Q}$ is arbitrary, we obtain $\alpha = \Psi_n^{(0)}(\sigma)$, as desired. Therefore, $(K)_n'$ is equivalent with $(K)_n$. This settles the proof of Proposition 1.

Proof of Theorem 1 By Proposition 1, it is enough to prove $(K)_n'$ when $\sigma \in G_{\mathbb{Q}}$. Recall that $G_{\mathbb{Q}}$ acts on M from the left via its action on the Puiseux coefficients, as $\sum_\nu a_\nu t^\nu \to \sum_\nu \sigma(a_\nu) t^\nu$, and note that this action commutes with the automorphism $\sum_\nu a_\nu t^\nu \to \sum_\nu a_\nu t^{n\nu}$; namely,

$$(1) \qquad\qquad \Phi_n(\sigma(F)) = \sigma\Phi_n(F), \text{ all } F \in M_n.$$

We want to show that $\varphi_n(\sigma(h)) = \sigma(\varphi_n(h))$ for any $\sigma \in G_{\mathbb{Q}}$ and $h \in H_n$; or equivalently,

$$(2) \qquad\qquad \varphi_n(\sigma \circ h \circ \sigma^{-1}) = \sigma \circ \varphi_n(h) \circ \sigma^{-1}.$$

as automorphisms of M. Take any $\Phi_n(F) \in M \quad (F \in M_n)$. Then

$$\varphi_n(\sigma \circ h \circ \sigma^{-1})(\Phi_n(F)) = \Phi_n(\sigma \circ h \circ \sigma^{-1} F) = \sigma\varphi_n(h)\sigma^{-1}\Phi_n(F)$$

by [Claim 1] and (1); whence (2). This settles the proof of Theorem 1.

We define GTK to be the subgroup of GT of all those $\sigma \in GT$ satisfying the equivalent conditions $(K)_n, (K)'_n$ for all $n \in \mathbb{N}$. By Theorem 1,

$$G_{\mathbb{Q}} \subseteq GTK \subseteq GT.$$

Remark. For $d, n \in \mathbb{N}$ and $m = nd$, there is a natural surjective homomorphism $\varphi_{m,n} : H_m \to H_n$ defined by $x^m \to x^n, x^{-j}yx^j \to 1$ $(j \not\equiv 0 \ (mod\ d)), x^{-dk}yx^{dk} \to x^{-k}yx^k$. One has $\varphi_{m,1} = \varphi_m$, and the restriction of the $G_{\mathbb{Q}}$-action on $\hat{F}_2$ to its subgroups H_m, H_n are compatible with $\varphi_{m,n}$. But we do not obtain any stronger conditions by using $\varphi_{m,n}$, because $\varphi_{m,n}$ is simply the restriction of $\varphi_d : H_d \to H_1 = \hat{F}_2$ to $H_m \subset H_d$.

§**2**

We shall consider another subgroup GTA$'$ of GT, and discuss its relations with GTA and GTK.

For each $n \in \mathbb{N}$, $i \in \mathbb{Z}/n$ and $\sigma \in$ GT, we defined in [BΓ] a certain element $\Psi_n^{(i)}(\sigma)$ of $\hat{\mathbb{Z}}$. It is the logarithm of $\kappa_{n,r}^{(i)}(\sigma)$ in the sense that $\kappa_{n,r}^{(i)}(\sigma) = e[r\Psi_n^{(i)}(\sigma)]$ for all $r \in \mathbb{Q}$. By [BΓ] §1.10, it is a "distribution", i.e.,

$$\sum_{\substack{j \pmod{nd} \\ j \to i}} \Psi_{nd}^{(j)}(\sigma) = \Psi_n^{(i)}(\sigma) \tag{1}$$

holds for any $d \in \mathbb{N}$, and it is a "quasi 1-cocycle", i.e.,

$$\Psi_n^{(i)}(\sigma\tau) = \Psi_n^{(i)}(\sigma) + \lambda_\sigma \Psi_n^{(\lambda_\sigma^{-1} i)}(\tau) \quad (\sigma, \tau \in \mathrm{GT}). \tag{2}$$

We shall consider the following condition (A$'$) for elements σ of GT;

$$\text{(A}'\text{)} \qquad \begin{cases} \text{(i)} & \Psi_{mn}^{(0)}(\sigma) = \Psi_m^{(0)}(\sigma) + \Psi_n^{(0)}(\sigma) \quad \text{for any } m, n \in \mathbb{N}, \\ \text{(ii)} & \Psi_{mn}^{(in)}(\sigma) = \Psi_m^{(i)}(\sigma) \quad \text{for any } m, n \in \mathbb{N}, i \in \mathbb{Z}/m, i \neq 0. \end{cases}$$

It is easy to see (by (2)) that the set of all $\sigma \in GT$ satisfying (A$'$) forms a subgroup of GT which we call GTA$'$. We shall show first that GTK $\subset$ GTA$'$, and then that the condition (A$'$) is equivalent with the logarithmic derivative of Anderson's multiplication formula for the gamma function $\Gamma_\sigma(s)$ associated with σ, and hence in particular that GTA $\subset$ GTA$'$. (That $\sigma \in$ GTA implies (A$'$)(i) was already noted in [BΓ]§1.10 (in terms of $\kappa_{n,r}^{(0)}(\sigma)$).)

Proposition 2. $\mathrm{GTK} \subset \mathrm{GTA}'$.

(Proof) By [BΓ]§1.8, for any $\sigma \in \mathrm{GT}$, the action of $f_\sigma \in \mathrm{Gal}(M/K)$ on $(1-\zeta_n^{-j}t^{1/n})^r \in M$ $(n \in \mathbb{N},\ j \in \mathbb{Z}/n,\ r \in \mathbb{Q},\ \zeta_n = e[\frac{1}{n}])$ is simply a scalar multiplication by $\kappa_{n,r}^{(j)}(\sigma) = e[\Psi_n^{(j)}(\sigma)r]$. Thus, for $m, n \in \mathbb{N}$ and $j \in \mathbb{Z}/mn$, one has

$$f_\sigma(1-\zeta_{mn}^{-j}t^{1/mn})^r = e[\Psi_{mn}^{(j)}(\sigma)r](1-\zeta_{mn}^{-j}t^{1/mn})^r. \tag{3}$$

Apply the automorphism $\Phi_n : M_n \xrightarrow{\sim} M$ of §1 to both sides of (3). Then by [Claim 1](§1), we obtain

$$\varphi_n(f_\sigma)(1-\zeta_{mn}^{-j}t^{1/m})^r = e[\Psi_{mn}^{(j)}(\sigma)r](1-\zeta_{mn}^{-j}t^{1/m})^r$$

for any $\sigma \in \mathrm{GT}$. Now assume that $\sigma \in \mathrm{GTK}$. Then $\varphi_n(f_\sigma) = y^{\Psi_n^{(0)}(\sigma)} f_\sigma$; hence

$$y^{\Psi_n^{(0)}(\sigma)} f_\sigma(1-\zeta_{mn}^{-j}t^{1/m})^r = e[\Psi_{mn}^{(j)}(\sigma)r](1-\zeta_{mn}^{-j}t^{1/m})^r. \tag{4}$$

Now, let $j = 0$. Then since

$$f_\sigma(1-t^{1/m})^r = e[\Psi_m^{(0)}(\sigma)r](1-t^{1/m})^r,$$

and

$$y^\alpha(1-t^{1/m})^r = e[\alpha r](1-t^{1/m})^r \quad (\alpha \in \hat{\mathbb{Z}}),$$

(4) gives

$$e[\Psi_m^{(0)}(\sigma)r]e[\Psi_n^{(0)}(\sigma)r] = e[\Psi_{mn}^{(0)}(\sigma)r]$$

for any $r \in \mathbb{Q}$; hence

$$\Psi_m^{(0)}(\sigma) + \Psi_n^{(0)}(\sigma) = \Psi_{mn}^{(0)}(\sigma).$$

Secondly, let $j = in$ with $i \not\equiv 0 (\mathrm{mod} m)$. Then (4) reads as

$$y^{\Psi_n^{(0)}(\sigma)} e[\Psi_m^{(i)}(\sigma)r](1-\zeta_m^{-i}t^{1/m})^r = e[\Psi_{mn}^{(in)}(\sigma)r](1-\zeta_m^{-i}t^{1/m})^r. \tag{5}$$

But since y leaves $(1-\zeta_m^{-i}t^{1/m})^r$ invariant (as $i \not\equiv 0(\mathrm{mod} m)$), we obtain directly from (5) (for all $r \in \mathbb{Q}$) that $\Psi_{mn}^{(in)}(\sigma) = \Psi_m^{(i)}(\sigma)$, as desired. $\square$

Our next task is to clarify the relation between GTA' and GTA (defined in [BΓ]§1.10). First, let us briefly recall the definition of GTA. Let $\sigma \in \mathrm{GT}$ and $\Gamma_\sigma = \Gamma_\sigma(s)$ be a branch

of the gamma function associated with σ. It is an invertible element of the ring $\tilde{A}_1 = \mathbb{W}[\![\hat{\mathbb{Z}}]\!]$ regarded as a $(\mathbb{W} \otimes \mathbb{Q}_{ab})$-valued function of $s \in \mathbb{Q}/\mathbb{Z}$ (cf. [BΓ] §1.7). The group GTA is the collection of all such $\sigma \in \mathrm{GT}$ that

$$\text{(A)} \qquad \Big(\prod_{nc=0} \Gamma_\sigma(s+c) \Big) \Big(\prod_{nc=0} \Gamma_\sigma(c) \Big)^{-1} \Gamma_\sigma(ns)^{-1} = 1 \otimes e[ns\Psi_n^{(0)}(\sigma)]$$

holds for all $s \in \mathbb{Q}/\mathbb{Z}$ and all $n \in \mathbb{N}$.

When $\sigma \in G_{\mathbb{Q}}$, this is Anderson's multiplication formula. We are going to show that our condition (A′) is equivalent to the "Dlog" of (A). To give this a precise sense, we first note that the embedding

$$\tilde{A}_1 \subset \mathrm{Fct}(\mathbb{Q}/\mathbb{Z},\ \mathbb{W} \otimes \mathbb{Q}_{ab}) \tag{6}$$

and the derivation

$$D : \tilde{A}_1 \to \tilde{A}_1, \quad D(\mathbf{x}^{-1}) = \mathbf{x}^{-1}$$

treated in [BΓ] §1.8 extend naturally to those for $\tilde{\tilde{A}}_1 = \tilde{A}_1 \otimes \mathbb{Q}_{ab}$. The extended $\mathbb{Q}_{ab}$-linear derivation will also be denoted by D. Thus, we have

$$\tilde{\tilde{A}}_1 \subset \mathrm{Fct}(\mathbb{Q}/\mathbb{Z},\ \mathbb{W} \otimes \mathbb{Q}_{ab}), \quad D : \tilde{\tilde{A}}_1 \to \tilde{\tilde{A}}_1.$$

(The injectivity of the $\mathbb{Q}_{ab}$-linear extension of the embedding (6) follows immediately from the fact that the matrix $\big(1 \otimes e\big[\frac{-ij}{N}\big]\big)$ $(i, j \in \mathbb{Z}/N)$ is invertible in $\mathbb{W} \otimes \mathbb{Q}_{ab}$ for all N.) Secondly, note that for any $f \in \tilde{\tilde{A}}_1$ regarded as a function $f(s)$ of $s \in \mathbb{Q}/\mathbb{Z}$, and for any $c \in \mathbb{Q}/\mathbb{Z}$, the function $f(s+c)$ of s corresponds to a certain element of $\tilde{\tilde{A}}_1$ described in terms of f and c as follows. For any $N \in \mathbb{N}$ such that $Nc = 0$, write

$$f \equiv \sum_{j (\mathrm{mod}\ N)} a_j \mathbf{x}^{-j} \mod (\mathbf{x}^N - 1)$$

with $a_j \in \mathbb{W} \otimes \mathbb{Q}_{ab}$. Then

$$\sum_{j (\mathrm{mod}\ N)} a_j (1 \otimes e[cj]) \mathbf{x}^{-j} \mod (\mathbf{x}^N - 1)$$

forms a projective system (when N runs over all positive integers with $Nc = 0$), and this gives the element of $\tilde{\tilde{A}}_1$ corresponding to $f(s+c)$. The translation $f(s) \to f(s+c)$ is

an automorphism of $\tilde{\tilde{A}}_1$, and it commutes with its derivation D (because D is the unique continuous derivation of $\tilde{\tilde{A}}_1$ such that $D(\mathbf{x}^{-1}) = \mathbf{x}^{-1}$, and translations multiply $\mathbf{x}^{-1}$ by constants). Finally, for any positive integer n, and $f = f(s) \in \tilde{\tilde{A}}_1$, the function $f(ns)$ corresponds to the image of f under the endomorphism of $\tilde{\tilde{A}}_1$ induced from $\mathbf{x} \to \mathbf{x}^n$. One has $D(f(ns)) = n(Df)(ns)$.

For each $f \in \tilde{\tilde{A}}_1^{\times}$, we define $\mathrm{D}\log f := Df/f \in \tilde{\tilde{A}}$. The operation D log transforms multiplication to addition, commutes with the translations $f(s) \to f(s+c)$, and one has $(\mathrm{D}\log)(f(ns)) = n \cdot (\mathrm{D}\log f)(ns)$.

Now put $\gamma_\sigma = \mathrm{D}\log\Gamma_\sigma \in \tilde{A}_1$, and write $\gamma_\sigma = \gamma_\sigma(s)$ when we regard this as a function of s. Then the D log-version of (A) reads as

$$\text{(D logA)} \qquad \sum_{nc=0} \gamma_\sigma(s+c) - n\gamma_\sigma(ns) = n\Psi_n^{(0)}(\sigma).$$

This is obtained by taking D log of the equality (A), and hence is a consequence of (A).

Proposition 3. *The two conditions* (A′) *and* (D logA) *for* $\sigma \in \mathrm{GT}$ *are equivalent.*

Corollary. $\mathrm{GTA} \subset \mathrm{GTA}'$.

Proof of Proposition 3. Let $\sigma \in \mathrm{GT}$, $n, N \in \mathbb{N}$ with $n|N$. Write $N = mn$. Then by Theorem 2 of [BΓ], we have

$$\gamma_\sigma(s) \equiv \gamma_\sigma(0) + \sum_{j(\mathrm{mod}\ N)} \Psi_N^{(j)}(\sigma)\mathbf{x}^{-j} \mod (\mathbf{x}^N - 1). \tag{7}$$

Let us compute the left hand side of (D logA) $\mathrm{mod}(\mathbf{x}^N - 1)$ using (7). First,

$$\begin{aligned}
\sum_{nc=0} \gamma_\sigma(s+c) &= \sum_{k(\mathrm{mod}\ n)} \gamma_\sigma\Big(s + \frac{k}{n}\Big) \\
&\equiv n\gamma_\sigma(0) + \sum_{\substack{k(\mathrm{mod}\ n) \\ j(\mathrm{mod}\ N)}} \Big(\Psi_N^{(j)}(\sigma) \otimes e\big[\tfrac{jk}{n}\big]\Big)\,\mathbf{x}^{-j} \\
&\equiv n\left(\gamma_\sigma(0) + \sum_{\substack{j(\mathrm{mod}\ N) \\ j \equiv 0(\mathrm{mod}\ n)}} \Psi_N^{(j)}(\sigma)\mathbf{x}^{-j}\right) \\
&\equiv n\left(\gamma_\sigma(0) + \sum_{i(\mathrm{mod}\ m)} \Psi_N^{(ni)}(\sigma)\mathbf{x}^{-ni}\right) \mod (\mathbf{x}^N - 1).
\end{aligned}$$

Secondly,

$$\begin{aligned}\gamma_\sigma(ns) &\equiv \gamma_\sigma(0) + \sum_{j(\mathrm{mod}\ N)} \Psi_N^{(j)}(\sigma)\mathbf{x}^{-nj} \mod (\mathbf{x}^N - 1)\\ &\equiv \gamma_\sigma(0) + \sum_{i(\mathrm{mod}\ m)} \left(\sum_{\substack{j\to i \\ j(\mathrm{mod}\ N)}} \Psi_N^{(j)}(\sigma) \right)\mathbf{x}^{-ni}\\ &\equiv \gamma_\sigma(0) + \sum_{i(\mathrm{mod}\ m)} \Psi_m^{(i)}(\sigma)\mathbf{x}^{-ni}.\end{aligned}$$

Therefore, the left hand side of (D logA) is congruent $\mathrm{mod}(\mathbf{x}^N - 1)$ to

$$n \sum_{i(\mathrm{mod}\ m)} \left(\Psi_{mn}^{(ni)}(\sigma) - \Psi_m^{(i)}(\sigma)\right)\mathbf{x}^{-ni},$$

while its right hand side is equal to the constant $n\Psi_n^{(0)}(\sigma)$. Therefore, (D logA) is equivalent with

$$\begin{cases} \Psi_{mn}^{(0)}(\sigma) - \Psi_m^{(0)}(\sigma) = \Psi_n^{(0)}(\sigma) & \\ \Psi_{mn}^{(in)}(\sigma) = \Psi_m^{(i)}(\sigma) & \text{if } \ i \not\equiv 0(\mathrm{mod} m) \end{cases}$$

for all $m, n \in \mathbb{N}$, which is nothing but (A′).

□

Remark. The kernel of the derivation D contains non-constant elements such as sum of elements of the form $c\mathbf{x}^{c'}$ with $c, c' \in \widehat{\mathbb{Z}}$ and $cc' = 0$ (e.g. c, c' belong to the $\mathbb{Z}_p$, $\mathbb{Z}_{p'}$ factors of $\widehat{\mathbb{Z}}$ for $p \neq p'$), and so (A) does not seem to follow directly from D logA.

On the other hand, the author has paid serious efforts in trying to prove GTK $\subset$ GTA, but without success so far.

Supplement

[Corrections for related previous articles]

Some direct consequences of Theorem 2 of [BΓ] did not quite agree with Theorem 10 of my older paper [PGC], nor with a consequence of Cor 8.6.3 of G. Anderson [A]. In each case, the difference is that of complex conjugation. It turned out that the mistake was not in our recent [BΓ] but in [PGC] (the author's "youthful folly") and because of an induced misinterpretation of [PGC] in [A]. At this opportunity, the author would like to apologize for the mistake and ask for the publishing of corrections.

[PGC] "Profinite braid groups, Galois representations and complex multiplications",
Ann. of Math. **123** (1986), 43-106

p 76 (ℓ 1$\sim$4) In this framework, in general, an automorphism α of a function field of one variable and the *corresponding* automorphism α^* of the corresponding curve *should both act from the left*, so that $(\alpha f)(\alpha^* P) = f(P)$ for any geometric point P and any rational function f of the curve. Thus, θ_n should map a geometric point of $X_n^{(a,b,c)}$ with coordinate (s_0, t_0) to the point $(\zeta_n^{-1} s_0, t_0)$. This is the "original sin."

p 77 (ℓ 2) σ_{-k} should be σ_k

(ℓ 3) $\sigma_{k'}$ should be $\sigma_{-k'}$

p 81 Theorem 7 $J_{l^n}^{(a,b,c)}(\mathfrak{p}_n)$ should be $J_{l^n}^{(-a,-b,-c)}(\mathfrak{p}_n)$

p 82 Remark 1 π' should be $\bar{\pi}'$

p 95 Theorem 10 (in the exponent) $-U^m - V^m - W^m$ should be $U^m + V^m + W^m$,

$$\sum_{\substack{i,j \geq 1 \\ i+j=m}} \frac{U^i V^j}{i!j!} \text{ should be } -\sum_{\substack{i,j \geq 1 \\ i+j=m}} \frac{U^i V^j}{i!j!}$$

p 97 (ℓ 2) R_n should be $-R_n$

p 105 Corollary (in the exponent) $U^m + V^m + W^m$ should be $-(U^m + V^m + W^m)$

$$\sum_{\substack{i,j \geq 1 \\ i+j=m}} \frac{U^i V^j}{i!j!} \text{ should be } -\sum_{\substack{i,j \geq 1 \\ i+j=m}} \frac{U^i V^j}{i!j!}$$

[A] G. Anderson "The hyperadelic gamma function",
Inv. math. **95** (1989), 63-131

p 125 (ℓ 11) z should go to the inverse of what is written here.

Another point to note in [A] is that the action of the roots of unity on the Fermat curve defined in p.85 is the inverse of what we did in [PGC].

References

[A] *G. W. Anderson*, The hyperadelic gamma function, Inv. math. **95** (1989), 63-131.

[I_1]=[PGC] *Y. Ihara*, Profinite braid groups, Galois representations and complex multiplications, Ann. of Math. **123** (1986), 43-106.

[I_2] ———, On the embedding of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ into $\widehat{GT}$; in "The Grothendieck Theory of Dessins d'Enfants", London Math. Soc. Lecture Notes Series **200**, Cambridge Univ. Press (1994), 289-305.

[I_3] = [BΓ] ———, On beta and gamma functions associated with the Grothendieck-Teichmüller group; to appear in Proceedings of a conference on Galois groups and fundamental groups held at Gainsville, Florida in October 1996, (H. Völklein (ed.), London Math. Soc. Lecture Notes Series).

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