EQUIVALENCE RELATIONS ON ALGEBRAIC CYCLES

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Algebraic cycles give important invariants of algebraic varieties, and it is common to study the groups of algebraic cycles via so-called adequate equivalence relations. For example, the basic Chow groups are defined by considering cycles modulo rational equivalence. Rational, algebraic, homological and numerical equivalence have been considered since long time, and it is still a most interesting task to understand the precise relationship between them. But there are other adequate equivalence relations, like the ℓ-cubical equivalence which coincides with algebraic equivalence for $\ell = 1$.

More recently, Bloch and Beilinson have proposed the existence of a filtration $F^0 \supset F^1 \supset F^2 \supset \ldots$ by adequate equivalence relations which would provide a good understanding of the Chow groups with rational coefficients of smooth projective varieties. In particular, the filtration should be finite on each fixed Chow group $CH^j(X)_{\mathbb{Q}}$, and each graded quotient $Gr_F^\nu CH^j(X)_{\mathbb{Q}}$ should only depend on homological equivalence, in fact, only on $H^{2j-\nu}(X)$. The first step F^1 is homological equivalence, but the following steps will differ considerably from the equivalence relations recalled above; e.g., F^2 is related to Abel-Jacobi equivalence.

It is the objective of this paper to propose simple candidates for the F^ν and to study the relationship between the F^ν and other equivalence relations in the literature, like algebraic equivalence or the more general niveau and coniveau filtrations on Chow groups. In the following, we always consider $\mathbb{Q}$ -rational cycle groups.

In sections 1 and 2 we review the basics on adequate equivalence relations and the conjectured Bloch-Beilinson filtration. In section 3 we recall

[0]To appear in the Proceedings of the Banff conference on "Arithmetic and Geometry of Algebraic Cycles", published by Kluwer.

a concept introduced by H. Saito: the product $E * E'$ of two adequate equivalence relations E and E'. In particular, this allows to define the ℓ-fold self-product $E^{*\ell}$ of an equivalence relation. Using this notion we prove in section 4: If the Bloch-Beilinson filtration $F^\cdot$ exists and the standard conjecture B holds, then

$$F^\nu = H^{*\nu} \tag{1}$$

where H is homological equivalence. Note that the right hand side is well and unconditionally defined; however, it seems hard to prove the desired properties of this filtration.

In section 5 we introduce niveau and coniveau filtrations on Chow groups: As for the latter one let

$$N^q CH^j(X)$$

be the subgroup generated by cycles which are homologous to zero on some q-codimensional subvariety. These groups arise in connection with the coniveau spectral sequence of Bloch and Ogus [3]. We show that, for fixed p, the groups $N_p CH^j(X) := N^{j-p} CH^j(X)$ define an adequate equivalence relation N_p, with $0 = N_0 \subseteq N_1 \subseteq \ldots$ and N_1 = algebraic equivalence, if one takes the usual ℓ-adic or singular cohomology. The following niveau filtration was introduced by Nori [21]. Let

$$A_r CH^j(X)$$

be the subgroup of those cycles which come via correspondences from r-dimensional cycles homologically equivalent to zero on some smooth, projective variety T. Again one obtains adequate equivalence relations $A_0 \subseteq A_1 \subseteq \ldots$, with A_0 = algebraic equivalence. Similarly we may define adequate equivalence relations $\tilde{N}_0 \subseteq \tilde{N}_1 \subseteq \ldots$ by letting $\tilde{N}_p CH^j(X)$ be the subgroup of cycles coming via correspondences from p-codimensional cycles homologically to zero on some smooth, projective variety T. By results of Nori one has

$$A_r \subseteq \tilde{N}_{r+1} \subseteq N_{r+1}.$$

The equality $\tilde{N}_p = N_p$ would follow from the Tate conjecture, and in section 6 we show the following relation between $F^\cdot, A.$ and $\tilde{N}.$: If the Bloch-Beilinson filtration $F^\cdot$ exists and the standard conjecture B holds, then

$$A_r = \tilde{N}_{r+1} \tag{2}$$

$$F^\nu CH^j(X) \subset A_{j-\nu} CH^j(X). \tag{3}$$

Note that equation (2) does not involve $F^\cdot$. The inclusion (3) would have remarkable consequences. For $\nu = 2$ it suggests the inclusion

$$CH^j(X)_{\mathbb{Q},AJ} \overset{?}{\subseteq} N^1 CH^j(X)_{\mathbb{Q}} \tag{4}$$

for complex projective varieties, i.e., cycles in the kernel of the Abel-Jacobi map should be homologous to zero on a divisor, at least modulo torsion. The validity of this inclusion was asked by H. Esnault. For $\nu = 2$ and $j = 2$ it would suggest the inclusion

$$CH^2(X)_{\mathbb{Q},AJ} \quad \stackrel{?}{\subseteq} \quad CH^2(X)_{alg}, \tag{5}$$

i.e., that codimension 2 cycles in the kernel of the Abel-Jacobi map are algebraically equivalent to zero. This was asked by M. Nori [21]. For $\nu = j$, the inclusion (3) predicts that there is a filtration (induced by $F^\cdot$) on the Griffiths group $Griff^j(X) = CH^j(X)_{hom}/CH^j(X)_{alg}$ such that the graded quotients only depend on the Grothendieck motives

$$h^{j+1}(X)/N^1, h^{j+2}(X)/N^2, \ldots, \ h^{2j-1}(X)/N^{j-1},$$

so that, e.g., we expect

$$H^3(X) = N^1H^3(X) \quad \stackrel{?}{\Rightarrow} \quad Griff^2(X)_{\mathbb{Q}} = 0, \tag{6}$$

where $N^\cdot$ denotes the coniveau filtration on cohomology. This corresponds to questions of R. Laterveer.

We prove the results in a more precise and more concrete form. For example we consider filtrations $F^\cdot$ 'of Bloch-Beilinson type' as introduced by S. Saito [24], and get a more precise statement than (3) by introducing the notion of niveau filtrations $\langle E \rangle_r$ and coniveau filtrations $\{E\}_p$ associated to any adequate equivalence relation E (One has $A_r = \langle H \rangle_r$ and $\tilde{N}_p = \{H\}_p$). In a following paper we will give some applications to ℓ-adic cycle maps. Finally, in section 7 we discuss some connection with the conjectural theory of mixed motives. We explain that our results support the expectation that the graded quotients of the Bloch-Beilinson filtration are related to Yoneda-Ext-groups of mixed motives. More concretely, however, our results positively answer a question raised in [15].

This work was done while the author was supported by the European TMR network 'Arithmetical Algebraic Geometry' (ERB FMRX 960006) and completed while the author visited the Research Institute for Mathematical Sciences in Kyoto. It is a pleasure to thank H. Esnault, R. Laterveer, J. Murre and S. Saito for stimulating discussions.

1. Adequate equivalence relations

Fix a field k. Following Samuel [25], an additive equivalence relation E on algebraic cycles (on smooth, projective varieties over k) is called adequate, if push-forward, pull-back and intersection of cycles are well-defined modulo E. More formally, the following is required.

(i) The equivalence relation is compatible with addition of cycles.
(ii) If α and β are cycles on X, then there exists a cycle α' equivalent to α such that α' and β intersect properly (i.e., this holds for the components).
(iii) Let α and β be cycles on X and $X \times Y$, respectively, such that $\beta.(\alpha \times Y)$ is defined. If α is equivalent to zero, then so is the cycle $\beta(\alpha) := (pr_Y)_*(\beta.(\alpha \times Y))$ on Y.

The following four adequate equivalence relations have been considered most classically.

1. R =rational equivalence: $\alpha \in Z^j(X)$(:= group of algebraic cycles of codimension j on X) is rationally equivalent to zero, if there is a $\gamma \in Z^j(X \times \mathbb{P}^1)$ such that for two k-rational points a,b of $\mathbb{P}^1$ (w.l.o.g we may take $a = 0$ and $b = 1$) the pull-backs $\gamma(a)$ and $\gamma(b) \in Z^j(X)$ exist, and $\alpha = \gamma(a) - \gamma(b)$. By definition, the Chow group of cycles of codimension i on X is $CH^j(X) := Z^j(X)$ /rational equivalence.
2. A =algebraic equivalence: similar, with the projective line $\mathbb{P}^1$ replaced by an arbitrary curve C.
3. H =homological equivalence: This depends on the choice of some Weil cohomology theory $X \rightsquigarrow H^*(X)$ ([18], e.g., we may take $H^*(X) = H^*(X \times_k \overline{k}, \mathbb{Q}_\ell)$ - étale ℓ-adic cohomology - for a prime $\ell \neq char(k)$ and an algebraic closure $\overline{k}$ of k). If cl is the corresponding cycle class map, then α is homologically equivalent to zero if $cl(\alpha) = 0$.
4. N =numerical equivalence: $\alpha \in Z^j(X)$, for X irreducible of dimension d, is called numerically equivalent to zero if the intersection number $(\alpha.\beta)$ is zero for all $\beta \in Z^{d-j}(X)$.

It is well-known that rational equivalence is the finest adequate equivalence relation and that numerical equivalence is the coarsest non-trivial one. Apart from this the relationship between these four relations can be depicted as follows:

$$\begin{array}{ccccccc} \text{rat.} & \overset{\subset}{\neq} & \text{alg.} & \overset{\subset}{\neq} & \text{hom.} & \overset{\subset}{=?} & \text{num.} \\ R & & A & & H & & N \end{array}$$

Rational equivalence is strictly finer than algebraic one, already for divisors on curves, and a famous counterexample of Griffiths [9] showed that algebraic equivalence is strictly finer than homological one, even modulo torsion, already for codimension 2 cycles on threefolds. Grothendieck's standard conjectures on algebraic cycles [18] predict that homological equivalence, a priori only finer than numerical one, agrees with the latter.

There are many more adequate equivalence relations. The following one was defined by Samuel [25].

Definition 1.1 *Let k be algebraically closed, and let ℓ be an integer ≥ 0. Two cycles $\xi_1, \xi_2 \in Z^j(X)$ are called ℓ-cubical equivalent ($\xi_1 \sim_{(\ell)} \xi_2$), if*

there are curves $C_1,\dots,C_\ell$, *a cycle* $z \in Z^j(C_1 \times \dots \times C_\ell \times X)$ *and closed points* $a_i^{(0)}, a_i^{(1)}$ *on* C_i $(i = 1,\dots,\ell)$ *such that* $z(a_1^{(i_1)},\dots a_\ell^{(i_1)}) \in Z^j(X)$ *(pull-back via the point* $\underline{a} = (a_1^{(i_1)},\dots,a_\ell^{(i_\ell)})$ *of* $C_1 \times \dots \times C_\ell$*) exists for all* $i_1,\dots,i_\ell \in \{0,1\}$ *and such that*

$$\xi_1 - \xi_2 = \sum_{i_1,\dots,i_\ell \in \{0,1\}} (-1)^{i_1+\dots+i_\ell} z(a_1^{(i_1)},\dots,a_\ell^{(i_\ell)}).$$

Remarks 1.2 (1) One gets the same equivalence relation if in the definition one replaces the curves C_i by arbitrary smooth projective varieties, or by abelian varieties; one may also take $C_1 = \dots = C_\ell$ in the definition. The reason is that finitely many points on a smooth, projective variety V can always be connected by a smooth curve C on V.

(2) Let $F_c^\ell CH^j(X) = \{\xi \in Z^j(X) | \xi \sim_{(\ell)} 0\}/\text{rational equivalence}$. Then $F_c^0 CH^j(X) = CH^j(X), F_c^1 CH^j(X) = ACH^j(X) := \{[\alpha] | \alpha \text{ is algebraically equivalent to zero}\}$, and

$$F_c^0 \supset F_c^1 \supset F_c^2 \supset \dots$$

The following well-known lemma is based on the fact that rational equivalence is the finest adequate equivalence relation.

Lemma 1.3 *Giving an adequate equivalence relation* E *is tantamount to giving abelian subgroups*

$$ECH^j(X) \subseteq CH^j(X)$$

for every smooth, projective X *and every* $j \geq 0$ *such that the following equivalent conditions hold, where we abbreviate* $CH(X) = \oplus_i CH^j(X)$*:*

(i) The subgroups $ECH(X) \subset CH(X)$ *are respected by algebraic correspondences, i.e., for* $\alpha \in ECH(X)$ *and* $\gamma \in CH(X \times Y)$ *we have*

$$\gamma(\alpha) := (pr_Y)_*(\gamma . pr_X^* \alpha) \in ECH(Y).$$

(ii) The subgroups are respected by pull-backs f^* *and push-forwards* f_* *for morphisms* $f : X \to Y$*, and for intersection product one has*

$$\alpha \in CH(X)\ ,\ \beta \in ECH(X) \Rightarrow \alpha.\beta \in CH(X).$$

(iii) Same as (ii), except that one restricts to pr_X^* *for* $pr_X : X \times Y \to X$ *and* $(pr_Y)_*$ *for* $pr_Y : X \times Y \to Y$*.*

In fact, the $1-1$-correspondence is given by assigning to E the subgroups

$$ECH^j(X) = \{[z] \in CH^j(X) \mid z \text{ is } E\text{-equivalent to zero}\}. \qquad (7)$$

The implications (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i) are obvious, and for (i) $\Rightarrow$ (ii) one notes that for $\alpha \in CH(X)$ and $\beta \in CH(Y)$ one has $f^*(\beta) = {}^t\Gamma_f(\beta)$ and $f_*(\alpha) = \Gamma_f(\alpha)$, where $\Gamma_f = \mathrm{Im}((id, f) : X \to X \times Y)$ is the graph of f and ${}^t\Gamma_f$ is its transpose in $CH(Y \times X)$. Moreover, $\alpha.\beta = \Delta_*(\alpha)(\beta)$, where $\Delta : X \hookrightarrow X \times X$ is the diagonal.

From now on we will use the notation of 1.3 and (7); in particular, we will denote (adequate) equivalence relations by capital letters, and we have defined the equivalence relations F_c^ℓ and groups like $HCH^j(X)$.

1.4 Recall that a correspondence from X to Y is a class in $CH(X \times Y)$, and that there is a well-defined action on Chow groups according to the formula in 1.3(i). More generally, let E be an adequate equivalence relation, and put

$$CH_E(X) = CH(X)/ECH(X).$$

Then there is an induced action

$$CH_E(X \times Y) \times CH_E(X) \to CH_E(Y).$$

Moreover, one can define a category $\mathcal{M}_k^E$ of motives modulo E over k as follows. Let

$$Corr_E^r(X,Y) := \bigoplus_{\nu=1}^{m} CH_E^{dim X_\nu + r}(X_\nu \times Y),$$

where $X_1, \ldots, X_m$ are the irreducible components of X, be the group of correspondences of degree r modulo E from X to Y. There are associative composition laws

$$\begin{aligned} Corr_E^r(Y,Z) \times Corr_E^s(X,Y) &\to Corr_E^{r+s}(X,Z) \\ (g,f) &\mapsto g \circ f := (pr_{XZ})_*(pr_{YZ}^* g . pr_{XY}^* f), \end{aligned}$$

and by taking these as compositions of morphisms one gets a category $\mathcal{M}_k^E$ of motives modulo E as follows. Objects are triples (X,p,a) where X is a smooth, projective variety, $p \in Corr^0(X,X)$ is an idempotent ($p^2 = p$) and a is an integer, and the morphism sets are

$$Hom((X,p,a),(Y,q,b)) = qCorr_E^{b-a}(X,Y)p.$$

$\mathcal{M}_k^E$ is a pseudo-abelian category (it is additive, and idempotent endomorphisms correspond to direct factors). Particular examples are the category of 'Chow motives' $\mathcal{CHM}_k := \mathcal{M}_k^R$ (motives modulo rational equivalence) and the category of 'Grothendieck motives' $\mathcal{M}_k := \mathcal{M}_k^H$ (motives modulo homological equivalence). For each E there is a contravariant functor

$$h^E : \mathcal{V}_k \to \mathcal{M}_k^E$$

from the category $\mathcal{V}_k$ of smooth projective varieties over k to $\mathcal{M}_k^E$ sending a variety X to $h(X) := (X, id, 0)$ and a morphism f to the class of ${}^t\Gamma_f$. Define the m-th Tate twist ($m \in \mathbb{Z}$) of a motive $M = (X, p, a)$ as $M(m) := (X, p, a+m)$. Then every motive is a direct factor of $h(X)(j)$ for some X in $\mathcal{M}_k^E$ and some $j \in \mathbb{Z}$.

1.5 $\mathcal{M}_k^E$ is an additive rigid tensor category. Recall that a tensor category (or symmetry monoidal category) is a category $\mathcal{C}$ with a tensor law, i.e., a functor

$$\mathcal{C} \times \mathcal{C} \to \mathcal{C} \quad , \quad (A, B) \mapsto A \otimes B,$$

together with a unit object 1 and associativity and commutativity constraints satisfying certain (obvious) compatibilities [6]. A tensor category $\mathcal{C}$ is rigid, if internal Hom's and in particular duals $A^\vee = \underline{Hom}(A, 1)$ exists, and if the natural morphism

$$A^\vee \otimes B \to \underline{Hom}(A, B)$$

is an isomorphism for all A, B. The tensor product in $\mathcal{M}_k^E$ is defined by $(X, p, a) \otimes (Y, q, b) := (X \times Y, p \times q, a + b)$.

Remark 1.6 There is a notion of an ideal I in an additive category $\mathcal{A}$ (cf. [16]): it is given by subgroups $IHom(A, B) \subseteq Hom(A, B)$ in all Hom sets such that $g \circ f \in I$ for composable morphisms f and g whenever one of them is in I. This corresponds to a usual ideal if one formally regards the sum of all Hom sets as a ring. If $\mathcal{A}$ is a rigid additive tensor category, let us call an ideal I in $\mathcal{A}$ a tensor ideal if $f \otimes g \in I$ for all morphisms f and g such that one of them is in I. Then one can reformulate lemma 1.3 as follows: There is a 1-1-correspondence between adequate equivalence relations and tensor ideals in $\mathcal{CHM}_k$.

2. The Bloch-Beilinson filtration

From now on we shall always consider cycle and Chow groups with $\mathbb{Q}$-coefficients, without changing the notation.

2.1 Bloch and Beilinson have proposed the existence of filtrations

$$F^0 \supseteq F^1 \supseteq F^2 \supseteq \dots$$

on all Chow groups of all smooth projective varieties such that

(a) $F^0 CH^j(X) = CH^j(X), F^1 CH^j(X) = HCH^j(X)$ (cycles homologically equivalent to zero).

(b) $F^r . F^s \subset F^{r+s}$ under intersection product.

(c) F^r is respected by f_*, f^* for morphisms $f : X \to Y$.
(d) (note that by (a) - (c), algebraic correspondences respect each F^r, and the action on $Gr_F^\cdot CH^j(X)$ factors through homological equivalence.) $Gr_F^\nu CH^j(X)$ only depends on the Grothendieck motive $h^{2j-\nu}(X)$.
(e) $F^\nu CH^j(X) = 0$ for $\nu \gg 0$.

Remarks 2.2 (1)Property (d) implicitly requires the algebraicity of the Künneth components $\Delta(r,s) \in H^r(X) \otimes H^s(X)$ of the class of the diagonal, $cl(\Delta) \in H^{2\dim(X)}(X \times X)$, and means

$$\Delta(r,s) \ \mid \ Gr_F^\nu CH^j(X) = \begin{cases} id, & s = 2j - \nu, \\ 0, & \text{else.} \end{cases}$$

The algebraicity of the $\Delta(r,s)$ is the standard conjecture $C(X)$, cf. [18].

(2)By definition, the F^ν define adequate equivalence relations - if they exist.

(3)A Bloch-Beilinson filtration is unique if it exists [13, 5.7].

(4)(cf. [13, 2.2]) Grothendieck's standard conjecture $B(X)$ would imply that the $\Delta(r,s)$ are algebraic and that one has

(f') $F^{j+1}CH^j(X) = 0$.

Recall that $B(X)$ postulates that the inverses of the Lefschetz isomorphisms

$$L^r : H^{\dim X - r}(X) \xrightarrow{\sim} H^{\dim X + r}(X)$$

are given by algebraic correspondences (acting on the cohomology).

(5)Most optimistically, one expects: There exists a suitable category $\mathcal{MM}_k$ of mixed motives over k such that "Beilinsons's formula" holds:

$$Gr_F^\nu CH^j(X) = Ext^\nu_{\mathcal{MM}_k}(1, h^{2j-\nu}(X)(j))$$

where $1 = h(pt)$ is the unit (or trivial) motive. This is explained in more detail in [13]; some more discussion can be found in section 7 below.

(6)Let $k = \mathbb{C}$. If the Hodge structures realization functor

$$\mathcal{MM}_k \to \text{Mixed Hodge structures}$$

is fully faithful, then $F^2 =$ Abel-Jacobi equivalence, i.e.,

$$F^2CH^j(X) = CH^j(X)_{AJ} := \ker\left(HCH^j(X) \xrightarrow{cl^{(1)}} \frac{H^{2j-1}(X, \mathbb{C})}{H^{2j-1}(X, \mathbb{Q}(j)) + F^2} \right)$$

where $cl^{(1)}$ is the $\mathbb{Q}$-rational Abel-Jacobi map.

2.3 In the remaining part of the paper, we want to address the following questions.

(1) Can one give a simple (conjectural) description of F^ν?
(2) Can one expect Beilinsons's formula?
(3) What is the relationship between F^ν and other adequate equivalence relations, like A, F^ℓ_c, or the coniveau filtration $N.$ (recalled below)?
(4) Is F^ν generated by products from F^1?

3. Products of adequate equivalence relations

The following definition and the results in this section are taken from a paper of Hiroshi Saito [22].

Definition 3.1 *For adequate equivalence relations E and E', define the equivalence relation $E * E'$ by*

$\alpha \in E * E'CH(X)$
$\Leftrightarrow$ *α is sum of cycles of the form $(pr_X)_*(\alpha_1 \cdot \alpha_2)$, where $\alpha_1 \in ECH(T \times X)$ and $\alpha_2 \in E'CH(T \times X)$ for some smooth, projective T*
$\Leftrightarrow$ *there exists T smooth, projective such that α is in the image of*

$$\begin{array}{ccc} ECH(T \times X) \otimes E'CH(T \times X) & \overset{intersection}{\longrightarrow} & CH(T \times X) \\ & & \downarrow (pr_X)_* \\ & & CH(X). \end{array}$$

Then $E * E'$ is again adequate. We skip the proof of this and of the equivalence of the two conditions [22]. Moreover, one has

Lemma 3.2 *The following conditions are equivalent for adequate equivalence relations E, E' and E'':*

(a) $x \in ECH(X), y \in E'CH(X) \Rightarrow x.y \in E''CH(X)$
(b) $x \in ECH(X), y \in E'CH(Y) \Rightarrow x \times y \in E''CH(X \times Y)$
(c) $E * E' \subset E''$

(Where by $E \subset E'$ we mean $ECH(X) \subset E'CH(X)$ for all X, i.e., E is finer than E'). Hence: $E * E'$ is the smallest adequate equivalence relation E'' satisfying (a) (or (b)).

There are several operations for adequate equivalence relations which resemble the case of ideals in a (commutative) ring (In fact, they correspond to the usual operations via the interpretation in remark 1.6). For adequate equivalence relations E, E' define $E + E'$ and $E \cap E'$ by

$$\begin{aligned} (E + E')CH(X) &= ECH(X) + E'CH(X) \\ (E \cap E')CH(X) &= ECH(X) \cap E'CH(X). \end{aligned}$$

Then one has formulae like

$$E * E' = E' * E \subset E \cap E'$$

$$(E * E') * E'' = E * (E' * E'')$$
$$E * (E' + E'') = E * E' + E * E''.$$

Let $E^{*\ell} = E * \ldots * E$ (ℓ times). Then

$$I = E^{*0} \supset E^{*1} \supset E^{*2} \supset \ldots,$$

where I is the trivial equivalence relation ($ICH(X) = CH(X)$ for all X).

Example 3.3 $A^{*\ell} = F_c^\ell$ is the ℓ-cubical equivalence, i.e., $(F_c^1)^{*\ell} = F_c^\ell$.

The following remarkable result should be compared with the contents of the following sections.

Theorem 3.4 *([22, 6.2, 6.4]) Let $k = \mathbb{C}$. Then*

$$\begin{aligned}(A * H)CH^j(X) &= ACH^j(X) \cap H^{*2}CH^j(X) \\ &= Ker(ACH^j(X) \xrightarrow{cl^1} J^j(X)_{\mathbb{Q}}).\end{aligned}$$

Recall that we use $\mathbb{Q}$-coefficients throughout. For $\mathbb{Z}$-coefficients the result remains true up to finite groups!

4. Bloch-Beilinson filtration and products

The following observation can be regarded as an answer to questions 2.3(1) and (4). If we use cohomology or homological equivalence H, the choice of some fixed Weil cohomology theory is understood.

Theorem 4.1 *Assume that the Bloch-Beilinson filtration $F^\cdot$ exists. If the standard conjecture $B(X)$ holds for the smooth, projective variety X, then*

$$F^\nu = (F^1)^{*\nu} = H^{*\nu}$$

on $CH(X)$ for all $\nu \geq 0$.

Remarks 4.2 (1) The formula $F^\nu = (F^1)^{*\nu}$ shows in which (non-trivial) way F^ν is generated by products from $F^1 = H$: via adding parameter varieties T and taking pushforwards to X.

(2) To show that $H^{*\nu}CH^j(X) = 0$ for $\nu \gg 0$ (viz., $\nu \geq j + 1$), one would like to bound the dimension of the required parameter varieties T.

(3) Let $F^\cdot$ be any descending filtration defined on the Chow groups of all smooth, projective varieties satisfying the properties (b) and (c) of 2.1. Then F^ν defines an adequate equivalence relation for every ν, and $F^r * F^s \subseteq F^{r+s}$ for all r and s, so that $(F^1)^{*\nu} \subseteq F^\nu$ for every $\nu > 0$. If in addition one has the property

(a') $F^0CH^j(X) = CH^j(X)$ and $F^1CH^j(X) \supseteq HCH^j(X)$,

then $H^{*\nu} \subseteq F^\nu$ for all $\nu \geq 0$. In other words, the filtration $H^{*\nu}$ is the smallest filtration satisfying (a') and 2.1 (b) and (c) (or, equivalently, the properties 2.1 (a),(b) and (c)).

In particular, the inclusion $H^{*\nu} = (F^1)^{*\nu} \subseteq F^\nu$ is obvious for the Bloch-Beilinson filtration without assuming $B(X)$. We treat the converse inclusion in a somewhat more general (and less conditional) setting. The following definition is due to Shuji Saito [24].

Definition 4.3 *A descending filtration $F^\cdot$ defined on the Chow groups of all smooth, projective varieties is called a filtration of Bloch-Beilinson type, if the following conditions hold.*

(a) $F^0CH^j(X) = CH^j(X)$.
(b) $F^r.F^s \subseteq F^{r+s}$ *under intersection product.*
(c) F^r *is respected by* f_* *and* f^* *for morphisms* $f : X \to Y$.
(d) Let $\gamma \in Corr^{j-i}(Y, X)$ *be a correspondence. If the induced map*

$$\gamma^{2j-\nu} : H^{2i-\nu}(Y) \to H^{2j-\nu}(X)$$

is zero, then so is the map

$$Gr_F^\nu\gamma : Gr_F^\nu CH^i(Y) \to Gr_F^\nu CH^j(X).$$

In other words, $F^\cdot$ satisfies 2.1 (b) and (c) and some weaker versions of 2.1 (a) and (d), while 2.1 (e) is omitted. The modified version of (d) avoids the standard conjecture $C(X)$; on the other hand, in the presence of $C(X)$, the conditions 2.1 (d) and 4.3 (d) are clearly equivalent. Since 2.1 (e) is omitted, the filtration $F^\cdot$ gives rise to a possibly non-zero adequate equivalence relation D_F defined by

$$D_FCH^j(X) = \cap_{\nu>0}\, F^\nu CH^j(X),$$

and to assoiciated cycle groups modulo D_F-equivalence

$$CH_F^j(X) := CH^j(X)/D_FCH^j(X).$$

Remarks 4.4 (a) One has $F^\nu CH^j(X) = D_FCH^j(X)$ for $\nu > 2j$, cf. [24, Remark (1-3)].

(b) Saito defined particular filtrations of Bloch-Beilinson type F_{BM} and F_B in [23] and [24]. Because $F^1 = H$ for these, one has $H^{*\nu} \subset F^\nu$ for them by remark 4.2 (3), and, as observed in [24], in the presence of standard conjecture C they give the Bloch-Beilinson filtration if and only if the associated D_F-equivalence is zero for them.

Since $D_F = 0$ for the Bloch-Beilinson filtration, the equality in 4.1 is now implied by the following result.

Theorem 4.5 *Let $F^\cdot$ be a filtration of Bloch-Beilinson type, and assume the standard conjecture $B(X)$ holds for the smooth projective variety X. Then*

$$F^\nu = (F^1)^{*\nu} \quad \text{for all } \nu \geq 0$$

for the induced filtrations on $CH^j_F(X)$.

This follows (in a more precise form) from lemma 4.6 and theorem 4.7 below.

Lemma 4.6 *Let E be an adequate equivalence relation, and define the filtration $E^{[\nu]}$ on Chow groups inductively by $E^{[1]} = E$ and*

$$E^{[\nu+1]} = \sum Im\ (\gamma : E^{[\nu]}CH(T) \to CH(X))$$

for all $\nu \geq 1$, where the sum is over all smooth, projective T and all correspondences $\gamma \in ECH(X \times T)$. Then $E^{[\nu]} = E^{*\nu}$.

Proof. By induction on ν. Obviously $E^{[\nu+1]} \subseteq E^{[\nu]} * E = E^{*(\nu+1)}$, the last equality holding by induction. Conversely, consider $(pr_X)_*(\alpha_1 \cdot \alpha_2) \in E^{[\nu]} * ECH(X)$, with $\alpha_1 \in E^{[\nu]}CH(T \times X)$ and $\alpha_2 \in ECH(T \times X)$. Then

$$\alpha_1 \cdot \alpha_2 = ((\Delta_{T \times X})_*(\alpha_2))(\alpha_1)$$

where $\Delta_{T \times X} : T \times X \to T \times X \times T \times X$ is the diagonal. Hence

$$\begin{aligned} (pr_X)_*(\alpha_1 \cdot \alpha_2) &= (\Gamma_{pr_X} \circ (\Delta_{T \times X})_*(\alpha_2))(\alpha_1) \\ &\in E^{[\nu+1]}CH(X), \end{aligned}$$

because $\Gamma_{pr_X} \circ (\Delta_{T \times X})_*(\alpha_2) \in ECH(X \times T \times X)$. Here we have used the fact that

$$f_*(\gamma(\alpha)) = (\Gamma_f \circ \gamma)(\alpha)$$

for $\alpha \in CH(Z)$, a correspondence $\gamma \in CH(Z \times Y)$ and a morphism $f : Y \to X$, where $\Gamma_f \circ \gamma$ is the composition of correspondences.

Theorem 4.7 *Let $F^\cdot$ be a filtration of Bloch-Beilinson type, and let X be a smooth projective variety for which the standard conjecture $B(X)$ holds. Then, for $\nu \geq 1$ and $j \geq 0$ one has*

$$F^{\nu+1}CH^j(X) = \sum\ Im\ (\gamma : F^\nu CH^{j-1}(T) \to CH^j(X)) + D_F CH^j(X)$$

where the sum is over all irreducible smooth, projective varieties T and all correspondences $\gamma \in F^1CH^{dim X+1}(X \times T)$.

We need the following slight generalization of a result of Nori.

Proposition 4.8 *Let X be a smooth projective variety of pure dimension d, and let $\alpha \in CH_r(X)$ be an r-cycle, for some $r \geq 0$.*
(a) If $d \geq 2r + m$ for some integer $m > 0$, then there is a smooth linear space section $i : Y \hookrightarrow X$ of codimension m such that α lies in the image of

$$i_* : CH_r(Y) \to CH_r(X).$$

(b) If $\alpha \in HCH_r(X)$ and $d \geq 2r + 1 + m$, then α lies in the image of

$$i_* : HCH_r(Y) \to HCH_r(X).$$

(c) Let $F^\cdot$ be a filtration of Bloch-Beilinson type and assume that $\alpha \in F^\nu CH^j(X)$ for some $\nu \geq 0$ with $d \geq 2r + \nu + m$. Assume moreover that $F^\cdot$ satisfies the following Lefschetz condition $L(X,Y,\mu)$ for $\mu = 0, 1, \ldots, \nu - 1$.

$L(X,Y,\mu)$: The map $i_* : Gr_F^\mu CH^{j-m}(Y) \to Gr_F^\mu CH^j(X)$ is injective for all integers j with $2j - \mu > \dim X + m$.

Then α lies in the image of

$$i_* : F^\nu CH_r(Y) \to F^\nu CH_r(X).$$

Proof. We follow the proof of [21, Proposition5.3], where Nori proves (a) and (b). If $d > 2r - 2$, one can find a cycle α' rationally equivalent to α with smooth prime components, by a result of Kleiman [17] (recall that we use Chow groups with rational coefficients). If $d \geq 2r + m$, one shows as in [21] that α' is supported on some smooth linear space section $i : Y \hookrightarrow X$ of codimension m. Under the assumptions of (c), the map

$$i_* : Gr_F^\mu CH^{d-r-m}(Y) \to Gr_F^\mu CH^{d-r}(X)$$

is injective for $\mu = 0, 1, \ldots, \nu - 1$, because $2(d - r) - \mu > d + m$. Hence $\alpha' \in F^\nu CH^{d-r-m}(Y)$ which proves (c).

Remarks 4.9 (1) If $F^1 = H$, then property L(X,Y,0) holds by hard Lefschetz. Letting $F^\nu = H$ for all $\nu \geq 1$, we obtain (b).

(2) If the standard conjecture $B(X)$ holds, then property L(X,Y,μ) holds for all filtrations of Bloch-Beilinson type, all linear space sections $Y \hookrightarrow X$ and all $\mu \geq 0$. In fact, by induction on m it suffices to consider the case $m = 1$, i.e., of a hyperplane section $i : Y \hookrightarrow X$. (Note that $B(X)$ implies $B(Y)$, cf. [18].) By $B(X)$ there is a correspondence $\lambda \in Corr^1(Y,X)$ such that $\lambda^{2(j-1)-\mu}$ is an inverse of the Lefschetz isomorphism

$$i_*^{2j-\mu} : H^{2j-2-\mu}(Y) \xrightarrow{\sim} H^{2j-\mu}(X) \qquad (2j - \mu - 1 > d).$$

By condition 4.3 (d), the map i_* in $\mathrm{L}(X,Y,\mu)$ has an inverse induced by λ.

(3) For a prime $\ell \neq char(k)$, let

$$cl_\ell : CH^j(X) \longrightarrow H^{2j}_{cont}(X, \mathbb{Q}_\ell(j))$$

be the cycle map, and let $F^\nu_\ell H^{2j}(X, \mathbb{Q}_\ell(j))$ be the canonical descending filtration associated to the Hochschild-Serre spectral sequence

$$E_2^{r,s} = H^r_{cont}(G_k, H^s(\overline{X}, \mathbb{Q}_\ell(j)) \Longrightarrow H^{r+s}_{cont}(X, \mathbb{Q}_\ell(j))$$

(so that $F^\nu_\ell / F^{\nu+1}_\ell = E_\infty^{\nu,2j-\nu}$). Then $F^\nu_\ell CH^j(X) := (cl_\ell)^{-1}(F^\nu_\ell)$ defines a filtration of Bloch-Beilinson type with F^1_ℓ = homological equivalence for ℓ-adic cohomology, which satisfies property $\mathrm{L}(X,Y,\mu)$ for all linear space sections $Y \subset X$ and all $\mu \geq 1$.

Proof of Theorem 4.7. We show by induction on $\dim X$ that

$$F^{\nu+1}CH^j(X) = \sum \mathrm{Im}\,(\gamma : F^\nu CH^{j-1}(T) \to CH^j(X)) + F^{\nu+2}CH^j(X).$$

for all $\nu \geq 1$. Then the claim follows by descending induction on ν,in view of remark 4.4 (a). The case $\dim X = 0$ is trivial; so let X be irreducible of dimension $d \geq 1$. Let $\nu \geq 1$, and let $\alpha \in F^{\nu+1}CH^j(X)$.

If $d > 2j - \nu - 1$, consider a smooth hyperplane section $i : Y \hookrightarrow X$. By $B(X)$ the injective morphism

$$i^* : H^{2j-\nu-1}(X) \to H^{2j-\nu-1}(Y)$$

has a left inverse induced by a correspondence λ. Hence, as in remark 4.8 (2), the same is true for the map

$$i^* : Gr_F^{\nu+1}CH^j(X) \to Gr_F^{\nu+1}CH^j(Y).$$

We conclude that α lies in the image of $F^{\nu+1}CH^j(Y)$ under λ modulo $F^{\nu+2}CH^j(X)$, and proceed by induction assumption on Y, as $B(Y)$ holds [18].

Hence we may assume that $d < 2j - \nu$. Then $d > 2(d-j)+\nu$, and by 4.8 (c) and 4.9 (2) one finds a linear space section $i : Y \hookrightarrow X$ of codimension $m = d - (2(d-j)+\nu) > 0$ such that α is in the image of

$$i_* : F^\nu CH^{j-m}(Y) \to CH^j(X).$$

By the following proposition, $\alpha \in \gamma'(\beta) + \gamma''(\beta') + D_F CH^j(X)$ for correspondences $\gamma', \gamma'' \in CH^d(Y \times X)$ and elements $\beta, \beta' \in CH^{j-m}(Y)$ with $\gamma \in F^1$, $\beta \in F^\nu$, and $\beta' \in F^{\nu+1}$, and we proceed by induction on $\dim X$.

Proposition 4.10 *Let X and Y be smooth projective varieties such that the standard conjectures $C(X)$ and $C(Y)$ hold. For integers $\mu,\nu,i,j \geq 0$ let $\alpha \in F^{\nu+\mu}CH^j(X)$, $\beta \in F^{\nu}CH^i(Y)$, and $\gamma \in CH^{\dim Y+j-i}(Y \times X)$ with $\alpha = \gamma(\beta)$ for the induced map*

$$\gamma : CH^i(Y) \to CH^j(X).$$

Then $\alpha = \gamma'(\beta) + \gamma''(\beta') + \alpha'$ for elements $\gamma' \in F^{\mu}CH^{\dim Y+j-i}(Y \times X)$, $\gamma'' \in CH^{\dim Y+j-i}(Y \times X)$, $\beta' \in F^{\nu+1}CH^i(Y)$, and $\alpha \in D_F CH^j(X)$.

Both for clarity and future reference, we will prove this result in a more general form (proposition 4.13 (c) below). We start with the following general setup. Let E be an adequate equivalence relation, and consider motives modulo E. The covariant functor

$$CH_E := Hom(1,-) \ : \ \mathcal{M}_k^E \longrightarrow (\mathbb{Q}\text{-vector spaces}) \tag{8}$$

defines Chow groups modulo E for every motive. One has $CH_E(h(X)(j)) = Hom_{\mathcal{M}_k^E}(1, h^E(X)(j)) = CH_E^j(X)$ for $X \in \mathcal{V}_k$ and $j \in \mathbb{Z}$. In the following assume that E is finer than homological equivalence. Then we have canonical covariant functors for all integers i,

$$H^i : \mathcal{M}_k^E \longrightarrow (K\text{-vector spaces}) \tag{9}$$

with $H^i((X,,p,m)) = pH^{i+2m}(X)$, where $K = H^0(Speck)$ is the coefficient field of $H^{\cdot}$.

Definition 4.11 *(a) A motive M is pure of weight w, if $H^i(M) = 0$ for $i \neq w$.*
(b) M has a weight decomposition, if there is a decomposition

$$M = \oplus_i M^i$$

with M^i pure of weight i. We abbreviate this property by $C^E(M)$.

The following properties are clear from the properties of a Weil cohomology theory.

Lemma 4.12 *(a) The Grothendieck motive $h(X) \in \mathcal{M}_k$ associated to a smooth, projective variety X has a weight decomposition iff the Künneth components of the cohomology class of the diagonal Δ_X are algebraic, i.e., iff the standard conjecture $C(X)$ holds. The weight decomposition of a Grothendieck motive is unique if it exists (i.e., the associated idempotents in $End(M)$ are unique).*
(b) The Chow motive $h^R(X) \in \mathcal{CHM}_k$ asociated to X has a weight decomposition iff X has a Chow-Künneth decomposition in the sense of [20]. A

weight decomposition of a Chow motive is not in general unique.
(c) If M has a weight decomposition $M = \oplus M^i$ and if $m \in \mathbb{Z}$, then $M(m)$ has a weight decomposition with $M(m)^i = M^{i+2m}(2m)$.
(d) If M has a weight decomposition $M = \oplus M^i$, then the dual $M^\vee = \underline{Hom}(M,1)$ has a weight decomposition, with $(M^\vee)^i = (M^{-i})^\vee$.
(e) If M and N have weight decompositions $M = \oplus M^i$ and $N = \oplus N^j$, respectively, then $M \otimes N$ and $\underline{Hom}(M,N)$ have weight decompositions with

$$\begin{aligned} (M \otimes N)^i &= \bigoplus_{p+q=i} M^p \otimes N^q \\ \underline{Hom}(M,N)^i &= \bigoplus_{q-p=i} \underline{Hom}(M^p, N^q). \end{aligned}$$

Following [24], we call a filtration of Bloch-Beilinson type admissible, if the associated D_F-equivalence is finer than homological equivalence. For such a filtation $F^\cdot$ we denote the induced filtration on the Chow groups $CH_F(-)$ again by $F^\cdot$, and by 4.3 (a) - (c) the functor $CH_F := CH_{D_F}$ induces functors

$$F^\nu CH_F \ , \ Gr_F^\nu CH_F \ : \ \mathcal{M}_k^{D_F} \to (\mathbb{Q}\text{-vector spaces}).$$

Explicitely one has $F^\nu CH_F(M) = pF^\nu CH_F^m(X)$ and $Gr_F^\nu CH_F(M) = pGr_F^\nu CH_F^m(X) = pGr_F^\nu CH^m(X)$ for $M = (X,p,m)$.

Proposition 4.13 *Let $F^\cdot$ be an admissible filtration of Bloch-Beilinson type, and consider motives modulo D_F-equivalence, i.e., in $\mathcal{M}_k^{D_F}$.*
(a) Then a motive M has a weight decomposition if and only if this holds for the associated Grothendieck motive $\overline{M}$. If $M = \oplus_i M^i$ is a weight decomposition, then

$$F^\nu CH_F(M) = \bigoplus_{i \le -\nu} CH_F(M^i).$$

(b) If $M = \oplus_i M^i$ and $N = \oplus_j N^j$ are weight decompositions, then

$$Hom(N^j, M^i) = 0 \quad \text{for } j < i$$

and

$$F^\nu Hom(N,M) = \bigoplus_{i-j \le -\nu} Hom(N^j, M^i)$$

via the isomorphisms $Hom(N,M) = Hom(1, N^\vee \otimes M) = CH_F(N^\vee \otimes M)$.
(c) Let $\gamma : N \to M$ be a morphism in $\mathcal{M}_k^{D_F}$, where M and N have weight decompositions. Assume we have an element $\beta \in F^\nu CH_F(N)$ such that $\alpha := \gamma(\beta) \in F^{\nu+\mu} CH_F(M)$ for some $\mu, \nu \ge 0$. Then there are some elements $\gamma' \in F^\mu Hom(N,M), \gamma'' \in Hom(N,M)$ and $\beta' \in F^{\nu+1} CH_F(N)$ such that

$$\alpha = \gamma'(\beta) + \gamma''(\beta').$$

We use the following lemma.

Lemma 4.14 *Let $F^\cdot$ be a filtration of Bloch-Beilinson type, and consider motives in $\mathcal{M}_k^{D_F}$.*
(a) Let $f : N \to M$ be a morphism of motives, and let $f^\mu : H^\mu(N) \to H^\mu(M)$ and $Gr_F^\mu(f) : Gr_F^\mu CH_F(N) \to Gr_F^\mu CH_F(M)$ be the induced maps. Then

$$f^{-\nu} = 0 \quad \Rightarrow \quad Gr_F^\nu(f) = 0.$$

(b) If $F^\cdot$ is admissible and the Grothendieck motive $\overline{M}$ associated to M has a weight decomposition, then $F^1 CH_F(M) = HCH_F(M)$.

Proof. For (a) let $M = (X, p, j)$ and $N = (Y, q, i)$, and $g = p \circ g \circ q$ with $g \in Corr_{D_F}^{j-i}(Y, X)$. Then one has a commutative diagram

$$\begin{array}{ccccccc} H^{2i-\nu}(Y) & \stackrel{q}{\to} & H^{2i-\nu}(Y) & \stackrel{g^{2j-\nu}}{\to} & H^{2j-\nu}(X) & \stackrel{p}{\to} & H^{2j-\nu}(X) \\ & \searrow & \uparrow & & \downarrow & \nearrow & \\ H^{-\nu}(N) & = & qH^{2i-\nu}(Y) & \stackrel{f^{2j-\nu}}{\to} & pH^{2j-\nu}(X) & = & H^{-\nu}(M), \end{array}$$

in which the composition in the bottom line is $f^{-\nu}$. If now $f^{-\nu} = 0$, then $(p \circ g \circ q)^{2j-\nu}$, the composition in the top line, is zero, and by 4.3 (d) the map

$$Gr_F^\nu(p \circ g \circ q) : Gr_F^\nu CH^i(Y) \to Gr_F^\nu CH^j(X)$$

is zero. A diagram analogous to the one above now shows that $Gr_F^\nu(f) : Gr_F^\nu CH_F(N) \to Gr_F^\nu CH_F(M)$ is zero.

For (b) let $id_{\overline{M}} = \sum \pi_i$ be a decomposition of the identity in $End(\overline{M})$ into a sum of orthogonal idempotents such that π_i corresponds to the factor M^i of weight i, and choose any elements $\tilde{\pi}_i \in End(M)$ lifting the π_i such that $id_M = \sum \pi_i$. Then it follows from (a) that on $Gr_F^\nu CH_F(M)$, $\pi_{-\nu}$ acts as the identity, and π_i acts as zero for $i \neq -\nu$. Going through the definitions one sees that the cycle map for M is a map

$$cl : CH_F(M) \to CH_H(M) \hookrightarrow H^0(M)$$

compatible with correspondences, so that $\pi_i = \delta_{i,0} \cdot id$, where $\delta_{i,0}$ is the Kronecker symbol. A descending induction on ν, starting with $\nu \gg 0$ where $F^\nu CH_F(M) = 0$ (cf. 4.4 (a)), now shows that $F^\nu CH_F(M) \subseteq HCH_F(M)$ for $\nu \geq 1$. On the other hand, noting that $CH_F(M) = Hom(1, M)$, we deduce also from (a) that $Gr_F^0 CH_F(M) \to H^0(M)$ is injective, and we have proved (b).

Proof of Proposition 4.13. If $\overline{M}$ has a weight decomposition, so has the Grothendieck motive associated to $M^\vee \otimes M$. Hence we have $HCH_F(M^\vee \otimes M) = F^1CH_F(M^\vee \otimes M)$ by 4.14 (b). By 4.3 (b) and 4.4 (a) this is a nilpotent ideal in $CH_F(M^\vee \otimes M) = End(M)$. This implies that one can lift any orthogonal set of idempotents, cf. [13, 5.4], and hence the first claim in (a). For the second claim it suffices to show

$$Gr_F^\nu CH_F(M^i) = 0 \quad \text{for } i \neq -\nu.$$

For this it suffices in turn to show

$$Gr_F^\nu CH_F(M) = 0 \quad \text{if } H^{-\nu}(M) = 0.$$

But this follows from 4.14 (a) applied to the identity of M.

Now (b) follows from (a) and 4.11 because

$$Hom(N, M) = Hom(1, \underline{Hom}(N, M)) = Hom(1, N^\vee \otimes M)$$

so that

$$Gr_F^\nu CH_F(\underline{Hom}(N, M)) = \bigoplus_{i-j=-\nu} Hom(N^j, M^i).$$

Note that the left hand side is zero for $\nu < 0$ by definition.

As for (c), let $M = \oplus M^i$ and $N = \oplus N^j$ be Künneth decompositions, and write $\beta = \sum_{j \leq -\nu} \beta_j$ with $\beta_j \in CH_F(N^j)$ according to (a), and $\gamma = (\gamma_{ij})$ as matrix with $\gamma_{ij} \in Hom(N^j, M^i)$. By (b) we have $\gamma_{ij} = 0$ for $j < i$, and by (a) and assumption on $\alpha = \gamma(\beta)$ we have $\gamma_{ij}(\beta_j) = 0$ for all $i > -\mu - \nu$. Let

$$\gamma' = \sum_{j \geq i+\mu} \gamma_{ij} \quad \text{and} \quad \gamma'' = \sum_{j < i+\mu \ , \ i \leq -\mu-\nu} \gamma_{ij}.$$

Then $\gamma' \in F^\mu Hom(N, M)$ by (b), and

$$\gamma(\beta) = \gamma'(\beta) + \gamma''(\beta')$$

with $\beta' = \sum_{j<-\nu} \beta_j \in F^{\nu+1}CH(N)$ as claimed.

Remarks 4.15 (a) Precursors of theorem 4.1 - relying on Beilinson's formula - were announced in [14] and at the Banff conference. The stronger result proved here has in fact implications for Beilinson's formula, cf. section 7 below.

(b) Define a filtration on Chow groups recursively by $F^0 = I, F^1 = H$, and

$$F^{\nu+1}CH^j(X) = \sum \text{ Im } (\gamma : F^\nu CH^{j-1}(T) \to CH^j(X))$$

for $\nu \geq 1$, where the sum is over all irreducible smooth, projective varieties T and all correspondences $\gamma \in F^1CH^{dimX+1}(X \times T)$. Then $F^{j+1}CH^j(X) = 0$ by induction on j, and by 4.7 this filtration should be the Bloch-Beilinson filtration. The problem is to show that this is an adequate equivalence relation to start with.

5. Filtrations by niveau and coniveau

For any equivalence relation one can define an associated niveau filtration:

Definition 5.1 *For an adequate equivalence relation E and an integer r define*

$$\langle E \rangle_r CH^j(X) = \sum Im(\gamma : ECH_r(T) \to CH^j(X))$$

where the sum is over all smooth, projective varieties T and all correspondences $\gamma \in CH^{j+r}(T \times X)$.

Remarks 5.2 (1) For $r = 0$ this definition is due to H. Saito [22].

(2) The equivalence relations $A_r := \langle H \rangle_r$ were defined by M. Nori [21]. One has $A_0 = \langle H \rangle_0 = A$ (algebraic equivalence), and $A_0 \subseteq A_1 \subseteq A_2 \subseteq \ldots \subseteq H$.

(3) Like in [21] one shows that, for any adequate E, one gets a chain of adequate equivalence relations $\langle E \rangle_0 \subseteq \langle E \rangle_1 \subseteq \ldots \subseteq E$.

Using his result recalled in 4.8 (b), Nori showed:

Proposition 5.3 *(cf. [21, Proposition 5.3]) One has*

$$A_r CH^j(X) = \sum Im(\gamma : HCH_r(T) \to CH^j(X))$$

where the sum is over all smooth, projective varieties T of pure dimension $2r+1$ and all correspondences $\gamma \in CH^{j+r}(T \times X)$.

We deduce the following generalization from proposition 4.8 (c).

Proposition 5.4 *Let $F^\cdot$ be a filtration of Bloch-Beilinson type, for a fixed Weil cohomology theory $X \mapsto H^*(X)$, and fix an integer $\nu \geq 1$. Assume that $F^\cdot$ satisfies the Lefschetz condition $L(Y, Z, \mu)$ of proposition 4.7 for all hyperplane sections $Z \hookrightarrow Y$ and all $\mu = 0, \ldots, \nu - 1$. Then, for any $r \geq 0$ one has*

$$\langle F^\nu \rangle_r CH^j(X) = \sum Im(\gamma : F^\nu CH_r(T) \to CH^j(X))$$

where the sum is over all smooth, projective varieties T of pure dimension $2r + \nu$ and all correspondences $\gamma \in CH^{j+r}(T \times X)$.

Proof. Let $\alpha = \gamma(\beta)$ for some $\alpha \in F^\nu CH_r(T)$ and some correspondence $\gamma \in CH^{j+r}(T \times X)$, where T is a smooth projective variety. We may assume that T is irreducible.

If $\dim T < 2r + \nu$, i.e., $\dim T = 2r + \nu - m$ with $m > 0$, consider $T' = \mathbb{P}^m \times T$ with the projection $p : T' \to T$ and the zero section $s : T \to T'$. Since $p_* s_*(\beta) = \beta$, we can replace (T, β, γ) by $(T', s_*(\beta), \gamma \circ p_*)$, where now $\dim T' = 2r + \nu$.

If $e := \dim T > 2r + \nu$, i.e., $e = 2r + \nu + m$ with positive m, then by 4.7 there is a smooth linear space section $i : T' \hookrightarrow T$ of codimension m such that β lies in the image of

$$i_* : F^\nu CH^{e-r-m}(T') \to F^\nu CH^{e-r}(T).$$

Let $\beta' \in F^\nu CH_r(T')$ with $i_*(\beta') = \beta$. Then we can replace (T, β, γ) by $(T', \beta', \gamma \circ i_*)$, where $dim T' = 2r + \nu$ as wanted.

We now consider coniveau filtrations. Fix a Bloch-Ogus theory on varieties over k

$$\begin{aligned} (Y \hookrightarrow X) &\mapsto H^i_Y(X, j) \quad (i, j \in \mathbb{Z}) \\ X &\mapsto H_a(X, b) \quad (a, b \in \mathbb{Z}) \ , \end{aligned}$$

i.e., a pair consisting of a cohomology theory with supports and a homology theory satisfying the axioms in [3]. For example, we may take

$$H^i_Y(X, j) = H^i_{Y,cont}(X, \mathbb{Q}_\ell(j)) \ , \quad H_a(X, j) = H^{cont}_a(X, \mathbb{Q}_\ell(b)) \tag{10}$$

(continuous étale cohomology and homology [7],[11]) for a prime ℓ invertible in k, or

$$H^i_{\overline{Y}}(\overline{X}, \mathbb{Q}_\ell(j)) \ , \quad H_a(\overline{X}, \mathbb{Q}_\ell(b)) \tag{11}$$

(geometric ℓ-adic étale cohomology and homology), where $\overline{X} = X \times_k \overline{k}$ for an algebraic closure k of k and $\ell \neq char\ k$). If $k = \mathbb{C}$, we may take

$$H^i_{Y(\mathbb{C})}(X(\mathbb{C}), \mathbb{Q}(j)) \ , \quad H_a(X(\mathbb{C}), \mathbb{Q}(b)) \tag{12}$$

(singular cohomology/Borel-Moore homology) where $\mathbb{Q}(j) = \mathbb{Q} \cdot (2\pi\sqrt{-1})^j$ is a certain Tate twist of coefficients which is important for Hodge theory and comparison isomorphisms, but will be less important for us. The coniveau filtration on cohomology is then defined as

$$N^p H^i(X, j) = \bigcup \text{ Im } (H^i_Y(X, j) \to H^i(X, j)) \tag{13}$$

where Y runs through the closed subschemes of codimension $\geq p$. It is descending, with $N^0 H^i(X, j) = H^i(X, j)$ and $N^p H^i(X, j) = 0$ for smooth

X and $i-2p<0$ by Poincaré duality (=purity) if we assume the following property (cf. [3, (7.7.1)])

$$\text{If } \dim X \leq d, \text{ then } H_a(X,b)=0 \text{ for } a>2d. \tag{14}$$

This property is satisfied for the cohomology theories (10), (11), (12) above.

Next we consider a coniveau filtration on Chow groups which was implicitly defined by Bloch and Ogus (cf. [3, (7.2.2)], where Ker (ζ_X^r) should read Ker (ζ_X^{r+1}), and the latter corresponds to our N_r). Recall that the axioms of loc. cit. provide canonical cycle maps

$$cl : Z_a(Y) \to H_{2a}(Y,a) \tag{15}$$

for every variety Y, where $Z_a(Y)$ denotes the group of algebraic cycles of dimension a on Y. If Y is a closed subscheme of a smooth variety X these induce cycle maps

$$cl_Y : Z_Y^j(X) \to H_Y^{2j}(X,2j), \tag{16}$$

where $Z_Y^j(X)$ is the group of codimension j cycles on X supported on Y. The image of a cycle α under the latter map is also called the local cycle class of α.

Definition 5.5 *For a smooth variety X let $N_pCH^j(X)=\{[\alpha]\in CH^j(X)|$ there is $Y\subsetneq X$ closed of codimension $\geq j-p$ with $|\alpha|\subseteq Y$ and α homologous to zero on $Y\}$. Here $|\alpha|$ is the support of a cycle α, and α is called homologous to zero on Y if $cl_Y(\alpha)=0$.*

Then $N.$ is an ascending filtration, with $N_0=0$ if the map

$$\mathbb{Z}\to H_{2d}(X,d)\cong H^0(X,0), 1\mapsto \eta_X \tag{17}$$

is injective for every variety X which is smooth and irreducible of dimension d. This property holds for the cohomology theories (10), (11), and (12).

Theorem 5.6 *Assume the following properties.*

(i) There are functional cupproducts

$$H_{Y_1}^{i_1}(X,j_1)\times H_{Y_2}^{i_2}(X,j_2)\to H_{Y_1\cap Y_2}^{i_1+i_2}(X,j_1+j_2)$$

for X smooth and proper and Y_1,Y_2 closed in X, which are compatible with the intersection of algebraic cycles on X via the cycle maps (16).

(ii) The cycle maps (15) are compatible with push-forward for proper maps.

(iii) The cycle maps (16) are compatible with pull-back for flat maps.

(These properties are satisfies for the theories (10), (11), (12)). Then for each p, N_p restricted to smooth, projective varieties gives an adequate equivalence relation.

Proof. We use the criterion 1.3 (iii). Let $pr_X : X \times Y \to X$ and $pr_Y : X \times Y \to Y$ be the projections. The inclusion $(pr_Y)_* N_p C H_a(X \times Y) \subseteq N_p C H_a(Y)$ follows from assumption (ii), because for a closed $Z \subseteq X \times Y$ one has $\operatorname{codim}_Y(pr_Y(Z)) \geq \operatorname{codim}_{X\times Y}(Z)$. Similarly, $pr_X^* N_p C H^j(X) \subseteq N_p C H^j(X \times Y)$ by (iii), because for a closed $W \subseteq X$ one has $\operatorname{codim}_X(W) = \operatorname{codim}_{X\times Y}(W \times Y)$. Finally consider cycles α and β of codimensions i and j on X, respectively, and assume that there is a closed $Z \subset X$ with codimension $i-p$ such that $|\alpha| \subseteq Z$ and the cycle class of α in $H_Z^{2i}(X,i)$ is zero. By the moving lemma we may assume that β intersects $|\alpha|$ and Z properly, so that $\operatorname{codim}_X(Z \cap |\beta|) = i-p+j$. By assumption, the cupproduct

$$H_Z^{2i}(X,i) \times H_{|\beta|}^{2j}(X,j) \to H_{Z\cap|\beta|}^{2(i+j)}(X, i+j)$$

maps $(0 = cl_Z(\alpha), cl_{|\beta|}(\beta))$ to $cl_{Z\cap|\beta|}(\alpha.\beta)$, so the latter is zero and $\alpha.\beta$ is in $N_p Z^{i+j}(X)$.

Next we identify N_1 for the theories (10), (11), and (12).

Lemma 5.7 *For the theories (11) and (12) one has $N_1 = A$ (algebraic equivalence).*

Proof. We generalize the proof of [3, (7.3)] where one can find a version with integral coefficients over fields of zero characteristic.

The inclusion $A \subseteq N_1$ can be proved as in loc.cit. Step 1 (cf. (18) below for a different proof). For the inclusion $N_1 \subseteq A$ we may assume that k is algebraically closed and that X is irreducible of dimension d, say. Let α be a cycle of codimension j on X, and let $Z \subseteq X$ be closed of pure codimension $j-1$ such that $|\alpha| \subseteq Z$ and the class of α in $H_Z^{2j}(X,j) \xrightarrow{\sim} H_{2d-2j}(Z, d-j)$ is zero. By deJong's resolution of singularities [4], there exists a proper, surjective, generically finite morphism $f : W \to Z$ with W smooth over k. Let $Z_1 \subseteq Z$ be of pure codimension 1 such that $|\alpha| \subseteq Z_1$, $Z - Z_1$ is smooth and $f : W - f^{-1}(Z_1) \to Z - Z_1$ is finite. We obtain a commutative exact diagram (omitting the 'Tate twists' $d-j$ in the notation)

$$\begin{array}{ccccc} H_{2d-2j+1}(W - f^{-1}(Z_1)) & \to & H_{2d-2j}(f^{-1}(Z_1)) & \to & H_{2d-2j}(W) \\ \downarrow f_* & & \downarrow f_* & & \downarrow f_* \\ H_{2d-2j+1}(Z - Z_1) & \to & H_{2d-2j}(Z_1) & \to & H_{2d-2j}(Z) \end{array}$$

where $d-j = \dim Z_1 = \dim W - 1$. The left-most f_* is surjective because our cohomology groups are $\mathbb{Q}$-vector spaces. This follows from the projection formula for the capproduct (cf. [3, (1.3.3)])

$$f_*(\eta_{W-f^{-1}(Z_1)} \cap f^* z) = f_* \eta_{W-f^{-1}(Z_1)} \cap z$$

for $z \in H^1(Z - Z_1, 1)$, together with the fact that $z \mapsto \eta_{Z-Z_1} \cap z$ is bijective and that $f_* \eta_{W - f^{-1}(Z_1)} \cap Z = m \cdot \eta_{Z-Z_1}$, where m is the degree of $W - f^{-1}(Z_1)$ over $Z - Z_1$ (cf. [3, (7.1.2)]). Via a diagram chase it suffices to show that homological and algebraic equivalence agree for divisors on the smooth, projective variety W (note that $H_{2(d-j)}(f^{-1}(Z_1), d-j) = H^2_{f^{-1}(Z_1)}(W, 1) = Z^1_{f^{-1}(Z_1)}(W) \otimes_{\mathbb{Z}} K$, where $K = \mathbb{Q}_\ell$ for (11) and $K = \mathbb{Q}$ for (12)). But this is classical.

Lemma 5.8 *For the theory (10) and a finitely generated field one has $N_1 = R$ (rational equivalence).*

Proof. Proceeding as in the proof of the previous lemma, we have to show that the kernel of the cycle map $Pic(W)_{\mathbb{Q}_\ell} \to H^2_{cont}(W, \mathbb{Q}_\ell(1))$ is trivial for smooth, projective varieties over k. This is known (cf. [11]).

Remark 5.9 It would be interesting to understand the filtration $0 = N_1 CH^j(X) \subseteq N_2 CH^j(X) \subseteq \ldots \subseteq N_j CH^j(X) = N_{j+1} CH^j(X) = \ldots = \mathrm{Ker}(CH^j(X) \to H^{2j}_{cont}(X, \mathbb{Q}_\ell(j)))$ better in the case of lemma 5.8. According to some optimistic conjectures, the last group is zero if k is finitely generated. Can one prove $N_i = N_{i+1}$ unconditionally in some cases?

Finally, we relate the niveau and coniveau filtration on Chow groups. As observed by Nori [21, Remark 5.4], proposition 5.3 implies the relation $A_r \subset N_{r+1}$. We want to reinterprete and generalize this.

Definition 5.10 *(a)For any adequate equivalence relation E define the associated coniveau filtration by*

$$\{E\}_p CH^j(X) = \sum Im(\gamma : ECH^p(Y) \to CH^j(X))$$

where the sum is over all smooth, projective varieties T and all correspondences $\gamma \in Corr^{j-p}(T \times X)$.
(b) Define $\tilde{N}_p = \{H\}_p$.

It is clear from this that one gets a chain $\{E\}_0 \subseteq \{E\}_1 \subseteq \ldots \subseteq E$ of adequate equivalence relations.

Lemma 5.11 *If X is equidimensional, then*

$$\{E\}_p CH^j(X) = \sum Im(g_* : HCH^p(Y) \to CH^j(X))$$

where the sum ranges over all morphisms $g : Y \to X$ with Y smooth, projective of dimension $\dim X - j + p$.

Proof. Let X be of pure dimension d, and let $\gamma \in Corr^{j-p}(T \times X)$ be as in 5.10 (a). We may assume that T is irreducible and that $\gamma = Z$ is a prime cycle, i.e., an irreducible subvariety of dimension $d - j + p$. By weak reslution of singularities [4] there is a smooth variety $\tilde{Z}$ and a surjective, projective, generically finite morphism

$$f : \tilde{Z} \to Z.$$

Let $p : \tilde{Z} \twoheadrightarrow Z \hookrightarrow T \times X \to T$ and $q : \tilde{Z} \twoheadrightarrow Z \hookrightarrow T \times X \to X$ be the natural projections. Then $m.\gamma = q_* \circ p^*$, where m is the generic degree of f. Thus Im γ = Im $q_* \circ p^*$, and the claim follows.

This description shows that $\tilde{N}_p \subseteq N_p$ so that

$$A_r \subseteq \tilde{N}_{r+1} \subseteq N_{r+1}. \tag{18}$$

where the first inclusion follows from 5.3. Proposition 5.4 implies the following generalization.

Proposition 5.12 *Under the assumptions of proposition 5.4, one has an inclusion*

$$\langle F^\nu \rangle_r \subseteq \{F^\nu\}_{r+\nu} \text{ for } \nu \geq 1.$$

Proof. By 5.4, any $\alpha \in \langle F^\nu \rangle_r CH^j(X)$ comes via a correspondence from an element $\beta \in F^\nu CH_r(T)$, where T is smooth projective of pure dimension $2r + \nu$. But then $\beta \in \{F^\nu\}_{r+\nu} CH^{r+\nu}(T)$, hence the claim for α because $\{F^\nu\}_{r+\nu}$ is adequate.

Remark 5.13 The inclusion $\tilde{N}_p \subseteq N_p$ is an equality if the following hypothesis is true:

(H) If $f : \tilde{Z} \to Z$ is a surjective, generically finite morphisms of irreducible projective varieties, with smooth $\tilde{Z}$, then the induced map

$$f_* : HCH^p(\tilde{Z}) \to \mathrm{Ker}(CH_{\dim Z - p}(Z) \to H_{2(\dim Z - p)}(Z, p))$$

is surjective.

One can show that this would follow from the Tate conjecture, or the existences of motivic sheaves [13]. Note that $f_* : CH^p(\tilde{Z}) \to CH^p(Z)$ is surjective (cf. the argument in [12, Lemma 7.6]).

6. Niveau and coniveau properties of the Bloch-Beilinson filtration

We now study the relation between the (conjectural) Bloch-Beilinson filtration $F^\cdot$, the niveau filtration $A.$, and the coniveau filtration $N.$ for the

geometric theories (11) or (12). In particular, we want to investigate the relation between $F^\cdot$ and algebraic equivalence $A = A_0 = N_1$.

Theorem 6.1 *Let $F^\cdot$ be a filtration of Bloch-Beilinson type, and assume that the smooth, projective variety X satisfies the standard conjecture $B(X)$. Then for all $\nu \geq 1$ one has*

$$F^\nu CH^j(X) = \langle F^\nu \rangle_{j-\nu} CH^j(X) + D_F^j(X),$$

where $D_F^j(X) := D_F CH^j(X) = \cap_{\nu>0} F^\nu CH^j(X)$. If $F^1 = H$, then

$$F^\nu CH^j(X) \subseteq A_{j-\nu} CH^j(X) + D_F^j(X) \subseteq N_{j-\nu+1} CH^j(X) + D_F^j(X).$$

Proof. For the first claim it suffices to show

$$F^\nu CH^j(X) = \langle F^\nu \rangle_{j-\nu} CH^j(X) + F^{\nu+1} CH^j(X) \tag{19}$$

for all $\nu \geq 1$. The required identity then follows by descending induction on ν, because $D_F^j(X) = F^{2j+1} CH^j(X)$ for a filtration of Bloch-Beilinson type (cf. 4.4 (a)) , and $\langle F^\nu \rangle_{j-\nu} \supseteq \langle F^{\nu+1} \rangle_{j-\nu+1}$. By $B(X)$ there are a smooth projective variety $Y_{2j-\nu}$ of dimension $2j-\nu$ and correspondences $\gamma_1 \in CH^{2j-\nu}(Y_{2j-\nu} \times X)$ and $\gamma_2 \in CH^d(X \times Y_{2j-\nu})$ such that the composition of the induced maps

$$H^{2j-\nu}(X) \xrightarrow{\gamma_2^{2j-\nu}} H^{2j-\nu}(Y_{2j-\nu}) \xrightarrow{\gamma_1^{2j-\nu}} H^{2j-\nu}(X)$$

is the identity map. (In fact, one may take a linear space section of dimension $2j-\nu$ for $Y_{2j-\nu}$ if $2j-\nu < \dim X$.) By 4.3 (d) the same is true for the composition

$$Gr_F^\nu CH^j(X) \xrightarrow{\gamma_2} Gr_F^\nu CH^j(Y_{2j-\nu}) \xrightarrow{\gamma_1} Gr_F^\nu CH^j(X).$$

Hence the last map γ_1 is surjective. But $F^\nu CH^j(Y_{2j-\nu}) = F^\nu CH_{j-\nu}(Y_{2j-\nu}) = \langle F^\nu \rangle_{j-\nu} CH^j(Y_{2j-\nu})$ for trivial reasons; so the equality (19) follows, because $\langle F^\nu \rangle_{j-\nu}$ is an adequate equivalence relation.

For the second claim we note that $\langle F^\nu \rangle_r \subseteq \langle F^1 \rangle_r = A_r$ for $\nu \geq 1$.

Corollary 6.2 *Assume the standard conjecture B holds. Then*

$$\{F^\nu\}_{r+\nu} = \langle F^\nu \rangle_r + \{D_F\}_{r+\nu}.$$

In particular, if $F^1 = H$, then

$$\tilde{N}_{r+1} = A_r + \{D_F\}_{r+1}.$$

Proof. By 5.12 and 4.9 (2) we already know that $\{F^\nu\}_{r+\nu} \subseteq \langle F^\nu\rangle_r$. Now let $\alpha \in \{F^\nu\}_{r+\nu}CH^j(X)$, for a variety X of pure dimension d. Then $\alpha = g_*(\beta)$ for a morphism $g : Y \to X$ with Y smooth projective of dimension $d-j+r+\nu$ and a cycle $\beta \in F^\nu CH^{r+\nu}(Y)$. By 6.1 and B(Y), $F^\nu CH^{r+\nu}(Y) = \langle F^\nu\rangle_r + D_F CH^{r+\nu}(Y)$; hence the result.

Theorem 6.3 *Let $F^\cdot$ be a filtration of Bloch-Beilinson type such that $F^1 = H$ and $F^2CH^1(C) = 0$ for smooth projective curves C. If the standard conjecture $C(X)$ holds, then*

$$A \cap F^\nu CH^j_F(X) = A * F^{\nu-1}CH^j_F(X).$$

Proof. The inclusion $A * F^{\nu-1} \subseteq A \cap F^\nu$ is obvious and holds on $CH^j(X)$ and without assuming $C(X)$. Conversely, let $\alpha \in ACH^j(X)$. Then there is a smooth projective curve C and a correspondence $\gamma \in CH^{j+1}(C \times X)$ such that α is in the image of

$$\gamma : HCH^1(C) = F^1CH^1(C) \to CH^j(X).$$

(This is classical, but cf. also 5.3.) By 4.10 there are correspondences $\gamma', \gamma'' \in CH^{j+1}(C \times X)$ with $\gamma' \in F^{\nu-1}$ and elements $\beta \in F^1CH^1(C), \beta' \in F^2CH^1(C)$ with

$$\alpha \in \gamma'(\beta) + \gamma''(\beta') + D_F CH^j(X).$$

This shows the claim, because $F^2CH^2_F CH^1(C) = 0$ and obviously $\gamma'(\beta) \in A * F^{\nu-1}CH^j(X)$.

We note that the assumption on $F^\cdot$ in 6.3 holds for the filtration $F^\cdot_\ell$ of 4.9 (3) if k is a finitely generated field. It also holds for the filtrations F_{BM} and F_B of [24], and for the Bloch-Beilinson filtration if it exists.

Corollary 6.4 *Assume that the Bloch-Beilinson filtration $F^\cdot$ exists, and let X be a smooth projective variety.*
(a) If the standard conjecture $C(X)$ holds, then

$$A \cap F^\nu = A * F^{\nu-1}.$$

(b) If the standard conjecture $B(X)$ holds, then

$$F^\nu CH^j(X) = \langle F^\nu\rangle_{j-\nu}CH^j(X) \quad \textit{for all } \nu \geq 0.$$

(c) Assume $B(Y)$ holds for all smooth projective varieties Y. Then

$$\langle F^\nu\rangle_r = \{F^\nu\}_{r+\nu} \quad \textit{for all } \nu, r \geq 0.$$

In particular, $A_r = \tilde{N}_{r+1}$ *for all* $r \geq 0$, *and*

$$F^\nu CH^j(X) \subseteq A_{j-\nu} CH^j(X) \subseteq N_{j-\nu+1} CH^j(X).$$

This is immediate from 6.1, 6.2, and 6.3, because by definition $D_F^j(X) = 0$ for the Bloch-Beilinson filtration. A special case ($\nu = j$) of 6.4 is the inclusion

$$F^j CH^j(X) \subseteq ACH^j(X), \tag{20}$$

i.e., cycles in the bottom part of the Bloch-Beilinson filtration are algebraically equivalent to zero. For $j = 2$ it suggests a positive answer to a question of Nori [21]: Nori asked if for a complex projective variety X all cycles in $CH^2_{AJ}(X)$, the kernel of the Abel-Jacobi map on codimenson 2 cycles, are alebraically equivalent to zero. This would follow from (20) and the expected equality $F^2 CH^j(X) = CH^j_{AJ}(X)$, cf. 2.2 (6). On the other hand Nori constructed cycles of codimension $j > 2$ which lie in $CH^j(X)_{AJ}$ but not in $ACH^j(X)$ [21]. This shows that (20) is in some sense sharp.

The general meaning of 6.4 (c) for the coniveau filtration might become clearer in a different numbering. Write

$$N^q CH^j(X) := N_{j-p} CH^j(X)$$

for the subgroup of cycles which are homologous to zero on some q-codimensional subscheme of X. Then 6.4 (c) implies

$$F^\nu CH^j(X) \quad \subseteq \quad N^{\nu-1} CH^j(X) \quad . \tag{21}$$

In particular, one should expect

$$CH^j_{AJ}(X) \quad \stackrel{?}{\subseteq} \quad N^1 CH^j(X) \tag{22}$$

for complex projective varieties (this was asked by H. Esnault).

We now derive further consequences of 6.4. For the rest of this section, assume the standard conjectures and the existence of the Bloch-Beilinson filtration $F^\cdot$.

Let X be a smooth, projective variety of pure dimension d. For each $i \geq 0$ and every $p \geq 0$ we define a submotive $N^p h^i(X)$ of the Grothendieck motive $h^i(X)$ by

$$N^p h^i(X) = \Sigma \text{ Im } (g_* : h^{i-2(d-e)}(Y)(-d+e) \to h^i(X))$$

where the sum is over all morphisms $g : Y \to X$ with Y smooth, projective of pure dimension $e \leq d - p$.

Remark 6.5 Applying our cohomology theory (11) or (12), the submotive $N^p h^i(X) \subset h^i(X)$ corresponds to $N^p H^i(X,\cdot) \subseteq H^i(X,\cdot)$. This follows from the fact that, for $Y \subset X$ closed and a proper surjective morphism $f : \tilde{Y} \to Y$ with $\tilde{Y}$ smooth of dimension e, the morphism induced by f,

$$\begin{array}{ccccc} H_{2d-i}(\tilde{Y}, d-j) & \to & \mathrm{Im}(H_{2d-i}(Y, d-j) & \to & H_{2d-i}(X, d-j)) \\ \| & & \| & & \| \\ H^{i-2d+2e}(\tilde{Y}, j-d+e) & \to & \mathrm{Im}(H^i_Y(X, j) & \to & H^i(X, j)) \end{array}$$

is surjective [12, 7.7]. Note that for a given Y there exists such an $f : \tilde{Y} \to Y$ with $e = \dim Y$ by [4].

Consider the covariant functor on Chow motives ((8) for $E = R$)

$$CH = Hom(1, -) : \mathcal{CHM}_k \longrightarrow (\mathbb{Q}\text{-vector spaces})$$

for which $CH(h^R(X)(j)) = CH^j(X)$. By definition of the Bloch-Beilinson filtration it induces a functor on Grothendieck motives

$$Gr^\nu_F CH : \mathcal{M}_k \longrightarrow (\mathbb{Q}\text{-vector spaces})$$

with $Gr^\nu_F CH(h(X)(j)) = Gr^\nu_F CH^j(X)$. If we assume Beilinson's formula 2.2 (5) then we have

$$Gr^\nu_F CH(M) = Ext^\nu_{\mathcal{MM}_k}(1, M^{-\nu})$$

for a motive M, where M^w is the factor of weight w of M, but we do not want to use this in the following. However we will use that by 2.1 (d) we have

$$Gr^\nu_F CH^j(X) = Gr^\nu_F CH(h(X)(j)) = Gr^\nu_F CH(h^{2j-\nu}(X)(j)). \tag{23}$$

Corollary 6.6 *Let $\nu \geq 1$. Via (23) one has*

$$\begin{aligned} Gr^\nu_F CH^j(N^{j-\nu-r} h^{2j-\nu}(X)(j)) &= (\langle F^\nu \rangle_r + F^{\nu+1}) CH^j(X)/F^{\nu+1} CH^j(X) \\ &\subseteq \frac{F^\nu \cap N_{r+1} + F^{\nu+1}}{F^{\nu+1}} CH^j(X). \end{aligned}$$

Proof. It suffices to show the equality and to consider X of pure dimension d, say. If α lies in the left hand side, then there is a morphism $g : Y \to X$ with Y smooth, proper of dimension $d - j + \nu + r$ such that α lies in the image of

$$\begin{array}{rccc} g_* : & Gr^\nu_F CH(h^{\nu-2r}(Y)(\nu+r)) & \to & Gr^\nu_F CH(h^{2j-\nu}(X)(j)) \\ & \| & & \| \\ & Gr^\nu_F CH^{\nu+r}(Y) & & Gr^\nu_F CH^j(X). \end{array}$$

Note that $\mathcal{M}_k$ is semi-simple by our assumptions. By theorem 6.1, we have $F^\nu CH^{\nu+r}(Y) \subseteq \langle F^\nu\rangle_r CH^{\nu+r}(Y) + F^{\nu+1}CH^{\nu+r}(Y)$; hence we obtain $\alpha \in (\langle F^\nu\rangle_r + F^{\nu+1})CH^j(X)/F^{\nu+1}CH^j(X)$ as wanted.

Conversely, if $\alpha \in \langle F^\nu\rangle_r CH^j(X)$, then $\alpha \in \{F^\nu\}_{\nu+r}CH^j(X)$ by 6.4 (c), so by lemma 5.11 there is a smooth projective variety Y of dimension $d-j+\nu+r$ and a morphism $g: Y \to X$ such that α lies in the image of $g_*: F^\nu CH^{\nu+r}(Y) \to F^\nu CH^j(X)$. We conclude that $\alpha \bmod F^{\nu+1} \in Gr_F^\nu CH^j(N^{j-\nu-r}h^{2j-\nu}(X)(j))$ as wanted.

From 6.4 and the equality $(F^\nu+N_p)/(F^{\nu+1}+N_p) \xleftarrow{\sim} F^\nu/(F^\nu\cap N_p+F^{\nu+1})$ we get

Corollary 6.7 *The Bloch-Beilinson filtration induces a descending filtration on the quotient $HCH^j(X)/N_pCH^j(X)$ such that the graded quotients only depend on the Grothendieck motives*

$$h^{j+p}(X)/N^1, h^{j+p+1}(X)/N^2, \ldots, h^{2j-1}(X)/N^{j-p}.$$

Note that this does not imply that N_p/N_{p-1} only depends on homological equivalence in general, since it does not depend on $h^{j+p+1}(X)/N^1$ alone. But it implies that the top quotient $HCH^j(X)/N_{j-1}CH^j(X)$ only depends on the Grothendieck motive $h^{2j-1}(X)/N^1$.

For $p=0$, i.e., the whole Chow group, the statement of 6.5 is known [12, 11.4 d)]. For $p=1$ we get the following result (where the first statement is a reformulation of (16)):

Corollary 6.8 *Let $F^\cdot$ be the filtration on the Griffiths group $Griff^j(X) = HCH^j(X)/ACH^j(X)$ induced by the Bloch-Beilinson filtration. Then one has $F^jGriff^j(X)=0$, and the graded quotients only depend on the motives $h^{j+1}(X)/N^1$, $h^{j+2}(X)/N^2$, $\ldots$, $h^{2j-1}(X)/N^{j-1}$.*

This result 'explains' why $Griff^d(X)$ is trivial for $d = \dim X$: since $h^{2d-\nu}(X) = N^{d-\nu}h^{2d-\nu}(X)$ for $\nu = 1, \ldots, d-1$. As another example, it lets us expect

$$H^3(X) = N^1H^3(X) \quad \overset{?}{\Rightarrow} \quad Griff^2(X) = 0. \tag{24}$$

(It should be noted that the converse does not hold in general, by work of M. Green [8] and C. Voisin (unpublished).) Related questions on the Griffiths group were also asked by R. Laterveer (private communication).

For $\nu = 1$ corollary 6.5 gives the following result which shows that, apart from the relation 6.2, the filtrations $F^\cdot$ and $N.$ tend to be independent.

Corollary 6.9 *Via (23) one has*

$$Gr_F^1(N^{j-p}h^{2j-1}(X)(j)) = (\tilde{N}_p + F^2)/F^2CH^j(X).$$

7. Mixed motives

Finally we comment on question 2.3 (2). A satisfactory abelian category of mixed motives has not yet been constructed, in particular none satisfying 2.2 (5). But by now there are several different and independent constructions of a triangulated motivic category [10], [19], [26]. By this we mean a triangulated rigid tensor category $\mathcal{D}_k$ together with a covariant functor

$$R : \mathcal{CHM}_k \longrightarrow \mathcal{D}_k$$

and isomorphisms for all smooth, projective varieties X and all $i, j \in \mathbb{Z}$

$$H^i_{\mathcal{M}}(X, \mathbb{Q}(j)) \xrightarrow{\sim} Hom_{\mathcal{D}_k}(1, R(X)(j)[i])$$

which are compatible with the action of correspondences on both sides. Here $H^i_{\mathcal{M}}(X, \mathbb{Q}(j)) := K_{2j-i}(X)^{(j)}_{\mathbb{Q}}$ is motivic cohomology, defined as the subspace of rational algebraic K-theory $K_{2j-i}(X)_{\mathbb{Q}}$ on which each Adams operator ψ^m acts as multiplication by m^j.

In particular, by Grothendieck's isomorphism $K_0(X)^{(j)} \tilde{=} CH^j(X)_{\mathbb{Q}}$ we have functorial isomorphisms

$$cl : CH^j(X) \xrightarrow{\sim} Hom_{\mathcal{D}_k}(1, R(X)(j)[2j]).$$

By the formalism of rigid tensor categories this implies that R is a full embedding.

As explained in [13], the existence of the Bloch-Beilinson filtration would follow from the existence of a suitable t-structure [2] τ on $\mathcal{D}_k$. Namely, let $\mathcal{MM}_k := \tau_{\leq 0}\tau_{\geq 0}\mathcal{D}_k \subseteq \mathcal{D}_k$ be the heart of the t-structure (which is an abelian category), and assume the following conditions

(i) For each $i \in \mathbb{Z}$ let ${}^{\tau}H^i : K \mapsto \tau_{\leq 0}\tau_{\geq 0}(K[-i])$ be the i-th t-cohomology functor. Then the composition

$$GrR : \mathcal{CHM}_k \xrightarrow{R} \mathcal{D}_k \xrightarrow{\oplus_i {}^{\tau}H^i} \mathcal{MM}_k$$

factors through homological equivalence and induces a full embedding $\mathcal{M}_k \hookrightarrow \mathcal{MM}_k$.

(ii) Via this embedding, the decomposition $GrR(h(X)) = \oplus\ {}^{\tau}H^i(R(X))$ gives the weight decomposition of the Grothendieck motive $h(X)$, for every smooth, projective variety X. Here we have written $R(X) := R(h^R(X))$, for short.

(In particular, this implies that the standard conjecture C holds.) Then we would get a (non-canonical) decomposition

$$R(X) \simeq \bigoplus_i h^i(X)[-i]$$

and would get the Bloch-Beilinson filtration by letting $F^\nu CH^j(X)$ be the preimage of

$$Hom_{\mathcal{D}_k}(1,(\tau_{\leq 2j-\nu}R(X))(j)[2j]) \subseteq Hom_{\mathcal{D}_k}(1,R(X)(j)[2j])$$

under cl. However, instead of Beilinson's formula 2.2 (5) one a priori only gets

$$Gr_F^\nu CH^j(X) \tilde{=} Hom_{\mathcal{D}_k}(1,h^{2j-\nu}(X)(j)[\nu]).$$

By the general theory of t-structures there is a canonical functor of triangulated categories

$$real : D^b(\mathcal{MM}_k) \longrightarrow \mathcal{D}_k,$$

and one has

$$Hom_{D^b(\mathcal{MM}_k)}(M,N[i]) = Ext^i_{\mathcal{MM}_k}(M,N) \quad \text{(Yoneda Ext-groups)}$$

for objects M and N in $\mathcal{MM}_k$. Hence the question is if *real* is fully faithful, or, when we are just interested in Beilinson's formula, if the map

$$Hom_{D^b(\mathcal{MM}_k)}(1,h^{2j-\nu}(X)(j)[\nu]) \to Hom_{\mathcal{D}_k}(1,h^{2j-\nu}(X)(j)[\nu]) \qquad (25)$$

induced by *real* is an isomorphism for smooth, projective X. From our results in section 4 we can deduce:

Proposition 7.1 *Let $\mathcal{D}_k$ be a triangulated rigid tensor category, and let*

$$R : \mathcal{CHM}_k \longrightarrow \mathcal{D}_k$$

be a full embedding of tensor categories. Let τ be t-structure on $\mathcal{D}_k$ such that the conditions (i) and (ii) above hold for the associated heart $\mathcal{MM}_k$. Assume that the standard conjecture B holds. Let $\nu \geq 0$ be an integer and let M and N be motives in $\mathcal{M}_k$ which are of pure of weights r and s, respectively, where $r - s = \nu$. Then the map

$$Hom_{D^b(\mathcal{MM}_k)}(M,N[\nu]) \to Hom_{\mathcal{D}_k}(M,N[\nu]) \qquad (26)$$

induced by real is surjective, and bijective for $\nu \leq 2$. In particular, (25) is surjective for all $j,\nu \geq 0$ and bijective for $\nu \leq 2$.

Proof. The last part follows from the first one, because 1 and $h^{2j-\nu}(X)$ are pure of weights 0 and $-\nu$, respectively. We prove the first statement by induction on ν. One knows from the general theory that the map (26) is an isomorphism for $\nu = 0,1$ and injective for $\nu = 2$ (for all objects M,N in the heart $\mathcal{MM}_k$) cf. [2]. Moreover, because $Hom(M,N) = Hom(1,M^\vee \otimes N)$ in $\mathcal{D} := \mathcal{D}_k$ and $\mathcal{D}' := D^b(\mathcal{MM}_k)$, it is equivalent to prove that, for any

$\nu \geq 1$ and any pure motive M of weight $-\nu$ the map $Hom_{\mathcal{D}'}(1, M[\nu]) \to Hom_{\mathcal{D}}(1, M[\nu])$ is surjective. Because such M is a direct factor of a motive $h^{2j-\nu}(X)(j)$ for a smooth, projective variety X and some $j \geq 0$, we may assume $M = h^{2j-\nu}(X)(j)$. Let α be an element in $Hom_{\mathcal{D}}(1, h^{2j-\nu}(X)(j)[\nu])$. Via the full embedding R, theorem 4.1 (or more directly, theorem 4.7) then implies that there are a smooth, projective variety T, and morphisms

$$\begin{aligned} \beta &\in Hom_{\mathcal{D}}(1, h^{2i-\nu+1}(T)(i)[\nu-1]) \\ \gamma &\in Hom_{\mathcal{D}}(h^{2i-\nu+1}(T)(i)[\nu-1], h^{2j-\nu}(X)(j)[\nu]) \end{aligned}$$

for some $i \geq 0$ such that $\alpha = \gamma \circ \beta$ (composition). By induction, we may assume that β and γ are in the image of *real*. In fact, this is obvious for β, while for γ we use the isomorphism $Hom(N[\nu-1], M[\nu]) = Hom(1, N^\vee \otimes M[1])$ in $\mathcal{D}'$ and $\mathcal{D}$. Hence α is in the image of *real* as claimed.

Remark 7.2 For $\nu = 2$ the argument in the proof of 7.1, applied to the filtration of Bloch-Beilinson type F^i_ℓ of 4.9 (3) and the triangulated category of $\mathbb{Q}_\ell$-sheaves gives a positive answer to a question raised in [15]: Consider the higher ℓ-adic cycle maps

$$cl^{(\nu)}_\ell : F^\nu_\ell CH^j(X) \to H^\nu(G_k, H^{2j-\nu}(\overline{X}, \mathbb{Q}_\ell(j)))$$

induced by the Hochschild-Serre sequence and the cycle map in 4.9 (3). If the standard conjecture $B(X)$ holds, then the 2-extension

$$cl^{(2)}(\alpha) \in Ext^2_{G_k}(\mathbb{Q}_\ell, H^{2j-2}(\overline{X}, \mathbb{Q}_\ell(j))) = H^2(G_k, H^{2j-2}(\overline{X}, \mathbb{Q}_\ell(j)))$$

associated to an algebraic cycle $\alpha \in F^2_\ell H^{2j}(X, \mathbb{Q}_\ell(j))$ is motivic, to wit, the Yoneda product of two cycle classes $cl^{(1)}(\beta)$ and $cl^{(1)}(\gamma)$, with $\beta \in F^1_\ell CH^i(T)$ and $\gamma \in F^1_\ell Corr^{j-i}(T \times X)$. In fact, (the proof of) theorem 4.7 tells us that we may take $i = j-1$ and for T a suitable hyperplane section.

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