The integral formula for the solutions of the quantum Knizhnik-Zamolodchikov equation associated with $U_q(\widehat{sl}_n)$ for $|q| = 1$

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Dedicated to the memory of Moshe Flato.

Abstract

We write the integral formula of Tarasov-Varchenko type for the solutions to the quantum Knizhnik-Zamolodchikov equation associated with a tensor product of the vector representations of sl_n. We consider the case where the deformation parameter q satisfies $|q| = 1$. We use the bosonization of the type II vertex operators in order to find the hypergeometric pairing in this setting.

1 Introduction

In this paper we construct a family of solutions to the quantum Knizhnik-Zamolodchikov equation [?] associated with the vector representation of sl_n. To be more precise, we consider the case

$$\lambda = \frac{4\pi}{n} \tag{1}$$

(called level 0) of the following difference equation:

$$\begin{aligned} f(\beta_1, \dots, \beta_r - \lambda i, \dots, \beta_N) &= R_{r,r-1}(\beta_r - \lambda i, \beta_{r-1}) \cdots R_{r,1}(\beta_r - \lambda i, \beta_1) \\ &\times D_r R_{r,N}(\beta_r, \beta_N) \cdots R_{r,r+1}(\beta_r, \beta_{r+1}) f(\beta_1, \dots, \beta_N). \end{aligned} \tag{2}$$

In this equation, the unknown function $f(\beta_1, \dots, \beta_N)$ $(\beta_1, \dots, \beta_N \in \mathbf{C})$ takes its value in $\underbrace{V \otimes \cdots \otimes V}_{N}$ where

$$V = \oplus_{j=0}^{n-1} \mathbf{C} v_j \tag{3}$$

is the vector representation of sl_n. The matrix $R_{rs}(\beta_r, \beta_s)$ acts on the r-th and s-th components of the tensor product. It is the R-matrix for $U_q(\widehat{sl}_n)$. The explicit formula of the R-matrix is given by (??). In this paper, we restrict to the case where

$$q = e^{-\frac{2\pi^2 i}{\rho n}} \quad (\rho \in \mathbf{R}_{>0}). \tag{4}$$

The matrix D_r is a diagonal matrix acting on the r-th component. In this paper, we restrict to the case

$$D_r = 1. \tag{5}$$

For $n = 2$, Smirnov [?] constructed a family of solutions to (??) which he identified with the form factors of the quantum sine-Gordon model. Later, Lukyanov [?] gave a construction of the form factors by using the bosonic vertex operators. In the sl_n case, we adapt Lukyanov's approach with the modification given in [?]. In this paper, however, we only construct the so-called type II vertex operators. This construction gives us the hypergeometric pairing for two functions w and W in the terminology of Tarasov and Varchenko [?, ?, ?]:

$$I(w, W) = \prod_{j=1}^{n-1} \prod_{1 \le r \le \nu_j} \int \frac{d\gamma_{j,r}}{2\pi i} K(\{\gamma_{j,r}\}) w(\{\gamma_{j,r}\}) W(\{\gamma_{j,r}\}). \tag{6}$$

In this formula, the dependence on $\beta_1, \ldots, \beta_N$ in the right hand side is implicit in the kernel K as well as w and W. We write the integral formula of Tarasov-Varchenko type for the solutions to (??) with $\lambda = \frac{4\pi}{n}$ by using this pairing:

$$f(\beta_1, \ldots, \beta_N) = \sum_{j_1, \ldots, j_r} I(w_{j_1, \ldots, j_r}, W) v_{j_1} \otimes \cdots \otimes v_{j_N}. \tag{7}$$

We prove the convergence of the integral (??) in the positive Weyl chamber, and show that the equation (??) (with $\lambda = \frac{4\pi}{n}$) is indeed satisfied.

Finally we remark that the construction for sl_n case with generic highest weights is considered in [?] using the Jackson integral, and in [?] in the setting of the hypergeometric pairing. (see [?, ?, ?, ?] for the conformal case, i.e., $q = 1$).

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2 Operator construction

In this section, we construct the vertex operators

$$\Psi_j^*(\beta) \quad (j = 0, 1, \ldots, n-1; \beta \in \mathbf{R}) \tag{8}$$

satisfying the commutation relations with the R-matrix for the quantum affine algebra $U_q(\widehat{sl}_n)$, i.e.,

$$\Psi_j^*(\beta_1)\Psi_j^*(\beta_2) = s(\beta_1 - \beta_2)\Psi_j^*(\beta_2)\Psi_j^*(\beta_1), \tag{9}$$

and for $j \neq k$

$$\begin{aligned} \Psi_j^*(\beta_1)\Psi_k^*(\beta_2) &= s(\beta_1 - \beta_2)\{\bar{R}(\beta_1, \beta_2)_{jk}^{jk}\Psi_k^*(\beta_2)\Psi_j^*(\beta_1) \\ &+ \bar{R}(\beta_1, \beta_2)_{jk}^{kj}\Psi_j^*(\beta_2)\Psi_k^*(\beta_1)\} \end{aligned} \tag{10}$$

where

$$s(\beta) = \frac{S_2(-i\beta|\rho, 2\pi)S_2(i\beta + \frac{2(n-1)\pi}{n}|\rho, 2\pi)}{S_2(i\beta|\rho, 2\pi)S_2(-i\beta + \frac{2(n-1)\pi}{n}|\rho, 2\pi)}, \tag{11}$$

$$\bar{R}(\beta_1, \beta_2)_{jk}^{jk} = -\frac{\mathrm{sh}\,\frac{\pi}{\rho}(\beta_1 - \beta_2)}{\mathrm{sh}\,\frac{\pi}{\rho}(\beta_1 - \beta_2 - \frac{2\pi i}{n})} \quad (j \neq k), \tag{12}$$

$$\bar{R}(\beta_1, \beta_2)_{jk}^{kj} = \begin{cases} -\dfrac{e^{\frac{\pi}{\rho}(\beta_1 - \beta_2)}\mathrm{sh}\,\frac{2\pi^2 i}{\rho n}}{\mathrm{sh}\,\frac{\pi}{\rho}(\beta_1 - \beta_2 - \frac{2\pi i}{n})} & (j > k); \\ -\dfrac{e^{\frac{\pi}{\rho}(\beta_2 - \beta_1)}\mathrm{sh}\,\frac{2\pi^2 i}{\rho n}}{\mathrm{sh}\,\frac{\pi}{\rho}(\beta_1 - \beta_2 - \frac{2\pi i}{n})} & (j < k). \end{cases} \tag{13}$$

We define $\bar{R}(\beta_1, \beta_2) \in \mathrm{End}(V \otimes V)$ by

$$\bar{R}(\beta_1, \beta_2)v_j \otimes v_k = \sum_{j'k'} \bar{R}(\beta_1, \beta_2)_{j'k'}^{jk} v_{j'} \otimes v_{k'}, \tag{14}$$

where $\bar{R}(\beta_1, \beta_2)_{j'k'}^{jk} = 0$ except for (??), (??) and $\bar{R}(\beta_1, \beta_2)_{jj}^{jj} = 1$.

We refer the reader to [?] for the double sine function $S_2(x|\omega_1, \omega_2)$. The deformation parameters q and ρ are identified by the relation (??). The construction of the vertex operators forces us to choose the normalization factor $s(\beta_1 - \beta_2)$.

The bosonization of the level 1 highest weight representations for $U_q(\widehat{sl}_n)$ and the vertex operators acting on them are given in [?] and [?], respectively. However, the present case is not connected to those works in which $|q| < 1$. The case $|q| = 1$ is obtained in the limit of the elliptic case considered in [?, ?]. We adapt the construction in [?] by using the method developed in [?, ?, ?].

Let $a_j(t)$ $(1 \leq j \leq n-1; t \in \mathbf{R})$ be the free bose fields satisfying the commutation relations

$$[a_j(t), a_k(t')] = A_{jk}(t)\delta(t+t'), \quad A_{jk}(t) = -\frac{1}{t}\frac{\mathrm{sh}\,\frac{a_{jk}\pi t}{n}\mathrm{sh}\,(\frac{\rho}{2}+\frac{\pi}{n})t}{\mathrm{sh}\,\frac{\pi t}{n}\mathrm{sh}\,\frac{\rho t}{2}}. \tag{15}$$

Here $(a_{jk})_{1\leq j,k\leq n-1}$ is the Cartan matrix of type A_n.

We consider the Fock space $\mathcal{F}$ generated by the vacuum vector $|\mathrm{vac}\rangle$ satisfying

$$a_j(t)|\mathrm{vac}\rangle = 0 \text{ if } t > 0. \tag{16}$$

Set

$$a_1^*(t) \quad = \quad -\sum_{j=1}^{n-1} a_j(t)\frac{\mathrm{sh}\,\frac{(n-j)\pi t}{n}}{\mathrm{sh}\,\pi t}. \tag{17}$$

We have

$$[a_1^*(t), a_j(t')] \quad = \quad \delta_{1j}\frac{1}{t}\frac{\mathrm{sh}\,(\frac{\rho}{2}+\frac{\pi}{n})t}{\mathrm{sh}\,\frac{\rho t}{2}}\delta(t+t'). \tag{18}$$

We introduce the currents

$$V_j(\beta) \quad = \quad :\exp\left(\int_{-\infty}^{\infty} a_j(t)e^{i\beta t}dt\right): \quad (1 \leq j \leq n-1). \tag{19}$$

They satisfy the following commutation relations.

$$V_j(\beta_1)V_k(\beta_2) \quad = \quad V_k(\beta_2)V_j(\beta_1) \quad (|j-k| \geq 2), \tag{20}$$

$$V_j(\beta_1)V_{j-1}(\beta_2) \quad = \quad \frac{\mathrm{sh}\,\frac{\pi}{\rho}(\beta_1-\beta_2+\frac{\pi i}{n})}{\mathrm{sh}\,\frac{\pi}{\rho}(\beta_2-\beta_1+\frac{\pi i}{n})}V_{j-1}(\beta_2)V_j(\beta_1) \quad (2 \leq j \leq n-1), \tag{21}$$

$$V_j(\beta_1)V_j(\beta_2) \quad = \quad -\frac{\mathrm{sh}\,\frac{\pi}{\rho}(\beta_1-\beta_2-\frac{2\pi i}{n})}{\mathrm{sh}\,\frac{\pi}{\rho}(\beta_2-\beta_1-\frac{2\pi i}{n})}V_j(\beta_2)V_j(\beta_1) \quad (1 \leq j \leq n-1). \tag{22}$$

Now we define the vertex operators.

$$\Psi_j^*(\beta) \quad = \quad \prod_{1\leq k\leq j}\int_{-\infty}^{\infty}\frac{d\alpha_k}{2\pi i}\frac{e^{\frac{\pi}{\rho}(\beta-\alpha_j)}V_0(\beta)V_1(\alpha_1)\cdots V_j(\alpha_j)}{\prod_{k=1}^{j}\mathrm{sh}\,\frac{\pi}{\rho}(\alpha_{k-1}-\alpha_k+\frac{\pi i}{n})}. \tag{23}$$

Here we use $\alpha_0 = \beta$ and

$$V_0(\beta) \quad = \quad :\exp\left(\int_{-\infty}^{\infty} a_1^*(t)e^{i\beta t}dt\right): . \tag{24}$$

The relations (??) and (??) are also valid for $V_0(\beta)$. However, the relation (??) is modified to

$$V_0(\beta_1)V_0(\beta_2) \quad = \quad s(\beta_1-\beta_2)V_0(\beta_2)V_0(\beta_1). \tag{25}$$

We give the proof of (??) and (??) later, which is the same thing with the proof of Lemma ?? in Section 3.

For a parameter $\lambda \in \mathbf{R}_{>0}$ we define the λ-expectation value by

$$\langle a_j(t)a_k(t')\rangle_\lambda = \frac{e^{\lambda t}}{e^{\lambda t}-1}A_{jk}(t)\delta(t+t'). \tag{26}$$

If $\lambda = \infty$ this reduces to the vacuum expectation value. The λ-expectation values of products of vertex operators can be calculated by using Wick's theorem. We recall the following formula (see [?]) for the two point function:

$$\langle : e^{\int_{-\infty}^{\infty} a(t)e^{i\beta_1 t}} :: e^{\int_{-\infty}^{\infty} b(t)e^{i\beta_2 t}} : \rangle_\lambda \quad = \quad \exp\Big(\int_0^\infty A(t)\frac{\mathrm{ch}(i(\beta_1-\beta_2)+\frac{\lambda}{2})t}{\mathrm{sh}\frac{\lambda t}{2}}dt\Big), \tag{27}$$

where $a(t)$ and $b(t)$ are bosons satisfying $[a(t),b(t')] = A(t)\delta(t+t')$ and $A(t) = -A(-t)$. We also list the two point functions of $V_j(\beta)$. In the formulas below, const. means a constant independent of the spectral parameters.

$$\langle V_0(\beta_1)V_0(\beta_2)\rangle_\lambda \quad = \quad E_\lambda(\beta_1-\beta_2), \tag{28}$$

$$E_\lambda(\beta) \quad = \quad \mathrm{const.}\frac{S_3(-i\beta)S_3(i\beta+\lambda)}{S_3(\frac{2\pi}{n}+\rho-i\beta)S_3(\frac{2\pi}{n}+\rho+i\beta+\lambda)}, \tag{29}$$

$$\text{where} \qquad S_3(\beta) = S_3(\beta|\rho,\lambda,2\pi). \tag{30}$$

We have

$$\frac{E_\lambda(\beta)}{E_\lambda(-\beta)} = s(\beta), \quad E_\lambda(\lambda i-\beta) = E_\lambda(\beta). \tag{31}$$

For $1 \le j \le n-1$ we have

$$\langle V_j(\beta_1)V_{j-1}(\beta_2)\rangle_\lambda \quad = \quad \langle V_{j-1}(\beta_1)V_j(\beta_2)\rangle_\lambda \tag{32}$$

$$= \quad \mathrm{const.}\,\mathrm{sh}\,\frac{\pi}{\rho}(\beta_1-\beta_2+\frac{\pi i}{n})\varphi(\beta_1-\beta_2), \tag{33}$$

$$\varphi(\beta) \quad = \quad \frac{1}{S_2(i\beta-\frac{\pi}{n}|\rho,\lambda)S_2(-i\beta-\frac{\pi}{n}|\rho,\lambda)}, \tag{34}$$

$$\langle V_j(\beta_1)V_j(\beta_2)\rangle_\lambda \quad = \quad \mathrm{const.}\,\psi(\beta_1-\beta_2)\mathrm{sh}\,\frac{\pi}{\rho}(\beta_1-\beta_2-\frac{2\pi i}{n})h(\beta_1-\beta_2,\lambda), \tag{35}$$

$$h(\beta,\lambda) \quad = \quad \mathrm{sh}\,\frac{\pi}{\lambda}\beta\mathrm{sh}\,\frac{\pi}{\lambda}(\beta-\frac{2\pi i}{n})\mathrm{sh}\,\frac{\pi}{\lambda}(\beta+\frac{2\pi i}{n}), \tag{36}$$

$$\psi(\beta) \quad = \quad \frac{1}{S_2(i\beta+\frac{2\pi}{n}|\rho,\lambda)S_2(-i\beta+\frac{2\pi}{n}|\rho,\lambda)}. \tag{37}$$

The rest of the two point functions of $V_j(\beta)$'s are 1. These formulas are to be understood as analytic continuations from the regions where the integrals (??) are convergent. Because of the existence of poles, if they are used as integrands, we must pay a special attention to the choice of integration contours.

To see this point closely, let us compute

$$\langle \Psi^*_{j_N}(\beta_N)\cdots\Psi^*_{j_1}(\beta_1)\rangle \quad = \quad E(\beta_1,\ldots,\beta_N)\prod_{j,r}\int_C \frac{d\alpha_{j,r}}{2\pi i} I_{j_1,\ldots,j_N}(\{\alpha_{j,r}\}). \tag{38}$$

We associated the integration variables $\alpha_{j,r}$ $(1 \le r \le N; 1 \le j \le j_r)$ to the current $V_j(\alpha_{j,r})$ contained in the vertex operator $\Psi^*_{j_r}(\beta_r)$ (see (??)). We also set

$$\alpha_{0,r} \quad = \quad \beta_r, \tag{39}$$

$$\mathcal{N}_j \quad = \quad \{r; j_r \ge j\}. \tag{40}$$

The factor $E(\beta_1,\ldots,\beta_N)$ is given by the pair product

$$E(\beta_1,\ldots,\beta_N) \quad = \quad \text{const.} \prod_{1\le r<s\le N} E_\lambda(\beta_s-\beta_r) \tag{41}$$

The integrand consists of three parts,

$$I_{j_1,\ldots,j_N}(\beta_1,\ldots,\beta_N) \quad = \quad K(\{\alpha_{j,r}\})g(\{\alpha_{j,r}\})W(\{\alpha_{j,r}\}), \tag{42}$$

where

$$K(\{\alpha_{j,r}\}) \quad = \quad \prod_{j=1}^{n-1}\Big\{ \prod_{\substack{r\in\mathcal{N}_j \\ s\in\mathcal{N}_{j-1}}} \varphi(\alpha_{j,r}-\alpha_{j-1,s}) \prod_{\substack{1\le r<s\le N \\ r,s\in\mathcal{N}_j}} \psi(\alpha_{j,s}-\alpha_{j,r})\Big\}, \tag{43}$$

$$g(\{\alpha_{j,r}\}) \quad = \quad \prod_{j=1}^{n-1}\Big\{ \prod_{r\in\mathcal{N}_j} e^{\frac{\pi}{\rho}(\alpha_{j-1,r}-\alpha_{j,r})} \prod_{\substack{r\in\mathcal{N}_j \\ s\in\mathcal{N}_{j-1} \\ r>s}} \mathrm{sh}\,\frac{\pi}{\rho}(\alpha_{j,r}-\alpha_{j-1,s}+\frac{\pi i}{n})$$

$$\times \prod_{\substack{r\in\mathcal{N}_j \\ s\in\mathcal{N}_{j-1} \\ r<s}} \mathrm{sh}\,\frac{\pi}{\rho}(\alpha_{j-1,s}-\alpha_{j,r}+\frac{\pi i}{n}) \prod_{\substack{r,s\in\mathcal{N}_j \\ r<s}} \mathrm{sh}\,\frac{\pi}{\rho}(\alpha_{j,s}-\alpha_{j,r}-\frac{2\pi i}{n})\Big\}, \tag{44}$$

$$W(\{\alpha_{j,r}\}) \quad = \quad \prod_{j=1}^{n-1}\prod_{\substack{r,s\in\mathcal{N}_j \\ r<s}} h(\alpha_{j,s}-\alpha_{j,r},\lambda). \tag{45}$$

The poles of the integrand come from those of $K(\{\alpha_{j,r}\})$. They are located at

$$\alpha_{j,r} - \alpha_{j-1,s} = \pm(\frac{\pi i}{n} - \rho i \mathbf{Z}_{\geq 0} - \lambda i \mathbf{Z}_{\geq 0}), \tag{46}$$

$$\alpha_{j,s} - \alpha_{j,r} = \pm(\frac{2\pi i}{n} + \rho i \mathbf{Z}_{\geq 0} + \lambda i \mathbf{Z}_{\geq 0}). \tag{47}$$

The contour $C_{j,r}$ for $\alpha_{j,r}$ is chosen so that

the poles at $\alpha_{j-1,s} + \frac{\pi i}{n} - \rho i \mathbf{Z}_{\geq 0} - \lambda i \mathbf{Z}_{\geq 0}$ are below $C_{j,r}$,

the poles at $\alpha_{j-1,s} - \frac{\pi i}{n} + \rho i \mathbf{Z}_{\geq 0} + \lambda i \mathbf{Z}_{\geq 0}$ are above $C_{j,r}$,

the poles at $\alpha_{j,s} + \frac{2\pi i}{n} + \rho i \mathbf{Z}_{\geq 0} + \lambda i \mathbf{Z}_{\geq 0}$ are above $C_{j,r}$,

the poles at $\alpha_{j,s} - \frac{2\pi i}{n} - \rho i \mathbf{Z}_{\geq 0} - \lambda i \mathbf{Z}_{\geq 0}$ are below $C_{j,r}$.

These conditions are not compatible if all the poles are really existent. However, in the actual situation for our matrix element $\langle \Psi^*_{j_N}(\beta_N) \cdots \Psi^*_{j_1}(\beta_1) \rangle_\lambda$, this is not the case: the poles at

$$\alpha_{j,s} - \alpha_{j,r} = \pm \frac{2\pi i}{n} \tag{48}$$

are canceled by the zeros of $W(\{\alpha_{j,r}\})$. It is easy to see that under this cancellation, a consistent choice of the contours is possible.

In this paper we finish the operator theory at this point. We will not discuss the type I vertex operators and their form factors ([?, ?]). In the next section, however, following the idea of Tarasov and Varchenko (see [?, ?, ?]), we will modify $W(\{\alpha_{j,r}\})$, and thereby construct a family of solutions to the quantum Knizhnik-Zamolodchiov equation in the special case where

$$\lambda = \frac{4\pi}{n}. \tag{49}$$

Note that, in this case, the function $\psi(\beta)$ simplifies to

$$\psi(\beta) = \frac{1}{2i \mathrm{sh}\, \frac{n}{4}(\beta - \frac{2\pi i}{n})}. \tag{50}$$

3 Integral formula

For non-negative integers $\nu_1, \cdots, \nu_{n-1}$ satisfying

$$N = \nu_0 \geq \nu_1 \geq \nu_2 \geq \cdots \geq \nu_{n-1} \geq \nu_n = 0, \tag{51}$$

we denote by $\mathcal{Z}_{\nu_1,\cdots,\nu_{n-1}}$ the set of all the N-tuples $J = (j_1, \cdots, j_N) \in (\mathbf{Z}_{\geq 0})^N$ such that

$$\nu_j = \#\mathcal{N}_j. \tag{52}$$

Define $r_{j,m}$ $(0 \leq j \leq n-1; 1 \leq m \leq \nu_j)$ as follows:

$$\mathcal{N}_j = \{r_{j,1}, \cdots, r_{j,\nu_j}\}, \quad r_{j,1} < \cdots < r_{j,\nu_j}. \tag{53}$$

We have, in particular, $r_{0,m} = m$. We make a correspondence between two sets of the integration variables:

$$\alpha^J_{j,r_{j,m}} \quad = \quad \gamma_{j,m}. \tag{54}$$

We set

$$\begin{aligned} w_J(\{\gamma_{j,m}\}) &= \mathrm{Skew}_{n-1} \circ \cdots \circ \mathrm{Skew}_1 g_J(\{\gamma_{j,m}\}), \qquad (55)\\ g_J(\{\gamma_{j,m}\}) &= \prod_{j=1}^{n-1} \Big\{ \prod_{m=1}^{\nu_j} \Big\{ \prod_{r \in \mathcal{N}_j} e^{\frac{\pi}{\rho}(\alpha^J_{j-1,r} - \alpha^J_{j,r})} \end{aligned}$$

$$\times \prod_{\substack{m' \\ r_{j-1,m'} < r_{j,m}}} \mathrm{sh}\,\frac{\pi}{\rho}(\gamma_{j,m} - \gamma_{j-1,m'} + \frac{\pi i}{n}) \prod_{\substack{m' \\ r_{j,m} < r_{j-1,m'}}} \mathrm{sh}\,\frac{\pi}{\rho}(\gamma_{j,m} - \gamma_{j-1,m'} - \frac{\pi i}{n}) \Big\}$$

$$\times \prod_{1 \leq m < m' \leq \nu_j} \mathrm{sh}\,\frac{\pi}{\rho}(\gamma_{j,m'} - \gamma_{j,m} - \frac{2\pi i}{n}) \Big\}, \tag{56}$$

where $\gamma_{0,m} = \beta_m$ and Skew_j is the skew-symmetrization with respect to the variables $\{\gamma_{j,m}\}_{m=1,\cdots,\nu_j}$:

$$\mathrm{Skew}_j X(\gamma_{j,1}, \ldots, \gamma_{j,\nu_j}) \quad = \quad \sum_{\sigma \in S_{\nu_j}} \mathrm{sgn}(\sigma) X(\gamma_{j,\sigma(1)}, \ldots, \gamma_{j,\sigma(\nu_j)}). \tag{57}$$

Note that the above definition differs from (??) by sign. This change is necessary because we will choose different W in the below. Accordingly, we change the R matrix:

$$R(\beta_1, \beta_2)^{j'k'}_{jk} = \begin{cases} -\bar{R}(\beta_1, \beta_2)^{jk}_{jk} & \text{if } (j',k') = (j,k) \text{ and } j \neq k; \\ \bar{R}(\beta_1, \beta_2)^{j'k'}_{jk} & \text{otherwise.} \end{cases} \tag{58}$$

In the following, we abbreviate

$$w_J(\{\gamma_{j,r}\}) = w_J(\beta_{k_1}, \cdots, \beta_{k_N}) \tag{59}$$

when the dependence on $\{\gamma_{j,m}\}$ $(j \neq 0)$ is irrelevant.

For the function w_J, the following equality holds.

Lemma 1

$$w_{(j_1,\cdots,j_{k+1},j_k,\cdots,j_N)}(\beta_1,\cdots,\beta_{k+1},\beta_k,\cdots,\beta_N) = \sum_{j'_k,j'_{k+1}} R(\beta_k,\beta_{k+1})^{j'_k j'_{k+1}}_{j_k j_{k+1}} w_{(j_1,\cdots,j'_k,j'_{k+1},\cdots,j_N)}(\beta_1,\cdots,\beta_k,\beta_{k+1},\cdots,\beta_N) \quad (60)$$

Proof. It is enough to prove (??) for $N = 2$. First, we prove

$$w_{(l,l)}(\beta_1,\beta_2) = w_{(l,l)}(\beta_2,\beta_1), \quad l = 0,1,\cdots,n-1. \quad (61)$$

Note that

$$g_{(l,l)}(\beta_1,\beta_2) = e^{\frac{\pi}{\rho}(\beta_1+\beta_2-\gamma_{l,1}-\gamma_{l,2})} \prod_{j=1}^{l} \Big\{ \mathrm{sh}\,\frac{\pi}{\rho}(\gamma_{j,2}-\gamma_{j-1,1}+\frac{\pi i}{n}) \times \mathrm{sh}\,\frac{\pi}{\rho}(\gamma_{j,1}-\gamma_{j-1,2}-\frac{\pi i}{n})\,\mathrm{sh}\,\frac{\pi}{\rho}(\gamma_{j,2}-\gamma_{j,1}-\frac{2\pi i}{n}) \Big\}. \quad (62)$$

By using

$$\mathrm{sh}\,\frac{\pi}{\rho}(\gamma_2-\beta_1+\frac{\pi i}{n})\,\mathrm{sh}\,\frac{\pi}{\rho}(\gamma_1-\beta_2-\frac{\pi i}{n}) - (\beta_1 \leftrightarrow \beta_2) = \mathrm{sh}\,\frac{\pi}{\rho}(\beta_1-\beta_2)\,\mathrm{sh}\,\frac{\pi}{\rho}(\gamma_2-\gamma_1+\frac{2\pi i}{n}) \quad (63)$$

repeatedly, we obtain

$$\mathrm{Skew}_{l-1} \circ \cdots \circ \mathrm{Skew}_0 g_{(l,l)}(\beta_1,\beta_2) = e^{\frac{\pi i}{\rho}(\beta_1+\beta_2-\gamma_{l,1}-\gamma_{l,2})}\,\mathrm{sh}\,\frac{\pi}{\rho}(\beta_1-\beta_2) \times \mathrm{sh}\,\frac{\pi}{\rho}(\gamma_{l,2}-\gamma_{l,1}+\frac{2\pi i}{n})\,\mathrm{sh}\,\frac{\pi}{\rho}(\gamma_{l,2}-\gamma_{l,1}-\frac{2\pi i}{n}) \prod_{j=1}^{l-1} h(\gamma_{j,1}-\gamma_{j,2}). \quad (64)$$

This is symmetric with respect to $\gamma_{l,1}, \gamma_{l,2}$, and therefore, we have (??).

Next, we show that

$$\mathrm{Skew}_l \circ \cdots \circ \mathrm{Skew}_1 X_{(l+1,l)}(\beta_2,\beta_1) = 0 \quad (65)$$

where

$$X_{(l+1,l)}(\beta_2,\beta_1) = g_{(l+1,l)}(\beta_2,\beta_1) - R(\beta_1,\beta_2)^{l+1,l}_{l,l+1} g_{(l+1,l)}(\beta_1,\beta_2) - R(\beta_1,\beta_2)^{l,l+1}_{l,l+1} g_{(l,l+1)}(\beta_1,\beta_2). \quad (66)$$

The proof for $g_{(l,l+1)}(\beta_2,\beta_1)$ is similar. The proof for $g_{(l,k)}$ for general j,k reduces to the case $k = l \pm 1$.

For $l=0$, one can check (??) directly. For $l \geq 1$ we proceed as follows. Set $\gamma_{l,1}=\gamma_1, \gamma_{l,2}=\gamma_2, \gamma_{l+1,1}=\gamma$. Noting that

$$g_{(l+1,l)}(\beta_1,\beta_2) = g_{(l,l)}(\beta_1,\beta_2)e^{\frac{\pi i}{\rho}(\gamma_1-\gamma)}\,\mathrm{sh}\,\frac{\pi}{\rho}(\gamma-\gamma_2-\frac{\pi i}{n}), \tag{67}$$

$$g_{(l,l+1)}(\beta_1,\beta_2) = g_{(l,l)}(\beta_1,\beta_2)e^{\frac{\pi i}{\rho}(\gamma_2-\gamma)}\,\mathrm{sh}\,\frac{\pi}{\rho}(\gamma-\gamma_1+\frac{\pi i}{n}), \tag{68}$$

we have

$$\begin{aligned} X_{l+1,l}(\beta_2,\beta_1) &= \Big(\frac{1}{2}e^{-\frac{2\pi\gamma}{\rho}} - e^{\frac{\pi}{\rho}(\gamma_1-\gamma_2-\frac{\pi i}{n})}\Big)\mathrm{Skew}_0 g_{(l,l)}(\beta_1,\beta_2) \\ &+ \frac{2\,\mathrm{sh}\,\frac{\pi}{\rho}(\beta_1-\beta_2)\,\mathrm{sh}\,\frac{\pi}{\rho}(\gamma_1-\gamma_2-\frac{2\pi i}{n})e^{-\frac{\pi^2 i}{\rho n}}}{\mathrm{sh}\,\frac{\pi}{\rho}(\beta_1-\beta_2-\frac{2\pi i}{n})} g_{(l,l)}(\beta_1,\beta_2). \end{aligned} \tag{69}$$

Note that

$$\begin{aligned} &\mathrm{Skew}_1\circ\cdots,\circ\mathrm{Skew}_l\Big\{\mathrm{sh}\,\frac{\pi}{\rho}(\gamma_1-\gamma_2-\frac{2\pi i}{n})g_{(l,l)}(\beta_1,\beta_2)\Big\} \\ &= \mathrm{sh}\,\frac{\pi}{\rho}(\beta_1-\beta_2-\frac{2\pi i}{n})e^{\frac{\pi i}{\rho}(\beta_1+\beta_2-\gamma_1-\gamma_2)}\prod_{j=1}^{l}h(\gamma_{j,1}-\gamma_{j,2}). \end{aligned} \tag{70}$$

Using (??) and (??), we obtain (??). □

Following [?, ?, ?], we define the hypergeometric pairing. Set

$$\mathcal{F}^{(\rho)}_{\nu_1,\cdots,\nu_{n-1}} = \sum_{J\in\mathcal{Z}_{\nu_1,\cdots,\nu_{n-1}}} \mathbf{C}w_J(\{\gamma_{j,m}\}). \tag{71}$$

In our setting, the hypergeometric pairing gives rise to the pairing given by (??) between $w\in\mathcal{F}^{(\rho)}_{\nu_1,\cdots,\nu_{n-1}}$ and $W\in\mathcal{F}^{(\lambda)}_{\nu_1,\cdots,\nu_{n-1}}$, where the kernel K is given by

$$\prod_{j=1}^{n-1}\Big\{\prod_{m=1}^{\nu_j}\prod_{m'=1}^{\nu_{j-1}}\varphi(\gamma_{j,m}-\gamma_{j-1,m'})\prod_{1\leq m<m'\leq\nu_j}\psi(\gamma_{j,m}-\gamma_{j,m'})\Big\}. \tag{72}$$

In this paper, we restrict to the level 0 case, i.e., $\lambda=\frac{4\pi}{n}$, where we have (??). In this case, it is convenient to move the ψ-part of the kernel to the space of W. Therefore, we set

$$\mathcal{W}_{\nu_1,\cdots,\nu_{n-1}} = \sum_{J\in\mathcal{Z}_{\nu_1,\cdots,\nu_{n-1}}} \mathbf{C}W_J(\{\gamma_{j,m}\}) \tag{73}$$

where

$$W_J(\{\gamma_{j,m}\}) = \mathrm{Skew}_{n-1}\circ\cdots\circ\mathrm{Skew}_1 G_J(\{\gamma_{j,m}\}), \tag{74}$$

$$G_J(\{\gamma_{j,m}\}) = \prod_{j=1}^{n-1}\Big\{\prod_{r\in\mathcal{N}_j} e^{\frac{n}{4}(\alpha^J_{j-1,r}-\alpha^J_{j,r})}$$

$$\times\prod_{m=1}^{\nu_j}\prod_{\substack{m'\\ r_{j-1,m'}<r_{j,m}}}\operatorname{sh}\frac{n}{4}(\gamma_{j,m}-\gamma_{j-1,m'}+\frac{\pi i}{n})\prod_{\substack{m'\\ r_{j,m}<r_{j-1,m'}}}\operatorname{sh}\frac{n}{4}(\gamma_{j,m}-\gamma_{j-1,m'}-\frac{\pi i}{n})\Big\}. \tag{75}$$

We define the pairing as follows. For $w\in\mathcal{F}^{(\rho)}_{\nu_1,\cdots,\nu_{n-1}}$ and $W\in\mathcal{W}_{\nu_1,\cdots,\nu_{n-1}}$, we set

$$I(w,W)(\beta_1,\cdots,\beta_N) = \left(\prod_{j=1}^{n-1}\prod_{m=1}^{\nu_j}\int_{C_j}d\gamma_{j,m}\right)\prod_{j=1}^{n-1}\prod_{m=1}^{\nu_j}\prod_{m'=1}^{\nu_{j-1}}\varphi(\gamma_{j,m}-\gamma_{j-1,m'})$$
$$\times w(\{\gamma_{j,m}\})W(\{\gamma_{j,m}\}). \tag{76}$$

In the above formula, φ is given by (??) with $\lambda=\frac{4\pi}{n}$. The contour C_j for $\gamma_{j,m}$ is $(-\infty,\infty)$ except that the poles at

$$\gamma_{j\pm1,m'}+\frac{\pi i}{n}-\rho\mathbf{Z}_{\geq0}-\frac{4\pi}{n}\mathbf{Z}_{\geq0} \tag{77}$$

are below C_j and the poles at

$$\gamma_{j\pm1,m'}-\frac{\pi i}{n}+\rho\mathbf{Z}_{\geq0}+\frac{4\pi}{n}\mathbf{Z}_{\geq0} \tag{78}$$

are above C_j.

The weight of the function $f(\beta_1,\ldots,\beta_N)$ given by (??) is

$$\sum_{j=0}^{n-1}(\nu_j-\nu_{j+1})\varepsilon_j. \tag{79}$$

The positive Weyl chamber is given by

$$\nu_{j-1}+\nu_{j+1}\geq2\nu_j,\quad \text{for all } j=1,\cdots,n-1. \tag{80}$$

Proposition 2 *Suppose that $\nu_1,\cdots,\nu_{n-1}$ satisfy (??) and (??). Then the integral (??) is absolutely convergent.*

Proof. It is easy to see that

$$|w(\{\gamma_{j,m}\})|\leq$$
$$\text{const.}\exp\Big\{\frac{\pi}{\rho}\sum_{j=1}^{n-1}\left(\sum_{m=1}^{\nu_j}\sum_{m'=1}^{\nu_{j-1}}|\gamma_{j,m}-\gamma_{j-1,m'}|+\sum_{1\leq m<m'\leq\nu_j}|\gamma_{j,m}-\gamma_{j,m'}|\right)\Big\},$$
$$|W(\{\gamma_{j,m}\})|\leq$$
$$\text{const.}\exp\Big\{\frac{n}{4}\sum_{j=1}^{n-1}\left(\sum_{m=1}^{\nu_j}\sum_{m'=1}^{\nu_{j-1}}|\gamma_{j,m}-\gamma_{j-1,m'}|\right)\Big\},$$

where const. is a constant independent of $\{\gamma_{j,m}\}$.

The asymptotic behaviour of φ is as follows (see [?]):

$$\varphi(\beta) \sim \exp\Big\{-\left(\frac{n}{4}+\frac{3\pi}{2\rho}\right)|\beta|\Big\}, \quad \beta \to \pm\infty. \tag{81}$$

Therefore, we have

$$|\text{the integrand of } I(w,W)| \le \text{const.}\exp\Big(-\frac{\pi}{2\rho}H_{\nu_1,\cdots,\nu_{n-1}}(\{\gamma_{j,m}\})\Big), \tag{82}$$

where

$$H_{\nu_1,\cdots,\nu_{n-1}}(\{\gamma_{j,m}\}) = \\ \sum_{j=1}^{n-1}\left(\sum_{m=1}^{\nu_j}\sum_{m'=1}^{\nu_{j-1}}|\gamma_{j,m}-\gamma_{j-1,m'}| - 2\sum_{1\le m<m'\le\nu_j}|\gamma_{j,m}-\gamma_{j,m'}|\right). \tag{83}$$

Set

$$\nu = \sum_{j=0}^{n-1}\nu_j\,. \tag{84}$$

Let $\gamma_1,\ldots,\gamma_\nu$ be a renumbering of the variables $\{\gamma_{j,m}\}_{\substack{0\le j\le n-1\\ 1\le m\le\nu_j}}$. We consider the integral in the region

$$\gamma_1 < \cdots < \gamma_\nu\,. \tag{85}$$

Set

$$\gamma_{k_1} = \min(\beta_1,\ldots,\beta_N), \tag{86}$$
$$\gamma_{k_2} = \max(\beta_1,\ldots,\beta_N). \tag{87}$$

One can rewrite (??) as

$$H_{\nu_1,\cdots,\nu_{n-1}}(\{\gamma_{j,m}\}) = \sum_{l=1}^{\nu-1}M_l(\gamma_{l+1}-\gamma_l). \tag{88}$$

Let

$$\tilde\gamma_1 < \cdots < \tilde\gamma_{\nu-N} \tag{89}$$

be the renumbering of the variables $\{\gamma_{j,m}\}_{\substack{1\le j\le n-1\\ 1\le m\le\nu_j}}$. We change the integration variables from $\tilde\gamma_k$ to t_k by setting

$$t_k = \begin{cases}\gamma_{k+1}-\gamma_k & \text{if } 1\le k<k_1;\\ \gamma_{k+N}-\gamma_{k+N-1} & \text{if } k_2-N<k\le\nu-N;\\ \tilde\gamma_k & \text{otherwise.}\end{cases} \tag{90}$$

For the convergence, it is enough to show that

$$M_k > 0 \quad (1 \leq k < k_1), \tag{91}$$

and

$$M_{k+N-1} > 0 \quad (k_2 - N < k \leq \nu - N). \tag{92}$$

We will show (??). The proof of (??) is similar.

Set

$$x_j = \sharp\{m; \gamma_{j,m} \leq \gamma_k\} \quad (0 \leq j \leq n-1). \tag{93}$$

Note that $x_0 = 0$ because $k < k_1$. Note also that not all x_j $(1 \leq j \leq n-1)$ are zero. We have

$$M_k = \sum_{j=1}^{n-1} (\nu_{j-1} + \nu_{j+1} - 2\nu_j) x_j + 2 \left(\sum_{j=1}^{n-1} x_j^2 - \sum_{j=1}^{n-2} x_j x_{j+1} \right). \tag{94}$$

The second term of the right hand side of the above formula is positive. Therefore, $M_k > 0$ if the condition (??) is satisfied. □

The hypergeometric pairing has the following property.

Lemma 3

$$\begin{aligned} & I\big(g_{\overline{J}}(\beta_N, \beta_1, \cdots, \beta_{N-1}), W(\beta_1, \cdots, \beta_{N-1}, \beta_N)\big)\Big|_{\beta_N \to \beta_N - \frac{4\pi i}{n}} \\ & \quad = I\big(g_J(\beta_1, \cdots, \beta_N), W(\beta_1, \cdots, \beta_N)\big), \end{aligned} \tag{95}$$

where $\overline{J} = (j_N, j_1, \cdots, j_{N-1})$ *for* $J = (j_1, \cdots, j_N)$, *and the left hand side is understood as the analytic continuation of the integral.*

Proof. We prove (??) in the form

$$\begin{aligned} & I\big(g_{\overline{J}}(\beta_N, \beta_1, \cdots, \beta_{N-1}), W(\beta_1, \cdots, \beta_{N-1}, \beta_N)\big) \\ & \quad = I\big(g_J(\beta_1, \cdots, \beta_N), W(\beta_1, \cdots, \beta_N)\big)\Big|_{\beta_N \to \beta_N + \frac{4\pi i}{n}}. \end{aligned} \tag{96}$$

Note that the integration variables γ_{j,ν_j} $(j = 1, \cdots, j_N)$ are associated with the N-th component in the sense of the operator construction in Section 2. In other words, we have

$$r_{j,\nu_j} \quad = \quad N. \tag{97}$$

Therefore, it is natural to shift the contours for these variables at the same time when we change β_N. In fact, by this simultaneous shift, no crossing of contours

by poles occurs. To see this, it is enough to observe that the function g_J has zeros at $\gamma_{j\pm1,m} = \gamma_{j,\nu_j} + \frac{\pi i}{n}$ if $r_{j\pm1,m} < N$.

We have

$$W(\{\gamma_{j,m}\}_{m\neq N})\Big|_{\substack{\gamma_{j,\nu_j}\to\gamma_{j,\nu_j}+\frac{4\pi i}{n} \\ (0\le j\le j_N)}} = (-1)^{N+\nu_{j_N}+\nu_{j_N+1}-1} W(\{\gamma_{j,m}\}). \quad (98)$$

Note also that (see [?])

$$\frac{\varphi(\beta+\frac{4\pi i}{n})}{\varphi(\beta)} = -\frac{\mathrm{sh}\,\frac{\pi}{\rho}(\beta-\frac{\pi i}{n})}{\mathrm{sh}\,\frac{\pi}{\rho}(\beta+\frac{5\pi i}{n})}. \quad (99)$$

We shift the contours in the integral (??) (with $w = g_J$), and then change the variables $\gamma_{j,r}$ $(1 \le j \le j_N)$: first by

$$\gamma_{j,\nu_j} \to \gamma_{j,\nu_j} + \frac{4\pi i}{n}, \quad (100)$$

and second by

$$\gamma_{j,r} \to \gamma_{j,r+1} \quad (1 \le r \le \nu_j - 1), \quad \gamma_{j,\nu_j} \to \gamma_{j,1}. \quad (101)$$

Rewriting the integrand by using (??) and (??), we get (??). □

From Lemma ?? and Lemma ??, it is easy to show the following theorem.

Theorem 4 *We asuume the condition* (??). *For* $W \in \mathcal{W}_{\nu_1,\cdots,\nu_{n-1}}$, *we set*

$$f_W(\beta_1,\cdots,\beta_N) = \sum_{J\in\mathcal{Z}_{\nu_1,\cdots,\nu_{n-1}}} I(w_J, W)(\beta_1,\cdots,\beta_N)v_J, \quad (102)$$

where $v_J = v_{j_1}\otimes\cdots\otimes v_{j_N}$ *for* $J = (j_1,\cdots,j_N)$. *Then* f_W *is a solution to* (??) *with the restriction* (??) *and* (??).

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