RIMS-1228

Perturbative or Path-Integral Approach *versus* Operator-Formalism Approach

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Abstract

In the conformal-gauge two-dimensional quantum gravity, the solution obtained by the perturbative or path-integral approach is compared with the one obtained by the operator-formalism approach. Treatments of the anomaly problem in both approaches are different. This difference is found to be essentially caused by the fact that the perturbative or path-integral approach is based on the T*-product (covariantized T-product), which generally violates field equations. Indeed, this fact induces some extra one-loop Feynman diagrams, which would not exist unless a nonzero contribution arose from a zero field. Some demerits of the path-integral approach are explicitly demonstrated.

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§1. Introduction

Quantum field theory is undoubtedly a very successful theory except for its mathematically rigorous foundation. The basic quantities of quantum field theory are quantum fields, generically denoted by $\varphi(x)$, which are operator-valued singular functions of N-dimensional space-time x^μ. In the standard formulation, their properties are governed by the action S, which is an N-dimensional integral of the Lagrangian density $\mathcal{L}$ (a function of quantum fields at the same space-time point). Field equations and canonical (anti)commutation relations are derived from S by the standard procedure. Equal-time (anti)commutation relations follow from the canonical ones.

Field equations and equal-time (anti)commutation relations define the operator properties of quantum fields, that is, it is supposed to determine the operator algebra of quantum fields. Next, this operator algebra is represented in terms of state vectors, whose totality forms an infinite dimensional complex linear space equipped with (indefinite) inner product. This way of formulating the theory is operator-formalism approach. Evidently, this way of thinking is most natural as the formulation of quantum fields. Nevertheless, this approach has never been seriously considered as the standard approach to quantum field theory. The main reasons for this are as follows. First, it has been unknown how to find the solution in this approach. Second, it has been unknown how to deal with the divergence problem which arises from the singular nature of quantum fields.

The present-day's standard approach is covariant perturbation theory. It is based on the action S, rather than field equations. Decomposing it artificially into its free part and its interaction part, one obtains all Green's functions explicitly in terms of Feynman diagrams. Furthermore, renormalization theory tells us how to deal with the divergence problem: divergent contributions are absorbed into counter terms.

Path-integral formalism gives us the generating functional, Z, of the Green's functions. If path integral is defined as the infinite multiple integrals with respect to the expansion coefficients of quantum fields in terms of their free wave functions, Z reproduces the perturbative expressions. If path integral is defined by using an expansion in terms of the functions other than free wave functions, it is not clear whether or not Z yields the Green's functions which reproduce the perturbative ones (if expanded), but it is usually supposed that there exists an appropriate path-integral measure giving the expected results. Although we cannot accept this assumption without reservation,*) we do not strictly distinguish perturbative approach and path-integral one in the present paper.

*) This problem is particularly relevant in quantum gravity because gravity's free wave functions are rather artificial quantities. For example, the sum over all possible manifolds having a particular metric signature cannot be realized by a simple infinite multiple integral with respect to expansion coefficients

By taking the logarithm of Z, one obtains the generating functional, W, of connected Green's functions. Moreover, the effective action Γ, which is the generating functional of amputated proper Feynman diagrams, is obtained as the functional Legendre transform of W. The renormalization is neatly carried out in Γ and therefore the anomaly problem is usually discussed also in Γ. It should be noted here that perturbation series is not unique because the decomposition of the action into its free part and its interaction part is generally altered by a nonlinear redefinition of fields. Accordingly, the effective action Γ is a quantity which generally changes under the redefinition of fields. Thus, in the path-integral approach, neither renormalization procedure nor the anomaly problem are quite independent of the choice of quantum fields. This point becomes crucial when unphysical fields, whose natural definitions are not necessarily unique, play important roles as in quantum gravity.

Now, we have recently succeeded in formulating the method of finding the solution in the operator-formalism approach.[1)] Our method is as follows. From the field equations and equal-time (anti)commutation relations, we explicitly construct all independent N-dimensional (anti)commutation relations, by expanding them, if necessary, into the power series with respect to the parameters involved. We then calculate independent N-dimensional multiple (anti)commutators. The representation of the field algebra in terms of state vectors is constructed by giving all n-point Wightman functions ($n = 1, 2, \ldots$), i.e., vacuum expectation values of simple products of n quantum fields, $\varphi_1(x_1)\varphi_2(x_2)\cdots\varphi_n(x_n)$, so as to be consistent with the $(n-1)$ple (anti)commutators and with the energy positivity conditions.[*)] Here, in contrast with the axiomatic field theory,[2)] we need the Wightman functions involving composite fields, where a composite field is a product of fields at the same space-time point. When we set some of space-time points coincident in a higher-point Wightman function, we generally encounter divergent terms, which must be simply discarded in such a way that the resultant be independent of the ordering of the constituent fields of the composite field ("generalized normal product rule"). In this procedure, we do not introduce anything like a counter term. This is because a well-defined representation of the field algebra should be free of divergence; our standpoint is similar to that of the Lehmann-Symanzik-Zimmermann formalism,[3)] in which they developed the renormalized perturbation theory without encountering any unrenormalized quantities.

Of course, it is extremely difficult to carry out our way of finding the solution in realistic models. But, fortunately, we can explicitly construct the exact solutions in some two-

because a definite metric signature requires to insert a product of θ functions of the determinant and principal minors of $g_{\mu\nu}$ into the path integral.

[*)] The Wightman function is a boundary value of an analytic function of the variables ${x_i}^0 - {x_j}^0 \quad (i < j)$ from the lower half-planes.

dimensional models by our method.[4)-8)] Our results are seen to be quite satisfactory, but we encounter an anomalous phenomenon, which we call "field-equation anomaly", in quantum-gravity models[4)-7)] (but not in gauge-theory models[8)]): By construction, our Wightman functions are consistent with all two-dimensional (anti)commutators but not necessarily consistent with nonlinear field equations because there we encounter products of fields at the same space-time point. In any of the quantum-gravity models which we have exactly solved so far, one of field equations is *slightly* violated at the level of representation.*) This is the field-equation anomaly. It is different from the conventional anomalies which arise in connection with particular symmetries. Rather, as clarified in our previous work,[9)] various conventional anomalies[10)] are systematically explained on the basis of the field-equation anomaly and their ambiguities are shown to be caused by the nonuniqueness of perturbation theory. In this sense, we regard the field-equation anomaly as a more fundamental concept.

The purpose of the present paper is to make comparison between the solution obtained by the perturbative approach and the one obtained by the operator-formalism approach explicitly in the conformal-gauge two-dimensional quantum gravity. The exact solution of this model obtained previously[6)] can be written in terms of *tree diagrams only*. If the same were true also in the perturbative solution, no anomaly could be present. As is well known, however, this model has the conformal anomaly except for $D = 26$, where D denotes the number of the scalar fields which can be interpreted as the string coordinates. We trace the cause of this paradox and find that the perturbative approach induces some one-loop Feynman diagrams, which would not exist unless a nonzero contribution arose from a zero field. The cause of this strange phenomenon is found to be the use of T*-product (covariantized T-product) of quantum fields**) in the perturbative or path-integral approach. More generally, in the present paper, we clarify that various anomalous behaviors of this model found in the perturbative approach are caused by the use of T*-product.

In the present paper, we compare the perturbative or path-integral approach with the operator-formalism approach in the conformal-gauge two-dimensional quantum gravity, whose Lagrangian density is given in §2. In §3, §4 and §5, respectively, we discuss this model by the operator-formalism approach, by the perturbative approach and by the path-integral approach. In §6, we criticize the so-called "FP-ghost number current anomaly". The final section is devoted to discussions.

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*) The violation is *slight* in the sense that an anomaly-free equation can be obtained by differentiating the original field equation once or twice.

**) T*-product is a T-product modified in such a way that $\partial_1{}^0 \mathrm{T}^*\varphi_1(x_1)\cdots\varphi_n(x_n) = \mathrm{T}^*\partial_1{}^0\varphi_1(x_1)\cdots\varphi_n(x_n)$.

§2. Preliminaries

Throughout the present paper, we consider the BRS formalism of the conformal-gauge two-dimensional quantum gravity, in which the conformal degree of freedom is already eliminated.[11] Quantum fields are contravariant tensor-density gravitational field $\tilde{g}^{\mu\nu}$, FP ghost c^{μ}, FP anti-ghost $\bar{c}_{\mu\nu}$, B field $\tilde{b}_{\mu\nu}$ and scalar fields ϕ_M ($M = 0, 1, \ldots, D-1$). Their BRS transforms are as follows:

$$\boldsymbol{\delta}_* \tilde{g}^{\mu\nu} = \tilde{g}^{\mu\sigma}\partial_\sigma c^\nu + \tilde{g}^{\nu\sigma}\partial_\sigma c^\mu - \partial_\sigma(\tilde{g}^{\mu\nu}c^\sigma), \tag{2·1}$$

$$\boldsymbol{\delta}_* c^\mu = -c^\sigma \partial_\sigma c^\mu, \tag{2·2}$$

$$\boldsymbol{\delta}_* \bar{c}_{\mu\nu} = i\tilde{b}_{\mu\nu}, \tag{2·3}$$

$$\boldsymbol{\delta}_* \tilde{b}_{\mu\nu} = 0, \tag{2·4}$$

$$\boldsymbol{\delta}_* \phi_M = -c^\sigma \partial_\sigma \phi_M. \tag{2·5}$$

Since $\tilde{g}^{\mu\nu}$ has only two degrees of freedom because $\det\tilde{g}^{\mu\nu} = -1$, it is parametrized as

$$\tilde{g}^{\mu\nu} = (\eta^{\mu\nu} + h^{\mu\nu})(1 - \det h^{\sigma\tau})^{-1/2}, \tag{2·6}$$

where $h^{\mu\nu}$ is symmetric and traceless ($\eta_{\mu\nu}h^{\mu\nu} = 0$). Correspondingly, $\bar{c}_{\mu\nu}$ and $\tilde{b}_{\mu\nu}$ are also symmetric and traceless. It is convenient to rewrite any traceless symmetric tensor $X_{\mu\nu}$ into a vector-like quantity X^λ by

$$X^\lambda = \frac{1}{\sqrt{2}}\xi^{\lambda\mu\nu}X_{\mu\nu}, \tag{2·7}$$

where $\xi^{\lambda\mu\nu} = 1$ for $\lambda+\mu+\nu$ =even, $= 0$ otherwise. According to (2·7), we introduce h_λ, $\bar{c}^\lambda$ and $\tilde{b}^\lambda$.

The BRS-invariant action $S = \int d^2x\,\mathcal{L}$ is given by the Lagrangian density

$$\mathcal{L} = \frac{1}{2}\tilde{g}^{\mu\nu}\partial_\mu\phi_M \cdot \partial_\nu\phi^M - \frac{1}{2}\tilde{b}^\lambda h_\lambda - \frac{i}{2}\bar{c}^\lambda \boldsymbol{\delta}_*(h_\lambda), \tag{2·8}$$

where $\tilde{g}^{\mu\nu}$ is given by (2·6), that is,

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{\rm I}, \tag{2·9}$$

with

$$\mathcal{L}_0 = \frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi_M \cdot \partial_\nu\phi^M - \frac{1}{2}\tilde{b}^\lambda h_\lambda - \frac{i}{\sqrt{2}}\xi_{\lambda\mu\nu}\bar{c}^\lambda\partial^\mu c^\nu, \tag{2·10}$$

$$\mathcal{L}_{\rm I} = h_\lambda\left(\frac{1}{2\sqrt{2}}\xi^{\lambda\mu\nu}\partial_\mu\phi_M \cdot \partial_\nu\phi^M - \frac{i}{2}\xi^{\lambda\mu\nu}\xi_{\rho\mu\sigma}\bar{c}^\rho\partial_\nu c^\sigma - \frac{i}{2}\partial_\sigma\bar{c}^\lambda \cdot c^\sigma\right) + O(h^2). \tag{2·11}$$

The higher-order terms $O(h^2)$ are unnecessary to be specified because they contribute neither to field equations nor to canonical (anti)commutation relations. Furthermore, they give no contribution to perturbation theory. Thus, we may discard them.

It should be noted that the action is invariant under the FP-ghost conjugation*)

$$\begin{aligned} c^\lambda &\longrightarrow \bar{c}^\lambda, \\ \bar{c}^\lambda &\longrightarrow c^\lambda, \\ \tilde{b}^\lambda &\longrightarrow \tilde{b}^\lambda - i\xi^{\lambda\mu\nu}\xi_{\rho\mu\sigma}\bar{c}^\rho\partial_\nu c^\sigma - i\partial_\sigma\bar{c}^\lambda\cdot c^\sigma \\ &\qquad + i\xi^{\lambda\mu\nu}\xi_{\rho\mu\sigma}c^\rho\partial_\nu\bar{c}^\sigma + i\partial_\sigma c^\lambda\cdot\bar{c}^\sigma \ + \ O(h), \end{aligned} \tag{2·12}$$

as in the de Donder-gauge case.

Analysis can be much simplified by introducing light-cone coordinates $x^\pm = (x^0 \pm x^1)/\sqrt{2}$, because then $\xi_{\mu\nu\lambda} = 0$ except $\xi_{+++} = \xi_{---} = \sqrt{2}$. From (2·10) and (2·11) ($O(h^2)$ is omitted), we have

$$\mathcal{L}_0 = \partial_+\phi_M\cdot\partial_-\phi^M + \left[-\frac{1}{2}\tilde{b}^+h_+ - i\bar{c}^+\partial_-c^+ + \quad (+ \longleftrightarrow -) \right], \tag{2·13}$$

$$\begin{aligned} \mathcal{L}_{\rm I} = h_+\Bigg[&\frac{1}{2}\partial_+\phi_M\cdot\partial_+\phi^M - i\bar{c}^+\partial_+c^+ - \frac{i}{2}\left(\partial_+\bar{c}^+\cdot c^+ + \partial_-\bar{c}^+\cdot c^-\right)\Bigg] \\ &+ \quad (+ \longleftrightarrow -). \end{aligned} \tag{2·14}$$

In subsequent sections, we start with (2·9) together with (2·13) and (2·14). For later convenience, we introduce the following notation.

$$\tilde{\mathcal{T}}^\pm \equiv \partial_\pm\phi_M\cdot\partial_\pm\phi^M - 2i\bar{c}^\pm\partial_\pm c^\pm - i\partial_\pm\bar{c}^\pm\cdot c^\pm, \tag{2·15}$$

$$\tilde{\tilde{\mathcal{T}}}^\pm \equiv \tilde{\mathcal{T}}^\pm - i\partial_\mp\bar{c}^\pm\cdot c^\mp. \tag{2·16}$$

Then we have

$$\frac{\partial\mathcal{L}_{\rm I}}{\partial h_\pm} = \frac{1}{2}\tilde{\tilde{\mathcal{T}}}^\pm. \tag{2·17}$$

The Noether currents of the BRS invariance and the FP-ghost number conservation are given by

$$j_b{}^\mp = -c^\pm\partial_\pm\phi_M\cdot\partial_\pm\phi^M - i\bar{c}^\pm c^\pm\partial_\pm c^\pm, \tag{2·18}$$

$$j_c{}^\mp = -i\bar{c}^\pm c^\pm, \tag{2·19}$$

respectively. They are of course conserved.

*) So far, this fact has been overlooked because the FP antighost was treated as a tensor.

§3. Operator-formalism Approach

For the sake of comparison, we briefly review our previous results of the exact solution obtained by the operator-formalism approach.[6)]

The field equations are as follows:

$$h_{\pm} = 0, \tag{3·1}$$

$$\tilde{b}^{\pm} = \tilde{\mathcal{T}}^{\pm}, \tag{3·2}$$

$$\partial_{\mp} X^{\pm} = 0 \quad \text{for } X^{\pm} = c^{\pm},\ \bar{c}^{\pm},\ \tilde{b}^{\pm}, \tag{3·3}$$

$$\partial_{+}\partial_{-}\phi_M = 0, \tag{3·4}$$

where (3·3) for $X^{\pm} = \tilde{b}^{\pm}$ is derived by differentiating (3·2). Note that $c^{\pm}$, $\bar{c}^{\pm}$, $\tilde{b}^{\pm}$ and $\partial_{\pm}\phi_M$ are the functions of a single variable $x^{\pm}$ only.

Canonical quantization is carried out by taking ϕ_M and $c^{\pm}$ only as the canonical variables. Since they are free fields, their nonvanishing two-dimensional (anti)commutators are easily obtained; we have

$$[\partial_{\pm}\phi_M(x),\ \phi^N(y)] = -\frac{i}{2}\delta_M{}^N\delta(x^{\pm} - y^{\pm}), \tag{3·5}$$

$$\{c^{\pm}(x),\ \bar{c}^{\pm}(y)\} = -\delta(x^{\pm} - y^{\pm}). \tag{3·6}$$

The commutation relations involving $\tilde{b}^{\pm}$ are calculated by using (3·2):

$$[\tilde{b}^{\pm}(x),\ \phi_M(y)] = -i\partial_{\pm}\phi_M(x)\cdot\delta(x^{\pm} - y^{\pm}), \tag{3·7}$$

$$[\tilde{b}^{\pm}(x),\ c^{\pm}(y)] = -ic^{\pm}(x)\delta'(x^{\pm} - y^{\pm}) - 2i\partial_{\pm}c^{\pm}(x)\cdot\delta(x^{\pm} - y^{\pm}), \tag{3·8}$$

$$[\tilde{b}^{\pm}(x),\ \bar{c}^{\pm}(y)] = 2i\bar{c}^{\pm}(x)\delta'(x^{\pm} - y^{\pm}) + i\partial_{\pm}\bar{c}^{\pm}(x)\cdot\delta(x^{\pm} - y^{\pm}), \tag{3·9}$$

$$[\tilde{b}^{\pm}(x),\ \tilde{b}^{\pm}(y)] = i[\tilde{b}^{\pm}(x) + \tilde{b}^{\pm}(y)]\delta'(x^{\pm} - y^{\pm}). \tag{3·10}$$

Evidently, (3·10) is the BRS transform of (3·9).

Since no new operators are encountered in the right-hand sides of (3·5)-(3·10), we can easily calculate all multiple (anti)commutators explicitly. We then construct all truncated*) Wightman functions so as to be consistent with all multiple (anti)commutators under the energy positivity condition.

The 1-point functions are, in principle, completely arbitrary. But we set all of them equal to zero because we should not deliberately violate any of (3·1), FP-ghost number conservation, BRS invariance and $O(D)$ symmetry.

*) Truncation means to drop the contributions from vacuum intermediate states. The *truncated* Wightman function corresponds to the *connected* Green's function. In the present model, the distinction between truncated and nontruncated appears only for $n \geqq 4$.

The nonvanishing *truncated* n-point Wightman functions are those which consist of $(n-2)$ $\tilde{b}^{\pm}$'s and of either $c^{\pm}$ and $\bar{c}^{\pm}$ or two ϕ_M's. Diagrammatically, they are represented by tree diagrams. Although we have explicitly constructed all of them,[6] we here quote 2-point and 3-point ones only.

Nonvanishing 2-point Wightman functions are

$$\langle\, \phi_M(x_1)\phi^N(x_2)\,\rangle = \delta_M{}^N D^{(+)}(x_1-x_2), \tag{3·11}$$

$$\langle\, c^{\pm}(x_1)\bar{c}^{\pm}(x_2)\,\rangle = \langle\, \bar{c}^{\pm}(x_1)c^{\pm}(x_2)\,\rangle = -2i\partial_{\pm}D^{(+)}(x_1-x_2), \tag{3·12}$$

where*)

$$\partial_{\pm}D^{(+)}(x) \equiv -\frac{1}{4\pi}\cdot\frac{1}{x^{\pm}-i0}. \tag{3·13}$$

Nonvanishing 3-point ones are

$$\langle\, \phi_M(x_1)\tilde{b}^{\pm}(x_2)\phi^N(x_3)\,\rangle = -2\delta_M{}^N\partial_{\pm}D^{(+)}(x_1-x_2)\cdot\partial_{\pm}D^{(+)}(x_2-x_3), \tag{3·14}$$

$$\begin{aligned}\langle\, c^{\pm}(x_1)\tilde{b}^{\pm}(x_2)\bar{c}^{\pm}(x_3)\,\rangle &= 8i\partial_{\pm}D^{(+)}(x_1-x_2)\cdot\partial_{\pm}{}^2D^{(+)}(x_2-x_3)\\ &\quad -4i\partial_{\pm}{}^2D^{(+)}(x_1-x_2)\cdot\partial_{\pm}D^{(+)}(x_2-x_3)\end{aligned} \tag{3·15}$$

and their permutated ones, whose expressions are obtained from the above by changing some of $D^{(+)}(x)$'s into $-[D^{(+)}(x)]^*$ so as to become consistent with the energy-positivity condition (and by changing the overall sign if the order of $c^{\pm}$ and $\bar{c}^{\pm}$ is reversed).

Our system of Wightman functions is, of course, consistent with the field algebra defined by (3·5)-(3·10). It is also consistent with the BRS invariance and the FP-ghost number conservation. It should be noted that we need the use of the generalized normal-product rule to check the BRS invariance. Our system of Wightman functions is also consistent with all *linear* field equations (3·1), (3·3) and (3·4), but *not* with the *nonlinear* field equation (3·2). Indeed, by using the generalized normal-product rule, we can show that

$$\langle\, \tilde{b}^{\pm}(x_1)\tilde{b}^{\pm}(x_2)\,\rangle = 0, \tag{3·16}$$

$$\begin{aligned}\langle\, \tilde{b}^{\pm}(x_1)\tilde{\mathcal{T}}^{\pm}(x_2)\,\rangle &= \langle\, \tilde{\mathcal{T}}^{\pm}(x_1)\tilde{\mathcal{T}}^{\pm}(x_2)\,\rangle\\ &= 2(D-26)[\partial_{\pm}{}^2D^{(+)}(x_1-x_2)]^2\end{aligned} \tag{3·17}$$

in contradiction with (3·2). Thus the field equation (3·2), *modulo* (3·3) for $X^{\pm}=\tilde{b}^{\pm}$, is violated at the level of the representation in terms of state vectors. We call this matter "field-equation anomaly". This phenomenon is encountered also in several two-dimensional quantum-gravity models.[4)-7)]

*) $D^{(+)}(x)$ itself is infrared divergent and therefore requires the introduction of infrared cutoff.[12)]

The BRS Noether current (2·18) can be rewritten as

$$j_b{}^{\mp} = j'_b{}^{\mp} + (\tilde{b}^{\pm} - \tilde{\mathcal{T}}^{\pm})c^{\pm}, \tag{3·18}$$

with

$$j'_b{}^{\mp} \equiv -\tilde{b}^{\pm}c^{\pm} + i\bar{c}^{\pm}c^{\pm}\partial_{\pm}c^{\pm}. \tag{3·19}$$

At the operator level, $j'_b{}^{\mp}$ strictly equals $j_b{}^{\mp}$. But this equality no longer holds at the representation level because of the appearance of the field-equation anomaly. Indeed, $j_b{}^{\mp}$ is anomalous for $D \neq 26$, while $j'_b{}^{\mp}$ is free of anomaly for any value of D.[6),7)] On the other hand, $j_c{}^{\mp}$ is free of anomaly without making any modification.

§4. Perturbative approach

The perturbative approach is so familiar to everybody that no explanation about it is necessary. Nevertheless, when compared with the operator-formalism approach, the perturbative approach is seen to yield some surprising results.

The Lagrangian density $\mathcal{L}$ is decomposed into the free one $\mathcal{L}_0$, which is quadratic with respect to the fields adopted as the basic ones, and the remainder $\mathcal{L}_{\rm I}$, called the interaction Lagrangian density.

The Feynman propagators are obtained by taking the inverse of the differential operator sandwiched by the fields in $\mathcal{L}_0$. Thus, from (2·13), we have the following nonvanishing Feynman propagators:[*)]

$$\langle\, \mathrm{T}^*\, \phi_M(x_1)\phi^N(x_2)\,\rangle_0 = \delta_M{}^N D_{\rm F}(x_1 - x_2), \tag{4·1}$$

$$\langle\, \mathrm{T}^*\, \tilde{b}^{\pm}(x_1)h_{\pm}(x_2)\,\rangle_0 = -2i\delta^2(x_1 - x_2), \tag{4·2}$$

$$\langle\, \mathrm{T}^*\, \bar{c}^{\pm}(x_1)c^{\pm}(x_2)\,\rangle_0 = -2i\partial_{\pm}D_{\rm F}(x_1 - x_2), \tag{4·3}$$

where

$$\begin{aligned}\partial_{\pm}D_{\rm F}(x) &\equiv \theta(x^0)\partial_{\pm}D^{(+)}(x) + \theta(-x^0)\partial_{\pm}[D^{(+)}(x)]^* \\ &= -\frac{1}{4\pi}\left[\frac{\theta(x^{\mp}+x^{\pm})}{x^{\pm}-i0} + \frac{\theta(-x^{\mp}-x^{\pm})}{x^{\pm}+i0}\right].\end{aligned} \tag{4·4}$$

It is quite remarkable that $\langle\, \mathrm{T}^*\, \tilde{b}^{\pm}\, h_{\pm}\,\rangle_0$ is *nonvanishing* in spite of the fact that $h_{\pm}$ is a *zero* operator as is seen from (3·1). In contrast with the Wightman functions, the T*-product does not respect the validity of the field equations. As is seen from (4·3), it is

*) The subscript 0 indicates that the propagators are free ones.

also inadmissible to set $\partial_{\mp}c^{\pm} = \partial_{\mp}\bar{c}^{\pm} = 0$ in the perturbative approach. Hence we cannot discard the terms involving $\partial_{\mp}\bar{c}^{\pm}$ in $\mathcal{L}_{\rm I}$, that is, we have to distinguish $\tilde{\tilde{\mathcal{T}}}^{\pm}$ from $\tilde{\mathcal{T}}^{\pm}$. Thus the beautiful result of the operator formalism that $c^{\pm}$, $\bar{c}^{\pm}$, $\tilde{b}^{\pm}$ and $\partial_{\pm}\phi_M$ are irrelevant to $x^{\mp}$ is no longer valid in the perturbative approach. This fact makes the perturbative calculation complicated and sometimes misleading, as we shall see later.

By using $\mathcal{L}_{\rm I}$ given by (2·14), we can easily calculate the n-point Green's functions. For example, we have

$$\langle\, \mathrm{T}^*\, \phi_M(x_1)\tilde{b}^{\pm}(x_2)\phi^N(x_3)\,\rangle = -2\delta_M{}^N\partial_{\pm}D_{\rm F}(x_1-x_2)\cdot\partial_{\pm}D_{\rm F}(x_2-x_3), \tag{4·5}$$

$$\begin{aligned}\langle\, \mathrm{T}^*\, c^{\pm}(x_1)\tilde{b}^{\pm}(x_2)\bar{c}^{\pm}(x_3)\,\rangle &= 8i\partial_{\pm}D_{\rm F}(x_1-x_2)\cdot\partial_{\pm}{}^2D_{\rm F}(x_2-x_3)\\ &\quad -4i\partial_{\pm}{}^2D_{\rm F}(x_1-x_2)\cdot\partial_{\pm}D_{\rm F}(x_2-x_3),\end{aligned} \tag{4·6}$$

$$\langle\, \mathrm{T}^*\, c^{\pm}(x_1)\tilde{b}^{\pm}(x_2)\bar{c}^{\mp}(x_3)\,\rangle = -2\delta^2(x_1-x_2)\partial_{\mp}D_{\rm F}(x_2-x_3). \tag{4·7}$$

Evidently, (4·5) and (4·6) correspond to (3·14) and to (3·15), respectively. However, (4·7) is a result peculiar to the T*-product. This result is seen to be consistent with the Ward-Takahashi identity

$$\langle\, \mathrm{T}^*\, \boldsymbol{\delta}_*\,(c^{\pm}(x_1)\bar{c}^{\pm}(x_2)\bar{c}^{\mp}(x_3))\,\rangle = 0, \tag{4·8}$$

because the second term of $\boldsymbol{\delta}_*\, c^{\pm} = -c^{\pm}\partial_{\pm}c^{\pm} - c^{\mp}\partial_{\mp}c^{\pm}$ contributes.

Now, we come to the crucial point. In sharp contrast with the case of the operator-formalism approach, the perturbative approach yields quite a nontrivial result for the n-point Green's function consisting of B-fields only. Indeed, its connected part is given by a sum over one-loop Feynman diagrams. For example, we consider $\langle\, \mathrm{T}^*\, \tilde{b}^{\lambda}(x_1)\tilde{b}^{\rho}(x_2)\,\rangle$. Because of the nonvanishing of (4·2), the second-order perturbation term yields

$$\langle\, \mathrm{T}^*\, \tilde{b}^{\lambda}(x_1)\tilde{b}^{\rho}(x_2)\,\rangle = \langle\, \mathrm{T}^*\, \tilde{\tilde{\mathcal{T}}}^{\lambda}(x_1)\tilde{\tilde{\mathcal{T}}}^{\rho}(x_2)\,\rangle_0. \tag{4·9}$$

Therefore, we have

$$\langle\, \mathrm{T}^*\, \tilde{b}^{\pm}(x_1)\tilde{b}^{\pm}(x_2)\,\rangle = 2(D-26)[\partial_{\pm}{}^2D_{\rm F}(x_1-x_2)]^2, \tag{4·10}$$

$$\langle\, \mathrm{T}^*\, \tilde{b}^{\pm}(x_1)\tilde{b}^{\mp}(x_2)\,\rangle = -\frac{D-2}{2}[\delta^2(x_1-x_2)]^2. \tag{4·11}$$

They are divergent and therefore require the introduction of counter terms. Note that the use of the T*-product is responsible for the appearance of these divergences.

The nonvanishing of the Green's functions consisting of B-fields only implies the violation of the BRS invariance. In the de Donder gauge case, Takahashi[13] proposed to convert the violation of the BRS invariance for $D \neq 26$ in the two-point B-field Green's function into

the conformal anomaly.*) We apply his line of thought to the present model. In addition to (4·10) and (4·11), we must take it into account the following exact two-point Green's functions:

$$\langle\, \mathrm{T}^*\, h_\lambda(x_1) h_\rho(x_2)\,\rangle = 0, \tag{4·12}$$

$$\langle\, \mathrm{T}^*\, \tilde{b}^\lambda(x_1) h_\rho(x_2)\,\rangle = -2i\delta_\rho{}^\lambda \delta^2(x_1 - x_2). \tag{4·13}$$

The two-point functions of the effective action Γ is obtained by taking the matrix inverse of (4·10)-(4·13). Accordingly, we have

$$\Gamma = \int d^2x_1 \int d^2x_2 \Bigg[-\frac{1}{2}\delta^2(x_1 - x_2)\tilde{b}^\lambda(x_1) h_\lambda(x_2) + \frac{D-26}{2} i \sum_{\alpha=\pm} ([\partial_\alpha{}^2 D_{\mathrm{F}}(x_1 - x_2)]^2)_{\mathrm{R}} h_\alpha(x_1) h_\alpha(x_2) + \cdots \Bigg], \tag{4·14}$$

where a subscript R indicates regularization. The BRS-violating term in (4·14) is converted into the conformal-anomaly term by adding the conformal degree of freedom. We do not work out this procedure in detail because it is not our aim to do so.

The important point is the violation of the BRS invariance in the B-field Green's functions. In the de Donder gauge case, the BRS violation has arisen by applying the dimensional regularization only to internal lines but *not to external lines*.[9),14)] In the present model, external lines are absent because (4·2) is local. Instead, as is seen from (4·9), perturbative approach makes use of the field-equation anomaly without being aware of this fact.

§5. Path-integral approach

The path integral directly gives us the generating function of the Green's functions, that is, it deals with the T*-product quantities only. The path integral Z is formally expressed as

$$Z(J) = \int \Big(\prod_i \mathcal{D}\varphi_i\Big)\, \exp\, i\int d^N x(\mathcal{L} + \sum_i J_i \varphi_i) \tag{5·1}$$

with $Z(0) = 1$, where J_i denotes the source function corresponding to the field φ_i.

It is possible to derive the path-integral formula [corrected by the Lee-Yang term proportional to $\delta^N(0)$] from the canonical operator formalism.[15)] In this sense, the path-integral formalism can be regarded as the one equivalent to the operator formalism. But, one should

*) He made no mention about how to remove the BRS violation in the *higher-point* functions.

note that, in this derivation, one must use the *field equations at the representation level*. This fact implies that the path-integral formalism cannot take care of the existence of the field-equation anomaly.

From the successful experience of discussing the anomaly problem in gauge theories, it has been customary to believe that any anomaly always arises from the non-invariance of the path-integral measure under the symmetry which leaves the action S invariant. But we point out that anomalies can arise also from the field-equation anomaly which is beyond the scope of the path-integral formalism.

Let $F(\varphi)$ be an arbitrary function of φ_i's. The path-integral measure is supposed to be invariant under the functional translation $\varphi_i \longrightarrow \varphi_i + \delta\varphi_i$. Hence, by considering a variation of a field φ_i in $F(i^{-1}\partial/\partial J)Z|_{J=0}$, we obtain

$$i\langle \, \mathrm{T}^* \, F(\varphi)\frac{\delta S}{\delta\varphi_i} \, \rangle + \langle \, \mathrm{T}^* \, \frac{\delta F(\varphi)}{\delta\varphi_i} \, \rangle = 0. \tag{5·2}$$

This equation corresponds to the field equation $\delta S/\delta\varphi_i = 0$ of the operator formalism. The second term of (5·2) is a field-equation violating term due to the use of the T*-product. One should never confuse it with the field-equation anomaly. For example, in the conformal-gauge two-dimensional quantum gravity, (4·14) is reproduced from (5·2) by setting $F = \tilde{b}^\lambda$ and $\varphi_i = \tilde{b}^\rho$. Likewise, if we set $F = \tilde{b}^\lambda$ and $\varphi_i = h_\rho$ in (5·2), we obtain

$$\langle \, \mathrm{T}^* \, \tilde{b}^\lambda(\tilde{b}^\rho - \tilde{\tilde{\mathcal{T}}}^\rho) \, \rangle = 0, \tag{5·3}$$

that is, we do not encounter the field-equation anomaly. Instead, as is shown in the perturbative approach, (5·3) induces the violation of the BRS invariance in the path-integral approach.

Historically, the anomaly problem in the conformal-gauge two-dimensional quantum gravity was discussed first by Fujikawa [16] in the path-integral formalism. His formulation is not, however, understandable in the framework stated in §4.

First, he takes all *three* degrees of freedom of $g_{\mu\nu}$ as path-integration variables; nevertheless, each of his ghost fields has only *two* degrees of freedom. The extra one degree of freedom is the conformal one, denoted by ρ, is *not allowed*[*)] to be integrated (until the Liouville action is derived) in spite of the fact that it is an independent path-integration variable.

Second, by introducing tilde fields $\tilde{\varphi}_i = \rho^{n_i}\varphi_i$, n_i being a certain fractional number, he claims that the path-integral measure becomes BRS invariant if it is expressed in terms

*) If one carries out the integration over ρ after introducing the conformal ghosts, then it becomes impossible to work out his analysis.

of the tilde fields. He then derives the Liouville action expressed in terms of ρ alone by calculating the variation of the path-integral measure under the conformal transformation. That is, according to his theory, the conformal anomaly is directly obtained *without passing through the BRS anomaly* in contradiction to the consideration presented in §4.

We are thus unable to reproduce his analysis in terms of the explicit solution.

§6. FP-ghost number current anomaly

As we emphasized previously,[6] there is no FP-ghost number anomaly in the conformal-gauge two-dimensional quantum gravity: The exact solution is completely consistent with the FP-ghost number conservation. The conservation of the FP-ghost number current $j_c{}^\mu$ is a simple consequence of the fact that $c^\pm(x)$ and $\bar{c}^\pm(x)$ are independent of $x^\mp$. This property is never violated at the representation level. Nevertheless, many authors have claimed that the FP-ghost number current has anomaly. The reasons for the occurrence of this belief are its correspondence to the Riemann-Roch theorem and the field-equation-violating property of the T*-product.

Fujikawa[16] was the first to claim the existence of the FP-ghost number current anomaly. He derived it by making the FP-ghost number transformation*) in his path-integral formalism described at the end of §5. His result is written as

$$\partial_\lambda \langle\!\langle\, j_c{}^\lambda \,\rangle\!\rangle = \frac{3}{4\pi} \langle\!\langle\, \partial^2 \log\rho \,\rangle\!\rangle, \tag{6·1}$$

where we denote the path integration by $\langle\!\langle\ \cdots\ \rangle\!\rangle$. If the degrees of the reparametrization freedom is suppressed, one may write $-\partial^2 \log\rho = \sqrt{g}R$ (Euclidean metric is used). Here we must note that Fujikawa's theory is formulated in the *flat* background metric and that ρ is the path-integration variable.

Shortly later, Friedan, Martinec and Shenker,[17] who formulated conformal field theory, quoted (6·1) in the disguised form. They consider a completely *curved* background metric $\hat{g}_{\mu\nu}$. The right-hand side of their equation is const.$\sqrt{\hat{g}}\hat{R}$, *a function of* $\hat{g}_{\mu\nu}$, which is nothing but the quantity required by the Riemann-Roch theorem under the prerequisite of the conformal covariance. It may be certainly *analogous* to (6·1), but we cannot find any *logical connection* between them.**)

*) This must be done in the original tilde-variable expression. No anomaly is derived by the naive calculation if this transformation is made in the equivalent expression having the Liouville action explicitly.

**) According to Fujikawa (private communication), the possible existence of the zero points of ρ can take care of the effect of the topological number $\int d^2x\,\sqrt{g}\hat{R}$. We do not see how this idea can be formulated.

In the perturbative approach, the Friedan-Martinec-Shenker version of (6·1) is interpreted, through the consideration based on the effective action, as the matter that[18),19)]

$$J^{\lambda}{}_{\mu\nu} \equiv \langle \, \mathrm{T}^{*} \, j_{c}{}^{\lambda} \frac{\delta S}{\delta \hat{g}^{\mu\nu}} \, \rangle |_{\hat{g}^{\mu\nu}=\eta^{\mu\nu}} \tag{6·2}$$

has a nonvanishing nonlocal term, where $\hat{g}_{\mu\nu}$ is a background metric introduced in such a way that the gauge-fixing plus FP-ghost Lagrangian density becomes background covariant. Note in (6·2) that the background metric is taken to be *flat* in the Feynman-diagram calculation.

In the conformal-gauge case, $J^{\lambda}{}_{\mu\nu}$ is essentially equal to

$$\langle \, \mathrm{T}^{*} \, j_{c}{}^{\pm}(x_1) \widetilde{\mathcal{T}}^{\pm}(x_2) \, \rangle = -12 \partial_{\pm} D_{\mathrm{F}}(x_1 - x_2) \cdot \partial_{\pm}{}^{2} D_{\mathrm{F}}(x_1 - x_2). \tag{6·3}$$

It is in this sense that the FP-ghost number current anomaly is claimed to be obtained in the perturbative approach. It should be noted, however, that *if the T*-product is not taken, that is, if D_{F} is replaced by $D^{(+)}$, (6·3) becomes consistent with* $\partial_{\lambda} j_{c}{}^{\lambda} = 0$.

Düsedau[18)] found that $J^{\lambda}{}_{\mu\nu}$ has no nonlocal term in the de Donder-gauge case. Kraemmer and Rebhan[19)] discussed the gauge dependence of $J^{\lambda}{}_{\mu\nu}$ and claimed that the gauge independence can be recovered if one adds a contribution from the "Lagrange-multiplier (or B-field) current".*) But the relation between this fact and the Riemann-Roch theorem was not discussed.

Recently, Takahashi[13)] has reconsidered Düsedau's analysis from his perturbative approach described in §4. In discussing $j_{c}{}^{\lambda}$, he regards the quantum gravitational field as the background metric, just as Fujikawa did. Rederiving Düsedau's result, he asserts that the vanishing of the FP-ghost number current anomaly can be explained by the existence of the FP-ghost conjugation invariance of the de Donder gauge two-dimensional quantum gravity. One should note, however, that, as pointed out in §2, the *FP-ghost conjugation invariance exists also in the conformal-gauge case*. Therefore, his standpoint would imply the absence of the FP-ghost number current anomaly also in the conformal-gauge case.

Finally, we note that the nonexistence of the FP-ghost number current anomaly can be shown even if the gauge-fixing background metric is a nonflat one given by $\hat{\rho}(x)\eta_{\mu\nu}$, where $\hat{\rho}^{-1}$ *is assumed to exist*. In this case, (2·6) is replaced by

$$\tilde{g}^{\mu\nu} = (\eta^{\mu\nu} + \hat{\rho}^{-1} h^{\mu\nu})(1 - \hat{\rho}^{-2} \det h^{\sigma\tau})^{-1/2}. \tag{6·4}$$

The Lagrangian density of this case is obtained from (2·8) by simply replacing h_{λ} by $\hat{\rho}^{-1} h_{\lambda}$. Since $\boldsymbol{\delta}_{*}(\hat{\rho}^{-1} h_{\lambda}) = \hat{\rho}^{-1} \boldsymbol{\delta}_{*}(h_{\lambda})$, we can absorb the factor $\hat{\rho}^{-1}$ into $\tilde{b}^{\lambda}$ and $\bar{c}^{\lambda}$ by redefining

*) Although we cannot regard their proof as adequate, their claim itself can be verified by explicit calculation.[20)]

them. Thus the ghost part of the Lagrangian density of the nonflat case becomes completely the same as that of the flat case. Thus nothing new can happen about the FP-ghost number current.

§7. Discussion

Nowadays, the path-integral approach and the counter-term business have become so fashionable that many physicists preclude the consideration based on other approaches from the outset. Certainly, the path-integral approach is convenient and successful in gauge theories, but we wish to emphasize that the same is not necessarily true in quantum gravity.

The path-integral formalism directly deals with the solution at the representation level. Accordingly, if one adopts the path-integral approach, one can no longer perceive what happens in the transition from the operator level to the representation level. Indeed, one cannot describe the existence of the field-equation anomaly in the path-integral approach.

The quantities describable by the path-integral formalism are those which can be written in terms of the T*-product. The T*-product is certainly a very convenient notion because we need not take care of the ordering problem even for the timelikely separated field operators. On the other hand, as emphasized in the present paper, the T*-product has a demerit of violating the field equations explicitly. As demonstrated in the present paper, this fact induces unpleasant complications and misleading expressions. Furthermore, since the T*-product contains θ-functions, the Green's function is more singular than the corresponding Wightman functions, that is, some singularities found in the perturbative or path-integral approach may be superficial. When this fact is combined with the counter-term business, one is led to introducing counter terms which are purely of the T*-product origin. In the present paper, we have demonstrated that "anomalies" also can be of the T*-product origin.

We hope that more physicists reinvestigate the anomaly problem in quantum gravity without adhering to the path-integral approach.

Acknowledgements

One of the present authors (N. N.) would like to express his sincere thanks to Professor K. Fujikawa and Dr. H. Kanno for the discussions concerning the FP-ghost number current anomaly.

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