COMPACT KÄHLER MANIFOLDS WHOSE UNIVERSAL COVERING SPACES ARE BIHOLOMORPHIC TO $\mathbb{C}^n$

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ABSTRACT. These manifolds M are complex tori up to finite étale coverings, or satisfy both inequalities $\kappa(M) < n-3$ and $q(M) < n-3$.

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INTRODUCTION

We shall study the following conjecture posed by Iitaka:

Conjecture U_n. If the universal covering space of a compact Kähler manifold M is biholomorphic to $\mathbb{C}^n$, then M admits a finite étale covering from a complex torus.

1991 *Mathematics Subject Classification.* Primary 32J18, 32J27; Secondary 14E20, 32G20.

Key words and phrases. classification of compact Kähler manifolds, torus fibration, variation of Hodge structure, fundamental group, Nevanlinna theory.

The U_1 is obviously true. We can prove U_2 by the classification theory of surfaces. On the other hand, there are some examples of non-Kähler manifolds whose universal covering spaces are $\mathbb{C}^n$.

There are three important things related to the conjecture. First one is a theorem of Kobayashi–Ochiai [KO1]: *There is no non-degenerate holomorphic mapping from $\mathbb{C}^n$ into a variety of general type.* Thus the manifolds M in U_n are not of general type.

Second one is Bogomolov's decomposition theorem [Bo], [Be]: *A compact Kähler manifold with numerically trivial first Chern class possesses a finite étale covering from a product of complex tori, symplectic manifolds and special unitary manifolds.* Therefore if the manifold M in U_n has numerically trivial first Chern class, then the conclusion of U_n for M is true.

The conjecture U_n is hence closely related to the notion of Kodaira dimension κ developed by Iitaka. In fact, in papers [Ii1], [Ii2], he considered projective threefolds with $\kappa = 1$ and applied the structure theorem, i.e., the existence of so-called Iitaka fibration. The general fiber is an abelian surface or a hyperelliptic surface and the singular fibers are not so degenerate by the existence of étale covering from a Stein space. Thus a kind of canonical bundle formula was obtained and it derived a contradiction to $\kappa = 1$.

The third important thing is the abundance conjecture: *If the canonical divisor is nef, then it is semi-ample.* The abundance theorem is established for minimal projective threefolds by Miyaoka [Mi1], [Mi2] and Kawamata [Kw3]. Assume that the manifold M in U_n is projective. Then by the theory of extremal rays by Mori [Mo], the canonical divisor K_M is nef, since it contains no rational curves. Thus the abundance implies that $|mK_M|$ is a free linear system for some $m > 0$. In particular, $\kappa(M) \geq 0$. A general fiber of the Iitaka fibration of M is a para-abelian variety, i.e., a variety admitting a finite étale covering from an abelian variety, by Bogomolov's decomposition theorem. In this case, it was shown in our previous paper [N4] that there exist a finite étale covering $M' \to M$ and $M' \simeq A \times S$, where A is an abelian variety and S is a projective variety with $\dim S = \kappa(M)$. Since there is a surjective holomorphic mapping $\mathbb{C}^n \to S$, S should be a point by Kobayashi–Ochiai's theorem. Hence M is a para-abelian variety. Therefore the conjecture U_n for a projective manifold M is derived from an affirmative answer of the abundance conjecture for M. As a consequence, U_3 is true for projective varieties.

A purpose of this paper is to extend the argument to Kähler case. It is not easy by the lack of hyperplane sections which is essential in the proof in projective case. For example, we can not expect the isomorphism $M' \simeq A \times S$ as above. This is related to the reason why the Poincaré reducibility theorem does not necessarily hold for complex tori. For the simplicity, we call a compact Kähler manifold which admits a finite étale covering from a complex torus by a *complex analytic Q-torus* (*complex Q-torus* or *Q-torus*, for short). Such a manifold is called a *hyperelliptic manifold* in some literature. In our previous paper [N4], a projective Q-torus is called a para-abelian variety. We shall give two methods for the reduction of U_n to the case $\kappa \leq 0$. The idea of the first one is as follows: Let $f\colon X \to S$ be a morphism between nonsingular varieties bimeromorphically equivalent to the Iitaka fibration of M. We assume that general fibers are complex Q-tori. Then the fibration has not so bad degenerate fibers. If it is actually smooth, then we have the canonical bundle formula: $K_X \sim_{\mathbb{Q}} f^*K_S$, where $\sim_{\mathbb{Q}}$ denotes the $\mathbb{Q}$-linear equivalence relation. Therefore, S is of general type. Then we have a meromorphic mapping from $\mathbb{C}^n$ into S which induces surjection at the tangent spaces of general points. This contradicts to Kobayashi–Ochiai's theorem above. In general case, S need not to be of general type, since there are some "branch" loci on S. However, we have a canonical bundle formula $K_X \sim_{\mathbb{Q}} f^*(K_S + \Delta)$ for an effective $\mathbb{Q}$-divisor relating to the branch loci. Here the pair (S, Δ) is of general type. We have a generalization of Kobayashi–Ochiai's theorem for such branched case by Sakai [Sa1]. Since this treats only the extension problem of holomorphic mappings to meromorphic one, we need to extend it to meromorphic case. This is done in 5.3 and further generalization A.6 is also obtained. In this way, we have the following

Theorem (5.4)**.** *Let M be a compact Kähler manifold whose universal covering space is biholomorphic to $\mathbb{C}^n$. Suppose that $\kappa(M) \geq 0$ and that a general fiber of the Iitaka fibration $\Phi\colon M \cdots\to S_0$ is bimeromorphically equivalent to a complex Q-torus. Then M is a complex Q-torus.*

The second method is an application of the ∂-étale cohomology theory developed in [N3]. We study a Kähler torus fibration with finite local monodromies for H_1 of smooth fibers. This is bimeromorphically equivalent to a torus fibration constructed canonically from a torsion element of a ∂-étale cohomolgy group. Further in some cases, there is

a finite étale covering of the total space such that the Stein factorization is bimeromorphically equivalent to a smooth torus fibration. Combining with the notion: *Q-smooth* and *homotopically Q-smooth*, we have another proof of 5.4, where we need not 5.3 but Kobayashi–Ochiai's theorem. Furthermore we can generalize the results of Kollár [Kl2, 6.3] to the following form:

Theorem (8.6). *Let $f\colon M \to S$ be a proper surjective morphism from a compact Kähler manifold onto a normal variety. Suppose that general fibers are bimeromorphically equivalent to complex Q-tori and that there is an étale covering $\tau\colon U \to M$ where $U_s = \tau^{-1}f^{-1}(s)$ is non-compact and simple for 'general' $s \in S$. Then there exist a finite étale covering $M' \to M$ and a smooth torus fibration $Y \to T$ such that Y and M' are bimeromorphically equivalent and $\dim T = \dim S$.*

Here, a non-compact complex manifold is called *simple*, if there is no covering family $\{B_t\}_{t\in T}$ of compact analytic subvarieties of positive dimension (cf. 8.4). We pose the following conjectures U_n^0, U_n^{00} related to U_n:

Conjectures. Let M be an n-dimensional compact Kähler manifold whose universal covering space is non-compact and simple. Under the following assumption U_n^0 or U_n^{00}, M should be bimeromorphically equivalent to a complex Q-torus.

U_n^0: $\kappa(M) \leq 0$.

U_n^{00}: $\kappa(M) \leq 0$ and $q(M) = 0$.

The conjecture U_n is derived from U_k^0 for all $k \leq n$ and U_n^0 is derived from U_k^{00} for all $k \leq n$.

By the methods above, we can reduce the conjecture U_3 to the case $\kappa(M) \leq 0$. We may assume that M is non-projective. Then a result of Gromov [G, 3.2.C′] implies that $\chi(M, \mathcal{O}_M) = 0$. In particular, the irregularity $q(M)$ is not zero. Threefolds with non-trivial Albanese tori are well studied by means of addition theorems and period mappings (cf. [U2]). Further it was shown that a compact Kähler threefold admitting an elliptic fibration is uniruled or has a good minimal model in Appendix B of [N2]. As a consequence, we can show U_3 (cf. 5.8). We can prove also U_3^{00} by the result of Gromov, minimal model theory for projective threefolds and by [NS] (cf. 8.10).

We shall consider the following general form of the conjecture U_n:

Conjecture U'_n. Let M be an n-dimensional compact Kähler manifold and let U be its universal covering space. Assume that there is a meromorphic mapping $Z \cdots\to U$ of maximal rank from a Zariski-open subset Z of a compact complex variety $\overline{Z}$. Then M is a Q-torus if U is Stein. If U is non-compact and simple, then M is bimeromorphically equivalent to a Q-torus.

The conjecture U'_n follows from U^{00}_k for any $k \leq n$ (cf. 8.8, 8.9). Further as in Theorem B of [N4], we have

Theorem (8.12). *Let M be a compact Kähler manifold satisfying the assumption of Conjecture U'_n. Then the Albanese mapping $\alpha\colon M \to \mathrm{Alb}(M)$ is a fiber space. Suppose that a general fiber of the Iitaka fibration of $M_t = \alpha^{-1}(t)$ for general $t \in \mathrm{Alb}(M)$ is bimeromorphically equivalent to a complex Q-torus. Then M is also bimeromorphically equivalent to a complex Q-torus.*

In particular, M is bimeromorphically equivalent to a complex Q-torus, or satisfies both inequalities $\kappa(M) < n-3$ and $q(M) < n-3$.

The construction of our paper is as follows: In section 1, we recall the definition of Kähler space and Kähler morphism due to Fujiki [Fk1]. We also note a result of Varouchas [Va1], [Va2]. We introduce a notion: *cohomologically Kähler morphism* which is important in latter sections. We shall give an extension theorem 1.6 for meromorphic mappings between Kähler morphisms by applying the closedness of Douady space by Fujiki [Fk1].

The structure of a smooth torus fibration over a normal variety is studied in section 2. A variation of Hodge structure of weight -1 and its construction by a monodromy representation and a period mapping are briefly explained. Especially, we shall give an explicit construction of a Kähler metric on the 'universal' torus fibration in 2.1. Its direct proof is written in Appendix B. A smooth torus fibration is called *basic*, if it admits a holomorphic section. It has a structure of a system of Lie groups (a group variety) over the base space, whose zero is the given section. This is reconstructed from the associated variation of Hodge structure. A general smooth torus fibration is considered to be a torsor of some basic fibration. The torsor is determined by an element of the first cohomology group of the sheaf of germs of sections of the basic fibration. We shall describe the element by

using Deligne groups [EZ]. This description and the cocycle obtained in 2.1 are important to characterize a cohomologically Kähler torus fibration (cf. 2.17).

The fundamental group of a complex Q-torus is an extension of a finite group by a lattice (free abelian group of finite rank). Further the group has no torsion elements (cf. 3.5). The group of this kind is called a *Q-lattice*. We study Q-lattices and complex Q-tori in section 3. In particular, we shall show there is a unique finite étale covering from a complex torus with minimal degree. We call the covering by a *toric reduction*. In projective case, it was called an abelian reduction in [N4]. Further we show the existence of the simultaneous toric reduction for a proper smooth morphism fibered by complex Q-tori, after a base change. This is important since we can reduce to usual torus fibrations. The argument here (cf. 3.15) improves that in [Kl2, 6.4].

What kind of fibers appear in a Q-torus fibration whose total space possesses an étale covering from a Stein space? We shall give an answer in section 4. As an application of the extension theorem 1.6, we can prove (cf. 4.3): *If it is a degeneration of complex tori with a section, then it is a smooth basic torus fibration*. We derive results on fibrations of Q-tori with only equi-dimensional fibers, enough for our purpose by combining with an elementary argument of group cohomology (cf. 4.1), which originated in [Vi1, 9.10]. Especially, such fibrations, locally over the base, have finite étale coverings which induce smooth fibrations (cf. 4.7, 4.8). This property is called *Q-smooth* and discussed in section 7.

In section 5, we shall give a first proof for U_3. As an extension of Kobayashi–Ochiai's theorem [KO1], we give a generalization of Sakai's extension theorem (Theorem 1 of [Sa1]) in 5.3. For the proof, we apply the Second Main Theorem of Nevanlinna theory in several complex variable case. We use the formula by Shiffman (cf. Theorem 3.1 of [Sh1]). We have a further generalization in A.6. The argument here is well-known in the Nevanlinna theory, but it seems not to be appearing the result of this form. We shall derive to the contradiction to the assumption $\kappa(M) > 0$ for a compact Kähler manifold M in U_n if a general fiber of the Iitaka fibration is bimeromorphically equivalent to a complex Q-torus (cf. 5.4). The proof is as follows: By changing the base space of Iitaka fibration, we have a situation the Q-torus fibration can be described as in section 4. Here we have a canonical bundle formula, which holds globally since the period mapping is constant (cf.

5.5). This property contradicts to the extension theorem 5.3. For M in the conjecture U_n, the Albanese mapping is a fiber space (cf. 5.2) by the results on characterization of abelian varieties due to Ueno, Kawamata and Viehweg (cf. [U1], [KV], [Kw1]) together with Kobayashi–Ochiai's theorem above. We extend the addition theorem by Kollár [Kl1] (cf. [Vi2]) in the case general fibers are of general type, to the complex analytic situation (cf. 5.7). After the preparation, we can prove the conjecture U_3 for non-projective case by using the result of Gromov [G] in 5.8.

We study in section 6 a bimeromorphic classification problem of torus fibration whose smooth part defines a variation of Hodge structure isomorphic to given one. We assume that the base space S has only quotient singularities, the open subset $S^\star$ over which the variation of Hodge structure H is given, defines a toroidal embedding $S^\star \subset S$ (cf. [KKMS]), and that the local monodromies of H around $S \setminus S^\star$ are finite. We have a theory of ∂-étale cohomologies ([N3]) which is useful in the situation, especially in the case of multiple fibers. Since the monodromies are locally finite, we can construct quite easily a particular basic torus fibration associated with H. We call this by the *characteristic* basic torus fibration. We shall prove that a torus fibration smooth over $S^\star$ inducing same H can be described as a kind of torsor in ∂-étale cohomology theory, provided that it is locally Kähler (cf. 6.5, 6.8). In particular, a locally Kähler torus fibration with only locally finite monodromies is bimeromorphically equivalent to that with only equi-dimensional fibers. For the proof, we apply the extension theorem 1.6 and a ∂-étale version of Deligne groups. The criterion of cohomologically Kähler morphisms is also generalized to this situation (cf. 6.7).

We introduce new notion: *Q-smooth* and *homotopically Q-smooth* morphisms in section 7. Roughly speaking, a Q-smooth morphism is a morphism which is smooth up to a finite étale covering of the total space, and a homotopically Q-smooth morphism is a morphism $X \to S$ such that $\pi_1(X_s) \to \pi_1(X)$ is injective, where X_s is a general fiber. A Kähler Q-torus fibration is locally Q-smooth if the total space admits an étale covering from a Stein space, by results in section 4 and by the equi-dimensionality given in section 6. A Kähler torus fibration with trivial local monodromies is locally Q-smooth, by results in section 6. For the case of Kähler Q-torus fibrations, Q-smoothness implies the homotopically Q-smoothness. We shall prove the converse (cf. 7.8): *If a Kähler Q-torus*

fibration is locally homotopically Q-smooth, then it is bimeromorphically equivalent to a Q-smooth Q-torus fibration.

Finally in section 8, we treat the case the base space is compact. Theorem 8.2 gives another proof of 5.4. The description of torsors by ∂-étale cohomologies works instead of Nevanlinna theory. Further this is a structure theorem for compact Kähler manifolds with Stein universal covering spaces. In [Kl2], Kollár investigated projective manifolds whose universal covering spaces is non-compact and simple. Even though the property is not that of the fundamental group, such manifold was called to have large fundamental group in [Kl2]. This condition is weaker than that the universal covering space is Stein, but is independent of the choice of birational models. We shall give a generalization of [Kl2, 6.3] in 8.6 by using the modification of [Kl2, 6.4] in 3.15 and results in section 7. We have assumed in Theorems 5.4, 8.2 that a general fiber of the Iitaka fibration is bimeromorphically equivalent to a complex Q-torus. For the purpose of reducing U_n to cases $\kappa \leq 0$, we want to show this property by some inductive argument on dimension. However, since Iitaka fibrations are only meromorphic mapping, the universal covering spaces of fibers may not be Stein. Thus we consider the weak condition above and pose conjectures U_n^0 and U_n^{00} for this direction. As in the case of Stein universal coverings, the structure of compact Kähler manifolds whose universal covering spaces is non-compact and simple is reduced to the conjecture U_n^0 and the case of general type (cf. 8.7).

In Appendix A, we shall give a proof of the extension theorem A.6 of meromorphic mappings, along the argument in the book [NO]. In Appendix B, we present a proof of Theorem 2.1 on Kähler metrics, by simple calculation.

The author thanks to Professor Y. Fujimoto for his careful reading of the preliminary version of the paper.

Notation

(1) A complex analytic space is always assumed to be paracompact.

(2) A complex (analytic) variety means a complex analytic space which is reduced and irreducible.

(3) A holomorphic mapping is called a morphism, since it is so in the category of complex analytic spaces.

(4) A *meromorphic mapping* $f\colon X \cdots\to Y$ of complex analytic spaces is defined to be a "graph" $\Gamma_f \subset X \times Y$ satisfying the following conditions:

   (a) The first projection $p_X\colon \Gamma_f \to X$ is proper.

   (b) For a Zariski-open dense subset $U \subset X$, $p_X^{-1}(U) \to U$ is isomorphic and $p_X^{-1}(U)$ is dense in Γ_f.

   The *domain* of f is the biggest open subset of X where p_X is an isomorphism. Thus f is a morphism over the domain. The meromorphic mapping is called *proper* if the second projection $\Gamma_f \to Y$ is proper. f is a *bimeromorphic mapping* if its inverse $f^{-1}$is also well-defined as a meromorphic mapping, i.e., $\Gamma_f = \Gamma_{f^{-1}}$. In particular, a bimeromorphic mapping always is proper. A *bimeromorphic morphism* is a holomorphic bimeromorphic mapping.

(5) A proper surjective morphism $f\colon X \to Y$ of complex analytic varieties is called a *fiber space* or a *fibration*, if $\mathcal{O}_Y \simeq f_*\mathcal{O}_X$. A general (resp. 'general') fiber $X_y = f^{-1}(y)$ of the fibration f means the fiber over the complement of a union of at most finitely (resp. countably) many proper closed analytic subsets of Y. A *torus* fibration is a fibration whose general fibers are isomorphic to complex tori. An elliptic fibration is a torus fibration of relative dimension one.

(6) A meromorphic mapping $f\colon X \cdots\to Y$ is called a *meromorphic fiber space* or a *meromorphic fibration*, if the projection $p_Y\colon \Gamma_f \to Y$ from the graph is a fibration. A general fiber X_y of f is defined to be $p_X(p_Y^{-1}(y))$ for general $y \in Y$.

(7) An étale morphism $f\colon X \to Y$ is defined to be a flat and unramified morphism. In other words, it is a local isomorphism in complex analytic category.

(8) A locally Stein morphism is a morphism $f\colon X \to Y$ such that for any point $y \in Y$, there is a Stein open neighborhood U such that $f^{-1}(U)$ is also Stein. For a locally Stein morphism $f\colon X \to Y$ over a Stein space Y, X is not necessarily a Stein space.

1. Kähler spaces and Kähler morphisms

Let X be a complex analytic space. We follow the same notation as in [Fk1]. The sheaf $\mathcal{E}_X$ of germs of C^∞-*functions* on X is locally defined to be $i^{-1}(\mathcal{E}_M/(I+\bar{I})\mathcal{E}_M)$, where $i\colon X \hookrightarrow M$ is a closed immersion to a nonsingular variety, I is the defining ideal of X and $\mathcal{E}_M$ is the sheaf of germs of C^∞-functions on M. The sheaf $\mathcal{E}_X^{p,q}$ of germs of C^∞-(p,q)-forms and the differentials $\mathrm{d}\,\phi$, $\partial\,\phi$, and $\overline{\partial}\,\phi$ are also defined by a similar way. A *pluri-harmonic* function is a real-valued C^∞-function which is locally written as the real part of a holomorphic function. A *strictly pluri-subharmonic* function is also defined to be locally the restriction of a strictly pluri-subharmonic function p, i.e., $\sqrt{-1}\,\partial\,\overline{\partial}\,p$ is a positive definite $(1,1)$-form, on a nonsingular variety.

Remark. Suppose that X is normal. Let ${}_{\mathbb{R}}\mathcal{E}_X$ be the sheaf of germs of real-valued C^∞-functions on X, ${}_{\mathbb{R}}\mathcal{E}_X^{1,1}$ that of real C^∞-(1, 1)-forms and let $\mathcal{O}_X$ be the structure sheaf, i.e., sheaf of germs of holomorphic functions. Then there is an exact sequence of sheaves:

$$0 \to \mathbb{R}_X \to \mathcal{O}_X \to {}_{\mathbb{R}}\mathcal{E}_X \to {}_{\mathbb{R}}\mathcal{E}_X^{1,1},$$

where $\mathbb{R}_X \to \mathcal{O}_X$ is the natural inclusion, $\mathcal{O}_X \to \mathcal{E}_X$ is given by taking the minus of imaginary part, and the last map is induced from $\sqrt{-1}\,\partial\,\overline{\partial}$.

Definition 1.1. (1) A real C^∞-form ω on X is called a *Kähler form* if there exist an open covering $\{X_\lambda\}$ of X and strictly plurisubharmonic functions p_λ on X_λ such that $\omega_{|X_\lambda} = \sqrt{-1}\,\partial\,\overline{\partial}\,p_\lambda$, and that $(p_\lambda - p_\mu)_{|X_\lambda \cap X_\mu}$ are pluriharmonic.

(2) If X has a Kähler form, then it is called a *Kähler space*.

(3) A *Kähler structure* is defined to be a pair (ω, θ) consisting of a Kähler form ω and an element $\theta \in H^1(X, \mathcal{O}_X/\mathbb{R})$ such that they induce same element in $H^1(X, \mathcal{H}'_X)$, where $\mathcal{H}'_X$ is the kernel of ${}_{\mathbb{R}}\mathcal{E}_X \to {}_{\mathbb{R}}\mathcal{E}_X^{1,1}$.

(4) A cohomology class $\xi \in H^2(X, \mathbb{R})$ is called a *Kähler class* if it is induced from the θ of a Kähler structure (ω, θ).

In the case X is normal, a real C^∞-form is Kähler, if it is locally the restriction of a Kähler form on a nonsingular variety. Further since the sheaf $\mathcal{H}'_X$ is canonically isomorphic to $\mathcal{O}_X/\mathbb{R}$, a Kähler structure is determined only by the Kähler form.

The following Lemma is a consequence of Theorem 1 of [Va2]:

Lemma 1.2. *Let X be a normal Kähler variety with a Kähler class ξ and let G be a finite group acting holomorphically on X. Suppose that ξ is G-invariant. Then ξ defines a Kähler class of the quotient space $G\backslash X$.*

Definition 1.3. Let $f\colon X \to S$ be a proper surjective morphism of normal complex analytic varieties.

(1) If there is an open covering $\{S_\lambda\}$ of S such that $f^{-1}(S_\lambda)$ admit Kähler forms, then f is called a *locally Kähler morphism.*

(2) A d-closed real $(1,1)$-form ω on X is called a *relative Kähler form* if there exist an open covering $\{S_\lambda\}$ of S and Kähler forms η_λ on S_λ such that $\omega_{|X_\lambda} + f^*\eta_\lambda$ is a Kähler form for any λ, where $X_\lambda = f^{-1}(S_\lambda)$.

(3) If there is a relative Kähler form on X, then f is called a *Kähler morphism.*

(4) An element ξ in $H^2(X,\mathbb{R})$ is called a *relative Kähler class* if there is an open covering $\{S_\lambda\}$ of S such that the restriction of ξ in $H^2(X_\lambda,\mathbb{R})$ is induced from a relative Kähler form for $X_\lambda \to S_\lambda$.

(5) If there is a relative Kähler class, then f is called a *cohomologically Kähler morphism.*

Lemma 1.4. *In the case f is a smooth morphism, an element $\xi \in H^2(X,\mathbb{R})$ is a relative Kähler class if the restriction to any fiber is a Kähler class.*

Proof. Every fibers $X_s = f^{-1}(s)$ are compact Kähler manifolds. Thus the Betti number $B_n(X_s) = \dim H^n(X_s,\mathbb{C})$ is equal to the sum $\sum_{p+q=n} h^{p,q}(X_s)$ of Hodge numbers. The $B_n(X_s)$ is independent of $s \in S$ and $h^{p,q}(X_s)$ are upper semi-continuous function on S. Hence $h^{p,q}(X_s)$ are constant and thus the sheaf $R^q f_*\Omega^p_{X/S}$ are locally free sheaves satisfying

$$R^q f_*\Omega^p_{X/S} \otimes \mathbb{C}(s) \simeq H^q(X_s, \Omega^p_{X_s})$$

for any $s \in S$. In particular, we have an isomorphism $R^2 f_*\mathcal{O}_X \otimes \mathbb{C}(s) \simeq H^2(X_s,\mathcal{O}_{X_s})$. Therefore ξ is sent to zero by $H^2(X,\mathbb{R}) \to H^2(X,\mathcal{O}_X) \to H^0(S,R^2 f_*\mathcal{O}_X)$. Thus locally on S, there is a 2-form ω in $H^0(X, \sqrt{-1}\,\partial\,\overline{\partial}\,{}_{\mathbb{R}}\mathcal{E}_X)$ representing ξ. For each point $s \in S$, there is a real-valued C^∞-function α on X_s such that $\omega_{|X_s} + \sqrt{-1}\,\partial\,\overline{\partial}\,\alpha$ is a Kähler form

on X_s. Thus if we replace S by an open neighborhood of s, we have a real-valued C^∞-function $\tilde{\alpha}$ on X and a Kähler form η on S such that $\tilde{\alpha}$ is an extension of α and that $\omega + \sqrt{-1}\,\partial\,\overline{\partial}\,\tilde{\alpha} + f^*\eta$ is a Kähler form. □

Remark. There is a cohomologically Kähler smooth elliptic fibration which is not a Kähler morphism [N3].

Definition 1.5. Let $f\colon X \to S$ be a proper surjective morphism between normal varieties.

(1) f is called a *BK-morphism*, if it is bimeromorphically equivalent to a Kähler morphism over S.

(2) f is called an *LBK-morphism*, if it is a BK-morphism locally over S.

(3) f is called a *BCK-morphism*, if it is bimeromorphically equivalent to a cohomologically Kähler morphism over S.

Proposition 1.6. *Let $f\colon X \to S$ and $g\colon Y \to S$ be proper surjective LBK-morphisms from complex analytic varieties onto a normal variety and let $\sigma\colon S \cdots\to X$ and $\tau\colon S \cdots\to Y$ be meromorphic sections of f and g, respectively. Let $\phi\colon f^{-1}(S^\star) \cdots\to g^{-1}(S^\star)$ be a meromorphic mapping over a Zariski-open subset $S^\star$ of S such that*

$$\tau_{|S^\star} = \phi \circ \sigma_{|S^\star}.$$

Then ϕ extends to a meromorphic mapping $X \cdots\to Y$ over S, if the following conditions are satisfied for a point $s \in S^\star$:

(1) *σ and τ are holomorphic at s and ϕ is holomorphic along $X_s = f^{-1}(s)$.*

(2) *$Y_s = g^{-1}(s)$ is smooth.*

(3) *Let $\mathfrak{m}_{\sigma(s)}$ be the maximal ideal at the point $\sigma(s)$ in X_s, $\Omega^1_{Y_s}$ the sheaf of germs of Kähler differentials of Y_s and let $\phi_s\colon X_s \to Y_s$ be the restriction of ϕ. Then*

$$\mathrm{Hom}_{X_s}(\phi_s^*\Omega^1_{Y_s}, \mathfrak{m}_{\sigma(s)}) = 0.$$

Proof. Let $\mathcal{D} = \mathcal{D}_{X\times_S Y/S}$ be the relative Douady space over S. By considering the graph of ϕ, we have a meromorphic section $i\colon S^\star \cdots\to \mathcal{D}$. For an analytic space T over S, let $\mathcal{F}(T) = \mathcal{F}(T/S)$ be the set of all the closed analytic subspaces $\Gamma \subset X \times_S Y \times_S T$ satisfying the following two conditions:

(1) $\Gamma \to T$ is flat.

(2) Γ contains $\sigma(S) \times_S \tau(S) \times_S T$ as a closed subspace.

Then $\mathcal{F}$ forms a contravariant functor from the category of analytic spaces over S to the category of sets and it is representable by a closed analytic subspace $\mathcal{D}_0 \subset \mathcal{D}$. In our situation, the section i factors through $\mathcal{D}_0$. We consider the relative Zariski tangent spaces of $\mathcal{D}$ and $\mathcal{D}_0$ over S at the point $i(s)$. These are isomorphic to

$$\mathrm{Hom}_{X_s}(\phi_s^*\Omega^1_{Y_s}, \mathcal{O}_{X_s}) \text{ and } \mathrm{Hom}_{X_s}(\phi_s^*\Omega^1_{Y_s}, \mathfrak{m}_{\sigma(s)}),$$

respectively. The second space is zero by assumption. Therefore $\mathcal{D}_0 \to S$ is unramified at $i(s)$. By Theorem 5.2 in [Fk1], $i(S^\star)$ extends to a closed analytic subspace of $\mathcal{D}_0$ which is proper over S. Thus there is an extension of ϕ as a meromorphic mapping. $\square$

2. Variation of Hodge structure of weight -1.

Let S be a normal complex analytic variety. A *variation of Hodge structure* H of weight -1 of rank $2g$ on S consists of a locally constant system H of the free abelian group of rank $2g$, and a holomorphic subbundle $\mathcal{F}^0 = \mathcal{F}^0(\mathcal{H}) \subset \mathcal{H} := H \otimes \mathcal{O}_S$ of rank g such that the filtration $\mathcal{F}^1 = 0 \subset \mathcal{F}^0 \subset \mathcal{F}^{-1} = \mathcal{H}$ defines Hodge structures of weight -1 on each fiber. A *polarization* of H is a non-degenerate skew-symmetric real bilinear form $q\colon H \times H \to \mathbb{R}_S$ satisfying the Hodge–Riemann bilinear relations:

$$q(\mathcal{F}^0, \mathcal{F}^0) = 0, \quad \sqrt{-1}q(x, \overline{x}) > 0 \text{ for } 0 \neq x \in \mathcal{F}^0.$$

Example. Let $\pi\colon X \to S$ be a smooth proper surjective morphism whose fibers are compact Kähler manifolds. Then the local constant system of first homology groups of fibers modulo torsion:

$$R_1\pi_*\mathbb{Z}_X/(\mathrm{tor}) \simeq \mathcal{H}om(R^1\pi_*\mathbb{Z}_X, \mathbb{Z}_S)$$

naturally admits a variation of Hodge structure of weight -1. Suppose that there is an element $q \in H^0(S, R^2\pi_*\mathbb{R})$ such that the restriction $q_s \in H^2(X_s, \mathbb{R})$ to any fiber $X_s = \pi^{-1}(s)$ is a Kähler class. Then q defines a polarization.

For a variation of Hodge structure H of weight -1 and of rank $2g$, we set

$$\mathcal{V}_H := \mathrm{Gr}_{\mathcal{F}}^{-1}(\mathcal{H}) = \mathcal{F}^{-1}/\mathcal{F}^0,$$

which is a locally free sheaf of rank g. We denote by $\mathfrak{S}_H$ the cokernel of the natural injection $H \to \mathcal{H} \to \mathcal{V}_H$. Let $\mathbb{V} \to S$ be the vector bundle associated with $\mathcal{V}_H$. Then there is a closed subspace $\mathbb{H} \subset \mathbb{V}$ étale over S corresponding to H as a local constant system. Here $\mathbb{V}$ and $\mathbb{H}$ have abelian group structures over S and the inclusion $\mathbb{H} \hookrightarrow \mathbb{V}$ is a group homomorphism. Therefore we can define the quotient space $B = B(H) = \mathbb{V}/\mathbb{H}$ as a group object, i.e., a system of Lie groups, over S. The structure morphism $p\colon B \to S$ is a *smooth torus fibration*, i.e., a proper smooth morphism whose fibers are complex tori. The section $S \to B$ corresponding to zero in the group structure is called the *zero section*. Here the locally constant system $R_1p_*\mathbb{Z}_B$ is canonically isomorphic to H. Further we have also a canonical isomorphism

$$\mathcal{V}_H \simeq \mathcal{H}om_S(p_*\Omega^1_{B/S}, \mathcal{O}_S) \simeq p_*T_{B/S},$$

where $T_{B/S}$ denotes the relative tangent sheaf. Therefore the variation of Hodge structure H is reconstructed from $p\colon B \to S$. The sheaf $\mathfrak{S}_H$ is isomorphic to the sheaf of germs of sections of the abelian group structure p. We call the fibration $p\colon B \to S$ by the *smooth basic torus fibration* associated with H.

Let $\mathbb{H}_g$ be the Siegel upper half space of degree g:

$$\{\tau \in \mathrm{M}_g(\mathbb{C}) \mid {}^t\tau = \tau, \mathrm{Im}\,\tau \text{ is positive definite}\},$$

where $\mathrm{M}_g(\mathbb{C})$ denotes the space of $(g \times g)$ complex valued matrices. The $\mathbb{H}_g$ is the classifying space of $\mathbb{R}$-polarized Hodge structures of weight -1 and of rank $2g$. In fact, there is a polarized variation of $\mathbb{R}$-Hodge structure $H_{\mathbb{R}} = \mathbb{R}^{2g}$ on $\mathbb{H}_g$ such that

$$F^0(H_{\mathbb{R},\tau} \otimes \mathbb{C}) = \{(m,n) \in \mathbb{C}^g \oplus \mathbb{C}^g \mid m\tau + n = 0\}$$

and the polarization q is given by $q((m,n),(m',n')) = n\,{}^tm' - m\,{}^tn'$ for row vectors m, n, m', n'. Let $\mathrm{Sp}(g,\mathbb{R})$ be the symplectic group of degree g, which consists of matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

composed by four $g \times g$ real matrices a, b, c, d satisfying

$$\begin{pmatrix} {}^ta & {}^tc \\ {}^tb & {}^td \end{pmatrix} \begin{pmatrix} 0 & 1_g \\ -1_g & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & 1_g \\ -1_g & 0 \end{pmatrix}.$$

If $\gamma \in \mathrm{Sp}(g,\mathbb{R})$ is a symplectic matrix composed by matrices a, b, c, d as above, then we write $a =: a_\gamma$, $b =: b_\gamma$, $c =: c_\gamma$ and $d =: d_\gamma$. For $\tau \in \mathbb{H}_g$ and for $\gamma \in \mathrm{Sp}(g,\mathbb{R})$, the matrices τ and $c_\gamma\tau + d_\gamma$ are invertible and

$$\mathrm{Im}(c_\gamma\tau + d_\gamma) = {}^t(c_\gamma\tau + d_\gamma)\,\mathrm{Im}((a_\gamma\tau + b_\gamma)(c_\gamma\tau + d_\gamma)^{-1})(c_\gamma\overline{\tau} + d_\gamma).$$

Thus $\mathrm{Sp}(g,\mathbb{R})$ acts holomorphically on $\mathbb{H}_g$ by:

$$\mathbb{H}_g \ni \tau \mapsto (a_\gamma\tau + b_\gamma)(c_\gamma\tau + d_\gamma)^{-1}.$$

In fact, $\mathrm{Sp}(g,\mathbb{R})/\{\pm 1\}$ is isomorphic to the holomorphic automorphism group $\mathrm{Aut}(\mathbb{H}_g)$. We note that there is the following action of the semi-direct product $\mathrm{Sp}(g,\mathbb{R}) \ltimes \mathbb{R}^{2g}$ on $\mathbb{H}_g \times \mathbb{C}^g$: For $\gamma \in \mathrm{Sp}(g,\mathbb{R})$ and for a pair of row vectors $(m,n) \in \mathbb{R}^g \times \mathbb{R}^g$,

$$(\tau, z) \mapsto ((a_\gamma\tau + b_\gamma)(c_\gamma\tau + d_\gamma)^{-1}, (z + m\tau + n)(c_\gamma\tau + d_\gamma)^{-1}).$$

We define over $(\tau, z) \in \mathbb{H}_g \times \mathbb{C}^g$, a real valued function $\Psi(\tau, z)$ and a holomorphic function $h_{\gamma,m,n}(\tau, z)$ modulo $\mathbb{Z}$ for $\gamma \in \mathrm{Sp}(g, \mathbb{R})$ and $(m, n) \in \mathbb{R}^g \oplus \mathbb{R}^g$, as follows:

$$\begin{aligned}
\Psi(\tau, z) &:= -\frac{1}{4\pi} \log\det(\mathrm{Im}\,\tau) + \frac{1}{2}(\mathrm{Im}\, z)(\mathrm{Im}\,\tau)^{-1}(\mathrm{Im}\,{}^t z), \\
h_{\gamma,m,n}(\tau, z) &:= \frac{\sqrt{-1}}{2\pi} \log\det(c_\gamma \tau + d_\gamma) \\
&\quad - \frac{1}{2}(z + m\tau + n)(c_\gamma \tau + d_\gamma)^{-1} c_\gamma\, {}^t(z + m\tau + n) \\
&\quad + z\,{}^t m + \frac{1}{2} m \tau\,{}^t m.
\end{aligned}$$

Note that $\mathbb{H}_g$ is simply connected. Thus for each γ, we can take a branch of the multi-valued function $\log\det(c_\gamma \tau + \delta_\gamma)$.

Theorem 2.1. (1) $\sqrt{-1}\,\partial\,\overline{\partial}\,\Psi$ *is a Kähler form on* $\mathbb{H}_g \times \mathbb{C}^g$.

(2) *For* $(\gamma, m, n) \in \mathrm{Sp}(g, \mathbb{R}) \ltimes \mathbb{R}^{2g}$,

$$\Psi((\gamma, m, n) \cdot (\tau, z)) - \Psi(\tau, z) = \mathrm{Im}\, h_{\gamma,m,n}(\tau, z).$$

In particular, the Kähler form is invariant under the action of $\mathrm{Sp}(g, \mathbb{R}) \ltimes \mathbb{R}^{2g}$.

(3) *There exist real numbers* $r(u, v)$ *for* $u, v \in \mathrm{Sp}(g, \mathbb{R}) \ltimes \mathbb{R}^{2g}$ *which form a cocycle in* $Z^2(\mathrm{Sp}(g, \mathbb{R}) \ltimes \mathbb{R}^{2g}, \mathbb{R})$ *satisfying*

$$r(u, v) \equiv h_u(v \cdot (\tau, z)) + h_v(\tau, z) - h_{uv}(\tau, z) \mod \mathbb{Z}.$$

If $u = (\gamma_1, m_1, n_1)$ *and* $v = (\gamma_2, m_2, n_2)$, *then we can define* $r(u, v)$ *by*:

$$r(u, v) := \frac{1}{2}(m_1 a_{\gamma_2} + n_1 c_{\gamma_2})\,{}^t(m_1 b_{\gamma_2} + n_1 d_{\gamma_2} + 2n_2).$$

We give a proof by simple calculation in Appendix B.

Remark. The classical theta function of characteristic $(0, 0)$ and of modulus τ is defined by:

$$\vartheta_{0,0}(\tau, z) = \sum_{k \in \mathbb{Z}^g} e(\frac{1}{2} k \tau\,{}^t k + z\,{}^t k),$$

where we denotes $e(z) = \exp(2\pi\sqrt{-1} z)$. On the other hand, we can write

$$\Psi(\tau, z) = \frac{1}{2\pi} \log \int_{r \in \mathbb{R}^g} |e(\frac{1}{2} r \tau\,{}^t r + z\,{}^t r)|\, \mathrm{d}\, r.$$

Let H be a variation of Hodge structure of weight -1 of rank $2g$ with a polarization q on S, $\widetilde{S} \to S$ the universal covering mapping, and let H_s be the fiber over a point $s \in S$. Since q is a skew-symmetric non-degenerate bilinear form, there is an isomorphism

$$H_s \otimes \mathbb{R} \ni l \mapsto (m(l), n(l)) \in \mathbb{R}^g \oplus \mathbb{R}^g$$

such that $q_s(l, l') = n(l)\,{}^t m(l') - m(l)\,{}^t n(l')$ for $l, l' \in H_s$, where $m(l), n(l)$ are row vectors and q_s is the fiber of q. By the Hodge–Riemann bilinear relations, there is a matrix $\tau_s \in \mathbb{H}_g$ such that the subspace $F^0(H_s \otimes \mathbb{C})$ is identified with

$$\{(m, n) \in \mathbb{C}^g \oplus \mathbb{C}^g \,|\, m\tau_s + n = 0\}$$

under the isomorphism $(m, n)\colon H_s \otimes \mathbb{C} \simeq \mathbb{C}^g \oplus \mathbb{C}^g$. Here τ_s depends on the choice of (m, n). The monodromy representation $\pi_1(S, s) \to \mathrm{Aut}(H_s)$ preserves the polarization q_s and moves the the Hodge filtration $F^\bullet(H_s \otimes \mathbb{C})$ to another one. Therefore the monodromy representation is isomorphic to $\rho\colon \pi_1(S, s) \to \mathrm{Sp}(g, \mathbb{R})$ such that

$$(\flat) \qquad (m(l \cdot \gamma), n(l \cdot \gamma)) = (m(l), n(l)) \begin{pmatrix} a_{\rho(\gamma)} & b_{\rho(\gamma)} \\ c_{\rho(\gamma)} & d_{\rho(\gamma)} \end{pmatrix},$$

for $l \in H_s$ and for $\gamma \in \pi_1(S, s)$, where we consider H_s as a right $\pi_1(S, s)$-module. Further $\{\tau_s\}$ give rise to a holomorphic mapping $\tau\colon \widetilde{S} \to \mathbb{H}_g$ satisfying

$$(\natural) \qquad \tau(\gamma t) = (a_{\rho(\gamma)}\tau(t) + b_{\rho(\gamma)})(c_{\rho(\gamma)}\tau(t) + d_{\rho(\gamma)})^{-1},$$

for $t \in \widetilde{S}$. Conversely, let $\tau\colon \widetilde{S} \to \mathbb{H}_g$ be a holomorphic function and let $\rho\colon \pi_1(S, s) \to \mathrm{Sp}(g, \mathbb{R})$ be a group homomorphism of the above form satisfying $(\natural)$ above for $\gamma \in \pi_1(S, s)$. Then these data define a variation of $\mathbb{R}$-Hodge structure of weight -1 of rank $2g$ together with a polarization on S. If there is an isomorphism $(m, n)\colon \mathbb{Z}^{2g} \otimes \mathbb{R} \simeq \mathbb{R}^g \oplus \mathbb{R}^g$ such that the image of $\mathbb{Z}^{2g}$ is preserved by the right action of $\pi_1(S, s)$, then $\mathbb{Z}^{2g}$ is a right $\pi_1(S, s)$-module by the relation $(\flat)$ above. Hence it defines an integral structure of the variation of Hodge structure. Let us consider the following action of the semi-direct product $\pi_1(S, s) \ltimes \mathbb{Z}^{2g}$ on $\widetilde{S} \times \mathbb{C}^g$: For $l \in \mathbb{Z}^{2g}$ and $\gamma \in \pi_1(S, s)$,

$$\widetilde{S} \times \mathbb{C}^g \ni (t, z) \mapsto \left(\gamma t, \left(z + m\tau(t) + n\right)(c_{\rho(\gamma)}\tau(t) + d_{\rho(\gamma)})^{-1}\right).$$

The action is properly discontinuous and fixed point free. Thus the quotient space is nonsingular and a smooth torus fibration with a section is induced. The fibration is isomorphic to the smooth basic torus fibration $p\colon B \to S$ over S, by construction.

Corollary 2.2. *Let $p\colon B \to S$ be the smooth basic torus fibration associated with a variation of Hodge structure H of weight -1 on S. Then for a polarization q, there is a relative Kähler form on B which induces q as an element of $H^0(S, R^2p_*\mathbb{R}_B) \simeq \mathrm{Hom}(\wedge^2 H, \mathbb{R}_S)$.*

We here recall the notion of *Deligne groups* (cf. [EZ]) and we shall extend it slightly in the case of maximal degree. Let M be a complex analytic space and let $\boldsymbol{\omega}_M^{\mathrm{top}\bullet}$ be the topological dualizing complex satisfying the Poincaré–Verdier duality, i.e., $f^{\mathrm{top}!}\mathbb{Z}$ for the trivial morphism $f^{\mathrm{top}}\colon M \to (\text{point})$. There is also the holomorphic dualizing complex $\boldsymbol{\omega}_M^{\bullet}$ by [RR1]. By the trace mapping $R\Gamma_c(M, \boldsymbol{\omega}_M^{\bullet}) \to \mathbb{C}$, we can associate a morphism $\boldsymbol{\omega}_M^{\bullet} \to \boldsymbol{\omega}_M^{\mathrm{top}\bullet} \otimes_{\mathbb{Z}} \mathbb{C}$ in the derived category $D(M, \mathbb{C})$. Let $\underline{\mathbb{D}}_M^{\bullet}$ be an object in $D(M, \mathbb{Z})$ whose shift $\underline{\mathbb{D}}_M^{\bullet}[1]$ by one is quasi-isomorphic to the mapping cone of

$$\boldsymbol{\omega}_M^{\bullet} \oplus \boldsymbol{\omega}_M^{\mathrm{top}\bullet} \to \boldsymbol{\omega}_M^{\mathrm{top}\bullet} \otimes_{\mathbb{Z}} \mathbb{C}.$$

We define the *Deligne group* $D_0(M)$ of degree zero by

$$D_0(M) := \mathbb{H}_c^0(M, \underline{\mathbb{D}}_M^{\bullet}).$$

Note that if M is compact and nonsingular, then $D_0(M)$ coincides with $D^{2d}(M)$ for $d = \dim M$ in [EZ]. We also define the complex $\underline{\mathbb{D}}_M^{h\bullet}$ so that its shift $\underline{\mathbb{D}}_M^{h\bullet}[1]$ by one is quasi-isomorphic to the mapping cone of $\underline{\mathbb{D}}_M^{\bullet} \to \boldsymbol{\omega}_M^{\mathrm{top}\bullet}$. We set $D_0^h(M) := \mathbb{H}_c^0(M, \underline{\mathbb{D}}_M^{h\bullet})$. There is also a triangle

$$\cdots \overset{+1}{\to} \underline{\mathbb{D}}_M^{h\bullet} \to \boldsymbol{\omega}_M^{\bullet} \to \boldsymbol{\omega}_M^{\mathrm{top}\bullet} \otimes_{\mathbb{Z}} \mathbb{C} \overset{+1}{\to} \cdots$$

From the identification of the homology groups $H_i(M, \mathbb{Z}) = \mathbb{H}_c^{-i}(M, \boldsymbol{\omega}_M^{\mathrm{top}\bullet})$ for $i \geq 0$, we have the following long exact sequences

$$\cdots \to H_1(M, \mathbb{C}) \to D_0(M) \to \mathbb{H}_c^0(M, \boldsymbol{\omega}_M^{\bullet}) \oplus H_0(M, \mathbb{Z}) \to H_0(M, \mathbb{C}) \to 0,$$

$$\cdots \to H_1(M, \mathbb{Z}) \to D_0^h(M) \to D_0(M) \to H_0(M, \mathbb{Z}) \to 0,$$

$$\cdots \to H_1(M, \mathbb{C}) \to D_0^h(M) \to \mathbb{H}_c^0(M, \boldsymbol{\omega}_M^{\bullet}) \to H_0(M, \mathbb{C}) \to 0.$$

We know further that the separated space associated with the QDFS space $\mathbb{H}^0_c(M, \boldsymbol{\omega}^\bullet_M)$ is topological dual to $H^0(M, \mathcal{O}_M)$ by [RR1]. Let

$$0 \to H^0(M, \mathbb{C}) \to H^0(M, \mathcal{O}_M) \to H^0(M, \mathrm{d}\,\mathcal{O}_M) \to H^1(M, \mathbb{C})$$

be the exact sequence derived from

$$0 \to \mathbb{C}_M \to \mathcal{O}_M \stackrel{\mathrm{d}}{\to} \mathrm{d}\,\mathcal{O}_M \to 0.$$

We have a natural commutative diagram

$$\begin{array}{ccccccc} H_1(M,\mathbb{C}) & \longrightarrow & D^h_0(M) & \longrightarrow & \mathbb{H}^0_c(M,\boldsymbol{\omega}^\bullet_M) & \longrightarrow & H_0(M,\mathbb{C}) \\ \| & & \downarrow & & \downarrow & & \| \\ H^1(M,\mathbb{C})^\vee & \longrightarrow & H^0(M,\mathrm{d}\,\mathcal{O}_M)^\vee & \longrightarrow & H^0(M,\mathcal{O}_M)^\vee & \longrightarrow & H^0(M,\mathbb{C})^\vee \end{array}$$

where $V^\vee$ denotes the (algebraic) dual space $\mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$ for a $\mathbb{C}$-vector space V.

Let $M_1 \to M_2$ be a holomorphic mapping of complex analytic varieties. Then there is a functorial homomorphism $D_0(M_1) \to D_0(M_2)$. If P is a point, then $D_0(P) \simeq H_0(P, \mathbb{Z}) \simeq \mathbb{Z}$. Thus for a point $P \in M$, we define $v(P) \in D_0(M)$ as the image of the positive generator $1 \in H_0(P, \mathbb{Z}) \simeq D_0(P)$ under $D_0(P) \to D_0(M)$. Let $Z = \sum n_i P_i$ be a 0-cycle of M, i.e., a formal finite combination of points $P_i \in M$ and integers n_i. Then we define $v(Z) := \sum n_i v(P_i)$. Hence we have a homomorphism $v\colon Z_0(M) \to D_0(M)$ from the group $Z_0(M)$ of 0-cycles on M. We also have $\deg v(Z) = \deg Z = \sum n_i$ for the homomorphism $\deg\colon D_0(M) \to H_0(M, \mathbb{Z}) \to \mathbb{Z}$. The image of $v(Z)$ under $D_0(M) \to \mathbb{H}^0_c(M, \boldsymbol{\omega}^\bullet_M) \to H^0(M, \mathcal{O}_M)^\vee$ corresponds to the evaluation functional:

$$H^0(M, \mathcal{O}_M) \ni \varphi \mapsto \sum n_i \varphi(P_i).$$

Lemma 2.3. *Suppose that M is a compact Kähler manifold. Let $\alpha\colon M \to \mathrm{Alb}(M)$ be the Albanese mapping, where we assume that $\alpha(P_0) = 0$ for a point P_0. Then there is a split exact sequence*:

$$0 \to \mathrm{Alb}(M) \to D_0(M) \stackrel{\deg}{\to} \mathbb{Z} \to 0$$

and $\alpha(P) = v(P - P_0) \in \mathrm{Alb}(M)$ for any $P \in M$.

Proof. The exact sequence is derived from

$$\mathrm{Hom}(H^1(M, \mathcal{O}_M), \mathbb{C}) \oplus H_1(M, \mathbb{Z}) \to H_1(M, \mathbb{C}) \to \mathrm{Alb}(M) \to 0.$$

It is split since $\deg v(P) = 1$ for a point $P \in M$. Let $U \to M$ be the universal covering mapping and let P' and P'_0 be points on U lying over P and P_0, respectively. Then $v(P' - P'_0) \in D_0^h(U)$, since $H_1(U, \mathbb{C}) = 0$. As an element of $D_0^h(U)$, it defines a functional

$$H^0(U, \mathrm{d}\,\mathcal{O}_U) \ni \eta \mapsto \int_\gamma \eta,$$

where γ is a path $\gamma\colon [0,1] \to U$ with $\gamma(0) = P'_0$ and $\gamma(1) = P'$. The integral does not depend on the choice of γ. Hence its image $v(P - P_0) \in \mathrm{Alb}(X) \subset D_0(M)$ is just the image $\alpha(P)$. □

In the case M is a compact Kähler manifold, we then have the following commutative diagram of exact sequences:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Z_0^h(M) & \longrightarrow & Z_0(M) & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 \\ & & {\scriptstyle\alpha}\downarrow & & {\scriptstyle v}\downarrow & & \| & & \\ 0 & \longrightarrow & \mathrm{Alb}(M) & \longrightarrow & D_0(M) & \longrightarrow & \mathbb{Z} & \longrightarrow & 0, \end{array}$$

where $Z_0^h(M)$ denotes the group of 0-cycles of degree zero. The Albanese homomorphism $\alpha\colon Z_0^h(M) \to \mathrm{Alb}(M)$ is called by the Abel–Jacobi map.

Lemma 2.4. *Let A be a complex torus and let $0 \in A$ be the origin. Then the kernel of the Albanese homomorphism $\alpha\colon Z_0^h(A) \to A = \mathrm{Alb}(A)$ is generated by $z - \mathrm{tr}(a)_* z$ for $z \in Z_0^h(A)$ and $a \in A$, where $\mathrm{tr}(a)\colon A \to A$ is the translation mapping. In other words, the kernel is generated by $([x] - [0]) + ([y] - [0]) - ([x+y] - [0])$ for $x, y \in A$, where $[x]$ denotes the element in $Z_0(A)$ corresponding to the point x, and $x + y$ denotes the sum under the group law of A.*

Proof. The element $[x] - [0] \in Z_0^h(A)$ goes to $x \in A$. Hence $([x] - [0]) + ([y] - [0])$ goes to $x + y$. Thus $[x] - [0] - ([x+y] - [y]) = [x] - [0] - \mathrm{tr}(y)_*([x] - [0])$ is contained in the kernel. For $z = \sum n_i [x_i] \in Z_0^h(A)$, we see that $z - [\sum n_i x_i] + [0]$ is written by a combination of $w - \mathrm{tr}(a)_* w$ for $w \in Z_0^h(A)$ and $a \in A$. Thus we are done. □

Next, we shall consider the relative but proper situation. Let $f\colon Y \to S$ be a proper surjective morphism of complex analytic varieties. Suppose that S is a normal variety. Now the direct image functor f_* coincides with that $f_!$ with proper supports. Hence $Rf_! = Rf_*$. We write by $f^{\mathrm{top}!}\colon D^+(S, \mathbb{Z}) \to D^+(Y, \mathbb{Z})$ the adjoint functor of $Rf_! = Rf_*$

satisfying the Verdier duality

$$Rf_*R\mathcal{H}om_{\mathbb{Z}_Y}(E^\bullet, f^{\mathrm{top}!}F^\bullet) \sim_{\mathrm{qis}} R\mathcal{H}om_{\mathbb{Z}_S}(Rf_!E^\bullet, F^\bullet)$$

for $E^\bullet \in D^-(Y,\mathbb{Z})$ and $F^\bullet \in D^+(S,\mathbb{Z})$. The holomorphic $f^!$ is defined by

$$f^!F^\bullet = R\mathcal{H}om_{\mathcal{O}_Y}(Lf^*R\mathcal{H}om_{\mathcal{O}_S}(F^\bullet, \boldsymbol{\omega}_S^\bullet), \boldsymbol{\omega}_Y^\bullet)$$

for $F^\bullet \in D^+(S,\mathcal{O}_S)$ with coherent cohomology sheaves. Since f is proper, we have the duality

$$Rf_*R\mathcal{H}om_{\mathcal{O}_Y}(E^\bullet, f^!F^\bullet) \sim_{\mathrm{qis}} R\mathcal{H}om_{\mathcal{O}_S}(Rf_!E^\bullet, F^\bullet)$$

by [RRV] (cf. [RR2]). There exist natural morphisms $f^!\mathcal{O}_S \to f^{\mathrm{top}!}\mathcal{O}_S$ and $f^{\mathrm{top}!}\mathbb{Z}_S \to f^{\mathrm{top}!}\mathcal{O}_S$ in $D^+(Y,\mathbb{Z})$. Let $\underline{\mathbb{D}}^\bullet_{Y/S}$ be an object in $D(Y,\mathbb{Z})$ whose shift $\underline{\mathbb{D}}^\bullet_{Y/S}[1]$ by one is quasi-isomorphic to the mapping cone of

$$f^!\mathcal{O}_S \oplus f^{\mathrm{top}!}\mathbb{Z}_S \to f^{\mathrm{top}!}\mathcal{O}_S.$$

We define $\underline{\mathrm{D}}_0(Y/S) := \mathbb{R}^0 f_!\underline{\mathbb{D}}^\bullet_{Y/S}$ and call it by the *relative Deligne homology group* of degree zero. We can identify $R_i f_*\mathbb{Z}_Y = \mathbb{R}^{-i} f_! f^{\mathrm{top}!}\mathbb{Z}_S$ for $i \geq 0$. Thus we have the long exact sequence

$$R_1 f_*\mathbb{Z}_Y \otimes \mathcal{O}_S \to \underline{\mathrm{D}}_0(Y/S) \to \mathbb{R}^0 f_! f^!\mathcal{O}_S \oplus R_0 f_*\mathbb{Z}_X \to R_0 f_*\mathbb{Z}_X \otimes \mathcal{O}_S.$$

Suppose that $f\colon Y \to S$ be a smooth proper surjective morphism whose fibers are g-dimensional compact Kähler manifolds. Let $\Omega^\bullet_{Y/S}$ be the relative de Rham complex and let $\omega_{Y/S} \simeq \Omega^g_{Y/S}$ be the relative dualizing sheaf. Then $\omega_{Y/S}[g] \sim_{\mathrm{qis}} f^!\mathcal{O}_S$, $\Omega^\bullet_{Y/S}[2g] \sim_{\mathrm{qis}} f^{-1}\mathcal{O}_S[2g] \sim_{\mathrm{qis}} f^{\mathrm{top}!}\mathcal{O}_S$ and the natural morphism $f^!\mathcal{O}_S \to f^{\mathrm{top}!}\mathcal{O}_S$ is induced from $\omega_{Y/S}[-g] \to \Omega^\bullet_{Y/S}$. Thus $\underline{\mathbb{D}}^\bullet_{Y/S}$ is quasi-isomorphic to the following complex of sheaves on Y:

$$\mathbb{Z}_Y \to \mathcal{O}_Y \overset{\mathrm{d}_{Y/S}}{\to} \Omega^1_{Y/S} \to \cdots \to \Omega^{g-1}_{Y/S} \to 0$$

starting at degree $-2g$. By the degeneration of Hodge spectral sequence (cf. 1.4), we see that $R^i f_*\mathcal{O}_Y$ are all locally free sheaves. Therefore,

$$\mathbb{R}^{-i} f_! f^!\mathcal{O}_S \simeq R^{g-i} f_*\omega_{Y/S} \simeq \mathcal{H}om_{\mathcal{O}_S}(R^i f_*\mathcal{O}_S, \mathcal{O}_S),$$

Thus the long exact sequence above is written as follows:

$$\cdots \to \mathcal{H}om_{\mathcal{O}_S}(R^1 f_* \mathcal{O}_Y, \mathcal{O}_S) \oplus R_1 f_* \mathbb{Z}_Y \to R_1 f_* \mathbb{Z}_Y \otimes \mathcal{O}_S \to$$

$$\underline{\mathrm{D}}_0(Y/S) \to \mathcal{H}om_{\mathcal{O}_S}(f_* \mathcal{O}_Y, \mathcal{O}_S) \oplus R_0 f_* \mathbb{Z}_Y \to R_0 f_* \mathbb{Z}_Y \otimes \mathcal{O}_S \to 0.$$

Example. Suppose that $g = 0$, i.e., $f \colon Y \to S$ is a finite étale morphism. Then we see that $\underline{\mathbb{D}}^{\bullet}_{Y/S} \sim_{\mathrm{qis}} \mathbb{Z}_Y$. Thus we can define an element $v(Y) \in H^0(S, \underline{\mathrm{D}}_0(Y/S))$ corresponding to the volume element of Y.

Example. Suppose that f has connected fibers only. For the variation of Hodge structure $H := R_1 f_* \mathbb{Z}_Y/(\mathrm{tor})$ of weight -1, we have the following short exact sequence:

$$0 \to \mathfrak{S}_H \to \underline{\mathrm{D}}_0(Y/S) \to \mathbb{Z}_S \to 0.$$

Lemma 2.5. *Let $\pi \colon X \to S$ be a proper smooth surjective morphism onto a normal variety whose fibers are compact Kähler manifolds. Suppose that there is a section $\sigma \colon S \to X$ of π. Let H be the variation of Hodge structure $R_1 \pi_* \mathbb{Z}_X/(\mathrm{tor})$ and let $p \colon B \to S$ be the smooth basic torus fibration associated with H. Then there exist a holomorphic mapping $\alpha \colon X \to B$ over S uniquely up to isomorphisms over S satisfying the following conditions*:

(1) *$\alpha \circ \sigma$ is the zero section of p.*

(2) *The restriction of α to any fiber $X_s = f^{-1}(s) \to B_s = p^{-1}(s)$ is isomorphic to the Albanese mapping of X_s.*

Proof. Let $f \colon X \times_S X \to X$ be the second projection and let $\Delta \subset X \times_S X$ be the diagonal locus. We denote by same $v(\Delta)$ the image of $v(\Delta) \in H^0(X, \underline{\mathrm{D}}_0(\Delta/X))$ under $\underline{\mathrm{D}}_0(\Delta/X) \to \underline{\mathrm{D}}_0(X \times_S X/X)$. Similarly for the section $\sigma(S) \times_S X \subset X \times_S X$ for f, we have an element $v(\sigma(S) \times_S X)$. Then $v_X := v(\Delta) - v(\sigma(S) \times_S X)$ is an element of $H^0(X, \mathfrak{S}_{\pi^{-1}H})$. Therefore v_X defines a holomorphic section $X \to B \times_S X$ and equivalently the holomorphic mapping $\alpha \colon X \to B$ over S. We see that $\alpha \circ \sigma = 0$, since $v_{X|\sigma(S)} = 0$. By 2.3, it is the Albanese mapping on each fibers. Suppose that there is another $\alpha' \colon X \to B$ satisfying the both conditions above. The morphism $\alpha' \times_S \alpha' \colon X \times_S X \to B \times_S B$ sends the diagonal Δ to the diagonal of $B \times_S B$ and also sends the section $\sigma(S) \times_S X$ to the zero section. Hence by the functoriality of $\underline{\mathrm{D}}_0$, there is a morphism $\varphi \colon B \to B$ over S such that $\alpha' = \varphi \circ \alpha$. Also by 2.3, we see that φ is identity. $\square$

Corollary 2.6. *A smooth family of complex tori with a holomorphic section, is a smooth basic torus fibration.*

Let $\pi\colon X \to S$ be a smooth torus fibration with an isomorphism $\phi\colon R_1\pi_*\mathbb{Z}_X \simeq H$ of variations of Hodge structure. Then there exist an open covering $\{S_\lambda\}$ of S and sections $\sigma_\lambda\colon S_\lambda \to X$ of π. Then there is a unique isomorphism $\varphi_\lambda\colon X_\lambda := \pi^{-1}(X_\lambda) \to B_\lambda := p^{-1}(S_\lambda)$ such that φ_λ sends σ_λ to the zero section and $\varphi_{\lambda *} = \phi_{|S_\lambda}$. The difference

$$\varphi_\lambda \circ \varphi_\mu^{-1}\colon B_\lambda \cap B_\mu \to B_\lambda \cap B_\mu$$

is induced by the translation of a section $\eta_{\lambda,\mu}$ of $p\colon B \to S$ over $S_\lambda \cap S_\mu$. These sections $\eta_{\lambda,\mu}$ define a cohomology class η in $H^1(S, \mathfrak{S}_H)$ which is independent of the choice of open coverings $\{S_\lambda\}$ and sections σ_λ. Conversely, for a cohomology class η in $H^1(S, \mathfrak{S}_H)$, we have a smooth torus fibration $\pi\colon X \to S$ such that $R_1 f_*\mathbb{Z}_X \simeq H$. Such π is called a *torsor* of $p\colon B \to S$, since B acts on X over S, where

$$B \times_S X \ni (b, x) \mapsto (bx, x) \in X \times_S X$$

is an isomorphism. We denote by $B^\eta \to S$ the torsor associated with $\eta \in H^1(S, \mathfrak{S}_H)$.

Lemma 2.7. *Let $\pi\colon X \to S$ be a proper smooth surjective morphism onto a normal variety whose fibers are compact Kähler manifolds, H the variation of Hodge structure $R_1\pi_*\mathbb{Z}_X/(\mathrm{tor})$ of weight -1, and let $p\colon B \to S$ be the smooth basic torus fibration associated with H. Then for the extension class $\eta \in H^1(S, \mathfrak{S}_H)$ corresponding to the exact sequence*

$$0 \to \mathfrak{S}_H \to \underline{\mathrm{D}}_0(X/S) \to \mathbb{Z}_S \to 0,$$

there exists a holomorphic mapping $\alpha\colon X \to B^\eta$ over S uniquely up to the action of B/S on the torsor B^η such that its restriction to any fiber is isomorphic to the Albanese mapping.

Proof. There exist an open covering $\{S_\lambda\}$ of S and local sections $\sigma_\lambda\colon S_\lambda \to X_\lambda = \pi^{-1}(S_\lambda)$ of π. Then we have an element $v(\sigma_\lambda(S_\lambda) \times_S X) \in H^0(X_\lambda, \underline{\mathrm{D}}_0(X \times_S X/X))$ as in the proof of 2.5. Let $v_\lambda := v(\Delta) - v(\sigma_\lambda(S_\lambda) \times_S X)$ be the element of $H^0(X_\lambda, \mathfrak{S}_{\pi^{-1}H})$, which defines the relative Albanese mapping $\alpha_\lambda\colon X_\lambda \to B_\lambda := p^{-1}(S_\lambda)$. The sections $\eta_{\lambda,\mu} := v_\mu - v_\lambda$ are

considered to be elements of $H^0(S_\lambda \cap S_\mu, \mathfrak{S}_H)$ for any λ, μ. Note that $\eta_{\lambda,\mu} \in Z^1(\{S_\lambda\}, \mathfrak{S}_H)$ defines the extension class η of the exact sequence

$$0 \to \mathfrak{S}_H \to \underline{\mathrm{D}}_0(X/S) \to \mathbb{Z}_S \to 0.$$

We see that $\alpha_\lambda = \mathrm{tr}(\eta_{\lambda,\mu}) \circ \alpha_\mu$, where $\mathrm{tr}(\eta_{\lambda,\mu})$ denotes the translation of $B_\lambda \cap B_\mu$ by the section $\eta_{\lambda,\mu}$. Therefore $\{\alpha_\lambda\}$ define a holomorphic mapping $\alpha\colon X \to B^\eta$. □

Corollary 2.8. *For $\eta \in H^1(S, \mathfrak{S}_H)$, the extension class of the exact sequence*

$$0 \to \mathfrak{S}_H \to \underline{\mathrm{D}}_0(B^\eta/S) \to \mathbb{Z}_S \to 0$$

is again η.

Definition 2.9. A normal singularity (S, P) is called B_1-*smooth*, if $H_1(\mu^{-1}(P), \mathbb{R}) = 0$ for a desingularization $\mu\colon M \to S$. This is equivalent to the condition $R^1\mu_*\mathbb{Z}_M = 0$. The (S, P) is called π_1-*smooth* if $\pi_1(\mu^{-1}P) = 0$. This is equivalent to: $\pi_1(\mu^{-1}(\mathcal{U})) = 0$ for a sufficiently small open neighborhood $\mathcal{U}$ of P.

Remark. A quotient singularity is π_1-smooth (cf. [Kl2, 7.5]). A 1-rational singularity, i.e., a normal singularity S with $R^1\mu_*\mathcal{O}_M = 0$ for a desingularization $\mu\colon M \to S$, is B_1-smooth. However a B_1-smooth singularity is not necessarily a 1-rational (cf. [N1]).

Lemma 2.10. *Let $\mu\colon M \to N$ be a bimeromorphic morphism of nonsingular varieties and let G be a finite group acting on both M and N compatible way with μ. If the action is free on M, then so is on N.*

Proof. Let $\nu\colon G\backslash M \to G\backslash N$ be the induced bimeromorphic morphism. Let $\mathcal{U} \subset G\backslash N$ be a sufficiently small open neighborhood of any point. Then $\nu^{-1}(\mathcal{U})$ is simply connected. Hence its pullback in M is a disjoint union of copies of $\nu^{-1}(\mathcal{U})$, the number of whose components is equal to the order of G. Hence the cardinal number of every fiber of $N \to G\backslash N$ is also the order of G. Thus G acts freely on N. □

Lemma 2.11. *Let $\pi\colon X \to S$ be a smooth torus fibration over a B_1-smooth normal variety S. Then any meromorphic section $S \cdots\to X$ of π is holomorphic.*

Proof. We can localize S and thus we may assume that S is simply connected and π admits a holomorphic section. Then the universal covering space $\widetilde{X}$ of X is isomorphic

to the product $S \times \mathbb{C}^g$ and the fundamental group $\pi_1(X)$ is isomorphic to $\mathbb{Z}^{\oplus 2g}$. Let $T \subset X$ be the image of a meromorphic section. If it is not isomorphic to S, then there is a positive dimensional fiber of $T \to S$. Let $\nu\colon M \to T$ be a desingularization and let Γ be a positive dimensional fiber of $\mu\colon M \to T \to S$. Since S is B_1-smooth, we have $H^1(\Gamma, \mathbb{Z}) = 0$. Hence the natural homomorphism $\pi_1(\Gamma) \to \pi_1(X)$ is zero. Therefore $\nu(\Gamma)$ is a point since $\mathbb{C}^d$ contains no positive dimensional compact subvarieties. □

Another proof of 2.11. We have only to show $\mathfrak{S}_H \simeq \mu_*\mathfrak{S}_{\mu^{-1}H}$ for any bimeromorphic morphism $\mu\colon M \to S$ from a nonsingular variety. Let us consider the exact sequence

$$0 \to H \to \mathcal{V}_H \to \mathfrak{S}_H \to 0$$

and similar one on M. Since S is normal, we have $H \simeq \mu_*\mu^{-1}H$ and $\mathcal{V}_H \simeq \mu_*\mu^*\mathcal{V}_H$. Thus

$$0 \to \mathfrak{S}_H \to \mu_*\mathfrak{S}_{\mu^{-1}H} \to H \otimes_{\mathbb{Z}} R^1\mu_*\mathbb{Z}_M$$

is exact. Hence the vanishing $R^1\mu_*\mathbb{Z}_M = 0$ implies the isomorphism. □

Example. (1). Let $T \to E$ be a $\mathbb{P}^1$-bundle over an elliptic curve E having a section $E_0 \subset T$ with $E_0^2 < 0$. Then there is a contraction morphism $T \to S$ of E_0. Thus there is a closed immersion $T \to S \times E_0$. It is a non-holomorphic meromorphic section of the elliptic fibration $S \times E_0 \to S$.

(2). Let T, S be the same surfaces as in (1). There is a finite surjective morphism $S \to \mathbb{P}^2$. Then the image T' of the composite $T \to S \times E_0 \to \mathbb{P}^2 \times E_0$ is not étale over $\mathbb{P}^2$. This is a non-étale multi-section.

(3). Let C be a irreducible plane cubic curve with one node. Then $H^1(C, \mathcal{O}_C) \simeq \mathbb{C}$. Hence there is a non-trivial torsor $X \to C$ of $C \times E \to C$ for an elliptic curve E. However its pullback to the normalization $\mathbb{P}^1 \to C$ is isomorphic to the trivial fibration $\mathbb{P}^1 \times E \to \mathbb{P}^1$.

Let S be a B_1-smooth normal variety, H a variation of Hodge structure of weight -1 and of rank $2g$ on S, and let $p\colon B \to S$ be the smooth basic torus fibration associated with H. Let $T \subset B$ be a closed subvariety such that $T \to S$ is a generically finite surjective morphism. Let m be the mapping degree $\deg T/S$. We denote by $B^{\times m}$ the m-times fiber product $B \times_S B \times_S \cdots \times_S B$. Let $\mu_m\colon B^{\times m} \to B$ be the multiplication

$$B^{\times m} \ni (b_1, b_2, \cdots, b_m) \mapsto \sum_{i=1}^{m} b_i \in B.$$

Let $T^{\times m}$ be the m-times fiber product of T over S. For the i-th projection $p_i\colon T^{\times m} \to T$ and for the diagonal locus $\Delta_T \subset T^{\times 2}$, let us define U_T as the complement of

$$\bigcup_{i\neq j} (p_i \times_S p_j)^{-1}(\Delta_T)$$

in $T^{\times m}$. Then the image $\mu_m(U_T)$ is contained in an irreducible component $\Sigma_m(T)$ of $\mu_m(T^{\times m})$, which is bimeromorphically equivalent to S. By 2.11, $\Sigma_m(T)$ is a holomorphic section of $B \to S$.

Lemma 2.12 (cf. [Kd, 11.5]). *Let $\pi\colon X \to S$ be a smooth torus fibration associated with H and a cohomology class $\eta \in H^1(S, \mathfrak{S}_H)$, i.e., $X \simeq B^\eta$. Suppose that there is a closed subvariety $T \subset X$ which is generically finite and surjective over S. Then $m\eta = 0$ in $H^1(S, \mathfrak{S}_H)$ for the mapping degree $m = \deg T/S$.*

Proof. Let $X^{\times m}$ and $T^{\times m}$ be the m-times fiber products of X and T, respectively over S. The multiplication mapping $\mu_m\colon B^{\times m} \to B$ induces a morphism $\mu_m\colon X^{\times m} \to B^{m\eta}$ over S. Then as in the case $\eta = 0$, the image $\mu_m(U_T)$ is contained in an irreducible component $\Sigma_m(T)$ of $\mu_m(T^{\times m})$, which is a section of $B^{m\eta} \to S$. Thus $m\mu = 0$ in $H^1(S, \mathfrak{S}_H)$. □

Corollary 2.13. *Let $\pi\colon X \simeq B^\eta \to S$ be a smooth torus fibration over a B_1-smooth normal variety associated with a variation of Hodge structure H and an element $\eta \in H^1(S, \mathfrak{S}_H)$. Then the following three conditions are equivalent*:

(1) *There is a closed subvariety $\Sigma \subset X$ which is finite étale over S.*

(2) *There is a closed subvariety $\Sigma \subset X$ which is generically finite and surjective over S.*

(3) *η is a torsion element.*

Proof. By Lemma 2.12 above, we have only to show (3) $\Longrightarrow$ (1). Let m be a positive integer with $m\eta = 0$. Let $B \ni b \mapsto mb \in B$ be the multiplication mapping by m. This is a finite étale surjective morphism. By patching these multiplication mapping $X_\lambda \simeq B_\lambda \to B_\lambda$, we can construct a finite étale surjective morphism $X = B^\eta \to B^{m\eta} = B$. Let Σ be an irreducible component of the pullback of the zero section of $B \to S$. Then Σ is finite étale over S. □

Let $\pi\colon X \to S$ be a proper smooth morphism onto the B_1-smooth normal variety S whose fibers are g-dimensional compact Kähler manifolds. We denote by $Z_0(X/S)$ the free group generated by subvarieties $T \subset X$ which are proper surjective and generically finite over S. An element of $Z = \sum n_i T_i \in Z_0(X/S)$, where $n_i \in \mathbb{Z}$, is called a *relative* 0-*cycle* of X/S. Every T_i in Z is étale over a Zariski-open subset $S' \subset S$. Thus we have $v(Z) \in H^0(S', \underline{\mathrm{D}}_0(X/S))$. Note that the restriction $H^0(S, \underline{\mathrm{D}}_0(X/S)) \to H^0(S', \underline{\mathrm{D}}_0(X/S))$ is injective.

Lemma 2.14. *The $v(Z)$ is globally defined on S. Namely, there is a canonical homomorphism $v\colon Z_0(X/S) \to H^0(S, \underline{\mathrm{D}}_0(X/S))$.*

Proof. First we reduce to the case S is nonsingular. Let $\mu\colon M \to S$ be a desingularization. Then we see that $\mu_* \underline{\mathrm{D}}_0(X \times_S M/M) \simeq \underline{\mathrm{D}}_0(X/S)$ from the exact sequence

$$0 \to \mathfrak{S}_{\mu^{-1}H} \to \underline{\mathrm{D}}_0(X \times_S M/M) \to \mathbb{Z}_M \to 0,$$

and an isomorphism $\mu_* \mathfrak{S}_{\mu^{-1}H} \simeq \mathfrak{S}_H$. Thus our v is derived from that on M.

Next, we assume S is nonsingular. For a subvariety $T \subset X$ which is proper surjective and generically finite over S, let $\nu\colon Y \to T$ be a desingularization. For $f = \pi \circ \nu\colon Y \to S$, let R_f be the ramification divisor, i.e., $K_Y \sim f^* K_S + R_f$. Then $\underline{\mathbb{D}}^{\bullet}_{Y/S}$ is quasi-isomorphic to the following complex starting at degree zero:

$$\mathcal{O}_Y(R_f) \oplus \mathbb{Z}_Y \to f^{-1}\mathcal{O}_S.$$

By natural inclusions $\mathbb{Z}_Y \to \mathcal{O}_Y \to \mathcal{O}_Y(R_f)$, we have a canonical element $v(Y) \in H^0(S, \underline{\mathrm{D}}_0(Y/S))$. We define $v(T) \in H^0(S, \underline{\mathrm{D}}_0(X/S))$ to be the image of $v(Y)$ under $\nu_*\colon \underline{\mathrm{D}}_0(Y/S) \to \underline{\mathrm{D}}_0(X/S)$. Thus we have a required homomorphism $v\colon Z_0(X/S) \to H^0(S, \underline{\mathrm{D}}_0(X/S))$. □

Corollary 2.15. *Let $p\colon B \to S$ be the smooth basic torus fibration over the S, $\Sigma \subset B$ the zero section and let $T \subset B$ is a subvariety which is proper surjective and generically finite over S with $\deg T/S = m$. Then the section $\Sigma_m(T) \subset B$ constructed in the argument before 2.12 corresponds to $v(T) - mv(\Sigma) \in H^0(S, \mathfrak{S}_H) \subset H^0(S, \underline{\mathrm{D}}_0(B/S))$.*

Proof. We can replace S by an open subset. Thus we may assume that T is a disjoint union of sections T_i, $1 \le i \le m$ over S. Then $\Sigma_m(T)$ corresponds to the sum $\sum T_i$ under the group law of B/S. Thus we have reduced to 2.4. □

Suppose that the variation of Hodge structure H is isomorphic to $R_1\pi_*\mathbb{Z}_X$. We denote by $c\colon H^1(S,\mathfrak{S}_H) \to H^2(S,H)$ the connecting homomorphism for the exact sequence:

$$0 \to H \to \mathcal{V}_H \to \mathfrak{S}_H \to 0.$$

Lemma 2.16. *Let $\eta \in H^1(S,\mathfrak{S}_H)$ be the extension class of*

$$0 \to \mathfrak{S}_H \to \underline{\mathrm{D}}_0(X/S) \to \mathbb{Z}_S \to 0.$$

Then the following two conditions are equivalent:

(1) *The natural surjective homomorphism of fundamental groups $\pi_1(X) \to \pi_1(S)$ has a section, i.e., $\pi_1(X)$ is isomorphic to the semi-direct product $\pi_1(S) \ltimes \mathbb{Z}^{\oplus 2g}$.*

(2) *$c(\eta) = 0$.*

Proof. Let $\alpha\colon X \to B^\eta$ be the relative Albanese mapping (cf. 2.7). Then by considering a commutative diagram of homotopy exact sequences for X/S and B^η/S, we see that $\alpha_*\colon \pi_1(X) \to \pi_1(B^\eta)$ is an isomorphism. Therefore, we may assume $X = B^\eta$.

(1) $\Longrightarrow$ (2). Let $W \to X$ be the finite étale covering corresponding to the subgroup $\pi_1(S) \to \pi_1(S) \ltimes \mathbb{Z}^{\oplus 2g} \simeq \pi_1(X)$. Then W is isomorphic to the product $S \times \mathbb{C}^g$ locally on S and $W \to S$ is a torsor of the vector bundle $\mathbb{V} \to S$. The η is derived from $H^1(S,\mathcal{V}_H)$.

(2) $\Longrightarrow$ (1). There is a torsor $W \to S$ associated with a cohomology class in $H^1(S,\mathcal{V}_H)$ inducing η. Then the homotopy sequence

$$\pi_2(S) \to \pi_1(X_s) \to \pi_1(X) \to \pi_1(S) \to 1,$$

has a splitting by $\pi_i(W) \simeq \pi_i(S)$. □

Proposition 2.17. *Let $\pi\colon X = B^\eta \to S$ be the torsor corresponding to an element $\eta \in H^1(S,\mathfrak{S}_H)$. Suppose that H admits a polarization. Then the following conditions are equivalent*:

(1) *π is cohomologically Kähler.*

(2) *The Leray's spectral sequence*

$$E_2^{p,q} = H^p(S, R^q\pi_*\mathbb{R}_X) \Longrightarrow E^{p+q} = H^{p+q}(X,\mathbb{R})$$

is degenerates at E_2.

(3) *The homomorphism*

$$E^2 = H^2(X, \mathbb{R}) \to E_2^{0,2} = H^0(S, R^2\pi_*\mathbb{R}_X)$$

is surjective.

(4) *$c(\eta)$ is a torsion element.*

If S is a compact Kähler manifold, then the conditions above are also equivalent to: X is a compact Kähler manifold.

Proof. (1) $\Longrightarrow$ (2). Let $\xi \in H^2(X, \mathbb{R})$ be an element whose restriction to $H^2(X_s, \mathbb{R})$ are Kähler class for any fibers $X_s = \pi^{-1}(s)$. Then ξ defines a morphism

$$L = \xi \wedge \bullet \colon R\pi_*\mathbb{R}_X \to R\pi_*\mathbb{R}_X[2],$$

which induces isomorphisms $R^{g-i}\pi_*\mathbb{R}_X \simeq R^{g+i}\pi_*\mathbb{R}_X$ for any i. Thus by [D1], we have

$$R\pi_*\mathbb{R}_X \sim_{\mathrm{qis}} \bigoplus_{i=0}^{2g} R^i\pi_*\mathbb{R}_X[-i].$$

This implies that the Leray's spectral sequence degenerates.

(2) $\Longrightarrow$ (3). Trivial.

(3) $\Longrightarrow$ (4). For the truncation $\tau_{\geq 2g-1}(R\pi_*\mathbb{Z}_X)$, we have a natural morphism $R\pi_*\mathbb{Z}_X \to \tau_{\geq 2g-1}(R\pi_*\mathbb{Z}_X)$ and a triangle

$$\cdots \overset{+1}{\to} \tau_{\geq 2g-1}(R\pi_*\mathbb{Z}_X)[2g-1] \to \mathcal{V}_H \to \underline{\mathrm{D}}_0(X/S) \overset{+1}{\to} \cdots$$

Thus the homomorphism

$$E_2^{0,2g} = H^0(S, R^{2g}\pi_*\mathbb{Z}_X) \to E_2^{2,2g-1} = H^2(S, R^{2g-1}\pi_*\mathbb{Z}_X)$$

appearing at the Leray's spectral sequence for π is isomorphic to the composite of connecting homomorphisms

$$H^0(S, \mathbb{Z}_S) \to H^1(S, \mathfrak{S}_H) \overset{c}{\to} H^2(S, H)$$

for the exact sequences

$$0 \longrightarrow \mathfrak{S}_H \longrightarrow \underline{\mathrm{D}}^g(X/S) \longrightarrow \mathbb{Z}_S \longrightarrow 0,$$

$$0 \longrightarrow H \longrightarrow \mathcal{V}_H \longrightarrow \mathfrak{S}_H \longrightarrow 0.$$

By the condition (3), $c(\eta)$ goes to zero under $H^2(S, H) \to H^2(S, H \otimes \mathbb{R})$. Thus $c(\eta)$ is a torsion element.

(4) $\Longrightarrow$ (1). Let m be the order of $c(\eta)$ and let us consider the multiplication mapping by m: $X = B^\eta \to B^{m\eta}$ over S. If $B^{m\eta} \to S$ is cohomologically Kähler, then so is $X \to S$, since $X \to B^{m\eta}$ is a finite morphism. Hence we may assume that $c(\eta) = 0$. From the polarization q of H, we can reconstruct H by the period mapping from the universal covering space of S to $\mathbb{H}_g$ and from the monodromy representation. Let $r(u,v)$ be the cocycle appearing at 2.1. The homomorphism $(m,n)\colon \mathbb{Z}^{\oplus 2g} \to \mathbb{R}^{\oplus 2g}$ induces a cocycle r in $Z^2(\pi_1(X),\mathbb{R})$ and a cohomology class ξ by the natural homomorphism $H^2(\pi_1(X),\mathbb{R}) \to H^2(X,\mathbb{R})$. For a fiber $X_s = \pi^{-1}(s)$, $\pi_1(X_s) \simeq \mathbb{Z}^{\oplus 2g}$ can be considered as a subgroup of $\pi_1(X)$ and r induces a cohomology class in $H^2(X_s,\mathbb{R}) \simeq \mathrm{Hom}(\wedge^2 H_s,\mathbb{R})$, which corresponds to q_s. Thus we have $\xi_{|X_s} = q_s$. Hence π is cohomologically Kähler.

Finally, we treat the case S is compact Kähler manifold. Suppose that $c(\eta) = 0$. There is a Hodge structure of weight 0 on $H^1(S,H)$ and $H^1(S,H\otimes\mathbb{R}) \to H^1(S,\mathcal{V}_H)$ is surjective by a theorem of Deligne [Z]. The Kähler form $\sqrt{-1}\,\partial\,\overline{\partial}\,\Psi$ in 2.1 is invariant under the transformation

$$(\tau, z) \mapsto (\tau, z + m\tau + n)$$

for constant real vectors m, n. Since η is derived from $H^1(S,H\otimes\mathbb{R}) \to H^1(S,\mathcal{V}_H) \to H^1(S,\mathfrak{S}_H)$, $X = B^\eta$ admits a relative Kähler form. Thus X is Kähler. In the case $c(\eta)$ is only a torsion element of order m, we have a finite étale morphism $X \to B^{m\eta}$ induced from the multiplication by m. Thus X is also Kähler. □

Corollary 2.18. *Suppose that the proper smooth surjective morphism $\pi\colon X \to S$ is a BCK-morphism. Then $\pi_2(S) \to \pi_1(X_s)$ is a zero map for a fiber $X_s = \pi^{-1}(s)$.*

Proof. Let $\alpha\colon X \to B^\eta$ be the relative Albanese mapping to a torsor B^η over S. Then $B^\eta \to S$ is cohomologically Kähler and $c(\eta)$ is a torsion element by 2.17. Since $\pi_1(X) \simeq \pi_1(B^\eta)$, we may assume that $X = B^\eta$. Let $X \to Y := B^{m\eta}$ be the finite étale morphism induced from the multiplication by the order m of $c(\eta)$. By 2.16, $\pi_2(S) \to \pi_1(Y_s)$ is a zero map for a fiber Y_s. Hence $\pi_2(S) \to \pi_1(X_s)$ is also zero. □

Lemma 2.19. *Let $\pi\colon X \to S$ be a smooth torus fibration between compact Kähler manifolds. Suppose that S is bimeromorphically equivalent to a complex torus. Then there is a finite étale morphism $S' \to S$ such that $X \times_S S'$ is bimeromorphically equivalent to a complex torus and $X \times_S S' \to S'$ is a holomorphic principal torus fiber bundle.*

Proof. The period mapping $\widetilde{S} \to \mathbb{H}_g$ of the associated variation of Hodge structure H from the universal covering space $\widetilde{S}$ is constant since it induces a meromorphic mapping $\mathbb{C}^m \cdots\to \mathbb{H}_g$ for $m = \dim S$. Thus every fiber of π are isomorphic to each other and H is determined only by the monodromy representation $\pi_1(S) \to \mathrm{Sp}(g, \mathbb{R})$ whose image is a finite group. By replacing S by a finite étale covering space over S, we may assume that the monodromy representation is trivial. By 2.17, we see that $B_1(X) = 2 \dim X$. Thus X is bimeromorphically equivalent to a complex torus. Let F be a fiber. The sheaf $\mathfrak{S}_H$ is isomorphic to the sheaf of germs of holomorphic sections of $F \times S \to S$. Therefore $X \to S$ is a usual principal fiber bundle. □

Lemma 2.20. *Let $\pi : X \to S$ be a smooth torus fibration over a compact Kähler manifold S. If the variation of Hodge structure $R_1\pi_*\mathbb{Z}_X$ admits an $\mathbb{R}$-polarization, then $\kappa(X) = \kappa(S)$.*

Proof. Let $\mathcal{F}^0$ be the Hodge filtration of degree 0 of the variation of Hodge structure. Then $\pi_*\omega_{X/S}^{\otimes k} \simeq (\det \mathcal{F}^0)^{\otimes k}$ for any $k \geq 0$. Thus $\kappa(X) = \kappa(\omega_S \otimes \det \mathcal{F}^0)$. Let Γ be the image of the monodromy representation $\pi_1(S, s) \to \mathrm{Sp}(g, \mathbb{R})$, where $g = \dim X - \dim S$. By Lemma of Serre [Se], there is a finite index subgroup $\Gamma' \subset \Gamma$ which acts properly discontinuously and fixed point freely on the Siegel upper-half space $\mathbb{H}_g$. By considering the corresponding finite étale covering $S' \to S$ to Γ' and the pullback $X \times_S S' \to S'$, we may assume that $\Gamma = \Gamma'$, since κ is invariant under finite étale coverings. Then we have a morphism $S \to \Gamma\backslash\mathbb{H}_g$ to a complex manifold. Let $\phi\colon S \to W$ be its Stein factorization. Then W is a variety of general type by the argument in [Kl2, 5.9.3]. Thus $\kappa(S) = \kappa(S_w) + \dim W$ for a general fiber $S_w = \phi^{-1}(w)$, by [Kw1], [Vi1]. Further there is an invertible sheaf $\mathcal{L}$ on W such that $\det \mathcal{F}^0 \simeq \phi^*\mathcal{L}$. Since $\mathcal{L}$ is also big, we have $\kappa(\omega_S \otimes \det \mathcal{F}^0) = \kappa(S_w) + \dim W$ by [Vi1, 7.4]. □

3. Q-lattice and complex Q-torus

Definition 3.1. A group is called *torsion free* if there is no element of finite order except for the identity. A finitely generated abelian group is called a *lattice* if it is torsion free. A *Q-lattice* is a torsion free group having a lattice as a finite index subgroup.

Lemma 3.2. *Let G be a group and let M, N be subgroups. We denote by $[M, N]$ the subgroup of G generated by $[a, b] := aba^{-1}b^{-1}$ for $a \in M, b \in N$ and we set*

$$C(M) := \{x \in G \mid [a, x] = 1 \text{ for any } a \in M\}.$$

If the following three conditions are satisfied, then $[M, N] = \{1\}$:

(1) $[M, N] \subset C(N)$.

(2) $[M, N]$ *is torsion free.*

(3) $C(M) \cap N$ *is a finite index subgroup of N.*

Proof. For $a \in M$, let us consider the map

$$\phi_a \colon N \ni b \mapsto [a, b] \in [M, N].$$

Then by the condition (1), ϕ_a is a group homomorphism, since $[a, xy] = [a, x]x[a, y]x^{-1}$ in general. The kernel contains $C(M) \cap N$. Thus ϕ_a is derived from a homomorphism from a finite group to $[M, N]$, which is trivial by the condition (2). Hence every $a \in M$ commutes with N. □

Lemma 3.3. *Let $\phi \colon G \to H$ be a homomorphism of groups and let*

$$0 \to L \to V \to A \to 0$$

be an exact sequence of left H-modules.

(1) *Let $A \rtimes H$ be the semi-direct product. Giving a lift $\varphi \colon G \to A \rtimes H$ of ϕ is equivalent to giving a cocycle f in $Z^1(G, A)$, i.e., the mapping $f \colon G \to A$ satisfying*

$$f(gh) = f(g) + \phi(g)f(h)$$

for any $g, h \in G$.

(2) *For a lift* $\varphi\colon G \to A \rtimes H$ *of* ϕ*, let* $F \to V \rtimes H$ *be the pullback of* φ *by* $V \rtimes H \to A \rtimes H$*. Then the induced extension of groups*

$$1 \to L \to F \to G \to 1$$

is associated with the image of the corresponding cocycle $f \in Z^1(G, A)$ *under*

$$Z^1(G, A) \to H^1(G, A) \to H^2(G, L).$$

(3) *If* $H^1(G, V) = 0$ *and if the extension in* (2) *above has a splitting* $G \to F$*, then there is an element* $a \in A$ *such that*

$$\varphi(g) = (a, 1)(0, \phi(g))(a, 1)^{-1} \in A \rtimes H$$

for any $g \in G$*.*

Proof. (1). A lift $\varphi\colon G \to A \rtimes H$ is determined by a mapping $f\colon G \to A$ such that $\varphi(g) = (f(g), \phi(g))$. Here φ is a group homomorphism if and only if $f \in Z^1(G, A)$.

(2). The F is identified with the following subgroup of $V \rtimes G$:

$$\{(v, g) \in V \rtimes G \mid \lambda(v) = f(g)\},$$

where λ is the homomorphism $V \to A$. Thus the extension class in $H^2(G, L)$ is the image of f.

(3). The homomorphism $H^1(G, A) \to H^2(G, L)$ is injective. Hence F admits a splitting $G \to F$ if and only if $f \in B^1(G, A)$. Thus there is an element $a \in A$ such that $f(g) = a - \phi(g)a$ for any $g \in G$. □

Lemma 3.4. *For a* Q*-lattice* F*, there is a unique lattice* $L_F \subset F$ *of finite index such that it is a normal subgroup and that the homomorphism*

$$G_F := F/L_F \to \mathrm{Aut}(L_F)$$

induced from the inner automorphism is injective. Further

(1) L_F *contains any abelian subgroup of finite index in* F *and*

(2) L_F *contains any abelian normal subgroup of* F*.*

Proof. First of all, we shall show the L_F satisfies the both properties (1) and (2), by assuming the existence of L_F. The uniqueness of L_F is derived from (1). Let $L \subset F$ be an abelian subgroup of finite index. Then $L_F \cap L$ is also of finite index. Hence $M = L$ and $N = L_F$ satisfy the conditions of 3.2. Hence $[L, L_F] = \{1\}$. Since $G_F \to \mathrm{Aut}(L_F)$ is injective, $L \subset L_F$. Let $L \subset F$ be a normal abelian subgroup. Then $[L_F, L] = \{1\}$ by 3.2. Hence $L \subset L_F$.

Next, we prove the existence of L_F. Let $L_0 \subset F$ be a lattice of finite index. Then the normal subgroup

$$L_1 := \bigcap_{a \in F} a L_0 a^{-1}$$

is also a lattice of finite index. Let G_1 be the quotient group F/L_1. Then the inner automorphism of F induces a G_1-module structure on L_1. By applying 3.3 to the exact sequence of G_1-modules:

$$0 \to L_1 \to L_1 \otimes \mathbb{Q} \to L_1 \otimes \mathbb{Q}/\mathbb{Z} \to 0,$$

we see that the extension

$$1 \to L_1 \to F \to G_1 \to 1$$

is determined by a cocycle $f \in Z^1(G_1, L_1 \otimes \mathbb{Q}/\mathbb{Z})$, where F is isomorphic to the subgroup

$$\{(v, g) \in (L_1 \otimes \mathbb{Q}) \rtimes G_1 \mid f(g) = v \mod L_1\}.$$

Let L_F be the kernel of the homomorphism $F \to \mathrm{Aut}(L_1)$ induced from the inner automorphism and let $G_2 = L_F/L_1 \subset G_1$. Here L_1 is a trivial G_2-module. We shall consider the composite

$$L_F \hookrightarrow (L_1 \otimes \mathbb{Q}) \times G_2 \to L_1 \otimes \mathbb{Q}.$$

The composition of $L_1 \subset L$ with the homomorphism is just $\mathrm{id} \otimes \mathbb{Q}$. Hence $L_F \to L_1 \otimes \mathbb{Q}$ is an injective homomorphism. Therefore L_F is also a lattice of finite index. If $L' \subset F$ is an abelian subgroup containing L_1, then it is also contained in the kernel L_F of $F \to \mathrm{Aut}(L_1)$. Thus L_F is the unique maximal abelian subgroup containing L_1. By starting the argument from $L_0 = L_F$, we see the homomorphism $G_F := F/L_F \to \mathrm{Aut}(L_F)$ is injective. □

Remark. (1) If a group F contains a lattice L as a normal finite index subgroup and if the induced homomorphism $F/L \to \mathrm{Aut}(L)$ is an injection, then F is called a *crystallographic space group* and the finite group F/L is called a *crystallographic group*.

(2) Let L be a lattice. Then any Q-lattice F with $L_F \simeq L$ is constructed from a finite subgroup $G \subset \mathrm{Aut}(L)$ and a cocycle $f \in Z^1(G, L \otimes \mathbb{Q}/\mathbb{Z})$ such that the cohomology class of $f_{|H} \in Z^1(H, L \otimes \mathbb{Q}/\mathbb{Z})$ in $H^1(H, L \otimes \mathbb{Q}/\mathbb{Z}) \simeq H^2(H, L)$ does not vanish for any non-trivial subgroup H of G.

Example. If F is a Q-lattice with $\mathrm{rank}\, L_F = 1$, then F is abelian. By $\mathbb{Z} \ni m \mapsto (-1)^m \in \mathrm{Aut}(\mathbb{Z})$, we consider $\mathbb{Z}$ as a $\mathbb{Z}$-module. Let F be the semi-direct product $\mathbb{Z} \rtimes \mathbb{Z}$. Then F is a Q-lattice such that $L_F = \mathbb{Z} \times 2\mathbb{Z}$.

Let T be a complex torus of dimension d and we denote by L the fundamental group $\pi_1(T)$, which is considered to act on the universal covering space $U := \mathbb{C}^d$ of T as translations. Especially, L is a subgroup of U such that $L \otimes \mathbb{R} \simeq U$. Let $\mathrm{GL}(U; L)$ be the subgroup of $\mathrm{GL}(d, \mathbb{C})$ consisting of all the matrices A satisfying $AL = L$. Then $\mathrm{GL}(U; L)$ is a subgroup of $\mathrm{Aut}(L)$. Let $\mathrm{Aff}(U)$ be the affine transformation group of U. This is isomorphic to the semi-direct product $U \rtimes \mathrm{GL}(d, \mathbb{C})$. Let $\mathrm{Aff}(U; L)$ be the subgroup of $\mathrm{Aff}(U)$ corresponding to the semi-direct product $U \rtimes \mathrm{GL}(U; L)$. Note that it is the normalizer of L in the holomorphic automorphism group $\mathrm{Aut}(U)$. For the holomorphic automorphism group $\mathrm{Aut}(T)$ of T, we have the following exact sequence:

$$1 \to L \to \mathrm{Aff}(U; L) \to \mathrm{Aut}(T) \to 1.$$

Since $U \rtimes \mathrm{GL}(U; L) \simeq \mathrm{Aff}(U; L)$, we have an isomorphism $T \rtimes \mathrm{GL}(U; L) \simeq \mathrm{Aut}(T)$, where T is considered to be the subgroup of $\mathrm{Aut}(T)$ generated by translations.

Lemma 3.5. *Let $G \subset \mathrm{Aut}(T)$ be a finite subgroup and let $F \subset \mathrm{Aff}(U; L)$ be the pullback by $\mathrm{Aff}(U; L) \to \mathrm{Aut}(T)$. Then the induced exact sequence*

$$1 \to L \to F \to G \to 1$$

has a splitting $G \to F$, if and only if G has a fixed point in T. In particular, the following conditions are equivalent:

(1) *F is a Q-lattice.*

(2) *G acts on T freely.*

Proof. We apply 3.3 to the composite $G \to \mathrm{Aut}(T) \to \mathrm{GL}(U;L)$ and the exact sequence of $\mathrm{GL}(U;L)$-modules

$$0 \to L \to U \to T \to 0.$$

Note that $\mathrm{Aff}(U;L) \simeq U \rtimes \mathrm{GL}(U;L)$ and $\mathrm{Aut}(T) \simeq T \rtimes \mathrm{GL}(U;L)$. We have $H^1(G,U) = 0$, since G is a finite group and since U is a $\mathbb{C}$-vector space,. Thus by (3) of 3.3, there is a splitting $G \to F$ if and only if there is an element $a \in T$ such that $g(a) = a$ for any $g \in G$. □

Definition 3.6. A compact complex manifold X is called a *complex Q-torus* if there is a finite unramified covering $T \to X$ from a complex torus T.

Corollary 3.7. *The fundamental group of a complex Q-torus is a Q-lattice.*

Definition 3.8. Let X be a complex Q-torus and let $F = \pi_1(X)$ be the fundamental group. We call the étale finite covering $X^\sharp \to X$ corresponding to L_F by the *toric reduction* of X.

For a para-abelian variety, toric reduction is called *abelian reduction* in [N4]. If $T \to X$ is a finite étale covering from a complex torus, then it factors through the toric reduction $X^\sharp \to X$.

Remark. Let X be a compact complex manifold. Then the following two conditions are equivalent by 2.10:

(1) X is bimeromorphically equivalent to a complex Q-torus.

(2) There is a finite étale covering $X' \to X$ such that X' is bimeromorphically equivalent to a complex torus.

Lemma 3.9. *Let $\pi\colon X \to S$ be a proper smooth morphism with connected fibers over a connected complex analytic space. For a point $x \in X$, let $s = \pi(x)$, $X_s = \pi^{-1}(s)$, and let*

$$\phi\colon \pi_1(X_s, x) \to \pi_1(X, x)$$

be the natural homomorphism of fundamental groups. Then there is a homomorphism

$$\rho\colon \pi_1(X,x) \ni a \mapsto \rho_a \in \mathrm{Aut}(\pi_1(X_s,x))$$

satisfying the following two conditions:

(1) $\phi(\rho_a(v)) = a\phi(v)a^{-1}$ *for any* $a \in \pi_1(X,x)$ *and for any* $v \in \pi_1(X_s,x)$.

(2) *The homomorphism* $\rho\circ\phi$ *corresponds to the inner automorphism of* $\pi_1(X_s,x)$.

We call the ρ above by a *monodromy action* of $\pi_1(X,x)$.

Proof. Let us consider the fiber product $X\times_S X$, first and second projections $p_1,p_2\colon X\times_S X\to X$, and the diagonal mapping $\delta\colon X\hookrightarrow X\times_S X$. From the homotopy exact sequence for p_2, we have a short exact sequence

$$1\to\pi_1(X_s,x)\to\pi_1(X\times_S X,(x,x))\to\pi_1(X,x)\to 1$$

which is split by $\delta_*\colon \pi_1(X,x)\to\pi_1(X\times_S X,(x,x))$. Let $\rho\colon \pi_1(X,x)\to\mathrm{Aut}(\pi_1(X_s,x))$ be the homomorphism defined by

$$\rho_a(v) = \delta_*(a)v\delta_*(a)^{-1}.$$

Then it satisfies the condition (1). Since the pullback of p_2 by $X_s\to X$ is also the second projection $X_s\times X_s\to X_s$, the condition (2) is also satisfied. □

Corollary 3.10. *Let* $\pi\colon X\to S$ *be a proper smooth morphism over a connected complex analytic space. Suppose that* $F := \pi_1(X_s)$ *is a* $\mathbb{Q}$*-lattice for a fiber* $X_s = \pi^{-1}(s)$. *Then the image of the homomorphism* $\pi_2(S,s)\to\pi_1(X_s,x)$ *appearing at the homotopy exact sequence is contained in the* G_F*-invariant part* $L_F^{G_F}$ *of the lattice part* L_F.

Proof. The image is the kernel of $\phi\colon F\to\pi_1(X,x)$. Since $\rho\circ\phi$ is induced from the inner automorphism of F, the kernel of $\rho\circ\phi$ is the center of F, which is nothing but $L_F^{G_F}$. □

Definition 3.11. Let $\pi\colon X\to S$ be a proper smooth family of complex Q-tori. A finite étale surjective morphism $Y\to X$ is called a *toric reduction* for π if any fiber $Y_s\to X_s$ is the toric reduction for $s\in S$. A pair $(S'\to S, Y'\to X' = X\times_S S')$ of finite étale surjective morphisms is called a *weak toric reduction* if $Y'\to X'$ is a toric reduction for $X'\to S'$. It is called *Galois* if $Y'\to X$ is a Galois covering.

If S is connected, then an isomorphism class of weak toric reductions over X is determined by a subgroup $E \subset \pi_1(X,x)$ of finite index such that $\phi^{-1}(E) = L_F$, where $\phi\colon F = \pi_1(X_s,x) \to \pi_1(X,x)$ is the natural homomorphism. The weak toric reduction associated with such E is a toric reduction, if and only if $E \to \pi_1(S,s)$ is surjective. Note that a toric reduction $Y \to X$ for $X \to S$ is not necessarily Galois. If it is not Galois, then there is a finite étale covering $S' \to S$ such that the pullback $X' = X \times_S S' \to S'$ has a Galois toric reduction.

Lemma 3.12. *Let $\pi\colon X \to S$ be a proper smooth morphism of complex Q-tori over a simply connected complex analytic space S. Then there is a Galois toric reduction for π uniquely up to isomorphisms over X.*

Proof. By 3.10, $E = \phi(L_F)$ is the unique subgroup with $\phi^{-1}(E) = L_F$. It is a normal subgroup of $\pi_1(X)$ by 3.9-(1). □

Lemma 3.13 ([N4, 2.5])**.** *Let $\pi\colon X \to S$ be a proper smooth morphism of complex Q-tori with a section $\sigma\colon S \to X$. Then there is a toric reduction $Y \to X$ for π such that the section σ lifts to a section of $Y \to S$, which is unique up to isomorphisms over X.*

Proof. The monodromy action $\rho\colon \pi_1(X,x) \to \operatorname{Aut} F$ is uniquely determined by σ, where $F = \pi_1(X_s,x)$. Let D be the image of $\sigma_*\colon \pi_1(S,s) \to \pi_1(X,x)$. Then the required toric reduction corresponds to a finite index subgroup $E \subset \pi_1(X,x)$ such that $\phi^{-1}(E) = L_F$ and that $D \subset E$. Now $\pi_1(X,x)$ is isomorphic to the semi-direct product $F \rtimes D$. Let E be the subgroup $L_F \rtimes D$. Then E is the unique subgroup of $\pi_1(X,x)$ satisfying both $\phi^{-1}(E) = L_F$ and $D \subset E$. □

Theorem 3.14. *Let $\pi\colon X \to S$ be a proper smooth morphism onto a connected analytic space S. Suppose that $F := \pi_1(X_s)$ is a Q-lattice for a fiber $X_s = f^{-1}(s)$. Then there is a finite index normal subgroup $E \lhd \pi_1(X)$ such that $\phi^{-1}(E) = L_F$, where $\phi\colon \pi_1(X_s) \to \pi_1(X)$. In particular, any smooth fibration of complex Q-tori admits a Galois weak toric reduction.*

Proof. Let $\rho\colon \pi_1(X,x) \to \mathrm{Aut}(F)$ be the monodromy action for $x \in X_s$. There is a natural homomorphism $\mathrm{Aut}(F) \to \mathrm{Aut}(L_F)$. We have the commutative diagram:

$$\begin{array}{ccccc} F & \xrightarrow{\phi} & \pi_1(X,x) & \xrightarrow{\rho} & \mathrm{Aut}(F) \\ \| & & & & \downarrow \\ F & \longrightarrow & G_F & \longrightarrow & \mathrm{Aut}(L_F). \end{array}$$

We can take a positive integer m so that the composite

$$G_F \to \mathrm{Aut}(L_F) \to \mathrm{Aut}(L_F \otimes \mathbb{Z}/m\mathbb{Z})$$

is still injective. Let E be the kernel of

$$\pi_1(X,x) \to \mathrm{Aut}(L_F \otimes \mathbb{Z}/m\mathbb{Z}).$$

Then $\phi^{-1}(E) = L_F$. □

There is a similar argument in [Kl2, 6.4]. However, the proof of Lemma 6.4.1 there is incomplete. The following proposition makes up for the gap in [Kl2, 6.4].

Proposition 3.15. *For a normal subgroup $F \lhd E$, assume that F contains a finitely generated abelian group L as an finite index subgroup. Then there is a subgroup $E_0 \subset E$ of finite index such that $E_0 \cap F$ is abelian.*

Proof. We may assume that L is a lattice. Since the set of all the homomorphisms from F to a fixed finite group is a finite set, we may assume that L is a normal subgroup of E. Let G be the quotient finite group F/L. We have homomorphisms $E \to \mathrm{Aut}(L)$, $E \to \mathrm{Aut}(G)$, and $G \to \mathrm{Aut}(L)$ induced from conjugate actions of E on L, G and that of F on L, respectively. There are natural commutative diagrams

$$\begin{array}{ccc} F & \longrightarrow & E \\ \downarrow & & \downarrow \\ G & \longrightarrow & \mathrm{Aut}(L), \end{array} \qquad \begin{array}{ccc} F & \longrightarrow & E \\ \downarrow & & \downarrow \\ G & \longrightarrow & \mathrm{Aut}(G). \end{array}$$

Let G_1 be the image of $G \to \mathrm{Aut}(L)$. Let m be a positive integer such that $G_1 \subset \mathrm{Aut}(L) \to \mathrm{Aut}(L \otimes \mathbb{Z}/m\mathbb{Z})$ is still injective. Let E_0 be the intersection of the kernel of the composite $E \to \mathrm{Aut}(L) \to \mathrm{Aut}(L \otimes \mathbb{Z}/m\mathbb{Z})$ and that of $E \to \mathrm{Aut}(G)$. Then $E_0 \subset E$ is a normal subgroup of finite index and $E_0 \cap F$ contains L. The quotient group $G_0 := (E_0 \cap F)/L$ is abelian, since G_0 is contained in the center of G. Further L is a

trivial G_0-module, i.e., L is contained in the center of $E_0 \cap F$. Applying 3.2 to the case $M = N = E_0 \cap F$, we see that $E_0 \cap F$ is abelian. $\square$

4. DEGENERATION OF COMPLEX TORI

Let $p\colon B \to S$ be a smooth basic torus fibration over a simply connected normal variety S and let G be a finite group acting on both B and S compatible way with p. We denote by $\varphi_g\colon B \to B$ and $\psi_g\colon S \to S$ the actions of $g \in G$ on B and S, respectively. Then for the variation of Hodge structure $H := R_1p_*\mathbb{Z}_B$, we have isomorphisms $\varphi_{g*}\colon \psi_{g*}H \to H$ satisfying $\varphi_{gh*} = \varphi_{g*} \circ \psi_{g*}(\varphi_{h*})$ for $g, h \in G$. Thus these data φ_{g*} form a G-linearization of H. It induces G-module structures on $L := H^0(S, H)$, $V := H^0(S, \mathcal{V}_H)$ and $A := H^0(S, \mathfrak{S}_H)$. We then obtain another action of G on B, which preserves the zero section. We denote it by $\phi_g\colon B \to B$. The subgroup of $\mathrm{Aut}(B)$ generated by ϕ_g for $g \in G$ and by the action of A as translation is naturally isomorphic to the semi-direct product $A \rtimes G$. Here ϕ_g give a splitting $G \to A \rtimes G$. Thus there are sections $\sigma_g \in A$ such that $\varphi_g = \mathrm{tr}(\sigma_g) \circ \phi_g$, where $\mathrm{tr}(a)\colon B \to B$ denotes the translation by $a \in A$. Hence $\sigma_{gh} = \sigma_g + g \cdot \sigma_h$ for any $g, h \in G$. Here we note that $\mathrm{tr}(g \cdot a) = \phi_g \circ \mathrm{tr}(a) \circ \phi_g^{-1}$ for $g \in G$ and $a \in A$. Thus the mapping $\sigma\colon G \ni g \mapsto \sigma_g \in A$ forms a cocycle in $Z^1(G, A)$. For a global section $a \in A$, let $\varphi_g^a\colon B \to B$ be the isomorphism defined by $\mathrm{tr}(a) \circ \varphi_g \circ \mathrm{tr}(a)^{-1}$. Then $\varphi_g^a = \mathrm{tr}(\sigma_g + a - g \cdot a) \circ \phi_g$. Thus the cocycle associated with the new action of G defined by φ_g^a is given by $\sigma_g + a - g \cdot a$. For the subgroup

$$F := \{(v, g) \in V \rtimes G \,|\, \sigma_g = v \mod L\},$$

we have an exact sequence

$$1 \to L \to F \to G \to 1.$$

By 3.3, it corresponds to the cohomology class $c(\sigma)$ induced by $\sigma \in Z^1(G, A)$ under the injection $H^1(G, A) \to H^2(G, L)$. The order m of $c(\sigma)$ is finite. Hence there is a global section $a \in A$ such that $\delta_g := \sigma_g + a - g \cdot a \in A$ is contained in the image of $m^{-1}L \subset V \to A$ for any $g \in G$. In other words, $\delta_g\colon S \to B$ is a constant section to $m^{-1}\mathbb{H}/\mathbb{H} \subset B$. By changing the zero section by a, we may assume that $\delta = \sigma$, i.e., σ_g is a constant section to torsion points.

Lemma 4.1 (cf. [Vi1, 9.10]). (1) *If $c(\sigma) = 0$, then there is a section $\Sigma \subset B$ of p preserved by the action of G.*

(2) *Let $G_1 \subset G$ be the normal subgroup consisting all the element $g \in G$ which acts trivially on L and which has a fixed point in B. Then $\sigma_g = 0$ for any $g \in G_1$. In particular, p is isomorphic to the pullback of a smooth basic torus fibration over the quotient space $G_1\backslash S$.*

(3) *If G fixes a point of B, then there is a lift $G \to F$. In particular, there is a G-invariant section of p.*

Proof. (1). There is $a \in A$ such that $\sigma_g = a - g \cdot a$ for any $g \in G$. Hence $\varphi_g = \mathrm{tr}(a) \circ \phi_g \circ \mathrm{tr}(a)^{-1}$. Thus the image of $a\colon S \to B$ is invariant.

(2). Let $b \in B$ be a fixed point of $g \in G_1$. Then G_1 acts on the fiber $B_{p(b)} = p^{-1}(p(b))$. Since $L \simeq H_1(B_{p(b)}, \mathbb{Z})$, the action is trivial. Therefore $\sigma_g(s) = 0$. Hence $\sigma_g = 0$ as an element of A. In particular, G_1 preserves the zero section. If the action of $g \in G_1$ on S is trivial, then so is on B. Thus $B \to S$ is isomorphic to the pullback of $G_1\backslash B \to G_1\backslash S$.

(3). Let $b \in B$ be a fixed point of G and let us consider the homomorphism $G \to \mathrm{Aut}(B_{p(b)})$. Let $U_{p(b)}$ be the universal covering space of $B_{p(b)}$. Then we also have the following commutative diagram of exact sequences:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & L & \longrightarrow & \mathrm{Aff}(U_{p(b)}) & \longrightarrow & \mathrm{Aut}(B_{p(b)}) & \longrightarrow & 1 \\ & & \| & & \uparrow & & \uparrow & & \\ 1 & \longrightarrow & L & \longrightarrow & F & \longrightarrow & G & \longrightarrow & 1. \end{array}$$

Let F' and G' be the images of $F \to \mathrm{Aff}(U_{p(b)})$ and $G \to \mathrm{Aut}(B_{p(b)})$, respectively. Then the exact sequence

$$1 \to L \to F' \to G' \to 1$$

has a splitting $G' \to F'$ by 3.5. Thus we also have a splitting $G \to F$. □

Corollary 4.2 (cf. Lemma 2.7 of [N4])**.** *Let $\tau\colon Y \to X$ be a finite surjective morphism of normal varieties and let $\pi\colon X \to S$ be a proper surjective morphism onto a normal variety such that $X_s = \pi^{-1}(s)$ is a complex $\mathbb{Q}$-torus and $Y_s = \tau^{-1}(X_s) \to X_s$ is the toric reduction for general $s \in S$. If the composite $\pi \circ \tau\colon Y \to X \to S$ is a smooth morphism, then $\pi\colon X \to S$ is also smooth.*

Proof. We may assume that S is simply connected. Let $S^\star$ be a Zariski-open subset of S over which π is smooth and let $X^\star = \pi^{-1}(S^\star)$ and $Y^\star = \tau^{-1}(X^\star)$. We may assume

that there is a section $\sigma\colon S \to Y$ of $\pi \circ \tau$. Let F be the Q-lattice $\pi_1(X_s)$ for a point $s \in S^\star$. Then $\pi_1(X^\star) \simeq F \rtimes \pi_1(S^\star)$ and $\pi_1(Y^\star) \simeq L_F \rtimes \pi_1(S^\star)$. Let $P \lhd \pi_1(X^\star)$ be the maximal normal subgroup contained in $\pi_1(Y^\star)$. Then P is a finite index subgroup containing L_F, since L_F is a normal subgroup of $\pi_1(X^\star)$. Let $Y'^\star \to X^\star$ be the finite étale covering corresponding to P and let $Y'^\star \to S'^\star \to S^\star$ be the Stein factorization. Then $\deg(Y'^\star/Y^\star) = \deg(S'^\star/S^\star)$. There exist finite morphisms $Y' \to Y$ and $S' \to S$ extending $Y'^\star \to Y^\star$ and $S'^\star \to S^\star$, respectively. Then $Y' \simeq Y \times_S S'$ and hence $Y' \to S'$ is a smooth basic torus fibration. Let G be the Galois group $\mathrm{Gal}(Y'/X) \simeq \pi_1(X^\star)/P$ and let $G_1 \subset G$ be the subgroup consisting $g \in G$ such that g has a fixed point in Y'. Suppose that $G_1 = \{1\}$. Then $Y' \to Y$ is étale and hence so is $S' \to S$. Thus $Y' = Y$ since S is simply connected. Thus $Y \to X$ is an étale covering. Next assume that $G_1 \neq \{1\}$. By 4.1, $Y' \to S'$ is the pullback of a smooth basic torus fibration $G_1\backslash Y' \to G_1\backslash S'$. Since $G_1 \subset G$ is a normal subgroup, $Y \simeq G_1\backslash Y'$ and $Y \to X$ is a Galois covering. Thus $G = \pi_1(X^\star)/\pi_1(Y^\star)$ and $G_1 = \{1\}$. This is a contradiction. Therefore $X \to S$ is a smooth morphism. □

Theorem 4.3. *Let $\pi\colon X \to S$ be a proper surjective morphism from a normal variety onto a unit disc $S = \Delta = \{z \in \mathbb{C} \mid |z| < 1\}$ satisfying the following conditions*:

(1) *π is smooth outside the origin $0 \in S$.*

(2) *Smooth fibers are complex tori.*

(3) *There is a section $\sigma\colon S \to X$ of π.*

(4) *There is an étale covering $U \to X$ from a connected Stein space U.*

(5) *π is a BK-morphism.*

Then π is a smooth morphism and π has an abelian group structure over S whose zero section is σ.

Proof. Assume first that Theorem is true when the central fiber $X_0 = \pi^*(0)$ is reduced. For a finite covering $S' = \Delta \ni z \mapsto z^m \in S$, let X' be the normalization of $X \times_S S'$. Then for a suitable positive integer m, the central fiber X'_0 is reduced. A connected component of $U \times_X X'$ is also Stein and is an étale covering space over X'. Hence $X' \to S'$ is smooth by the assumption. Especially, X_0 is irreducible. Since there is a section σ, X_0 is reduced. Therefore, we may assume that X_0 is reduced for the proof.

Let $S^\star := S \setminus \{0\}$ be the punctured disc. Then $\pi^\star\colon X^\star := \pi^{-1}(S^\star) \to S^\star$ is a smooth morphism with a section $\sigma^\star := \sigma_{|S^\star}$. Thus $\pi^\star$ is the smooth basic torus fibration associated with a variation of Hodge structure of weight -1 over $S^\star$. In particular, there is a vector bundle $V^\star \to S^\star$ together with an étale covering $V^\star \to X^\star$ whose fibers form a discrete locally constant system isomorphic to the variation of Hodge structure. Let $\mu^\star\colon X^\star \times_{S^\star} X^\star \to X^\star$ and $\iota^\star\colon X^\star \to X^\star$ be the multiplication map and the inverse map of the group structure of $\pi^\star$, respectively.

The open subset $U^\star := U \times_S S^\star \subset U$ is also a Stein space. Since the section $\sigma(S) \subset X$ is simply connected, we have a lift $\sigma'\colon S \to U$ of σ. Thus the smooth morphism $U^\star \to S^\star$ have a section $\sigma'^\star = \sigma'_{|S^\star}$. Therefore, every fibers of $U^\star \to S^\star$ are connected. This implies that there is a covering map $V^\star \to U^\star$ over $X^\star$, where the zero section is sent to $\sigma'^\star$. Hence there is also a compatible abelian group structure on the composite $U^\star \to X^\star \to S^\star$.

The morphisms $\mu^\star$ and $\iota^\star$ extend to meromorphic mappings $\mu\colon X \times_S X \cdots\to X$ and $\iota\colon X \cdots\to X$ over S, by 1.6. Therefore, they also induce meromorphic mappings $\mu_U\colon U \times_S U \cdots\to U$ and $\iota_U\colon U \cdots\to U$. Since U is Stein, ι_U is holomorphic and μ_U induces a morphism from the normalization of $U \times_S U$. Therefore, ι is an automorphism and μ induces a morphism $\mu'\colon W \to X$ from the normalization W of $X \times_S X$. Let $\widetilde{X_0} \to X_0$ be the normalization of the central fiber $X_0 = \pi^*(0)$.

Claim 4.4. The μ', ι, and σ give a holomorphic group structure on $\widetilde{X_0}$.

Proof. Let W_0 be the fiber of $W \to S$ over $0 \in S$. Then there is a natural finite morphism $\widetilde{X_0} \times \widetilde{X_0} \to W_0$, since $W_0 \to X_0 \times X_0$ is a finite morphism isomorphic over a Zariski-open subset of X_0. Hence the fibers of μ' and ι over $0 \in S$ induce morphisms $\tilde{\mu}\colon \widetilde{X_0} \times \widetilde{X_0} \to \widetilde{X_0}$ and $\tilde{\iota}\colon \widetilde{X_0} \to \widetilde{X_0}$, respectively.

Let $\alpha\colon X \hookrightarrow X \times_S X$ be a closed embedding defined by $\alpha(x) = (x, \iota(x))$ and let $X' \subset W$ be the pullback of $\alpha(X)$ by the normalization $W \to X \times_S X$. Since X is normal and $X' \to X$ is a finite bimeromorphic morphism, we have a lift $\alpha'\colon X \hookrightarrow W$ of α. We note that there is a commutative diagram:

$$\begin{array}{ccc} X & \xrightarrow{\alpha'} & W \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\mu'} \\ S & \xrightarrow{\sigma} & X. \end{array}$$

Let $\beta\colon \widetilde{X_0} \hookrightarrow \widetilde{X_0} \times \widetilde{X_0}$ be a closed embedding defined by $\beta(y) = (y, \tilde{\iota}(y))$. There is a commutative diagram:

$$\begin{array}{ccccc} \widetilde{X_0} & \xrightarrow{\beta} & \widetilde{X_0} \times \widetilde{X_0} & \xrightarrow{\tilde{\mu}} & \widetilde{X_0} \\ \downarrow & & \downarrow & & \downarrow \\ X_0 & \xrightarrow{\alpha_0'} & W_0 & \xrightarrow{\mu_0'} & X_0, \end{array}$$

where α_0' and μ_0' are the fibers of α and μ' over $0 \in S$, respectively. Note that the composite $\mu_0' \circ \alpha_0'$ of bottom morphisms is a constant map to the point $\sigma(0)$. Thus the composite $\tilde{\mu} \circ \beta$ is also a constant map to a point $e \in \widetilde{X_0}$ lying over $\sigma(0)$. The associative law of μ induces that of $\tilde{\mu}$. We denote $x \cdot y := \tilde{\mu}(x, y)$ for $x, y \in \widetilde{X_0}$. Then $x \cdot \tilde{\iota}(x) = e$ for any x. We shall show e is a unit element, i.e., $e \cdot y = y \cdot e = y$ for any $y \in \widetilde{X_0}$. For the purpose, we have only to check $y \cdot x \cdot \iota(x) = x \cdot \iota(x) \cdot y = y$ for general x, y. Let us consider the composite

$$X \times_S X \overset{\alpha \times_S \mathrm{id}}{\longrightarrow} X \times_S X \times_S X \cdots \to X,$$

where the first morphism sends $(x, y) \mapsto (x, \iota(x), y)$ and the second meromorphic mapping sends $(x, y, z) \mapsto \mu(x, \mu(y, z)) = \mu(\mu(x, y), z)$. The composite is identified with the second projection. Therefore $x \cdot \iota(x) \cdot y = y$ for general $x, y \in \widetilde{X_0}$. The opposite side is similarly shown. Hence e is a unit. Since $\tilde{\iota}(x)$ is a right inverse, it is also a left inverse. Thus $\widetilde{X_0}$ has a natural group structure. □

Proof of 4.3 continued. Therefore, $\widetilde{X_0}$ is a disjoint union of complex tori. Since $X \to S$ is a BK-morphism, we have

$$1 = p_g(X_s) \geq \sum_{B \subset X_0} p_g(B),$$

where $s \in S$ is a general point, $B \subset X_0$ is any irreducible component and $p_g(B)$ denotes the geometric genus of the nonsingular model of B. Thus $\widetilde{X_0}$ and X_0 are irreducible.

Let us consider the evaluation homomorphism $\pi^*\pi_*\omega_X \to \omega_X$, where ω_X is the dualizing sheaf of X. This is surjective on $X^\star$ and hence, there is an effective Weil divisor D such that $\operatorname{Supp} D \subset X_0$ and $\omega_X \simeq \mathcal{O}_X(D)$. Since X_0 is irreducible, we see that $D = 0$. Therefore $\omega_X \simeq \mathcal{O}_X$ and $\omega_{X_0} \simeq \mathcal{O}_{X_0}$. Since the normalization of X_0 is a complex torus, the conductor is zero. Therefore X_0 itself is a complex torus. □

Proposition 4.5. *Let $\pi : X \to S$ be a proper surjective morphism from a normal variety onto a unit disc $S = \Delta$ such that π is smooth outside the origin $0 \in S$ and every smooth fibers are complex Q-tori. Suppose that X is Kähler and there is an étale covering $U \to X$ from a Stein space. Then X is nonsingular and the central fiber is a multiple of a complex Q-torus.*

Proof. First of all, we assume that π admits a section. Then there is a toric reduction $Y^\star \to X^\star$ of $X^\star = X \setminus \pi^{-1}(0) \to S^\star = S \setminus \{0\}$ as in 3.13. There is a finite morphism $Y \to X$ extending $Y^\star \to X^\star$. There is a lift of the section of π to Y by the construction of toric reduction. Thus $Y \to S$ is a torus fibration with a section. Since $U \times_X Y$ is also a Stein space, $Y \to S$ is a smooth toric fibration by 4.3. Hence $X \to S$ is a smooth morphism by 4.2.

Next, we treat the general case. Let m be the smallest positive integer such that for the ramified covering $S' = \Delta \ni t \mapsto t^m \in S$, the normalization X' of the fiber product $X \times_S S'$ admits a section of $X' \to S'$. Since $U \times_X X'$ is a Stein space étale over X', $X' \to S'$ is smooth by the previous argument. Therefore the central fiber $X_0 = \pi^*(0)$ is irreducible and m is the multiplicity of the fiber. In particular, $X' \to X$ is étale outside the singular locus of X. Let G' be the Galois group $\mathrm{Gal}(S'/S) \simeq \mathbb{Z}/m\mathbb{Z}$ and let F be the fundamental group of a smooth fiber $X_s = \pi^{-1}(s)$. The toric reduction $Y' \to X'$ of $X' \to S'$ is uniquely determined by the lattice part L_F of $F \simeq \pi_1(X')$. Thus the composite $Y' \to X' \to X$ is a Galois covering. We denote by G the Galois group and let $f : Y' \to S'$ be the smooth basic torus fibration. Suppose that a subgroup $G_0 \subset G$ fixes a point of Y'. Then there is a G_0-invariant section of f by 4.1. Since $G_0 \cap \mathrm{Gal}(Y'/X') = \{1\}$, the morphism $G_0 \backslash Y' \to G_0 \backslash S'$ admits a section and $G_0 \backslash X'$ is the normalization of $X \times_S (G_0 \backslash S')$. By the definition of m, we have $G_0 = \{1\}$. Therefore G acts freely on Y' and $X' \to X$ is étale. □

Lemma 4.6. *Let $f : V \to W$ be a proper surjective morphism between complex analytic varieties. Suppose that there is an irreducible component Γ of a fiber $f^{-1}(w)$ with $\dim \Gamma = \dim V - \dim W$. Then there exist an open neighborhood $W' \subset W$ of w and a closed subvariety V' of $f^{-1}(W')$ such that the restriction $V' \to W'$ is a finite surjective morphism.*

Proof. Let us take a general point $v \in \Gamma$ and let $v \in U \subset V$ be a Stein open neighborhood. Let $D_1, D_2, \ldots, D_k$ be general hypersurfaces of U containing v, where $k = \dim V - \dim W$.

Then v is an isolated point of $f^{-1}(w) \cap \bigcap D_j$. Therefore, for an open neighborhood W' of w and for an open neighborhood $V' \subset f^{-1}(W')$ of v in $\bigcap D_j$, the morphism $V' \to W'$ is proper. □

Theorem 4.7. *Let $\pi\colon X \to S$ be a proper surjective morphism from a normal variety onto a d-dimensional unit polydisc $S = \Delta^d$ and let $D = \sum_{i=1}^{l} D_i$ be a divisor of S satisfying the following conditions*:

(1) *$D_i = \{t_i = 0\}$ for a coordinate system $(t_1, t_2, \dots, t_d)$ of S.*

(2) *π is smooth outside D.*

(3) *A general fiber is a complex Q-torus.*

(4) *π is an equi-dimensional morphism.*

(5) *There is an étale covering $U \to X$ from a Stein space U.*

(6) *X is Kähler.*

Then $\pi^{-1}(D_i) = \operatorname{Supp} \pi^*(D_i)$ are irreducible divisors. Let m_i be the multiplicity $\pi^*(D_i)$ and let $S' = \Delta^d \to S$ be a Kummer covering defined by

$$S' \ni (t_1, t_2, \dots, t_d) \mapsto (t_1^{m_1}, t_2^{m_2}, \dots, t_l^{m_l}, t_{l+1}, \dots, t_d) \in S.$$

Then for the normalization X' of the main component of $X \times_S S'$, the induced morphism $X' \to S'$ is smooth and another induced morphism $X' \to X$ is étale outside the singular locus of X. Consequently, X has only toric quotient singularities, π is flat and the following canonical bundle formula holds:

$$K_X \sim_{\mathbb{Q}} \pi^*(K_S + \sum_{i=1}^{l}(1 - \frac{1}{m_i})D_i).$$

Proof. Let $C_i \subset S$ be a general smooth curve intersecting with the divisor D_i transversely at one general point of D_i. By applying 4.5 to the pullback $X \times_S C_i \to C_i$, we see that $\pi^* D_i = m_i \Gamma_i + E_i$ for a prime divisor Γ_i with $\pi(\Gamma_i) = D_i$ and another effective divisor E_i with $\pi(E_i) \subsetneq D_i$. Since π is equi-dimensional, $E_i = 0$ and thus $\Gamma_i = \pi^{-1}(D_i)$.

We define $S^\star := S \setminus D \simeq (\Delta^\star)^l \times \Delta^{d-l}$, where $\Delta^\star := \Delta \setminus \{0\}$ is the punctured disc. Let $S' \to S$ and $\pi'\colon X' \to S'$ be the Kummer covering and the induced fibration as above. There is a section of $X' \to S'$ over an open neighborhood $\mathcal{U}$ of a general point of any $\{t_i = 0\} \subset S'$. There is a Zariski-open subset $S^\circ \subset S'$ such that $\operatorname{codim}(S' \setminus S^\circ) \geq 2$ and $X' \to S'$ is smooth over S° by 4.5. Since S° is simply connected, we have a finite

morphism $Y' \to X'$ which gives a toric reduction for $X' \to S'$ over S° by 3.12. The smooth torus fibration $Y^\circ := Y' \times_{S'} S^\circ \to S^\circ$ defines a variation of Hodge structure H on S' since S° is simply connected. There is an element $\eta \in H^1(S^\circ, \mathfrak{S}_H)$ which corresponds to the torsor Y°/S° of the smooth basic torus fibration $p\colon B \to S'$ associated with H. From the exact sequence

$$0 \to H \to \mathcal{V}_H \to \mathfrak{S}_H \to 0,$$

and by the vanishing $H^2(S^\circ, \mathbb{Z}) = 0$, we have an isomorphism $H^1(S^\circ, \mathcal{V}_H) \simeq H^1(S^\circ, \mathfrak{S}_H)$. Thus $H^1(S^\circ, \mathfrak{S}_H)$ is torsion free. On the other hand, by 4.6, if we replace S' by an open neighborhood of the origin, then there is closed subvariety $\Sigma \subset Y'$ which is finite and surjective over S'. Thus $\eta = 0$ by 2.12. Hence $Y^\circ \to S^\circ$ admits a section. Two fibrations of complex tori $Y' \to S'$ and $B \to S'$ are isomorphic to each other over S°. Since the universal covering spaces of Y' and B are both Stein, we have an isomorphism $Y' \simeq B$ over S'. Therefore $Y' \to S'$ is a smooth family of complex tori. By 4.2, $X' \to S'$ is also a smooth family of complex Q-tori. □

Corollary 4.8. *Let S be a germ of normal variety with only quotient singularities and let D be an effective reduced divisor such that $S \setminus D \subset S$ is a toroidal embedding. Let $\pi\colon X \to S$ be a proper surjective morphism from a normal Kähler variety satisfying the following conditions*:

(1) *π is smooth over $S \setminus D$.*

(2) *A general fiber is a complex Q-torus.*

(3) *π is an equi-dimensional morphism.*

(4) *There is an étale covering $U \to X$ from a Stein space U.*

Then there is a finite covering $S' \to S$ branched only along D such that the normalization X' of $X \times_S S'$ is smooth over S' and étale over X. In particular, X has only quotient singularities and $X \setminus \pi^{-1}D \subset X$ is a toroidal embedding [KKMS].

Proof. Let $S^\sharp := \Delta^d$ be a unit polydisc with a coordinate $(t_1, t_2, \dots, t_d)$, and let $L_i := \{t_i = 0\}$ for $1 \le i \le l$ be divisors on $S^\sharp$. For a positive integer m, let Γ_m be the group $(\mathbb{Z}/m\mathbb{Z})^{\oplus l}$, which acts on $S^\sharp$ by

$$(t_1, t_2, \dots, t_d) \mapsto (\mu^{n_1} t_1, \mu^{n_2} t_2, \dots, \mu^{n_l} t_l, t_{l+1}, \dots, t_d)$$

for $(n_1, n_2, \dots, n_d) \in \Gamma_m$, where $\mu := \exp(2\pi\sqrt{-1}/m)$. For $d = \dim S$, there exist positive integers l, m and a finite subgroup $\Gamma \subset \Gamma_m$ such that S is isomorphic to the quotient space $\Gamma\backslash S^\sharp$ and D is the image of $L = \cup L_i$. Let $\tau\colon S^\sharp \to S$ be the quotient morphism and let $X^\sharp$ be the normalization of $X \times_S S^\sharp$. After replacing $S^\sharp$ by a further covering $\Delta^d \to S^\sharp$, we may assume $X^\sharp \to S^\sharp$ is a smooth morphism with a section by 4.7. Let $Y^\sharp \to X^\sharp$ be the toric reduction for $X^\sharp/S^\sharp$. Note that $Y^\sharp \to X$ is a finite Galois covering.

Let $S' \to S$ be the finite Galois covering corresponding to a quotient group G of $\mathrm{Gal}(S^\sharp/S)$ such that for the normalization X' of $X \times_S S'$, $X' \to S'$ admits a section. Suppose that the degree of $S' \to S$ is minimal among such coverings. Let $Y' \to X'$ be the finite surjective morphism extending the toric reduction for the smooth part of $X' \to S'$ with respect to the section. Similarly to the argument in 4.7, there is a Zariski-open subset $S^\circ \subset S'$ of $\mathrm{codim}(S' \setminus S^\circ) \geq 2$ over which $Y' \to S'$ and $X' \to S'$ are smooth. Thus the Galois group $\mathrm{Gal}(S^\sharp/S')$ acts trivially on $H^0(S^\sharp, H^\sharp)$, where $H^\sharp$ denotes the variation of Hodge structure associated with $Y^\sharp \to S^\sharp$. Since $\mathrm{Gal}(S^\sharp/S')$ preserves a section of $Y^\sharp \to S^\sharp$, $Y' \to S'$ is a smooth basic torus fibration by 4.1. Thus $X' \to S'$ is also smooth by 4.2.

Suppose that a subgroup $G_0 \subset G = \mathrm{Gal}(S'/S)$ fixes a point of Y'. Then $G_0 \to \mathrm{Aut}(S')$ is injective and there is a G_0-invariant section of $Y' \to S'$ by 4.1. Then $X'' \to S''$ admits a section where $S'' := G_0\backslash S'$ and X'' denotes the normalization of $X \times_S S''$. Thus $G_0 = \{1\}$ by the choice of $S' \to S$. Therefore, $Y' \to X$ is étale. □

Theorem 4.9. *Let $\pi\colon X \to S$ be a proper surjective LBK-morphism of normal varieties and let D be a reduced effective divisor of S satisfying the following conditions*:

(1) *$S \setminus D \subset S$ is a toroidal embedding and S has only quotient singularities.*
(2) *π is smooth outside D.*
(3) *A general fiber is a complex Q-torus.*
(4) *π is an equi-dimensional morphism.*
(5) *There is an étale covering $U \to X$ such that $U \to X \to S$ is a locally Stein morphism.*

Then we have the following properties:

(a) *If π admits meromorphic sections locally on S, then π is a smooth morphism.*
(b) *$X \setminus \pi^{-1}D \subset X$ is a toroidal embedding and X has only quotient singularities.*

(c) $\pi^*(D_k)$ *is irreducible for any irreducible component* $D_k \subset D$.

(d) *Let* m_k *be the multiplicity of* $\pi^*(D_k)$, *i.e.,* $\pi^*(D_k) = m_k\pi^{-1}(D_k)$ *and let* Z *be the normalization of* $X \times_S X$. *Then the both projections* $\pi_1, \pi_2 \colon Z \to X$ *are smooth and there is a* $\mathbb{Q}$*-linear equivalence relation*

$$K_{Z/X} \sim_{\mathbb{Q}} \pi_1^*(K_{X/S} - \pi^*(\sum(1 - \frac{1}{m_k})D_k)),$$

where $K_{Z/X}$ *denotes the relative canonical divisor for* π_2 *and* $K_{X/S}$ *denotes the* $\mathbb{Q}$*-divisor* $K_X - \pi^*K_S$.

Proof. (a), (b), (c) are derived from 4.8.

(d). Let W be the fiber product $X \times_S X$ and let $p_1, p_2 \colon W \to X$ be projections. For the normalization $\nu \colon Z \to W$, $\pi_i = p_i \circ \nu$ for $i = 1, 2$. Since π_2 admits a meromorphic section which corresponds to the diagonal locus, there is a finite morphism $\tau \colon Y \to Z$ such that the composite $\pi_Y := \pi_2 \circ \tau \colon Y \to Z \to X$ is a torus fibration smooth over $\pi^{-1}(S \setminus D)$ and that the fiber $\tau_x \colon Y_x \to Z_x$ is isomorphic to the toric reduction for any $x \in \pi^{-1}(S \setminus D)$. Let $U_Y \to Y$ be the pullback of $U \to X$ by the composite $\pi_1 \circ \tau \colon Y \to Z \to X$. Then the composition of $U_Y \to Y$ and π_Y is also a locally Stein morphism. Hence π_Y is a smooth torus fibration by (a) and hence $\pi_2 \colon Z \to X$ is also smooth by 4.2. Let $S^\circ \subset S$ be a nonsingular Zariski-open subset such that $\operatorname{codim}(S \setminus S^\circ) \geq 2$ and that $X^\circ := \pi^{-1}(S^\circ)$ is nonsingular. Let $W^\circ \subset W$, $Y^\circ \subset Y$ and $Z^\circ \subset Z$ be the pullbacks of S°, respectively. Then W° is Gorenstein and

$$\omega_{W^\circ/S^\circ} \simeq p_1^*\omega_{X^\circ/S^\circ} \otimes p_2^*\omega_{X^\circ/S^\circ}.$$

From the trace map $\nu_*\omega_Z \to \omega_W$, we have an injection $\omega_{Z^\circ/X^\circ} \hookrightarrow \nu^*p_1^*\omega_{X^\circ/S^\circ}$, where we consider $Z^\circ \to X^\circ$ to be the restriction of π_2. The cokernel is supported in the divisor $\pi_2^{-1}\pi^{-1}D$. For a general point $w \in p_2^{-1}\pi^{-1}(D)$, its local ring $\mathcal{O}_{W,w}$ is smooth over

$$\mathbb{C}[x, y]/(x^m - y^m),$$

where m is the multiplicity of π^*D at the point $p_2(w)$. Thus by considering its conductor, we have

$$K_{Z/X} \sim_{\mathbb{Q}} \pi_1^*(K_{X/S} - \pi^*(\sum(1 - \frac{1}{m_k})D_k)). \qquad \square$$

5. First method

Definition 5.1. Let $f\colon Z \cdots\to M$ be a meromorphic mapping of nonsingular varieties.

(1) Let D be a Cartier divisor of M such that $f(Z) \not\subset \operatorname{Supp} D$. The pullback f^*D is defined to be $\mu_* g^* D$, where $\mu\colon Y \to Z$ be a bimeromorphic morphism from a nonsingular variety such that the composite $g := f \circ \mu\colon Y \to Z \cdots\to M$ is a morphism. The pullback is independent of the choice of $\mu\colon Y \to Z$.

(2) The f is called *of maximal rank*, if there is a point $z \in Z$ such that f is a holomorphic at z and the tangential map $T_{Z,z} \to T_{M,f(z)}$ is surjective.

(3) A meromorphic mapping f of maximal rank is called *non-degenerate*, if $\dim Z = \dim M$.

(4) For a non-degenerate meromorphic mapping, we have the following ramification formula:

$$K_Z = f^* K_M + R_f,$$

where K_Z and K_M denotes the canonical bundles of Z and M, respectively and R_f is an effective divisor called the *ramification divisor* of f.

The following result is due to Kobayashi–Ochiai [KO1] (cf. [KO2]).

Theorem. (I) *Let $f\colon \mathbb{C}^n \cdots\to M$ be a meromorphic mapping of maximal rank into a nonsingular projective variety M. Then M is not of general type, i.e., $\kappa(M) < \dim M$.*

(II) *Let $f\colon Z \cdots\to M$ be a meromorphic mapping from a Zariski-open subset Z of a complex variety $\overline{Z}$ into a nonsingular projective variety M of general type. If f is of maximal rank, then f extends to a meromorphic mapping from $\overline{Z}$.*

Combining with results of Ueno, Kawamata and Viehweg on characterization of abelian varieties, [U1], [KV], [Kw1], we have the following application:

Proposition 5.2. *The Albanese mapping $X \to \operatorname{Alb} X$ is a fiber space for a compact Kähler manifold X satisfying one of the following conditions*:

(I) *There is a meromorphic mapping $\mathbb{C}^n \cdots\to X$ of maximal rank.*

(II) *There is a meromorphic mapping $Z \cdots\to U$ of maximal rank from a Zariski-open subset Z of a compact complex variety $\overline{Z}$ into the universal covering space U of X.*

Proof. Let $X \to Y \to \operatorname{Alb} X$ be the Stein factorization and suppose that $Y \to \operatorname{Alb} X$ is not an isomorphism. Then by Theorem 23 of [Kw1], there exist a subtorus $T \subset \operatorname{Alb} X$, finite étale coverings $Y' \to Y$ and $T' \to T$, and a finite morphism $B \to \operatorname{Alb} X/T$ from a variety B of general type such that Y' is an analytic fiber bundle over B with T' as a fiber. In the case (I), we have a meromorphic mapping $\mathbb{C}^n \cdots\to B$ of maximal rank. Hence B is a point by Kobayashi–Ochiai's theorem. In the case (II), we have a meromorphic mapping $\phi\colon Z \cdots\to \widetilde{B}$ of maximal rank into the universal covering space $\widetilde{B}$ of B. Hence the composite $Z \cdots\to B$ extends to a meromorphic mapping $\overline{Z} \cdots\to B$ by Kobayashi–Ochiai's theorem. Thus we also have a meromorphic mapping $\overline{Z} \cdots\to \widetilde{B}$ extending ϕ. Therefore B is a point. In both cases, we see that Y is a torus by Theorem 22 of [Kw1]. This contradicts to the uniqueness of the Albanese mapping. □

A generalization of Kobayashi–Ochiai's theorem in the case (I) is given by Sakai in Theorem 1 of [Sa1], which is identical to the following result except on the condition: f is only meromorphic.

Theorem 5.3. *Let $f\colon \mathbb{C}^n \cdots\to M$ be a meromorphic mapping of maximal rank into a nonsingular projective variety M. Suppose that there is a simple normal crossing divisor $\sum D_k$ of M such that $f^*D_k \geq m_k f^{-1}(D_k)$ for some positive integers m_k for any irreducible components D_k, where $f^{-1}(D_k)$ denotes $f^*(D_k)_{\mathrm{red}}$. Then*

$$\kappa(K_M + \sum (1 - \frac{1}{m_k})D_k, M) < \dim M.$$

Proof. By considering general affine subspaces of $\mathbb{C}^n$ of dimension equal to $\dim M$, we may assume f is non-degenerate. Then a version of Second Main Theorem in Nevanlinna theory [Sh1, 3.1] (cf. [Sh2, 2.4]), states that

$$\| \quad T(K_M + \sum D_k, r) \leq N(\sum D_k, r) - N(R_f, r) + O(\log T(L, r)) + o(\log r)$$

for an ample divisor L on M, where T and N denotes the characteristic and the counting functions, respectively. For the meaning of other things $\|$, O, and o, we refer to [Sh1]. We have the inequality

$$R_f \geq f^*(\sum (1 - \frac{1}{m_k})D_k)$$

from the logarithmic ramification formula. By the First Main Theorem, we have

$$\| \quad N(\sum(\frac{1}{m_k})D_k, r) \leq T(\sum(\frac{1}{m_k})D_k, r) + O(1).$$

Therefore:

$$\begin{aligned} \| \quad & T(K_M + \sum(1 - \frac{1}{m_k})D_k, r) \\ & \leq -T(\sum(\frac{1}{m_k})D_k, r) + N(\sum(\frac{1}{m_k})D_k, r) - N(R_f - \sum(1 - \frac{1}{m_k})D_k, r) \\ & \quad + O(\log T(L, r)) + o(\log r) \\ & \leq O(\log T(L, r)) + o(\log r). \end{aligned}$$

If $\kappa(K_M + \sum(1 - (1/m_k))D_k, M) = \dim M$, then

$$\| \quad T(K_M + \sum(1 - \frac{1}{m_k})D_k, r) \geq cT(L, r)$$

for a positive constant $c > 0$. This contradicts to the inequality above. □

Even though the proof above is essentially an application of Nevanlinna theory for several complex variables, the author could not find any articles discussing theorem of this kind. Most related works concern defect relations, where usually D_k are supposed to be mutually linearly equivalent and ample. We shall give a generalization of this kind of Kobayashi–Ochiai's theorem in the case (II) in A.6 in Appendix A.

Theorem 5.4. *Let M be a compact Kähler manifold whose universal covering space is biholomorphic to $\mathbb{C}^n$. Suppose that $\kappa(M) \geq 0$ and that a general fiber of the Iitaka fibration $\Phi\colon M \cdots\to S_0$ is bimeromorphically equivalent to a complex Q-torus. Then $\kappa(M) = 0$ and thus M is a complex Q-torus.*

Proof. Suppose that $\kappa(M) > 0$. Let $M_1 \to M \times S_0$ be the normalization of the graph of the meromorphic mapping Φ. Let $U \simeq \mathbb{C}^n \to M$ be the universal covering mapping and let $U_1 := U \times_M M_1$. Then the composite of $U_1 \to M_1$ and $\Phi_1\colon M_1 \to S_0$ is a locally Stein morphism. Therefore a general fiber of Φ_1 is isomorphic to a complex Q-torus. Let $S \to S_0$ be a birational morphism from a nonsingular projective variety and let $X \to M_1 \times_{S_0} S$ be the normalization of the main component. We may assume that the induced morphism $\pi\colon X \to S$ is equi-dimensional by considering the flattening of Φ_1. Furthermore, we may assume that the morphism π is smooth outside a simple normal crossing divisor $D \subset S$.

For the fiber product $U_X := U_1 \times_{M_1} X$, the composite $U_X \to X \to S$ is also a locally Stein morphism. Thus locally on S, the morphism π satisfies the conditions of 4.7. Therefore, for the prime decomposition $D = \sum D_k$, $\pi^{-1}(D_k)$ are irreducible, $\pi^* D_k = m_k \pi^{-1}(D_k)$ for some positive integers m_k, X has only quotient singularities and the formula

$$(\sharp) \qquad K_X \sim_{\mathbb{Q}} \pi^*(K_S + \sum (1 - \frac{1}{m_k}) D_k)$$

holds at least locally on S. Further π is a flat morphism since X is Cohen-Macaulay.

Claim 5.5. The formula $(\sharp)$ holds globally on S.

Proof. Let Z be the normalization of the fiber product $X \times_S X$ and let $\pi_1, \pi_2 \colon Z \to X$ be the morphisms induced from projections. Then π_2 is smooth and

$$K_{Z/X} \sim_{\mathbb{Q}} \pi_1^*(K_{X/S} - \pi^*(\sum (1 - \frac{1}{m_k}) D_k))$$

holds by 4.9, where $K_{Z/X}$ is the relative canonical divisor for π_2. Let $\tau \colon Y \to Z$ be the toric reduction for $\pi_2 \colon Z \to X$ with respect to the diagonal section. Since τ is étale, we have an isomorphism $\tau^* \omega_{Z/X} \simeq \omega_{Y/X}$. The period mapping of the torus fibration $\pi_Y := \pi_2 \circ \tau \colon Y \to X$ is a morphism from the universal covering space U_X to the Siegel upper half space. Since U_X and $\mathbb{C}^n$ is bimeromorphically equivalent, the period mapping is constant. Therefore the monodromy representation of the associated variation of Hodge structure with π_Y is finite. In particular, $(\pi_{Y*} \omega_{Y/X})^{\otimes k} \simeq \mathcal{O}_X$ for some integer k. Thus, $K_{Z/X} \sim_{\mathbb{Q}} 0$. Therefore we have the formula $(\sharp)$ globally. □

Proof of 5.4 continued. We have now

$$\kappa(X) = \kappa(M) = \kappa(K_S + \sum (1 - \frac{1}{m_k}) D_k) = \dim S.$$

By applying 5.3 to the composite $f \colon U \cdots\to U_X \to X \to S$, we derive a contradiction to $\kappa(M) > 0$. □

Corollary 5.6. *Let M be a compact Kähler manifold whose universal covering space is biholomorphic to $\mathbb{C}^n$.*

(1) *If the canonical divisor K_M is semi-ample, then M is biholomorphic to a complex Q-torus.*

(2) *If $\dim M \geq 3$, then $\kappa(M) < \dim M - 2$.*

Proof. (1) The Iitaka fibration is now a morphism. Then for a general fiber V, the first Chern class is numerically trivial. There is an étale covering $\mathbb{C}^n \times_M V \to V$ from a Stein space. Thus V is a complex Q-torus by Bogomolov's decomposition theorem [Bo], [Be]. Therefore we have $K_M \sim_{\mathbb{Q}} 0$ by applying 5.4. Hence M is a complex Q-torus again by Bogomolov's decomposition theorem.

(2) It is enough to consider the case $\kappa(M) = \dim M - 2$ and a general fiber of the Iitaka fibration is not a complex Q-torus. The fiber is bimeromorphically equivalent to a $K3$ surface or an Enriques surface. Since the universal covering space of the fiber is not compact, this case does not occur. □

The following result is proved for algebraic varieties in [Kl1], [Vi2].

Theorem 5.7. *Let $f\colon X \to Y$ be a fiber space of compact complex manifolds whose general fibers $X_y = f^{-1}(y)$ are of general type. If $\kappa(\mathcal{M}, Y) \geq 0$ for an invertible sheaf $\mathcal{M}$, then*

$$\kappa(\omega_{X/Y} \otimes f^*\mathcal{M}) \geq \kappa(X_y) + \kappa(\mathcal{M}).$$

Proof. Let m be a sufficiently large positive integer and let $r = r_m$ be the rank of the direct image sheaf $\mathcal{F} := f_*(\omega_{X/Y}^{\otimes m})$. Let $\mu\colon Y' \to Y$ be a bimeromorphic morphism such that $\mathcal{F}' := \mu^*\mathcal{F}/(\text{tor})$ is a locally free sheaf. Then for a nonsingular model X' of $X \times_Y Y'$ and for the induced morphism $f'\colon X' \to Y'$, we have a natural injection $\mathcal{F}' \to f'_*(\omega_{X'/Y'}^{\otimes m}) \otimes \omega_{Y'/Y}^{\otimes m}$. Let $p\colon \mathbb{P} \to Y'$ be the projective bundle associated with $\mathcal{H}om(\mathcal{F}', \mathcal{O}_{Y'}^{\oplus r})$. Then for the tautological invertible sheaf $\mathcal{O}_{\mathbb{P}}(1)$, there is a natural homomorphism

$$p^*\mathcal{F}'^{\vee} \to \mathcal{O}_{\mathbb{P}}(1)^{\oplus r} \quad \text{and its dual} \quad \delta\colon \mathcal{O}_{\mathbb{P}}(-1)^{\oplus r} \to p^*\mathcal{F}'.$$

Let $X \cdots\to \mathbb{P}(\mathcal{F})$ be the meromorphic mapping associated with $f^*f_*(\omega_{X/Y}^{\otimes m}) \to \omega_{X/Y}^{\otimes m}$. It is a bimeromorphic mapping onto its image, if m is sufficiently large. For the fiber product $Z := X' \times_{Y'} \mathbb{P}$, we have also a meromorphic mapping $Z \cdots\to \mathbb{P}(p^*\mathcal{F}')$, which also induces a meromorphic mapping $Z \cdots\to \mathbb{P}^{r-1} \times \mathbb{P}$ by the generically isomorphic homomorphism δ. Thus there is a meromorphic mapping $\mathbb{P} \cdots\to \mathrm{Hilb}(\mathbb{P}^{r-1})$ such that $Z \to \mathbb{P}$ is bimeromorphically equivalent to the pullback of the universal family. We have the following

commutative diagram of nonsingular varieties as in [Vi1]:

$$\begin{array}{ccccccc} X & \longleftarrow & Z & \longleftarrow & Z' & \longrightarrow & V \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ Y & \longleftarrow & \mathbb{P} & \longleftarrow & \mathbb{P}' & \longrightarrow & W, \end{array}$$

where V, W are projective varieties, $\mathbb{P}' \to W$ is semi-stable in codimension one, $Z' \to Z$ and $\mathbb{P}' \to \mathbb{P}$ are generically finite surjective morphisms, and every squares in the diagram are bimeromorphically equivalent to Cartesian. We have $\kappa(\omega_{X/Y} \otimes f^*\mathcal{M}) \geq \kappa(\omega_{Z/\mathbb{P}} \otimes p^*\mu^*\mathcal{M}) \geq \kappa(\omega_{Z'/\mathbb{P}'} \otimes p'^*\mathcal{M})$, where p' is the composite $\mathbb{P}' \to \mathbb{P} \to Y' \to Y$. By the same argument as in Proof of (7.4) of [Vi1], we have

$$\kappa(\omega_{Z'/\mathbb{P}'} \otimes p'^*\mathcal{M}) \geq \kappa(V_w) + \max\{\mathrm{Var}(V/W), \kappa(\mathcal{M})\}. \qquad \square$$

Theorem 5.8. *Let X be a compact Kähler threefold whose universal covering space is biholomorphic to $\mathbb{C}^3$. Then X is a complex Q-torus.*

Proof. If X is projective, then this is proved in [N4]. Another proof is given by 5.6 and the abundance theorem [Mi1], [Mi2], [Kw3]. Next, we assume that X is not projective. As an application of Atiyah's index theorem [A], Gromov has proved the following result [G, 3.2.C$'$]: *If the universal covering space of a compact Kähler non-projective manifold X does not contain any positive dimensional compact subvarieties, then $\chi(X, \mathcal{O}_X) = 0$.* Thus we have an equality $2 \leq 1 + h^{2,0}(X) = q(X) + p_g(X)$, where q and p_g denote the irregularity and the geometric genus, respectively, whereas $h^{2,0}$ is the Hodge number of type $(2,0)$. Since $\kappa(X) \leq 0$, we have $p_g(X) \leq 1$. Thus $q(X) \geq 1$. The Albanese mapping $\alpha\colon X \to \mathrm{Alb}\, X$ is a fiber space by 5.2. Hence $q(X) \leq 3$. If $q(X) = 3$, then X is biholomorphic to a complex torus. If $q(X) = 2$, then a general fiber F of α is an elliptic curve. Because, if it is a curve of genus greater than one, then $\kappa(X) \geq \kappa(F) = 1 > 0$ by 5.7. A compact Kähler threefolds having a structure of elliptic fibration admits a good minimal model if it is not uniruled, by Appendix B of [N2]. Thus X itself is minimal and K_X is semi-ample. Hence $K_X \sim_{\mathbb{Q}} 0$ and thus X is a complex Q-torus by Bogomolov's decomposition theorem. Next suppose that $q(X) = 1$. Then $p_g(X) = 1$ and $\kappa(X) = 0$. Thus a general fiber F of α has also $p_g(F) \geq 1$. We have $\kappa(F) < 2$ by 5.7. If $\kappa(F) = 1$, then we have a relative Iitaka fibration $X \cdots\to S$ with $\dim S = 2$ over $\mathrm{Alb}(X)$. Thus

X is bimeromorphically equivalent to a threefold admitting an elliptic fibration. Hence X is a minimal model and K_X is semi-ample. Since K_F is also numerically trivial, this is a contradiction. Therefore $\kappa(F) = 0$. Since F has an étale covering from a Stein space, F is a two-dimensional complex torus. Hence by 4.5 and by the argument in 5.4, $K_X \sim_{\mathbb{Q}} \alpha^*(K_C + \Delta)$ for an effective $\mathbb{Q}$-divisor Δ on the elliptic curve $C = \operatorname{Alb} X$. Since $\kappa(X) = 0$, we have $\Delta = 0$. Therefore α is smooth and $K_X \sim_{\mathbb{Q}} 0$. By Bogomolov's decomposition theorem, X is a complex Q-torus. □

Corollary 5.9. *If the universal covering space of a compact Kähler manifold X with $n = \dim X \leq 3$ is Stein, and if $\kappa(X) \leq 0$, then X is a complex Q-torus.*

Proof. Suppose that X is a projective variety. Then K_X is nef by [Mo]. The abundance theorem [Mi1], [Mi2], [Kw3] implies $K_X \sim_{\mathbb{Q}} 0$. Thus X is a Q-torus by Bogomolov's decomposition theorem. Next suppose that X is not projective. Then $\chi(X, \mathcal{O}_X) = 0$ by [G, 3.2.C′]. Hence $2 \leq 1 + h^{2,0}(X) = q(X) + p_g(X)$ and thus $q(X) \geq 1$. Let $\alpha\colon X \to \operatorname{Alb} X$ be the Albanese mapping and let $X \to Y \to \operatorname{Alb} X$ be the Stein factorization. A general fiber of $X \to Y$ is not of general type by 5.7. If $\dim Y = 3$, then X is a complex torus by Theorem 22 of [Kw1]. If $\dim Y = 2$, then $X \to Y$ is an elliptic fibration. Thus X is a minimal model and $K_X \sim_{\mathbb{Q}} 0$ by Appendix B of [N2]. Hence X is a Q-torus. If $\dim Y = 1$ and if $\kappa(F) = 1$, then X is bimeromorphically equivalent to a threefold admitting an elliptic fibration. Thus X is a minimal model and K_X is semi-ample. Since K_F is numerically trivial, this is a contradiction. Thus if $\dim Y = 1$, then $\kappa(F) = 0$. If further $\kappa(Y) = 1$, then $\kappa(X) \geq \kappa(F) + 1$ by [Ft], [Kw1], [Vi1]. A contradiction. Hence $Y = \operatorname{Alb} X$ is an elliptic curve and $p_g(X) = h^{2,0}(X) = 1$. Thus F is a complex torus of dimension two. By the same argument as in the proof of 5.8, we see that X is a Q-torus. □

6. Kähler torus fibration with finite local monodromies.

Proposition 6.1. *Let $p\colon B \to S$ be a smooth basic torus fibration over a normal variety associated with a polarizable variation of Hodge structure H of weight -1 and let $\sigma\colon S^\circ \to B$ be a section over a Zariski-open subset S° with $\operatorname{codim}(S \setminus S^\circ) \geq 2$. Then σ extends to a meromorphic section over S.*

Proof. We may assume S is Stein and simply connected. Let $\mu\colon M \to S$ be a projective bimeromorphic morphism from a nonsingular variety such that $M^\circ := \mu^{-1}(S^\circ)$ is isomorphic to S° and that the complement $D := M \setminus M^\circ$ is a simple normal crossing divisor.

Claim 6.2. (1) $H^1(M, \mathbb{Z}) \to H^1(M^\circ, \mathbb{Z})$ is an isomorphism.

(2) $H^1(M, \mathcal{O}_M) \to H^1(M^\circ, \mathcal{O}_M)$ is injective.

Proof. (1). Let $j\colon M^\circ \hookrightarrow M$ be the immersion. Then $R^1 j_* \mathbb{Z}_{M^\circ} \simeq \bigoplus \mathbb{Z}_{D_i}$, where D_i are irreducible components of D. Thus $H^2_D(M, \mathbb{Z}) \simeq \bigoplus H^0(D_i, \mathbb{Z})$. Note that the first Chern classes $c_1(D_i)$ are linearly independent, since they are μ-exceptional. Thus $H^2_D(M, \mathbb{Z}) \to H^2(M, \mathbb{Z})$ is injective. Since $H^1_D(M, \mathbb{Z}) = 0$, we have an isomorphism $H^1(M, \mathbb{Z}) \simeq H^1(M^\circ, \mathbb{Z})$.

(2). Let us consider an exact sequence

$$0 \to \mathcal{O}_M \to \mathcal{E} \to \mathcal{O}_M \to 0$$

on M which is split over M°. We have an exact sequence

$$0 \to \mathcal{O}_S \to \mu_* \mathcal{E} \to \mathcal{O}_S \to R^1 \mu_* \mathcal{O}_M$$

by applying μ_*. Let $I \subset \mathcal{O}_S$ be the image of $\mu_* \mathcal{E} \to \mathcal{O}_S$. We have a splitting as a section of the sheaf $\mathcal{H}om_{\mathcal{O}_S}(\mu_* \mathcal{E}, \mathcal{O}_S)$ over S°. Since the sheaf is reflexive, the section extends to a global section over S. Thus

$$0 \to \mathcal{O}_S \to \mu_* \mathcal{E} \to I \to 0$$

is a split exact sequence. By applying μ^*, we have a commutative diagram of exact sequences:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}_M & \longrightarrow & \mu^*\mu_*\mathcal{E} & \longrightarrow & \mu^*I & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{O}_M & \longrightarrow & \mathcal{E} & \longrightarrow & \mathcal{O}_M & \longrightarrow & 0. \end{array}$$

Let E be an effective divisor of M supported in D whose defining ideal $\mathcal{O}_M(-E)$ is contained in $\mu^*I/(\mathrm{tor}) \subset \mathcal{O}_M$. Then for the element β in $H^1(M, \mathcal{O}_M)$ associated with the extension $\mathcal{E}$, its image in $H^1(M, \mathcal{O}_M(E))$ is zero. We know $\mu_*\mathcal{O}_E(E) = 0$ by Fujita's lemma [KMM, 1-3-2]. Applying μ_* to the exact sequence

$$0 \to \mathcal{O}_M \to \mathcal{O}_M(E) \to \mathcal{O}_E(E) \to 0,$$

we have $\beta = 0$. □

Proof of 6.1 continued. From the exact sequence

$$0 \to \mu^{-1}H \to \mu^*\mathcal{V}_H \to \mathfrak{S}_{\mu^{-1}H} \to 0,$$

we have the following commutative diagram:

$$\begin{array}{ccccccc} H^0(M, \mu^{-1}H) & \longrightarrow & H^0(M, \mu^*\mathcal{V}_H) & \longrightarrow & H^0(M, \mathfrak{S}_{\mu^{-1}H}) & \longrightarrow & \\ \| & & \| & & \downarrow & & \\ H^0(M^\circ, \mu^{-1}H) & \longrightarrow & H^0(M^\circ, \mu^*\mathcal{V}_H) & \longrightarrow & H^0(M^\circ, \mathfrak{S}_{\mu^{-1}H}) & \longrightarrow & \\ & & & & \longrightarrow H^1(M, \mu^{-1}H) & \longrightarrow & H^1(M, \mu^*\mathcal{V}_H) \\ & & & & \downarrow & & \downarrow \\ & & & & \longrightarrow H^1(M^\circ, \mu^{-1}H) & \longrightarrow & H^1(M^\circ, \mu^*\mathcal{V}_H). \end{array}$$

By the Claim above, we see that $H^0(M, \mathfrak{S}_{\mu^{-1}H}) \to H^0(M^\circ, \mathfrak{S}_{\mu^{-1}H})$ is an isomorphism. Therefore σ in $H^0(S^\circ, \mathfrak{S}_H)$ extends to a holomorphic mapping $M \to B$ over S. □

Theorem 6.3. *Let $\pi\colon X \to S$ be a proper surjective morphism from a normal variety onto a unit polydisc $S = \Delta^n$ with a coordinate system $(t_1, t_2, \dots, t_n)$ and let D be the normal crossing divisor $\sum_{1\le i\le l} D_i$, where $D_i := \{t_i = 0\}$ and $1 \le l \le n$. Suppose the following conditions are satisfied*:

(1) *π is smooth outside D.*

(2) *A smooth fiber is bimeromorphically equivalent to a complex torus.*

(3) *π is equi-dimensional.*

(4) *π is a BK-morphism.*

(5) *For a general point s of any D_i for $1 \leq i \leq l$, the fiber $X_s = f^{-1}(s)$ has a reduced component.*

(6) *$(R_1 f_* \mathbb{Z}_X)_{|S \setminus D}$ has only trivial monodromies.*

Then π is bimeromorphically equivalent to a smooth basic torus fibration.

Proof. Let H be the natural extension of the local constant system $(R_1\pi_*\mathbb{Z}_X)_{|S\setminus D}$ to S. Then H is the underlying local constant system of a polarizable variation of Hodge structure. Let $p\colon B \to S$ be the smooth basic torus fibration associated with H. There is a Zariski-open subset $S^\circ \subset S$ such that $S^\star := S \setminus D \subset S^\circ$, $\operatorname{codim}(S \setminus S^\circ) \geq 2$, and that $X^\circ := \pi^{-1}(S^\circ) \to S^\circ$ admits a local section. We may assume that X° is nonsingular. Let $\{S_\lambda\}$ be an open covering of S° and let $\sigma_\lambda\colon S_\lambda \to X_\lambda := \pi^{-1}(S_\lambda)$ be sections. We have the relative Albanese mapping $\pi^{-1}(S_\lambda \cap S^\star) \to p^{-1}(S_\lambda \cap S^\star)$ over $S_\lambda \cap S^\star$ which sends σ_λ to the zero section. By 1.6, it extends to a meromorphic mapping $\varphi_\lambda\colon X_\lambda \cdots\to p^{-1}(S_\lambda)$. The difference

$$\varphi_\lambda \circ \varphi_\mu^{-1}\colon p^{-1}(S_\lambda \cap S_\mu) \cdots\to p^{-1}(S_\lambda \cap S_\mu)$$

is a holomorphic automorphism. Since the difference does not change the local constant system H, it is the translation $\operatorname{tr}(\eta_{\lambda,\mu})$ of a section $\eta_{\lambda,\mu}$ of $p\colon B \to S$ over $S_\lambda \cap S_\mu$. Therefore, X° is bimeromorphically equivalent to the torsor B^η over S° for an element η in $H^1(S^\circ, \mathfrak{S}_H)$. From the exact sequence

$$0 \to H \to \mathcal{V}_H \to \mathfrak{S}_H \to 0,$$

we see that $H^1(S^\circ, \mathcal{V}_H) \simeq H^1(S^\circ, \mathfrak{S}_H)$ is a vector space. On the other hand we know that η is a torsion element by 2.12 and 4.6. Therefore $\eta = 0$. Hence we have a bimeromorphic mapping $\varphi\colon X^\circ \cdots\to p^{-1}(S^\circ)$ over S°. This is holomorphic since X° is nonsingular. This also gives a section of $B \times_S X \to X$ over X°. Since $\operatorname{codim}(X \setminus X^\circ) \geq 2$, this extends to a meromorphic section over X, by 6.1. Thus φ extends to a bimeromorphic mapping $X \cdots\to B$ over S. □

Let S be a normal variety with only quotient singularities and let D be an effective reduced divisor such that $S^\star := S \setminus D$ is nonsingular and that $S^\star \subset S$ is a toroidal embedding. We assume that a polarizable variation of Hodge structure H of weight -1,

rank $2g$ is given on $S^\star$ and that the local monodromies of H around D are finite. For a positive integer $k \geq 3$, let $N = N_k$ be the kernel of the homomorphism

$$\pi_1(S^\star, s) \to \mathrm{Aut}(H_s) \to \mathrm{Aut}(H_s \otimes \mathbb{Z}/k\mathbb{Z})$$

induced from the monodromy representation of H. Let $\delta\colon S^\natural \to S$ be the finite Galois covering from a normal variety such that it is étale over $S^\star$ and that $S^{\natural\star} := \delta^{-1}(S^\star) \to S^\star$ corresponds to the finite index subgroup N. The $S^\natural$ has only quotient singularities and $S^{\natural\star} \subset S^\natural$ is a toroidal embedding. Since the local monodromies of H around D are finite, those of $\delta^{-1}H$ around $\delta^{-1}D$ are trivial by Lemma of Serre [Se]. Let $H^\natural$ be the natural extension of $\delta^{-1}H$ to $S^\natural$ and let $B^\natural \to S^\natural$ be the associated smooth basic torus fibration. The Galois group N acts on $B^\natural$ compatible way with $S^\natural$ preserving the zero section. Let $B = N\backslash B^\natural$ be the quotient variety and let $p\colon B \to S$ be the induced torus fibration. The restriction of p to $S^\star$ is isomorphic to the smooth basic torus fibration $p^\star\colon B^\star \to S^\star$ associated with H. By 1.2 and 2.2, $p\colon B \to S$ is a Kähler morphism. Any meromorphic section $S \cdots\to B$ is holomorphic, since its pullback by $S^\natural \to S$ is holomorphic and since $S^\natural$ has only quotient singularities (cf. 2.11). The construction of $B =: B(H)$ does not depend on the choice of k. We call p by the the *characteristic basic torus fibration* associated with H.

Lemma 6.4. *A torus fibration $p'\colon B' \to S$ satisfying the following conditions is isomorphic to the characteristic basic torus fibration over S:*

(1) *p' is an extension of $p^\star$.*

(2) *There is a section of p' which is an extension of the zero section.*

(3) *p' is an LBK-morphism.*

(4) *For any point $s \in S$, there is an open neighborhood $\mathcal{U}$ and a finite surjective morphism $T \to \mathcal{U}$ such that it is étale outside D and that the normalization of $B \times_S T$ is smooth over T.*

Proof. We have a bimeromorphic mapping $\varphi\colon B' \cdots\to B$ preserving the zero sections by 1.6. Locally, we can take a common finite surjective morphism $T \to S$ satisfying the condition (4) above for both p' and p. Since φ induces an isomorphism for the pullback, φ should be an isomorphism. □

We apply the theory of ∂-étale cohomology [N3] to S and D. Let $\underline{S}$ be the complex analytic ∂-space (S, D). We denote by $\underline{j}\colon S^\star \hookrightarrow \underline{S}$ the morphism of ∂-spaces corresponding to the open immersion $j\colon S^\star \hookrightarrow S$. We have the lower canonical extension $\mathcal{H}$ of $H \otimes_{\mathbb{Z}} \mathcal{O}_{S^\star}$ to S and have a filtration $\mathcal{F}^\bullet$ of $\mathcal{H}$ by subbundles extending the Hodge filtration. We define $\mathcal{V}_{H/S} = \mathcal{F}^{-1}/\mathcal{F}^0$. Let $\underline{T} = (T, D_T) \to \underline{S}$ be a ∂-étale morphism, where T is nonsingular and D_T is a normal crossing divisor, and let H_T be the pullback of H to $T \setminus D_T$. We attach the vector space $H^0(T, \mathcal{V}_{H_T/T})$. Then this defines a sheaf $\mathcal{V}_{H/\underline{S}}$ on $\underline{S}$, i.e., a sheaf in the ∂-étale topology. Then for the natural morphism $\varepsilon_S\colon \underline{S} \to S = (S, \emptyset)$, we have $\varepsilon_{S*}\mathcal{V}_{H/\underline{S}} \simeq \mathcal{V}_{H/S}$. Let $\mathfrak{S}_{H/\underline{S}}$ be the sheaf of germs of meromorphic sections of $p\colon B \to S$ in the ∂-étale topology on $\underline{S}$. Then we have the following short exact sequence of sheaves on $\underline{S}$:

$$(\diamondsuit) \qquad 0 \to \underline{j}_* H \to \mathcal{V}_{H/\underline{S}} \to \mathfrak{S}_{H/\underline{S}} \to 0.$$

We denote by $\mathcal{T}(S, D, H)$ the set of all the pairs $(\pi\colon X \to S, \phi)$ consisting of a proper surjective morphism $\pi\colon X \to S$ smooth over $S^\star$, whose smooth fibers are bimeromorphically equivalent to complex tori, and an isomorphism

$$\phi\colon (R_1\pi_*\mathbb{Z}_X)_{|S^\star} \simeq H$$

modulo the following bimeromorphic equivalence relation $\sim_{\mathrm{bim}}$:

$$(\pi_1\colon X_1 \to S, \phi_1) \sim_{\mathrm{bim}} (\pi_2\colon X_2 \to S, \phi_2)$$

if there is a bimeromorphic mapping $\varphi\colon X_1 \cdots\to X_2$ over S such that $\phi_1 = \phi_2 \circ \varphi_*$.

We shall define subsets $\mathcal{K}(S, D, H)$, $\mathcal{LK}(S, D, H)$, and $\mathcal{CK}(S, D, H)$ of $\mathcal{T}(S, D, H)$ to be those consisting of pairs $(X \to S, \phi)$ where $X \to S$ is a BK-morphism, an LBK-morphism, and a BCK-morphism, respectively. Let $\mathcal{T}^e(S, D, H)$ be the subset of $\mathcal{T}(S, D, H)$ consisting of pairs $(X \to S, \phi)$ where $X \to S$ is equi-dimensional. We define also subsets $\mathcal{K}^e(S, D, H)$, $\mathcal{LK}^e(S, D, H)$, and $\mathcal{CK}^e(S, D, H)$ to be the intersections of $\mathcal{T}^e(S, D, H)$ with $\mathcal{K}(S, D, H)$, $\mathcal{LK}(S, D, H)$, and $\mathcal{CK}(S, D, H)$, respectively.

Proposition 6.5. *There is a one to one correspondence with $\mathcal{LK}^e(S, D, H)$ and the ∂-étale cohomology group $H^1(\underline{S}, \mathfrak{S}_{H/\underline{S}})$. Moreover, for each element η in $H^1(\underline{S}, \mathfrak{S}_{H/\underline{S}})$, there is a locally Kähler torus fibration $\pi\colon B^\eta \to S$ uniquely up to isomorphisms over S satisfying the following conditions*:

(1) *π is smooth outside D and induces an isomorphism of variation of Hodge structure to H over $S^\star$.*

(2) *(π, ϕ) corresponds to η for an isomorphism ϕ.*

(3) *For any point $s \in S$, there exist an open neighborhood $\mathcal{U}$ and a finite surjective morphism $T \to \mathcal{U}$ from a nonsingular variety étale outside D such that the normalization of $B^\eta \times_S T$ is smooth over T.*

Proof. Let $p\colon B \to S$ be the characteristic basic torus fibration associated with H and let $(\pi\colon X \to S, \phi)$ be an element of $\mathcal{LK}^e(S, D, H)$. Then by 6.3, there exist an open covering $\{S_\lambda\}$ of S and finite surjective morphisms $\{T_\lambda \to S_\lambda\}$ from nonsingular varieties étale outside D such that $X \times_S T_\lambda \to T_\lambda$ is bimeromorphically equivalent to a smooth basic torus fibration. Since $B \times_S T_\lambda$ is also bimeromorphically equivalent to the smooth fibration, the normalization B_λ of $B \times_S T_\lambda$ is also smooth over T_λ. Thus we have a collection of bimeromorphic mappings $\varphi_\lambda\colon X \times_S T_\lambda \cdots\to B_\lambda$ over T_λ such that $\varphi_{\lambda *}$ induces the original isomorphism $\phi\colon R^1\pi_*\mathbb{Z}_{|S^\star} \simeq H$. Let $T_{\lambda,\mu}$, $B_{\lambda,\mu}$, and $X_{\lambda,\mu}$ be the normalizations of $T_\lambda \times_S T_\mu$, $B \times_S T_{\lambda,\mu}$ and $X \times_S T_{\lambda,\mu}$, respectively. Then the meromorphic mapping $B_{\lambda,\mu} \cdots\to B_{\lambda,\mu}$ over $T_{\lambda,\mu}$ induced by $\varphi_\mu \circ \varphi_\lambda^{-1}$ is an isomorphism and is the translation mapping of a section $\eta_{\lambda,\mu}$ by 2.11. The collection $\{\eta_{\lambda,\mu}\}$ defines a cohomology class η in $H^1(\underline{S}, \mathfrak{S}_{H/\underline{S}})$. By the construction, η is independent of the choices of $\{S_\lambda\}, \{T_\lambda\}$, and $\{\varphi_\lambda\}$, since the difference appears only as translations; this is ignored in the cohomology. Further it is only determined by the bimeromorphic equivalence class of (π, ϕ) by the same reason. Thus we have a map $\mathcal{LK}^e(S, D, H) \to H^1(\underline{S}, \mathfrak{S}_{H/\underline{S}})$.

Conversely, let η be an element in $H^1(\underline{S}, \mathfrak{S}_{H/\underline{S}})$. Then there exist a ∂-étale covering $\{\underline{T}_\lambda \to \underline{S}\}$ and sections $\eta_{\lambda,\mu} \in H^0(\underline{T}_\lambda \times_{\underline{S}} \underline{T}_\mu, \mathfrak{S}_{H/\underline{S}})$ representing η. By taking a refinement, we may assume that there is an open covering $\{S_\lambda\}$ of S and the following conditions are satisfied for any λ:

- $\underline{T}_\lambda \to \underline{S}$ is induced from a finite Galois surjective morphism $T_\lambda \to S_\lambda$ which is étale outside D.
- T_λ is nonsingular.
- The normalization B_λ of $B \times_S T_\lambda$ is smooth over T_λ.

We define $T_{\lambda,\mu}$ and $B_{\lambda,\mu}$ as in the previous argument. Then $\eta_{\lambda,\mu}$ is considered to be a section of the smooth basic torus fibration $B_{\lambda,\mu} \to T_{\lambda,\mu}$. From the collection $\eta_{\lambda,\mu}$, we

shall construct a torus fibration as follows: First consider the case $\lambda = \mu$. Then $T_{\lambda,\lambda}$ is isomorphic to $G_\lambda \times T_\lambda$, where G_λ denotes the Galois group of $T_\lambda \to S_\lambda$. Then $\eta_{\lambda,\lambda}$ defines a new action of G_λ on B_λ by the cocycle condition. Let X_λ be the quotient variety by the action. Thus we have a Kähler torus fibration $\pi_\lambda \colon X_\lambda \to S_\lambda$ satisfying the conditions (1) and (3). For $\lambda \neq \mu$, by other cocycle conditions, the $\eta_{\lambda,\mu}$ defines an isomorphism between X_λ and X_μ over $S_\lambda \cap S_\mu$. Moreover these isomorphisms are gluing data of a torus fibration $X =: B^\eta \to S$ and an isomorphism ϕ of variation of Hodge structure to H is naturally derived. The uniqueness of $B^\eta \to S$ is proved by using a similar argument to that in 6.4. This construction gives the inverse of $\mathcal{LK}^e(S, D, H) \to H^1(\underline{S}, \mathfrak{S}_{H/\underline{S}})$. □

We call the torus fibration $\pi \colon B^\eta \to S$ above by the *characteristic torus fibration* associated with η.

Lemma 6.6. *Let $c \colon H^1(\underline{S}, \mathfrak{S}_{H/\underline{S}}) \to H^2(\underline{S}, \underline{j}_* H)$ be a connecting homomorphism of the exact sequence $(\diamondsuit)$. Suppose that $c(\eta) = 0$ in $H^2(\underline{S}, \underline{j}_* H)$. Then the characteristic fibration $\pi \colon B^\eta \to S$ is cohomologically Kähler.*

Proof. Let $\delta \colon S^\natural \to S$ be the finite covering defined before. Then the normalization $B^{\natural\eta}$ of $B^\eta \times_S S^\natural$ defines also the characteristic torus fibration over $S^\natural$ associated with the element $\delta^*\eta$ in $H^1(\underline{S}^\natural, \mathfrak{S}_{H/\underline{S}^\natural})$, where $\underline{S}^\natural = (S^\natural, \delta^{-1}D)$. Since $c(\delta^*\eta) = 0$, $\delta^*\eta$ is coming from $H^1(\underline{S}^\natural, \mathcal{V}_{H/\underline{S}^\natural}) \simeq H^1(S^\natural, \mathcal{V}_{H^\natural})$. Thus $B^{\natural\eta} \to S^\natural$ is smooth and cohomologically Kähler by 2.17. Thus $B^\eta \to S$ is also cohomologically Kähler by 1.2. □

Let $\pi \colon X = B^\eta \to S$ be the characteristic torus fibration for an element η. We set $D_X := \pi^{-1}D$ and define $\underline{X}$ to be the ∂-space (X, D_X). For any ∂-étale morphism $\underline{Y} = (Y, D_Y) \to \underline{X} = (X, D_X)$, we can choose Y to be normal since $X \setminus \tau^{-1}D$ is so. The sheaf $\Omega^p_{\underline{X}}$ on $\underline{X}$ is defined by the sheafification of

$$\underline{Y} \mapsto \Gamma(Y, \Omega^p_Y).$$

For the morphism $\underline{\pi} \colon \underline{X} \to \underline{S}$, we have the pullback $\underline{\pi}^*\Omega^p_{\underline{S}}$ as well as a homomorphism $\underline{\pi}^*\Omega^p_{\underline{S}} \to \Omega^p_{\underline{X}}$. These sheaves are ∂-étale locally free sheaves and the homomorphism above is injective. Furthermore its cokernel $\Omega^p_{\underline{X}/\underline{S}}$ is also ∂-étale locally free since $\underline{X} \to \underline{S}$ is ∂-étale locally a smooth morphism. This is shown as follows: We consider a ∂-étale morphism $\underline{T} = (T, D_T) \to \underline{S}$ such that T is isomorphic to an n-dimensional unit polydisc

with a coordinate system $(t_1, t_2, \dots, t_n)$, $D_T = \{t_1 t_2 \cdots t_l = 0\}$ and that the normalization X' of $X \times_S T$ is smooth over T. Let D' be the pullback of D_T by $X' \to T$. For any point of X', it has an open neighborhood isomorphic to $T \times \Delta^g$, hence it has a coordinate system $(u_1, u_2, \dots, u_{n+g})$ with $t_i = u_i$ for $1 \leq i \leq n$. Let $\underline{Y} = (Y, D_Y) \to \underline{X'} := (X', D')$ be a ∂-étale morphism such that Y is isomorphic to an $(n+g)$-dimensional unit polydisc with a coordinate $(y_1, y_2, \dots, y_{n+g})$ and $D_Y = \{y_1 y_2 \cdots y_m = 0\}$. By replacing $\underline{Y}$ by a ∂-étale morphism, we may write $t_i = u_i = y_i^{k_i}$ for $1 \leq i \leq l$ for positive integers k_i and $u_i = y_i$ for $l+1 \leq i \leq n+g$. Hence by replacing $\underline{T}$ by a ∂-étale morphism, we can make the morphism $Y \to T$ to be smooth. Thus $\underline{\pi}\colon \underline{X} \to \underline{S}$ is ∂-étale locally a smooth morphism and $\Omega^p_{\underline{X}/\underline{S}}$ is ∂-étale locally free. In particular, we have the de Rham complex

$$0 \to \mathcal{O}_{\underline{X}} \to \Omega^1_{\underline{X}/\underline{S}} \to \Omega^2_{\underline{X}/\underline{S}} \to \cdots \to \Omega^g_{\underline{X}/\underline{S}} \to 0,$$

which is quasi-isomorphic to $\underline{\pi}^{-1}\mathcal{O}_{\underline{S}}$. As in the usual case, we can define the Deligne homology group $\underline{\mathrm{D}}_0(\underline{X}/\underline{S})$ of degree zero to be $\mathbb{R}^0\underline{\pi}_*\underline{\mathbb{D}}^\bullet_{\underline{X}/\underline{S}}$, where the complex $\underline{\mathbb{D}}^\bullet_{\underline{X}/\underline{S}}$ is defined to be:

$$\mathbb{Z}_{\underline{X}} \to \mathcal{O}_{\underline{X}} \to \Omega^1_{\underline{X}/\underline{S}} \to \cdots \to \Omega^{g-1}_{\underline{X}/\underline{S}} \to 0,$$

where $\mathbb{Z}_{\underline{X}}$ is lying at degree $-2g$. We have a triangle

$$\cdots \overset{+1}{\to} \underline{\mathbb{D}}^\bullet_{\underline{X}/\underline{S}} \to \Omega^g_{\underline{X}/\underline{S}}[g] \oplus \mathbb{Z}_{\underline{X}}[2g] \to \underline{\pi}^{-1}\mathcal{O}_{\underline{S}} \overset{+1}{\to} \cdots$$

By taking higher direct images, we have an exact sequence

$$\cdots \to R^{g-1}\underline{\pi}_*\Omega^g_{\underline{X}/\underline{S}} \oplus R^{2g-1}\underline{\pi}_*\mathbb{Z}_{\underline{X}} \to R^{2g-1}\underline{\pi}_*\mathbb{Z}_{\underline{X}} \otimes \mathcal{O}_{\underline{S}} \to$$
$$\underline{\mathrm{D}}_0(\underline{X}/\underline{S}) \to R^{2g}\underline{\pi}_*\Omega^g_{\underline{X}/\underline{S}} \oplus R^{2g}\underline{\pi}_*\mathbb{Z}_{\underline{X}} \to R^{2g}\underline{\pi}_*\mathbb{Z}_{\underline{X}} \otimes \mathcal{O}_{\underline{S}} \to \cdots$$

Let $\underline{T} = (T, D_T) \to \underline{S}$ be a ∂-étale morphism such that for the normalization X_T of $X \times_S T$, the induced morphism $\pi_T\colon X_T \to T$ is smooth. Then we have the following isomorphisms.

$$(R^{g-1}\underline{\pi}_*\Omega^g_{\underline{X}/\underline{S}})_{|\underline{T}} \simeq R^{g-1}\pi_{T*}\Omega^g_{X_T/T} \otimes_{\mathcal{O}_T} \mathcal{O}_{\underline{T}},\ (R^g\underline{\pi}_*\Omega^g_{\underline{X}/\underline{S}})_{|\underline{T}} \simeq R^g\pi_{T*}\Omega^g_{X_T/T} \otimes_{\mathcal{O}_T} \mathcal{O}_{\underline{T}} \simeq \mathcal{O}_{\underline{T}},$$
$$(R^{2g-1}\underline{\pi}_*\mathbb{Z}_{\underline{X}} \otimes \mathcal{O}_{\underline{S}})_{|\underline{T}} \simeq R_1\pi_{T*}\mathbb{Z}_{X_T} \otimes \mathcal{O}_{\underline{T}}, \quad (R^{2g}\underline{\pi}_*\mathbb{Z}_{\underline{X}} \otimes \mathcal{O}_{\underline{S}})_{|\underline{T}} \simeq R_0\pi_{T*}\mathbb{Z}_{X_T} \otimes \mathcal{O}_{\underline{T}} \simeq \mathcal{O}_{\underline{T}},$$
$$R^{2g-1}\underline{\pi}_*\mathbb{Z}_{\underline{X}} \simeq \underline{j}_*H, \qquad R^g\underline{\pi}_*\mathbb{Z}_{\underline{X}} \simeq \mathbb{Z}_{\underline{S}}.$$

Further, we see that $\mathcal{V}_{H/\underline{S}}$ is isomorphic to the cokernel of

$$R^{g-1}\underline{\pi}_*\Omega^g_{\underline{X}/\underline{S}} \to R^{2g-1}\underline{\pi}_*\mathbb{Z}_{\underline{X}} \otimes \mathcal{O}_{\underline{S}}.$$

Therefore we have the following short exact sequence

$$0 \to \mathfrak{S}_{H/\underline{S}} \to \underline{\mathrm{D}}_0(\underline{X}/\underline{S}) \to \mathbb{Z}_{\underline{S}} \to 0.$$

As in 2.8, its extension class in $H^1(\underline{S}, \mathfrak{S}_{H/\underline{S}})$ is just η.

Theorem 6.7. *Let $(\pi\colon X \to S, \phi)$ be an element of $\mathcal{LK}^e(S, D, H)$. For the corresponding $\eta \in H^1(\underline{S}, \mathfrak{S}_{H/\underline{S}})$ by 6.5, let $B^\eta \to S$ be the characteristic torus fibration and let $c(\eta)$ be the image of η under*

$$H^1(\underline{S}, \mathfrak{S}_{H/\underline{S}}) \to H^2(\underline{S}, \underline{j}_*H).$$

Then the following conditions are equivalent:

(1) *$B^\eta \to S$ is cohomologically Kähler.*

(2) *π is a BCK-morphism.*

(3) *The restriction map $H^{2g}(X, \mathbb{R}) \to H^{2g}(X_s, \mathbb{R})$ is not zero for a general fiber $X_s = \pi^{-1}(s)$.*

(4) *$c(\eta)$ is a torsion element.*

(5) *There exists a finite surjective morphism $\tau\colon T \to S$ étale outside D such that $c(\tau^*\eta) = 0$.*

In particular, the subset $\mathcal{CK}^e(S, D, H)$ is identified with the kernel of $H^1(\underline{S}, \mathfrak{S}_{H/\underline{S}}) \to H^2(\underline{S}, \underline{j}_*H) \otimes \mathbb{Q}$.

Proof. (1) $\Longrightarrow$ (2). Trivial.

(2) $\Longrightarrow$ (3). There is an element ξ in $H^2(X, \mathbb{R})$ induced from a relative Kähler class of a bimeromorphically equivalent morphism. Then its cup product $\wedge^g\xi$ in $H^{2g}(X, \mathbb{R})$ is mapped to the positive generator of $H^{2g}(X_s, \mathbb{R})$ for any smooth fiber X_s.

(3) $\Longrightarrow$ (4). We may assume $X = B^\eta$. The truncation $\tau_{\geq 2g-1}(R\underline{\pi}_*\mathbb{Z}_{\underline{X}})$ of $R\underline{\pi}_*\mathbb{Z}_{\underline{X}}$ is quasi-isomorphic to the complex:

$$\cdots \to 0 \to \mathcal{V}_{H/\underline{S}} \to \underline{\mathrm{D}}_0(\underline{X}/\underline{S}) \to 0 \to \cdots$$

where $\mathcal{V}_{H/\underline{S}}$ is lying at degree $2g-1$. Therefore $c(\eta)$ corresponds to the image of 1 under the homomorphism

$$\mathbb{Z} \simeq E_2^{0,2g} = H^0(\underline{S}, R^{2g}\underline{\pi}_*\mathbb{Z}_X) \to E_2^{2,2g-1} = H^2(\underline{S}, R^{2g-1}\underline{\pi}_*\mathbb{Z}_X) \simeq H^2(\underline{S}, \underline{j}_*H)$$

appearing at Leray's spectral sequence. Then

$$E^{2g} \otimes \mathbb{Q} = H^{2g}(X, \mathbb{Q}) \to E_2^{0,2g} \otimes \mathbb{Q} = H^0(S, R^{2g}\pi_*\mathbb{Q}_X) \simeq \mathbb{Q}$$

is surjective by (3). Thus $c(\eta) \otimes \mathbb{Q} = 0$.

(4) $\Longrightarrow$ (5). We may also assume that $X = B^\eta$. Let us consider the exact sequence

$$H^1(\underline{S}, \mathcal{V}_{H/\underline{S}}) \to H^1(\underline{S}, \mathfrak{S}_{H/\underline{S}}) \to H^2(\underline{S}, \underline{j}_*H).$$

Then there is an element β in $H^1(\underline{S}, \mathfrak{S}_{H/\underline{S}})$ such that β is a torsion element and $\beta - \eta$ is coming from $H^1(\underline{S}, \mathcal{V}_{H/\underline{S}})$. Let $p\colon B \to S$ be the characteristic basic torus fibration associated with H and let $Z \to S$ be the characteristic torus fibration associated with β. Then Z and X are isomorphic locally on S, since $H^1(U, \mathcal{V}_{H/S}) = 0$ for a Stein open subset U. Let m be the order of β. By patching the multiplication mapping $B \to B$ by m locally over S, we have a finite surjective morphism $\mu_m\colon Z \to B$ over S which is étale over $S^\star$. Let $T' \subset Z$ be an irreducible component of the pullback of the zero section of p by μ_m and let $\tau\colon T \to T' \to S$ be the Galois closure of $T' \to S$. The normalization of $Z \times_S T$ defines a characteristic basic torus fibration and thus isomorphic to the normalization of $B \times_S T$ over T. Thus $\tau^*\beta = 0$ in $H^1(\underline{T}, \mathfrak{S}_{H/\underline{T}})$, where $\underline{T} = (T, \tau^{-1}D)$. Let Y be the normalization of $X \times_S T$. Then $Y \to T$ is the characteristic torus fibration associated with $\tau^*\eta$, where $c(\tau^*\eta) = 0$. Note that the condition $c(\tau^*\eta) = 0$ implies that $Y \to T$ is locally isomorphic to the characteristic basic torus fibration over T.

(5) $\Longrightarrow$ (1). The characteristic torus fibration associated with $\tau^*\eta$ is a cohomologically Kähler morphism by 6.6, since $c(\tau^*\eta) = 0$. Thus so is $B^\eta \to S$ by 1.2. □

Theorem 6.8.

$$\mathcal{LK}^e(S, D, H) = \mathcal{LK}(S, D, H).$$

In particular, any Kähler torus fibration smooth over $S^\star$ is bimeromorphically equivalent to an equi-dimensional morphism over S, if the associated variation of Hodge structure on $S^\star$ has only locally finite monodromies.

Proof. We can localize S to be a Stein variety. Let $(\pi\colon X \to S, \phi)$ be an element of $\mathcal{K}(S, D, H)$. We can take a flattening $\mu\colon M \to S$ of π such that μ is an isomorphism over $S^\star$ and $D_M = \mu^{-1}D$ is a simple normal crossing divisor. Then for the normalization Y of the fiber product $X \times_S M$, $\pi_M\colon Y \to M$ is an equi-dimensional torus fibration. Thus it is bimeromorphically equivalent over M to the characteristic torus fibration $B^\eta \to M$ associated with an element $\eta \in H^1(\underline{M}, \mathfrak{S}_{H/\underline{M}})$, where $\underline{M} = (M, D_M)$. By 6.7, $c(\eta)$ is a torsion element. We have the vanishing $R^i\underline{\mu}_*\mathcal{V}_{H/\underline{M}} = R^i\mu_*\mathcal{O}_M \otimes \mathcal{V}_{H/\underline{S}} = 0$ for $i > 0$, since $\mathcal{V}_{H/\underline{S}}$ is ∂-étale locally free. Thus $H^i(\underline{M}, \mathcal{V}_{H/\underline{M}}) = 0$ for $i > 0$ and hence $c\colon H^1(\underline{M}, \mathfrak{S}_{H/\underline{M}}) \to H^2(\underline{M}, \underline{j}_{M*}H)$ is an isomorphism, where $\underline{j}_M$ denotes the morphism $M \setminus D_M \hookrightarrow \underline{M}$. Therefore η is a torsion element. Let m be its order and let $B^\eta \to B^0$ be the morphism induced from multiplication by m, where $B^0 \to M$ is the characteristic basic torus fibration. Since B^0 is bimeromorphically equivalent to $B \times_S M$, there is a generically finite meromorphic mapping $B^\eta \cdots\to B$ over S. By considering the Stein factorization, we have an equi-dimensional torus fibration bimeromorphically equivalent to π over S. □

Corollary 6.9. *Let $\pi\colon X \to S$ be a proper surjective morphism. Assume that*

(1) *π is smooth outside D,*

(2) *smooth fibers are bimeromorphically equivalent to complex tori,*

(3) *π is a BCK-morphism,*

(4) *$(R_1\pi_*\mathbb{Z}_X)_{|S^\star}$ has only finite local monodromies around D.*

Then there exists a finite Galois covering $\tau\colon T \to S$ from a normal variety satisfying the following conditions:

(a) *τ is étale outside D.*

(b) *Let Y be the main component of $X \times_S T$. Then the induced morphism $Y \to T$ is bimeromorphically equivalent to a cohomologically Kähler smooth torus fibration over T.*

Lemma 7.1. *Let $f\colon V \to W$ be a proper surjective morphism of complex analytic varieties which is a smooth fibration over a Zariski-open subset $W^\star \subset W$. Assume that there is a simply connected subvariety $T \subset V$ which is generically finite and proper over W and which is étale over $W^\star$. Then the image of $\pi_1(V_w) \to \pi_1(V)$ is a finite index subgroup, where $V_w := f^{-1}(w)$ for $w \in W^\star$. If $\pi_1(V_w) \to \pi_1(V)$ is injective, then the homomorphism $\pi_1(V^\star) \to \mathrm{Aut}(\pi_1(V_w)) \to \mathrm{Out}(\pi_1(V_w))$ induced from the monodromy action* (cf. 3.9) *is finite, where $V^\star := f^{-1}(W^\star)$ and* Out *denotes the outer automorphism group.*

Proof. We denote $T^\star := T \cap V^\star$. Then $\pi_1(T^\star) \subset \pi_1(W^\star)$ is a finite index subgroup. The composite $\pi_1(T^\star) \to \pi_1(V^\star) \to \pi_1(V)$ is trivial, since $\pi_1(T) = \{1\}$. Let N be the kernel of the surjection $\pi_1(V^\star) \to \pi_1(V)$. Then the image of $N \to \pi_1(W^\star)$ is a normal finite index subgroup. Thus the image of $\pi_1(V_w) \to \pi_1(V)$ is also a finite index subgroup. If $\pi_1(V_w) \to \pi_1(V)$ is injective, then the monodromy action is induced from the inner automorphism of $\pi_1(V^\star)$. Let $N' \subset \pi_1(V^\star)$ be the subgroup generated by N and $\pi_1(V_w)$. Then $N' \simeq \pi_1(V_w) \times N$, since the image of $N' \to \pi_1(V)$ is isomorphic to $\pi_1(V_w)$. Therefore $N \to \pi_1(V^\star) \to \mathrm{Aut}(\pi_1(V_w))$ is trivial. In particular, the image of $\pi_1(V^\star) \to \mathrm{Out}(\pi_1(V_w))$ is finite. □

Let S be a normal variety with only quotient singularities and let D be a reduced effective divisor such that $S \setminus D \subset S$ is a toroidal embedding. Similarly to the previous section, we denote $S^\star = S \setminus D$, $\underline{S} = (S, D)$ and $\underline{j}\colon S^\star \hookrightarrow \underline{S}$.

Definition 7.2. Let $f\colon X \to S$ be a proper surjective morphism with connected fibers from a normal variety such that it is smooth over $S^\star$ and smooth fibers are bimeromorphically equivalent to Q-tori.

(1) f is called *homotopically Q-smooth* if X is π_1-smooth and if for a point $s \in S^\star$ and a point $x \in X_s = f^{-1}(s)$, the homomorphism $\pi_1(X_s, x) \to \pi_1(X, x)$ is injective.

(2) f is called *Q-smooth*, if there is a finite surjective morphism $T \to S$ étale outside D such that the normalization X_T of $X \times_S T$ is smooth over T and étale over X.

(3) f is called *locally homotopically Q-smooth*, if there is an open covering $\{S_\lambda\}$ of S such that $f^{-1}(S_\lambda) \to S_\lambda$ is homotopically Q-smooth for any λ.

(4) f is called *locally Q-smooth*, if there is an open covering $\{S_\lambda\}$ of S such that $f^{-1}(S_\lambda) \to S_\lambda$ is Q-smooth for any λ.

If f is Q-smooth or homotopically Q-smooth, then it is locally Q-smooth or locally homotopically Q-smooth, respectively. The condition: homotopically Q-smooth, does not depend on the choice of bimeromorphically equivalent π_1-smooth varieties over S.

Lemma 7.3. *If f is a Q-smooth Kähler morphism, then it is homotopically Q-smooth.*

Proof. Let $Z := X_T \to T$ be the smooth morphism as above. By 3.14, if we replace T by a further finite étale covering, then there is a finite étale Galois covering $Y \to Z$ such that $\pi_1(Y_t) \subset \pi_1(Z_t)$ is the lattice part of the Q-lattice $\pi_1(Z_t)$ for any $t \in T$, where Z_t and Y_t denote the fibers over t. In particular, fibers of $Y \to T$ are bimeromorphically equivalent to complex tori. Since $Y \to T$ is also a Kähler morphism, $\pi_2(T) \to \pi_1(Y_t)$ is zero by 2.18. Thus $\pi_1(Y_t) \to \pi_1(Y) \hookrightarrow \pi_1(X)$ is injective. Hence the homomorphism $\pi_1(X_s) \to \pi_1(X)$ is also injective for $s \in S^\star$, since $\pi_1(X_s)$ is a Q-lattice. □

Lemma 7.4. *Let $X \to S$ be a proper surjective morphism smooth over $S^\star$ such that smooth fibers are bimeromorphically equivalent to complex tori and that it is homotopically Q-smooth. Then local monodromies around D are finite.*

Proof. We may assume that S is a one-dimensional unit disc and D is the origin. Then there is a closed subvariety $T \subset X$ isomorphic to a unit disc such that $T \to S$ is finite surjective and étale over $S^\star$. By 7.1, the monodromy is finite. □

Lemma 7.5. *Let $f\colon X \to S$ be a proper surjective BK-morphism from a normal variety smooth over $S^\star$ such that smooth fibers are bimeromorphically equivalent to Q-tori. Suppose that there is an étale covering $U \to X$ such that $U \to X \to S$ is a locally Stein morphism. Then f is locally Q-smooth.*

Proof. Since the fiber U_s is Stein, smooth fibers are isomorphic to Q-tori. If f is equi-dimensional, then it is locally Q-smooth by 4.8. We shall show f is equi-dimensional. There is a finite étale covering $S'^\star \to S^\star$ and a toric reduction $Y^\star \to X^\star \times_{S^\star} S'^\star$ by 3.14. Let $Y \to X$ and $S' \to S$ be the finite coverings extending $Y^\star \to X^\star$ and $S'^\star \to S^\star$. Here $S'^\star \subset S'$ is a toroidal embedding and S' has only quotient singularities. Since $U \times_X Y \to S'$

is locally Stein, local monodromies around $S' \setminus S'^{\star}$ is finite by 4.5 or 4.8. Thus if we replace S' by a further finite covering étale over $S^{\star}$, we see that $Y \to S'$ is bimeromorphically equivalent to a smooth torus fibration by 6.9. Since $U \times_X Y \to S'$ is locally Stein, $Y \to S'$ itself is a smooth morphism. In particular, f is equi-dimensional. □

Lemma 7.6. *Let $f\colon X \to S$ be a proper surjective LBK-morphism smooth over $S^{\star}$ such that smooth fibers are bimeromorphically equivalent to complex tori and that the variation of Hodge structure $(R_1 f_* \mathbb{Z}_X)_{|S^{\star}} = H$ has only trivial local monodromies around D. Then f is locally bimeromorphically equivalent to a Q-smooth torus fibration.*

Proof. Since we can localize S, we may assume that there exists a finite surjective morphism $\tau\colon S^{\sharp} \to S$ from a unit polydisc $S^{\sharp} \simeq \Delta^n$ satisfying the following conditions:

- For a coordinate system $(t_1, t_2, \dots, t_n)$, D is the image of $L = \cup_{1 \leq i \leq l} L_i$, where $L_i = \{t_i = 0\}$ and $1 \leq l \leq n$.
- τ is Galois and the Galois group is a subgroup of the transformation group of $S^{\sharp}$ of the diagonal type:

  $$S^{\sharp} \ni (t_1, t_2, \dots, t_n) \mapsto (\mu_1 t_1, \mu_2 t_2, \dots, \mu_l t_l, t_{l+1}, \dots, t_n),$$

  where μ_i are roots of unity.

Then by 6.9, we can choose $S^{\sharp}$ so that $X \times_S S^{\sharp} \to S^{\sharp}$ is bimeromorphically equivalent to a smooth basic torus fibration $X^{\sharp} \to S^{\sharp}$, since $S^{\sharp}$ is a unit polydisc. We consider the action of the Galois group G of τ on $X^{\sharp}$. Let $G_1 \subset G$ be the subgroup generated by elements which have fixed points in $X^{\sharp}$. Since the monodromy of H is trivial, $X^{\sharp} \to S^{\sharp}$ is isomorphic to the pullback of the smooth basic torus fibration $X' := G_1 \backslash X^{\sharp} \to S' := G_1 \backslash S^{\sharp}$ by 4.1. By the definition of G_1, the action of G/G_1 on X' is free. Thus f is bimeromorphically equivalent to the Q-smooth torus fibration $G \backslash X^{\sharp} \to S$. □

Lemma 7.7. *Let $f\colon X \to S$ be a proper surjective LBK-morphism from a nonsingular variety smooth over $S^{\star}$ such that smooth fibers are bimeromorphically equivalent to complex tori and that f is homotopically Q-smooth. Suppose that f admits a meromorphic section and that S is simply connected. Then f is bimeromorphically equivalent to a smooth basic torus fibration.*

Proof. Let $\Sigma \subset X$ be a meromorphic section for f and let $\Sigma^\star := \Sigma \times_S S^\star$. Then Σ is simply connected and $\pi_1(\Sigma^\star) \to \pi_1(S^\star)$ is an isomorphism. Thus we have $\pi_1(X^\star) \simeq \pi_1(X_s) \times \pi_1(S^\star)$ by the argument of 7.1, where $X_s = f^{-1}(s)$ for $s \in S^\star$. In particular, the monodromy of $(R_1 f_* \mathbb{Z}_X)_{|S^\star}$ is trivial. Thus by 6.9, f is bimeromorphically equivalent to a smooth basic torus fibration. □

Theorem 7.8. *Let $f\colon X \to S$ be a proper surjective BCK-morphism smooth over $S^\star$ whose smooth fibers are bimeromorphically equivalent to complex Q-tori. Suppose that f is locally homotopically Q-smooth morphism. Then f is bimeromorphically equivalent to a Q-smooth Kähler morphism over S.*

Proof. By applying 3.14 to the smooth fibration $X^\star = f^{-1}(S^\star) \to S^\star$, we have finite Galois étale morphisms $T^\star \to S^\star$ and $Y^\star \to X^\star \times_{S^\star} T^\star$ such that $Y^\star \to X^\star$ is also a Galois morphism and that $\pi_1(Y_t) \subset \pi_1(X_s)$ is the lattice part of the Q-lattice $\pi_1(X_s)$ for any $t \in T^\star$ and the image s of t, where Y_t and X_s denote the fibers of $Y^\star \to T^\star$ and $X^\star \to S^\star$ over t and s, respectively. Let $\tau\colon T \to S$ and $Y' \to X \times_S T$ be finite surjective morphisms from normal varieties extending the étale morphisms above. Then $T^\star \subset T$ is also a toroidal embedding. For a nonsingular model Y'' of Y', the fibration $Y'' \to T$ is a locally homotopically Q-smooth BCK-morphism. In particular, the local monodromies around $\tau^{-1}D$ of the variation of Hodge structure associated with $Y^\star \to T^\star$ are finite by 7.4. If we replace T by a further finite covering étale over $S^\star$, then $Y' \to T$ is bimeromorphically equivalent to a smooth torus fibration $\pi\colon Y \to T$. By 4.2, there exist a smooth Q-torus fibration $X_T \to T$ bimeromorphically equivalent to $X \times_S T \to T$ such that π factors through X_T by a finite étale morphism $Y \to X_T$. Let us consider the action of $G = \mathrm{Gal}(Y'/X)$ on Y.

Claim 7.9. If an element $g \in G$ fixes a point $y \in Y$, then it fixes all the points of the fiber $Y_{\pi(y)} = \pi^{-1}\pi(y)$.

Proof. Let $s = \tau(\pi(y))$ and let $\mathcal{U} \subset S$ be an open neighborhood of s such that $\mathcal{U}$ and the connected component $\mathcal{V}$ of $\tau^{-1}(\mathcal{U})$ containing $\pi(y)$ are simply connected Stein varieties. Then $\mathcal{Y} := \pi^{-1}(\mathcal{V}) \to \mathcal{V}$ is a smooth basic torus fibration. Let $G_0 \subset G$ be the subgroup generated by g. Then there is a G_0-invariant section of $\mathcal{Y} \to \mathcal{V}$. Thus the induced morphism $G_0\backslash\mathcal{Y} \to G_0\backslash\mathcal{V}$ between quotient varieties is a torus fibration admitting a

section. It is also a homotopically Q-smooth morphism. Hence by 7.7, it is actually a smooth morphism. Therefore $\mathcal{Y} \to \mathcal{V}$ is the pullback of the smooth morphism by $\mathcal{V} \to G_0\backslash\mathcal{V}$ and G_0 fixes all the points in the fiber $Y_{\pi(y)}$. □

Proof of 7.8 continued. Let $G_1 \subset G$ be the subgroup consisting of $g \in G$ which has a fixed point in Y. Then by Claim above and by 4.1, the induced morphism $G_1\backslash Y \to G_1\backslash T$ between quotient varieties are smooth torus fibration and $Y \to T$ is its pullback. Thus if we choose $T \to S$ minimal in the degree, then the action of G on Y is fixed point free. Therefore $X \to S$ is bimeromorphically equivalent to the Q-smooth Q-torus fibration $G\backslash Y \to S$. □

Corollary 7.10. *Let $f\colon X \to S$ be a proper surjective BCK-morphism from a nonsingular variety smooth over $S^\star$ such that smooth fibers are bimeromorphically equivalent to complex tori and that the local monodromies of $(R_1 f_* \mathbb{Z}_X)_{|S^\star}$ around D are trivial. Then f is bimeromorphically equivalent to a Q-smooth torus fibration.*

Corollary 7.11. *Let $f\colon X \to S$ be a proper surjective BCK-morphism from a normal variety smooth over $S^\star$ whose smooth fibers are bimeromorphically equivalent to complex Q-tori. If f admits an étale covering $U \to X$ such that $U \to X \to S$ is a locally Stein morphism, then f is Q-smooth.*

8. Second method

Lemma 8.1. *Let $\tau\colon U \to M$ be an étale covering over a compact complex manifold. Suppose that there is a bimeromorphic morphism $\nu\colon U \to V$ onto a Stein space. Then there is a bimeromorphic morphism $\mu\colon M \to N$ onto a normal variety such that V is a ramified covering space over N. If further V is nonsingular, then $V \to N$ is étale.*

Proof. By considering the Galois closure, we may assume that $U \to M$ is Galois. Let G be the Galois group. Then it also acts holomorphically on V. Let $C \subset V$ be a connected compact neighborhood of a point and let I be the subset $\{g \in G \mid g(C) \cap C \neq \emptyset\}$. Then for $g \in I$, $g(\nu^{-1}C) \cap \nu^{-1}C$ is not empty. Since $\nu^{-1}C$ is compact, I is a finite set. Therefore the action of G on V is properly discontinuous and there is a quotient space $N = G\backslash V$ as a normal variety. Let $M \to N' \to N$ be the Stein factorization. Then for any $v \in V$, the connected compact subset $\tau(\nu^{-1}(v))$ is mapped to a point of N'. Thus $N' \simeq N$ and hence $M \to N$ is a bimeromorphic morphism. In the case, V is nonsingular, the assertion follows from 2.10. □

Theorem 8.2. *Let M be a compact Kähler manifold whose universal covering space U is Stein. Suppose that there is a meromorphic fiber space $M \cdots\to S$ whose general fibers are bimeromorphically equivalent to complex Q-tori of dimension $g > 0$. Then there exist a finite étale morphism $M' \to M$ and a smooth torus fibration $M' \to S'$ onto a compact Kähler manifold S' with $\dim S' = \dim S$, where the universal covering space V of S' is Stein. In particular, $U \simeq \mathbb{C}^g \times V$.*

Proof. Let M_0 be the normalization of the graph of meromorphic mapping $M \cdots\to S$. As in the proof of 5.4, general fibers of $M_0 \to S$ are biholomorphic to complex Q-tori. Let $S_1 \to S$ be a bimeromorphic morphism from a nonsingular variety and let M_1 be the normalization of $M_0 \times_S S_1$. We can assume that $M_1 \to S_1$ is smooth outside a normal crossing divisor D of S_1. The pullback $U_1 := U \times_M M_1$ defines an étale covering $U_1 \to M_1$ such that $U_1 \to M_1 \to S_1$ is a locally Stein morphism. Therefore $M_1 \to S_1$ is a Q-smooth morphism by 7.11. It follows that there exist a finite étale covering $M' \to M$, a smooth torus fibration $Y \to T$ over a compact Kähler manifold and a bimeromorphic morphism $\mu\colon Y \to M'$. We may assume that the torsor $Y \to T$ is associated with an element η in $H^1(T, \mathfrak{S}_H)$ satisfying $c(\eta) = 0$ for the associated variation of Hodge structure H (cf.

2.17). The universal covering space is isomorphic to $U_Y := U \times_{M'} Y$ since $\pi_1(Y) \simeq \pi_1(M')$. On the other hand, we have an isomorphism $U_Y \simeq \mathbb{C}^g \times \widetilde{T}$, where $\widetilde{T}$ denotes the universal covering space of T, by 2.16.

Claim 8.3. (1) There is a bimeromorphic morphism $\widetilde{T} \to V$ onto a Stein manifold V.

(2) There is an isomorphism $U \simeq \mathbb{C}^g \times V$ such that the bimeromorphic morphism $U_Y \to U$ is isomorphic to the pullback of $\widetilde{T} \to V$.

Proof. Let $\varphi\colon \mathbb{C}^g \times \widetilde{T} \to U$ be the bimeromorphic morphism above and let $U^\circ \subset U$ be the domain of the inverse mapping φ^{-1}. The image of $\{v\} \times \widetilde{T}$ is a closed analytic subvariety of U since φ is proper, and is bimeromorphically equivalent to $\widetilde{T}$ for 'general' v. Thus $\widetilde{T}$ is holomorphically convex and its Remmert reduction $\widetilde{T} \to V$ is bimeromorphic. Since $\operatorname{codim}(U \setminus U^\circ) \geq 2$, the meromorphic mapping $U \cdots\to \mathbb{C}^g \times \widetilde{T} \to \mathbb{C}^g \times V$ is holomorphic. Hence $U \simeq \mathbb{C}^g \times V$. In particular, V is smooth. □

Proof of 8.2 continued. By 8.1, the action of $\pi_1(T)$ on V is free and hence $S' := \pi_1(T)\backslash V$ is nonsingular. Therefore, we have a smooth torus fibration $M' \to S'$. The S' is also a compact Kähler manifold since so is M' (cf. [Va1]). □

We have another proof of Theorem 5.4 by using 8.2 as follows: Suppose that $\kappa(M) > 0$ and apply 8.2 to the Iitaka fibration. Then we have a finite étale covering $M' \to M$ and a smooth torus fibration $M' \to S'$ with $\dim S' = \kappa(M)$. Thus there is a holomorphic mapping $\mathbb{C}^n \to S'$ of maximal rank. On the other hand, S' is of general type since $\kappa(S') = \kappa(M')$ by 2.20. This contradicts to Kobayashi–Ochiai's theorem.

Definition 8.4 (cf. [Fk2])**.** Let U be a non-compact nonsingular complex analytic variety. U is said to be *simple*, if there is no covering family $\{B_t\}_{t\in T}$ of compact analytic subvarieties of positive dimension. Here a *covering family* means that $\bigcup B_t = U$.

Let M be a compact Kähler manifold admitting an étale covering $\tau\colon U \to M$ from a non-compact and simple variety U. Then the universal covering space of M is also simple. If M has a large fundamental group in the sense of [Kl2], then its universal covering space is simple. If U is a Stein variety of positive dimension, then it is simple. Moreover if U is a holomorphically convex positive dimensional variety whose Remmert reduction is

bimeromorphic, then it is simple by 8.1. The proof of the following result is similar to the argument of Kollár [Kl2, 6.6, 6.7]:

Proposition 8.5. *Let $f\colon M \to S$ be a proper surjective morphism from a compact Kähler manifold onto a nonsingular variety. Suppose that f is smooth outside a normal crossing divisor D, a general smooth fiber $M_s = f^{-1}(s)$ is bimeromorphically equivalent to a complex torus, and that the kernel of $H_1(M_s, \mathbb{Z}) \simeq \pi_1(M_s) \to \pi_1(M)$ contains no non-trivial sub-Hodge structures of $H_1(M_s, \mathbb{Z})$. Then we have the following properties*:

(1) *The local monodromies of $(R_1 f_* \mathbb{Z}_M)_{|S^\star}$ around D are finite, where $S^\star := S \setminus D$.*

(2) *There is a finite étale morphism $M' \to M$ such that for the Stein factorization $M' \to S' \to S$, the local monodromies of the associated variation of Hodge structure on $S'^\star := S' \times_S S^\star$ around $S' \setminus S'^\star$ are trivial.*

(3) *f is bimeromorphically equivalent over S to a $\mathbb{Q}$-smooth torus fibration.*

Proof. (1). Letting $M^\star := f^{-1}(S^\star)$, we consider the homotopy exact sequence:

$$\pi_2(S^\star) \to \pi_1(M_s) \to \pi_1(M^\star) \to \pi_1(S^\star) \to 1.$$

Since M is Kähler, $\pi_2(S^\star) \to \pi_1(M_s)$ is zero by 2.18. Let $\phi\colon \pi_1(M^\star) \to \pi_1(M)$ be the surjection, W_0 the kernel of the composite $\pi_1(M_s) \to \pi_1(M^\star) \to \pi_1(M)$ and let $W_0 \subset W$ be the saturation in $\pi_1(M_s)$. The monodromy representation

$$\rho\colon \pi_1(S^\star, s) \ni a \mapsto \rho_a \in \mathrm{Aut}(\pi_1(M_s))$$

is induced from the conjugation of $\pi_1(M^\star)$. Thus for $a \in \pi_1(S^\star)$, there is an element $b \in \pi_1(X^\star)$ such that

$$\phi(\rho_a(x)) = \phi(b)\phi(x)\phi(b)^{-1},$$

for any $x \in \pi_1(M_s)$. Thus W_0 and also W are invariant under the action of $\pi_1(S^\star)$ by ρ. Since $W_0 \subset \pi_1(M^\star)$ is a normal subgroup, we have an exact sequence:

$$1 \to \pi_1(M_s)/W_0 \to \pi_1(M^\star)/W_0 \to \pi_1(S^\star) \to 1$$

and an injection $\pi_1(M_s)/W_0 \hookrightarrow \pi_1(M)$.

Let $\mathcal{U} \subset S$ be an open subset isomorphic to a unit polydisc where $D \cap \mathcal{U}$ is isomorphic to a union of hyperplanes. Further let $C \subset \mathcal{U}$ be a general smooth curve such that it intersects

with D transversely at one point and that $f^{-1}C$ is also smooth. Then there is a smooth curve $C' \subset f^{-1}\mathcal{U}$ dominating C. Then the image of $\pi_1(M_s) \to \pi_1(f^{-1}C)$ is a finite index subgroup by 7.1. Thus the kernel of $\pi_1(f^{-1}C \cap M^\star) \to \pi_1(f^{-1}C)$ dominates a finite index subgroup of $\pi_1(C \cap S^\star)$. Hence the kernel of $\pi_1(f^{-1}\mathcal{U} \cap M^\star) \to \pi_1(f^{-1}\mathcal{U})$ is dominating a finite index subgroup of $\pi_1(\mathcal{U} \cap S^\star)$. Therefore the image of $\pi_1(M_s) \to \pi_1(f^{-1}\mathcal{U})$ is also a finite index subgroup. Hence the local constant systems associated with $\pi_1(M_s)/W_0$ and $\pi_1(M_s)/W$ have only finite local monodromies around D.

The underlying local constant system of the variation of Hodge structure $R_1f_*\mathbb{Z}_{M^\star}$ is defined over $\mathbb{Z}$, but its polarization is defined only over $\mathbb{R}$. Even in the situation, we can apply the argument of Deligne [D2, 4.2.6]. Thus the representation of $\pi_1(S^\star, s)$ on $H_1(M_s, \mathbb{C})$ is completely reducible. Suppose that the local monodromies of the local constant system associated with W around D are not finite. Then there is an irreducible $\mathbb{C}[\pi_1(S^\star)]$-module $I \subset W \otimes \mathbb{C}$ such that the local monodromies of the associated local system $\widetilde{I}$ are not finite. Every such irreducible $\mathbb{C}[\pi_1(S^\star)]$-submodules of $H_1(M_s, \mathbb{C})$ are contained in $W \otimes \mathbb{C}$ and let $W_{1,\mathbb{C}}$ be the subspace generated by such modules. Then there is a saturated submodule $W_1 \subset W$ such that $W_1 \otimes \mathbb{C} = W_{1,\mathbb{C}}$. Further $W_{1,\mathbb{C}}$ has a sub-Hodge structure, since it is invariant under the Weil operator. This contradicts to our assumption. Hence the local monodromies of $(R_1f_*\mathbb{Z}_M)_{|S^\star}$ are finite.

(2). Next, we consider the image L of $\pi_1(M_s) \to \pi_1(M)$. It is a normal subgroup. Let $\pi_1(M) \to \mathrm{Aut}(L)$ be the homomorphism defined by conjugation and let N be the kernel of $\pi_1(M) \to \mathrm{Aut}(L \otimes \mathbb{Z}/m\mathbb{Z})$ for $m \geq 3$. By Lemma of Serre [Se], non-trivial image of a subgroup of N in $\mathrm{Aut}(L)$ is infinite. Let $M' \to M$ be the finite étale covering corresponding to N and let $M' \to S' \to S$ be the Stein factorization. Since the image of $\pi_1(M_s)$ is contained in N, there is a bimeromorphic morphism $M' \to M \times_S S'$ over S'. The finite morphism $\tau\colon S' \to S$ is étale over $S^\star$. Thus $\tau^{-1}(S^\star) \subset S'$ is a toroidal embedding. Let $f'\colon M' \to S'$ be the induced morphism. Letting $\mathcal{U}' = \tau^{-1}\mathcal{U}$ and $\mathcal{U}'^\star = \tau^{-1}(\mathcal{U}^\star)$, we consider the homotopy exact sequence

$$1 \to \pi_1(M_s) \to \pi_1(f'^{-1}(\mathcal{U}'^\star)) \to \pi_1(\mathcal{U}'^\star) \to 1.$$

Since the image of $\pi_1(M_s) \to \pi_1(f'^{-1}(\mathcal{U}'))$ is a finite index subgroup, $\pi_1(f'^{-1}(\mathcal{U}')) \to \mathrm{Aut}(L)$ is trivial. Thus L defines a local constant system on $S'^\star = \tau^{-1}(S^\star)$ whose local

monodromies around $S' \setminus S'^{\star}$ are trivial. Let W'_0 be the kernel of $\pi_1(M_s) \to L$, $W'_0 \subset W'$ the saturation and suppose that the local constant system associated with W' has non-trivial local monodromies. Then there is an irreducible $\mathbb{C}[\pi_1(S'^{\star})]$-submodule $J \subset W' \otimes \mathbb{C}$ the local monodromies which induces are not trivial. Any such irreducible $\mathbb{C}[\pi_1(S'^{\star})]$-submodules of $H_1(M_s, \mathbb{C})$ are contained in $W' \otimes \mathbb{C}$. The submodule $W'_{1,\mathbb{C}}$ generated by all such modules has an integral structure. Further it is a sub-Hodge structure of $H_1(M_s, \mathbb{Z})$. This contradicts to our assumption. Hence the local monodromies of $(R_1 f'_* \mathbb{Z}_{M'})_{|S'^{\star}}$ are trivial.

(3). Let $M' \to S'$ be the morphism appearing in (2) above. Then it is bimeromorphically equivalent to a $\mathbb{Q}$-smooth torus fibration over S' by 7.10. Therefore, f is homotopically smooth. Thus the proof follows from 7.8. □

We shall show the following generalization of [Kl2, 6.3] to compact Kähler case:

Theorem 8.6. *Let $f\colon M \to S$ be a proper surjective morphism from a compact Kähler manifold onto a normal variety. Suppose that general fibers are bimeromorphically equivalent to complex $\mathbb{Q}$-tori and that there is an étale covering $\tau\colon U \to M$, where $U_s = \tau^{-1} f^{-1}(s)$ is non-compact and simple for 'general' $s \in S$. Then there exist a finite étale covering $M' \to M$ and a smooth torus fibration $Y \to T$ such that Y and M' are bimeromorphically equivalent and $\dim T = \dim S$. If further S is nonsingular and f is smooth outside a normal crossing divisor of S, then f is bimeromorphically equivalent to a $\mathbb{Q}$-smooth $\mathbb{Q}$-torus fibration.*

Proof. By applying 3.15 to the image of $\pi_1(M_s) \to \pi_1(M)$, we see that there is a finite étale covering $M_1 \to M$ such that the Stein factorization $M_1 \to S_1 \to S$ induces a proper surjective morphism whose general fibers are bimeromorphically equivalent to complex tori. Hence, for the first assertion, we may assume that general fibers are bimeromorphically equivalent to complex tori. We shall show that the kernel of $H_1(M_s, \mathbb{Z}) \simeq \pi_1(M_s) \to \pi_1(M)$ contains no non-trivial sub-Hodge structures of $H_1(M_s, \mathbb{Z})$. Assume the contrary. Let A be the Albanese torus of M_s. Then there is a complex subtorus $B \subset A$ which defines the sub-Hodge structure. Let $Z \subset M_s$ be the pullback of a translate of B in A. Then $\pi_1(Z) \to \pi_1(M_s) \to \pi_1(M)$ is zero. Thus connected components of $\tau^{-1}(Z)$ are isomorphic to Z. This contradicts to the condition on U_s.

By applying 8.5 to the base change of f by a suitable bimeromorphic morphism $S'' \to S$, we infer that there is a finite étale morphism $M' \to M$ such that M' is bimeromorphically equivalent to the total space Y of a smooth torus fibration $Y \to T$ over a compact Kähler manifold T of $\dim T = \dim S$.

The second assertion is derived from 7.8 since f is homotopically Q-smooth. □

Corollary 8.7. *Let M be a compact Kähler manifold whose universal covering space is non-compact and simple. Suppose that there is a meromorphic fiber space $M \cdots\to S$ whose general fibers are bimeromorphically equivalent to complex Q-tori of dimension $g > 0$. Then there exist a finite étale morphism $M' \to M$, a smooth torus fibration $Y \to T$ onto a compact Kähler manifold T with $\dim T = \dim S$ such that Y and M' are bimeromorphically equivalent and that the universal covering space of T is non-compact and simple.*

Proof. There exist a finite étale covering $M' \to M$, a smooth torus fibration $Y \to T$ over a compact Kähler manifold T, and a bimeromorphic mapping $Y \cdots\to M'$ by 8.6. The 'pullback' $U_Y := U \times_{M'} Y$ is isomorphic to the universal covering space of Y, which is non-compact and simple. Thus the universal covering space of T is also non-compact and simple. □

We pose the following two conjectures.

Conjectures. Let M be an n-dimensional compact Kähler manifold whose universal covering space is non-compact and simple. Under the following assumption U_n^0 or U_n^{00}, M should be bimeromorphically equivalent to a complex Q-torus.

U_n^0: $\kappa(M) \leq 0$.

U_n^{00}: $\kappa(M) \leq 0$ and $q(M) = 0$.

We also consider the following general form of conjecture U_n as in [N4]:

Conjecture U_n'. Let M be an n-dimensional compact Kähler manifold and let U be its universal covering space. Assume that there is a meromorphic mapping $Z \cdots\to U$ of maximal rank from a Zariski-open subset Z of a compact complex variety $\overline{Z}$. Then M is a Q-torus if U is Stein. If U is non-compact and simple, then M is bimeromorphically equivalent to a Q-torus.

The condition of U'_n is satisfied, for example, if U is biholomorphic to an affine variety.

Proposition 8.8. *The conjecture U'_n follows from U^0_k for any $k \leq n$.*

Proof. Suppose that $k := \kappa(M) > 0$. If $k = n$, then the composite $Z \cdots\to U \to M$ extends to a meromorphic mapping from the compact variety $\overline{Z}$ by [KO2]. Thus we have a meromorphic mapping $\overline{Z} \cdots\to U$ of maximal rank. This is impossible since U is non-compact. Therefore $k < n$. Let $M \cdots\to S_0$ be the Iitaka fibration. Then a nonsingular model of a general fiber satisfies the condition of U^0_{n-k}. Suppose the U^0_{n-k} is true. Then general fibers are bimeromorphic to Q-tori. By 8.2 and 8.7, there exist a finite étale covering $M' \to M$, a smooth torus fibration $Y \to T$ over a compact Kähler manifold T of $\dim T = k$ and a bimeromorphic mapping $Y \cdots\to T$. Here U is also bimeromorphically equivalent to $\mathbb{C}^{n-k} \times \widetilde{T}$, where $\widetilde{T}$ is the universal covering space of T. Thus $\widetilde{T}$ also satisfied the condition of U'_k. By induction on k, we see that T is bimeromorphically equivalent to a Q-torus. Therefore X is bimeromorphically equivalent to a Q-torus by 2.19. □

Proposition 8.9. *The conjecture U^0_n follows from U^0_k for any $k < n$ and U^{00}_n.*

Proof. Let M be a such compact Kähler manifold in Conjecture U^0_n with $q(M) > 0$. Let $\alpha\colon M \to \mathrm{Alb}(M)$ be the Albanese mapping and let $f\colon M \to Z$ be the Stein factorization. First, suppose that f is a bimeromorphic morphism. Then $\kappa(M) = 0$ and $Z = \mathrm{Alb}(M)$ by Theorem 22 of [Kw1]. Thus M is bimeromorphically equivalent to a complex torus. Next, suppose that general fibers F of f are of general type. Then by 5.7, we have $\kappa(M) > \kappa(Z) \geq 0$. This is a contradiction. Hence $\kappa(F) < \dim F$. There is an étale covering $F' \to F$ from a non-compact and simple manifold F'. Thus $\kappa(F) \geq 0$, by the affirmative solution of U^0_m for $m = \dim F < n$. Let $\phi\colon M \cdots\to S$ be the relative Iitaka fibration over Z. Then a nonsingular model V of a general fiber of ϕ has an étale covering from a non-compact and simple manifold. Since $\kappa(V) = 0$, the affirmative answer to U^0_k for $k = \dim V < n$ implies that V is bimeromorphically equivalent to a Q-torus. By 8.7, there exist a finite étale covering $M' \to M$, a smooth torus fibration $Y \to T$ with $\dim T = \dim S$ and a bimeromorphic mapping $Y \cdots\to M'$, where T is a compact Kähler manifold whose universal covering space is non-compact and simple. Here $\kappa(M) = \kappa(T) \leq 0$ by 2.20. Hence by the affirmative answer to U^0_l for $l = \dim T < n$, we

see that T is bimeromorphically equivalent to a Q-torus. Hence M is bimeromorphically equivalent to a complex Q-torus by 2.19. □

Theorem 8.10. *Conjecture U_n^0 is true for $n \leq 3$.*

Proof. By 8.9, it is enough to show U_n^{00} is true for $n \leq 3$. If $n = 1, 2$, then M in question does not exist. In the case $n = 3$, we first assume that M is not projective. Then by a similar argument to [G, 3.2.C′], we have $\chi(M, \mathcal{O}_M) = 0$. Hence $p_g(M) = p_g(M) + q(M) = h^{2,0}(M) + 1 \geq 2$. This contradicts to $\kappa(M) \leq 0$. Hence M is projective. Then there is a minimal model X of M whose canonical divisor is numerically trivial by the abundance theorem [Kw3]. Further the result of [NS, 1.4] implies that X is nonsingular and is a Q-torus. □

Lemma 8.11. *Let $f\colon X \to S$ be a fiber space from a compact Kähler manifold X such that S is finite over a complex torus. If a general fiber of f is positive dimensional and of general type, then $\kappa(X) > 0$ and a general fiber of the Iitaka fibration of X is bimeromorphically equivalent to a complex torus.*

Proof. By applying 5.7, we have $\kappa(X) > \kappa(S) \geq 0$. Let $X \cdots\to Y$ be the Iitaka fibration and let $X \cdots\to W \to S \times Y$ be the Stein factorization of the morphism into the product. For a general point $y \in Y$, the fiber W_y is a normal variety finite over a complex torus. Let $X_y \cdots\to W_y$ be the fiber of $X \cdots\to W$ and let X_w be its 'general' fiber. Since K_X is relatively big over S, it is so over W. Thus by 5.7, we have

$$0 = \kappa(X_y) \geq \kappa(X_w) + \kappa(W_y) \geq 0.$$

Hence $\kappa(W_y) = 0$ and $X \cdots\to W$ is bimeromorphic. Thus X_y is bimeromorphically equivalent to a complex torus by Theorem 22 of [Kw1]. □

Theorem 8.12 (cf. Theorem B of [N4])**.** *Let M be a compact Kähler manifold satisfying the assumption of Conjecture U_n'. Then the Albanese mapping $\alpha\colon M \to \mathrm{Alb}(M)$ is a fiber space. Suppose that a general fiber of the Iitaka fibration of $M_t = \alpha^{-1}(t)$ for general $t \in \mathrm{Alb}(M)$ is bimeromorphically equivalent to a complex Q-torus. Then M is also bimeromorphically equivalent to a complex Q-torus.*

Proof. The Albanese mapping of M is a fiber space by 5.2. Let $f\colon M \cdots\to Y$ be the relative Iitaka fibration over $\mathrm{Alb}(M)$. Then by 8.7, there exist a finite étale covering $M' \to M$ and a smooth torus fibration $X \to Y'$ with $\dim Y' = \dim Y$ over $\mathrm{Alb}(M)$ such that X and M' are bimeromorphically equivalent over $\mathrm{Alb}(M)$ and that the universal covering space $\widetilde{Y'}$ of Y' is also non-compact and simple. Further there is a meromorphic mapping $Z \cdots\to \widetilde{Y'}$ of maximal rank from a Zariski-open subset Z of a compact complex variety $\overline{Z}$. Let $Y' \to S' \to \mathrm{Alb}(M)$ be the Stein factorization. Then a general fiber of $Y' \to S'$ is of general type by 2.20. If $Y' \to S'$ is not bimeromorphic, then a general fiber of the Iitaka fibration of Y' is bimeromorphically equivalent to a complex torus, by 8.11. Thus if we replace M' and Y' by further finite étale coverings, respectively, then we have a smooth torus fibration $X \to V$ onto a variety of general type such that X and M' are bimeromorphically equivalent and that the universal covering space of V is non-compact and simple by 2.19 and 8.7. By an argument of 5.2, we see that V is a point. Thus M is bimeromorphically equivalent to a Q-torus.

Next suppose that $Y' \to S'$ is bimeromorphic. Since Y' is generically finite over $\mathrm{Alb}(M)$, a finite étale covering space of Y' is bimeromorphically equivalent to the total space of a fiber bundle of a complex torus over a variety of general type, by Theorem 23 of [Kw1]. Thus after replacing Y' by a finite étale covering space, we have a meromorphic fiber space $M' \cdots\to W$ onto a variety of general type whose general fibers are bimeromorphically equivalent to complex Q-tori, by 2.19. By an argument of 5.2 and 8.7, W is a point. Therefore, M is bimeromorphically equivalent to a complex Q-torus. □

Corollary 8.13. *U'_n is true for $n \leq 3$. For $n \geq 4$, let M be a compact Kähler manifold satisfying the assumption of Conjecture U'_n. Then M is bimeromorphically equivalent to a complex Q-torus or satisfies both inequalities $\kappa(M) < n-3$ and $q(M) < n-3$.*

Theorem 8.14. *Let M be a compact Kähler manifold whose universal covering space is Stein and let $f\colon M \to S$ be the Stein factorization of the Albanese mapping. Suppose that a general fiber of the Iitaka fibration of $M_s = f^{-1}(s)$ for general $s \in S$ is bimeromorphically equivalent to a complex Q-torus. Then K_M is semi-ample.*

Proof. Let $M \cdots\to Y$ be the relative Iitaka fibration over S. Then by 8.2, there exist a finite étale morphism $M' \to M$ and a smooth torus fibration $M' \to Y'$ with $\dim Y' = \dim Y$

such that the universal covering space of Y' is also Stein. Let $Y' \to S' \to S$ be the Stein factorization. Then $K_{Y'}$ is relatively big over S' by 2.20. Thus $\kappa(Y') \geq \kappa(Y'/S') + \kappa(S')$ by 5.7. If $\dim Y' = \dim S'$, then Y' is generically finite over $\mathrm{Alb}(M)$. By Theorem 23 of [Kw1], if we replace Y' by a finite étale covering manifold, then there is a fiber space $Y' \to B$ onto a variety of general type whose general fibers are constant complex torus. By 8.2, we may assume that $Y' \to B$ is a smooth torus fibration and the universal covering space of B is Stein. Hence K_B is ample by [Kw2]. Since $K_{M'}$ is a pullback of $K_B + L$ for a nef divisor L by 2.19 and 2.20, K_M is semi-ample.

Next suppose that $\dim Y' > \dim S'$, i.e., $\kappa(M/S) > 0$. Then $\kappa(Y') > 0$ and a general fiber of the Iitaka fibration $Y' \cdots\to V$ of Y' is bimeromorphically equivalent to a complex torus by 8.11. Applying 8.2 to $Y' \cdots\to V$, we have a finite étale morphism $Y'' \to Y'$ and a smooth torus fibration $Y'' \to V'$ with $\dim V' = \dim V$ where the universal covering space of V' is also Stein. Here V' is of general type, since $\kappa(V') = \kappa(Y') = \dim V'$ by 2.20. Therefore $K_{V'}$ is ample by [Kw2]. Hence if we replace M by a suitable finite étale covering space, we have a smooth torus fibration $\pi\colon M \to V$ over a projective variety V with ample canonical divisor. Since $K_M \sim \pi^*(K_V + L)$ for a nef divisor L, K_M is semi-ample. □

Corollary 8.15. *Let M be an n-dimensional compact Kähler manifold whose universal covering space is Stein. If $n \leq 3$, $\kappa(M) \geq n-3$, or if $\dim \alpha(M) \geq n-3$, then K_M is semi-ample, where $\alpha\colon M \to \mathrm{Alb}(M)$ denotes the Albanese mapping.*

Appendix A. Meromorphic mapping into a variety of general type

We shall consider an extension problem of meromorphic mappings from not only $\mathbb{C}^n$ but also Zariski-open subsets of complex manifolds. For the purpose, we recall some results on Nevanlinna theory along the book [NO].

Let M be an n-dimensional complex manifold and let Ω be a continuous real $2n$-form on M. For an open subset U with a local coordinate $z_1, z_2, \dots, z_n$, we can write

$$\Omega_{|U} = a_U(\frac{\sqrt{-1}}{2}dz_1 \wedge d\overline{z_1}) \wedge \cdots \wedge (\frac{\sqrt{-1}}{2}dz_n \wedge d\overline{z_n})$$

for a continuous real-valued function a_U on U. For a point $x \in U$, we write $\Omega(x) > 0$, $\Omega(x) < 0$, or $\Omega(x) = 0$ according as $a_U(x) > 0$, $a_U(x) < 0$, or $a_U(x) = 0$. We define

$$\mathrm{Zero}(\Omega) := \{x \in M \,|\, \Omega(x) = 0\}.$$

The Ω is called a *pseudo-volume form* if $\Omega(x) \geq 0$ for any $x \in M$. It is called a *pseudo-volume element* if it is a pseudo-volume form and it is a C^∞-form on $M \setminus \mathrm{Zero}(\Omega)$. If further $\mathrm{Zero}(\Omega) = \emptyset$, then it is called a *volume element*. If Ω is a pseudo-volume element, then the *Ricci form* $\mathrm{Ric}(\Omega)$ is defined over $M \setminus \mathrm{Zero}(\Omega)$ locally by:

$$\mathrm{Ric}(\Omega)_{|U\setminus\mathrm{Zero}(\Omega)} := -\sqrt{-1}\partial\overline{\partial}\log(a_{|U\setminus\mathrm{Zero}(\Omega)}).$$

The *curvature* function $K_\Omega \colon M \to [-\infty, 0)$ is defined by:

$$K_\Omega(x) := \begin{cases} -\infty, & x \in \mathrm{Zero}(\Omega), \\ -\dfrac{(-\mathrm{Ric}(\Omega))^n}{n!\,\Omega}, & x \notin \mathrm{Zero}(\Omega). \end{cases}$$

Definition A.1. The *Poincaré volume element* on

$$\Delta^n = \{(z_1, z_2, \dots, z_n) \in \mathbb{C}^n \,|\, |z_i| < 1 \text{ for any } i\}$$

is defined by:

$$\Omega^{\mathrm{P}} := \bigwedge_{j=1}^{n} \left(\frac{4}{(1-|z_j|^2)^2}\frac{\sqrt{-1}}{2}dz_j \wedge d\overline{z_j}\right).$$

The Poincaré volume element Ω^{P} satisfies $-\mathrm{Ric}\,\Omega^{\mathrm{P}} > 0$, $K_{\Omega^{\mathrm{P}}} = -1$, and $\varphi^*\Omega^{\mathrm{P}} = \Omega^{\mathrm{P}}$ for any holomorphic automorphism φ of Δ^n.

Definition A.2. The Poincaré volume element on

$$\Delta^\star \times \Delta^{n-1} = \{(z_1, z_2, \dots, z_n) \in \mathbb{C}^n \mid z_1 \neq 0, |z_i| < 1 \text{ for any } i\}$$

is defined by:

$$\Omega^{\star \mathrm{P}} := \left(\frac{4}{|z_1|^2 (\log |z_1|)^2} \frac{\sqrt{-1}}{2} dz_1 \wedge d\overline{z_1} \right) \wedge \bigwedge_{j=2}^{n} \left(\frac{4}{(1-|z_j|^2)^2} \frac{\sqrt{-1}}{2} dz_j \wedge d\overline{z_j} \right).$$

We recall the following Schwarz lemma (cf. Lemma 2.2.8 and Corollary 2.2.10 of [NO]):

Lemma A.3. *Let Ω be a pseudo-volume element on Δ^n (resp. $\Delta^\star \times \Delta^{n-1}$) satisfying the following conditions*:

(1) $-\operatorname{Ric} \Omega \geq 0$ *outside* $\operatorname{Zero}(\Omega)$.

(2) *There is a positive number A such that $K_\Omega(x) \leq -A$ for any x.*

Then $\Omega \leq A^{-1} \Omega^{\mathrm{P}}$ (resp. $\Omega \leq A^{-1} \Omega^{\star \mathrm{P}}$).

Let M be an n-dimensional nonsingular projective variety and $D = \sum_{k=1}^{l} D_k$ a simple normal crossing divisor, and let $e_1, e_2, \dots, e_l$ be rational numbers greater than one. Suppose that

$$\kappa(K_M + \sum_{k=1}^{l} (1 - \frac{1}{e_k}) D_k, M) = \dim M.$$

Then there exist a positive integer m, a very ample divisor A, and an effective divisor E such that $m/e_k \in \mathbb{Z}$ for any k and that

$$A + E \sim m(K_M + \sum_{k=1}^{l} (1 - \frac{1}{e_k}) D_k).$$

Let $\sigma \in H^0(M, \mathcal{O}_M(E))$ and $\delta_k \in H^0(M, \mathcal{O}_M(D_k))$ be defining equations of E and of D_k, respectively, and let $\varphi_0, \varphi_1, \dots, \varphi_N \in H^0(M, \mathcal{O}_M(A))$ be sections which induce a closed embedding $M \hookrightarrow \mathbb{P}^N$. Let $M = \bigcup_{\lambda \in \Lambda} M_\lambda$ be an open covering of M and suppose that M_λ has a local coordinate system $(z_1^\lambda, z_2^\lambda, \dots, z_n^\lambda)$ for any $\lambda \in \Lambda$. By replacing it with a finer covering, we may assume that invertible sheaves $\mathcal{O}_M(D_k)$, $\mathcal{O}_M(A)$, and $\mathcal{O}_M(E)$ are defined by transition functions

$$\varepsilon_k^{\lambda,\mu}, \alpha^{\lambda,\mu}, \quad \text{and} \quad u^{\lambda,\mu} \in H^0(M_\lambda \cap M_\mu, \mathcal{O}_M^\star),$$

respectively. The defining equation δ_k of D_k is given by holomorphic functions δ_k^λ on M_λ satisfying

$$\delta_k^\lambda = \varepsilon_k^{\lambda,\mu}\delta_k^\mu$$

on $M_\lambda \cap M_\mu$. Similarly, φ_i $(0 \leq i \leq N)$ and σ are given by holomorphic functions φ_i^λ and σ^λ on M_λ satisfying

$$\varphi_i^\lambda = \alpha^{\lambda,\mu}\varphi_i^\mu, \quad \text{and} \quad \sigma^\lambda = u^{\lambda,\mu}\sigma^\mu$$

on $M_\lambda \cap M_\mu$, respectively. Let $J^{\lambda,\mu} \in H^0(M_\lambda \cap M_\mu, \mathcal{O}_M^\star)$ be the transition function for $\mathcal{O}(-K_M)$ defined by

$$dz_1^\lambda \wedge dz_2^\lambda \wedge \cdots \wedge dz_n^\lambda = J^{\lambda,\mu} dz_1^\mu \wedge dz_2^\mu \wedge \cdots \wedge dz_n^\mu.$$

For each $0 \leq i \leq N$, we have the following equality on $M_\lambda \cap M_\mu$:

$$\frac{\sigma^\lambda \varphi_i^\lambda}{\left(\prod_{k=1}^{l}(\delta_k^\lambda)^{(1-\frac{1}{e_k})}\right)^m}(dz_1^\lambda \wedge \cdots \wedge dz_n^\lambda)^{\otimes m} = \frac{\sigma^\mu \varphi_i^\mu}{\left(\prod_{k=1}^{l}(\delta_k^\mu)^{(1-\frac{1}{e_k})}\right)^m}(dz_1^\mu \wedge \cdots \wedge dz_n^\mu)^{\otimes m}.$$

This defines a global meromorphic section η_i of $\mathcal{O}(mK_M)$ which is holomorphic on $M \setminus D$. The $2n$-form $(\eta_i \wedge \overline{\eta_i})^{1/m}$ on $M \setminus D$ is locally written by

$$(\eta_i \wedge \overline{\eta_i})^{\frac{1}{m}} = \frac{|\sigma^\lambda|^{\frac{2}{m}}|\varphi_i^\lambda|^{\frac{2}{m}}}{\prod_{k=1}^{l}|\delta_k^\lambda|^{2(1-\frac{1}{e_k})}} dz_1^\lambda \wedge \cdots \wedge dz_n^\lambda \wedge d\overline{z_1^\lambda} \wedge \cdots \wedge d\overline{z_n^\lambda}.$$

We define

$$\Phi^\lambda := \bigwedge_{j=1}^{n}\left(\frac{\sqrt{-1}}{2} dz_j^\lambda \wedge d\overline{z_j^\lambda}\right), \quad s_n := \frac{(\sqrt{-1})^n}{2^n}(-1)^{\frac{n(n-1)}{2}}$$

so that

$$\Phi^\lambda = s_n dz_1^\lambda \wedge \cdots \wedge dz_n^\lambda \wedge d\overline{z_1^\lambda} \wedge \cdots \wedge d\overline{z_n^\lambda}.$$

Let us define

$$\Psi' := s_n \left(\sum_{i=0}^{N} \eta_i \wedge \overline{\eta_i}\right)^{\frac{1}{m}}.$$

This is locally written by

$$\Psi'_{|M_\lambda} = \frac{|\sigma^\lambda|^{\frac{2}{m}} \left(\sum_{i=0}^{l} |\varphi_i^\lambda|^2\right)^{\frac{1}{m}}}{\prod_{k=1}^{l} |\delta_k^\lambda|^{2(1-\frac{1}{e_k})}} \Phi^\lambda.$$

Thus Ψ' is a pseudo-volume element on $M \setminus D$. Let us consider Hermitian metrics h_k of each $\mathcal{O}(D_k)$. There is a C^∞ positive valued function h_k^λ on M_λ such that

$$h_k^\lambda |\varepsilon_k^{\lambda,\mu}|^2 = h_k^\mu$$

on $M_\lambda \cap M_\mu$ and $\{h_k^\lambda\}$ defines the metric h_k. Let us denote $\|\delta_k\|^2 := h_k(\delta_k, \delta_k)$. Then

$$\|\delta_k\|^2_{|M_\lambda} = h_k^\lambda |\delta_k^\lambda|^2.$$

There is a d-closed positive C^∞-$(1,1)$-form ω_{FS} on M which is defined locally by

$$\omega_{\mathrm{FS}|M_\lambda} := \sqrt{-1}\partial\overline{\partial} \log \left(\sum_{i=0}^{N} |\varphi_i^\lambda|^2\right).$$

For $1 \leq k \leq l$, let ω_k be a d-closed C^∞-$(1,1)$-form locally defined by:

$$\omega_{k|M_\lambda} := -\sqrt{-1}\partial\overline{\partial} \log h_k^\lambda.$$

Let β_k $(1 \leq k \leq l)$ be sufficiently small positive rational numbers such that $\beta_k < 1/e_k$ and

$$\frac{1}{m}\omega_{\mathrm{FS}} + \sum_{k=1}^{l} \beta_k \omega_k$$

is a positive $(1,1)$-form on M. By multiplying some positive constants to h_k, we may assume that

$$\prod_{k=1}^{l} \left(\log \|\delta_k\|^2\right)^2 \|\delta_k\|^{2\beta_k} \leq 1$$

over M. For a positive real number c, let us define

$$\Psi := \frac{c}{\prod_{k=1}^{l} (\log \|\delta_k\|^2)^2 \|\delta_k\|^{2\beta_k}} \Psi'.$$

On M_λ, the Ψ is written as follows:

$$\frac{c\left(\prod_{k=1}^{l}(h_k^\lambda)^{(1-\frac{1}{e_k})}\right)|\sigma^\lambda|^{\frac{2}{m}}\left(\sum_{i=0}^{N}|\varphi_i^\lambda|^2\right)^{\frac{1}{m}}}{\prod_{k=1}^{l}(\log\|\delta_k\|^2)^2\|\delta_k\|^{2(1-\frac{1}{e_k}+\beta_k)}}\Phi^\lambda.$$

As in the arguments of Lemma 5.4.1 (5.5.1 in english edition) of [NO], Proposition 1 of [Sa1], and Proposition 4.1 of [Sa2] (cf. Lemma 3.4 of [Sh1]), we have:

Lemma A.4. *For a suitable choice of the Hermitian metrics h_k and the positive constant c, Ψ is a pseudo-volume element on $M \setminus D$ satisfying the following three conditions over $M \setminus (D \cup E)$:*

(1) $\mathrm{Ric}\,\Psi < 0$.

(2) $(-\mathrm{Ric}\,\Psi)^d \geq \Psi$.

(3) $\int_{M\setminus(D\cup E)}(-\mathrm{Ric}\,\Psi)^d < \infty$.

Let $f : U = \Delta^n \cdots\to M$ be a non-degenerate meromorphic mapping to the M. Suppose that there exist positive integers m_k for $1 \leq k \leq l$ such that

(1) $f^*D_k \geq m_k f^{-1}(D_k)$ for any k,

(2) $\kappa(K_M + \sum_{k=1}^{l}(1-(1/m_k))D_k) = \dim M$.

Let $0 < e_k < m_k$ be positive rational numbers such that $\kappa(K_M + \sum_{k=1}^{l}(1-(1/e_k))D_k) = \dim M$ and let us consider the same pseudo-volume element Ψ as above, where we assume further that for any k,

$$1 - \frac{1}{m_k} > 1 - \frac{1}{e_k} + \beta_k.$$

Note that there is an inequality of divisors

$$R_f \geq f^*(\sum_{k=1}^{l}(1-\frac{1}{m_k})D_k),$$

by the logarithmic ramification formula. Let $U_0 \subset U$ be the domain of f and let us denote by f_0 the restriction $f_{|U_0}$. Let $w_1, w_2, \ldots, w_n$ be the coordinate of U and let

$$\Phi := \bigwedge_{i=1}^{n}(\frac{\sqrt{-1}}{2}dw_i \wedge d\overline{w_i}).$$

Then there is a C^∞-function ξ defined over $f_0^{-1}(M \setminus D)$ such that

$$(f_0^* \Psi)_{|f_0^{-1}(M \setminus D)} = \xi \Phi.$$

By a similar argument to Lemma 5.4.13 (5.5.13 in english edition) of [NO] (cf. Lemma 3.5 of [Sh1]), we have:

Lemma A.5. (1) $\log \xi$ *extends to a pluri-subharmonic function on* U.

(2) $f_0^*(-\operatorname{Ric} \Psi)$ *extends to a locally square integrable* $(1,1)$*-form* $f^*(-\operatorname{Ric} \Psi)$ *on* U*, the current* $[f^*(-\operatorname{Ric} \Psi)]$ *which defines is positive.*

(3) *We have the following equality of currents*:

$$\frac{\sqrt{-1}}{2\pi} \partial\overline{\partial} [\log \xi] = \frac{1}{2\pi} [f^*(-\operatorname{Ric} \Psi)] + f^* E + R_f - f^* \left(\sum_{k=1}^{l} (1 - \frac{1}{e_k} + \beta_k) D_k \right).$$

In particular, the pluri-subharmonic function ξ is zero along $\operatorname{Supp} f^* D \cup \operatorname{Supp} f^* E$, since the Lelong numbers of the current in (3) along the loci are positive. The meromorphic mapping $f \colon U \cdots\to M \hookrightarrow \mathbb{P}^N$ is given by sections $f^*\varphi_0, f^*\varphi_1, \ldots, f^*\varphi_N$ of $\mathcal{O}_U(f^*A)$. Thus by considering the Lelong number of the current $[f^*(-\operatorname{Ric} \Psi)]$, we see that ξ is zero also along $U \setminus U_0$. Therefore ξ is C^∞ outside $\{\xi = 0\}$ and hence $f^*\Psi := \xi\Phi$ is a pseudo-volume element on U. By A.5 and A.3, we have $f^*\Psi \leq \Omega^{\mathrm{P}}$.

We shall prove the following generalization of results of Theorem 2 in [KO2], Proposition 3 in [Sa1], Proposition 4.2 in [Sa2] and our Theorem 5.3.

Theorem A.6. *Let* $U^\star$ *be a Zariski-open subset of a nonsingular complex analytic variety* U *and let* $f \colon U^\star \cdots\to M$ *be a meromorphic mapping of maximal rank to a nonsingular projective variety* M*. Suppose that there exist a simple normal crossing divisor* $D = \sum D_k$ *of* M *and positive integers* m_k *satisfying the following conditions*:

(1) $f^* D_k \geq m_k f^{-1}(D_k)$ *for any irreducible component* D_k.

(2) $\kappa(K_S + \sum (1 - (1/m_k) D_k)) = \dim M$.

Then the meromorphic mapping $U^\star \cdots\to S$ *extends to a meromorphic mapping* $U \cdots\to S$.

Proof. We may assume that U is isomorphic to an n-dimensional polydisc Δ^n and $U^\star$ is isomorphic to $\Delta^\star \times \Delta^{n-1}$, since the statement is local on U, and since f extends to a meromorphic mappings from U if $\operatorname{codim}(U \setminus U^\star) \geq 2$. We further can assume that f is non-degenerate as in the proof of Theorem 5.3. Let e_k be positive rational numbers

such that $e_k < m_k$ for every k and $\kappa(K_M + \sum(1-(1/e_k)D_k), M) = \dim M$. Let β_k are sufficiently small positive numbers as in the argument above and let Ψ be also the same pseudo-volume element on $M \setminus D$ as in A.4. Let $\widetilde{U} \to U^\star$ be the universal covering. Then $\widetilde{U}$ is biholomorphic to a unit polydisc U. By applying the previous argument and A.5 to the composite $\widetilde{U} \to U^\star \cdots\to M$, we have $f^*\Psi \leq \Omega^{\star \mathrm{P}}$. In particular,

$$s_n \left(f^*(\eta_i) \wedge \overline{f^*(\eta_i)}\right)^{\frac{1}{m}} \leq \frac{s_n \left(f^*(\eta_i) \wedge \overline{f^*(\eta_i)}\right)^{\frac{1}{m}}}{\prod_{k=1}^{l} \left(\log \|\delta_k\|^2\right)^2 \|\delta_k\|^{2\beta_k}} \leq \frac{1}{c}\Omega^{\star \mathrm{P}}.$$

Thus

$$(*) \qquad s_n \int_{U^\star} \left(f^*(\eta_i) \wedge \overline{f^*(\eta_i)}\right)^{\frac{1}{m}} < +\infty.$$

There is a meromorphic function b_i on $U^\star$ such that

$$f^*\eta_i = b_i dw_1 \wedge dw_2 \wedge \cdots \wedge dw_n.$$

By the same argument as in Lemma 4 in Addendum of [KO1], Lemma 2.6.1 of [NO], the inequality $(*)$ implies that b_i extends to a meromorphic function on U. Since the meromorphic mapping $f\colon U^\star \cdots\to M$ is given by $(b_0 : b_1 : \cdots : b_n)$, it extends to a meromorphic mapping $U \cdots\to M$. □

Appendix B. Proof of Theorem 2.1

(1). We shall show $\sqrt{-1}\,\partial\,\overline{\partial}\,\Psi$ is a Kähler metric. Starting from the calculation

$$\overline{\partial}\log\det(\operatorname{Im}\tau) = \operatorname{Tr}((\operatorname{Im}\tau)^{-1}\,\overline{\partial}(\operatorname{Im}\tau)),$$

we have

$$\begin{aligned}\partial\,\overline{\partial}\log\det(\operatorname{Im}\tau) &= \operatorname{Tr}(-(\operatorname{Im}\tau)^{-1}\,\partial(\operatorname{Im}\tau)\wedge(\operatorname{Im}\tau)^{-1}\,\overline{\partial}(\operatorname{Im}\tau)) + \operatorname{Tr}((\operatorname{Im}\tau)^{-1}\,\partial\,\overline{\partial}(\operatorname{Im}\tau))\\ &= -\frac{1}{4}\operatorname{Tr}((\operatorname{Im}\tau)^{-1}\,\mathrm{d}\,\tau\wedge(\operatorname{Im}\tau)^{-1}\,\mathrm{d}\,\overline{\tau}).\end{aligned}$$

Let P be a real symmetric positive definite matrix such that $P^2 = \operatorname{Im}\tau$. For any complex symmetric matrix η, we have

$$\operatorname{Tr}((\operatorname{Im}\tau)^{-1}\eta(\operatorname{Im}\tau)^{-1}\overline{\eta}) = \operatorname{Tr}(P^{-1}\eta P^{-1}P^{-1}\overline{\eta}P^{-1}) = \operatorname{Tr}(P^{-1}\eta P^{-1}\,{}^{\mathrm{t}}\overline{(P^{-1}\eta P^{-1})}) \geq 0,$$

where equality holds only when $\eta = 0$. Thus $-\sqrt{-1}\,\partial\,\overline{\partial}\log\det\operatorname{Im}\tau$ is a Kähler form on $\mathbb{H}_g$. Next, we consider $(\operatorname{Im} z)(\operatorname{Im}\tau)^{-1}(\operatorname{Im}\,{}^{\mathrm{t}}z)$. We have $\overline{\partial}((\operatorname{Im} z)(\operatorname{Im}\tau)^{-1}(\operatorname{Im}\,{}^{\mathrm{t}}z)) = A + B$, where

$$A = \sqrt{-1}(\operatorname{Im} z)(\operatorname{Im}\tau)^{-1}\,\mathrm{d}\,{}^{\mathrm{t}}\overline{z}, \quad B = -\frac{\sqrt{-1}}{2}(\operatorname{Im} z)(\operatorname{Im}\tau)^{-1}\,\mathrm{d}\,\overline{\tau}(\operatorname{Im}\tau)^{-1}(\operatorname{Im}\,{}^{\mathrm{t}}z).$$

Further $\partial\,A = A_1 + A_2$, where

$$A_1 = \frac{1}{2}\,\mathrm{d}\,z(\operatorname{Im}\tau)^{-1}\wedge\mathrm{d}\,{}^{\mathrm{t}}\overline{z}, \quad A_2 = -\frac{1}{2}(\operatorname{Im} z)(\operatorname{Im}\tau)^{-1}\,\mathrm{d}\,\tau(\operatorname{Im}\tau)^{-1}\wedge\mathrm{d}\,{}^{\mathrm{t}}\overline{z}.$$

Similarly, $\partial\,B = B_1 + B_2 + B_3 + B_4$, where

$$\begin{aligned}B_1 &= -\frac{1}{4}\,\mathrm{d}\,z\wedge(\operatorname{Im}\tau)^{-1}\,\mathrm{d}\,\overline{\tau}(\operatorname{Im}\tau)^{-1}(\operatorname{Im}\,{}^{\mathrm{t}}z),\\ B_2 &= \frac{1}{4}(\operatorname{Im} z)(\operatorname{Im}\tau)^{-1}\,\mathrm{d}\,\tau(\operatorname{Im}\tau)^{-1}\wedge\mathrm{d}\,\overline{\tau}(\operatorname{Im}\tau)^{-1}(\operatorname{Im}\,{}^{\mathrm{t}}z),\\ B_3 &= -\frac{1}{4}(\operatorname{Im} z)(\operatorname{Im}\tau)^{-1}\,\mathrm{d}\,\overline{\tau}\wedge(\operatorname{Im}\tau)^{-1}\,\mathrm{d}\,\tau(\operatorname{Im}\tau)^{-1}(\operatorname{Im}\,{}^{\mathrm{t}}z)\\ &= \frac{1}{4}(\operatorname{Im} z)(\operatorname{Im}\tau)^{-1}\,\mathrm{d}\,\tau(\operatorname{Im}\tau)^{-1}\wedge\mathrm{d}\,\overline{\tau}(\operatorname{Im}\tau)^{-1}(\operatorname{Im}\,{}^{\mathrm{t}}z),\\ B_4 &= \frac{1}{4}(\operatorname{Im} z)(\operatorname{Im}\tau)^{-1}\,\mathrm{d}\,\overline{\tau}(\operatorname{Im}\tau)^{-1}\wedge\mathrm{d}\,{}^{\mathrm{t}}z\\ &= -\frac{1}{4}\,\mathrm{d}\,z(\operatorname{Im}\tau)^{-1}\wedge\mathrm{d}\,\overline{\tau}(\operatorname{Im}\tau)^{-1}(\operatorname{Im}\,{}^{\mathrm{t}}z).\end{aligned}$$

Here we see that $B_1 = B_4 = -(1/2)\overline{A_2}$ and $B_2 = B_3$. Therefore

$$\begin{aligned}\partial\,\overline{\partial}((\operatorname{Im} z)(\operatorname{Im}\tau)^{-1}(\operatorname{Im}\,{}^t z)) &= \frac{1}{2}\,\mathrm{d}\,z(\operatorname{Im}\tau)^{-1}\wedge \mathrm{d}\,{}^t\overline{z}\\ &+(-\frac{1}{2})(\operatorname{Im} z)(\operatorname{Im}\tau)^{-1}\,\mathrm{d}\,\tau(\operatorname{Im}\tau)^{-1}\wedge \mathrm{d}\,{}^t\overline{z}+(-\frac{1}{2})\,\mathrm{d}\,z\wedge(\operatorname{Im}\tau)^{-1}\,\mathrm{d}\,\overline{\tau}(\operatorname{Im}\tau)^{-1}(\operatorname{Im}\,{}^t z)\\ &+\frac{1}{2}(\operatorname{Im} z)(\operatorname{Im}\tau)^{-1}\,\mathrm{d}\,\tau(\operatorname{Im}\tau)^{-1}\wedge\mathrm{d}\,\overline{\tau}(\operatorname{Im}\tau)^{-1}(\operatorname{Im}\,{}^t z).\end{aligned}$$

Let η be a complex symmetric matrix of degree g and let ξ be a row vector in $\mathbb{C}^g$. Then

$$\begin{aligned}\partial\,\overline{\partial}((\operatorname{Im} z)(\operatorname{Im}\tau)^{-1}(\operatorname{Im}\,{}^t z))_{|(\eta,\xi)} &= \frac{1}{2}\xi(\operatorname{Im}\tau)^{-1}\,{}^t\overline{\xi}\\ &+(-\frac{1}{2})(\operatorname{Im} z)(\operatorname{Im}\tau)^{-1}\eta(\operatorname{Im}\tau)^{-1}\wedge\,{}^t\overline{\xi}+(-\frac{1}{2})\xi(\operatorname{Im}\tau)^{-1}\overline{\eta}(\operatorname{Im}\tau)^{-1}(\operatorname{Im}\,{}^t z)\\ &+\frac{1}{2}(\operatorname{Im} z)(\operatorname{Im}\tau)^{-1}\eta(\operatorname{Im}\tau)^{-1}\overline{\eta}(\operatorname{Im}\tau)^{-1}(\operatorname{Im}\,{}^t z)\end{aligned}$$

and this is equal to

$$\frac{1}{2}(\xi P^{-1},(\operatorname{Im} z)(\operatorname{Im}\tau)^{-1}\eta P^{-1})\begin{pmatrix}1_g & -1_g\\ -1_g & 1_g\end{pmatrix}\begin{pmatrix}P^{-1}\,{}^t\overline{\xi}\\ P^{-1}\overline{\eta}(\operatorname{Im}\tau)^{-1}(\operatorname{Im}\,{}^t z)\end{pmatrix}\geq 0,$$

where P is the same symmetric matrix as before. This is equal to zero only when $\xi = (\operatorname{Im} z)(\operatorname{Im}\tau)^{-1}\eta$. Therefore $\sqrt{-1}\,\partial\,\overline{\partial}\,\Psi$ is a Kähler form.

(2). Let $a = a_\gamma$, $b = b_\gamma$, $c = c_\gamma$, and $d = d_\gamma$. First of all, we note that $\operatorname{Im}((a\tau+b)(c\tau+d)^{-1}) = {}^t(c\tau+d)^{-1}\operatorname{Im}\tau(c\overline{\tau}+d)^{-1}$. Thus

$$\begin{aligned}-\frac{1}{4\pi}\log\det\operatorname{Im}((a\tau+b)(c\tau+d)^{-1}) &= -\frac{1}{4\pi}\log\det({}^t(c\tau+d)^{-1}\operatorname{Im}\tau(c\overline{\tau}+d)^{-1})\\ &= -\frac{1}{4\pi}\log\det(\operatorname{Im}\tau)+\frac{1}{4\pi}\log|\det(c\tau+d)|^2\\ &= -\frac{1}{4\pi}\log\det(\operatorname{Im}\tau)+\operatorname{Im}(\frac{\sqrt{-1}}{2\pi}\log\det(c\tau+d)).\end{aligned}$$

Here is appearing the first term of $\operatorname{Im} h_{\gamma,m,n}(\tau,z)$. By using equalities

$$\begin{aligned}\operatorname{Im}(z(c\tau+d)^{-1}) &= (\operatorname{Im} z)(c\tau+d)^{-1}+\overline{z}\operatorname{Im}((c\tau+d)^{-1}),\\ \operatorname{Im}((c\tau+d)^{-1})(c\tau+d) &= -(c\overline{\tau}+d)^{-1}c\operatorname{Im}\tau,\end{aligned}$$

and their conjugates, we have

$$\begin{aligned}
&\operatorname{Im}(z(c\tau+d)^{-1})(\operatorname{Im}(a\tau+b)(c\tau+d)^{-1})^{-1}\operatorname{Im}{}^{\mathrm{t}}(z(c\tau+d)^{-1})\\
&=\big((\operatorname{Im}z)(c\tau+d)^{-1}+\overline{z}\operatorname{Im}((c\tau+d)^{-1})\big)(c\tau+d)(\operatorname{Im}\tau)^{-1}{}^{\mathrm{t}}(c\overline{\tau}+d)\operatorname{Im}{}^{\mathrm{t}}(z(c\tau+d)^{-1})\\
&=(\operatorname{Im}z)(\operatorname{Im}\tau)^{-1}{}^{\mathrm{t}}(c\overline{\tau}+d)\,{}^{\mathrm{t}}\big(z\operatorname{Im}((c\tau+d)^{-1})+(\operatorname{Im}z)(c\overline{\tau}+d)^{-1}\big)\\
&\quad+\overline{z}\operatorname{Im}((c\tau+d)^{-1})(c\tau+d)(\operatorname{Im}\tau)^{-1}{}^{\mathrm{t}}(c\overline{\tau}+d)\,{}^{\mathrm{t}}\big(z\operatorname{Im}((c\tau+d)^{-1})+(\operatorname{Im}z)(c\overline{\tau}+d)^{-1}\big)\\
&=(\operatorname{Im}z)(\operatorname{Im}\tau)^{-1}(\operatorname{Im}{}^{\mathrm{t}}z)+C_1+C_2+C_3,
\end{aligned}$$

where we denote

$$\begin{aligned}
C_1&=(\operatorname{Im}z)(\operatorname{Im}\tau)^{-1}{}^{\mathrm{t}}(c\overline{\tau}+d)\,{}^{\mathrm{t}}\operatorname{Im}((c\tau+d)^{-1})\,{}^{\mathrm{t}}z=-z(c\tau+d)^{-1}c(\operatorname{Im}{}^{\mathrm{t}}z),\\
C_2&=\overline{z}\operatorname{Im}((c\tau+d)^{-1})(c\tau+d)(\operatorname{Im}\tau)^{-1}(\operatorname{Im}{}^{\mathrm{t}}z)=-\overline{z}(c\overline{\tau}+d)^{-1}c(\operatorname{Im}{}^{\mathrm{t}}z),\\
C_3&=\overline{z}(\operatorname{Im}\tau)^{-1}(c\tau+d)(\operatorname{Im}\tau)^{-1}{}^{\mathrm{t}}(c\overline{\tau}+d)\,{}^{\mathrm{t}}\operatorname{Im}((c\tau+d)^{-1})\,{}^{\mathrm{t}}z\\
&=\overline{z}(c\overline{\tau}+d)^{-1}c(\operatorname{Im}\tau)\,{}^{\mathrm{t}}c\,{}^{\mathrm{t}}(c\tau+d)^{-1}\,{}^{\mathrm{t}}z.
\end{aligned}$$

We have $(c\tau+d)\,{}^{\mathrm{t}}c=c\,{}^{\mathrm{t}}(c\tau+d)$ by $d\,{}^{\mathrm{t}}c=c\,{}^{\mathrm{t}}d$. Thus $(c\tau+d)^{-1}c$ is a symmetric matrix. In fact,

$$(c\tau+d)\,{}^{\mathrm{t}}c\,{}^{\mathrm{t}}(c\tau+d)^{-1}=c\,{}^{\mathrm{t}}(c\tau+d)\,{}^{\mathrm{t}}(c\tau+d)^{-1}=c.$$

Therefore

$$\begin{aligned}
(c\overline{\tau}+d)^{-1}c-(c\tau+d)^{-1}c&=(c\overline{\tau}+d)^{-1}(c\,{}^{\mathrm{t}}(c\tau+d)-(c\overline{\tau}+d)\,{}^{\mathrm{t}}c)\,{}^{\mathrm{t}}(c\tau+d)^{-1}\\
&=2\sqrt{-1}(c\overline{\tau}+d)^{-1}c(\operatorname{Im}\tau)\,{}^{\mathrm{t}}c\,{}^{\mathrm{t}}(c\tau+d)^{-1}.
\end{aligned}$$

We can write $C_1=D_1+D_2$, $C_2=D_3+D_4$, where

$$\begin{aligned}
D_1&=\frac{\sqrt{-1}}{2}z(c\tau+d)^{-1}c\,{}^{\mathrm{t}}z, & D_2&=(-\frac{\sqrt{-1}}{2})z(c\tau+d)^{-1}c\,{}^{\mathrm{t}}\overline{z},\\
D_3&=\frac{\sqrt{-1}}{2}\overline{z}(c\overline{\tau}+d)^{-1}c\,{}^{\mathrm{t}}z, & D_4&=(-\frac{\sqrt{-1}}{2})\overline{z}(c\overline{\tau}+d)^{-1}c\,{}^{\mathrm{t}}\overline{z}.
\end{aligned}$$

Hence

$$\begin{aligned}
D_2+D_4&=\frac{\sqrt{-1}}{2}\overline{z}((c\overline{\tau}+d)^{-1}c-(c\tau+d)^{-1}c)\,{}^{\mathrm{t}}z\\
&=-\overline{z}(c\overline{\tau}+d)^{-1}c(\operatorname{Im}\tau)\,{}^{\mathrm{t}}c\,{}^{\mathrm{t}}(c\tau+d)^{-1}\,{}^{\mathrm{t}}z\\
&=-C_3.
\end{aligned}$$

Therefore, $C_1 + C_2 + C_3 = -\operatorname{Im}(z(c\tau+d)^{-1}c\,{}^t\!z)$. Hence

$$\Psi((\gamma,0,0)\cdot(\tau,z)) - \Psi(\tau,z) = \operatorname{Im} h_{\gamma,0,0}(\tau,z).$$

For $(\gamma,m,n) = (\gamma,0,0)(1,m,n)$, we note that

$$\begin{aligned}\Psi((\gamma,m,n)\cdot(\tau,z)) - \Psi(\tau,z) &= \Psi((\gamma,0,0)(\tau,z+m\tau+n)) - \Psi(\tau,z+m\tau+n)\\ &\quad + \Psi(\tau,z+m\tau+n) - \Psi(\tau,z)\\ &= \operatorname{Im} h_{\gamma,0,0}(\tau,z+m\tau+n) + \Psi(\tau,z+m\tau+n) - \Psi(\tau,z).\end{aligned}$$

The last two terms are calculated by

$$\begin{aligned}&\Psi(\tau,z+m\tau+n) - \Psi(\tau,z)\\ &\quad= \frac{1}{2}(\operatorname{Im} z + m\operatorname{Im}\tau)(\operatorname{Im}\tau)^{-1}\,{}^t\!(\operatorname{Im} z + m\operatorname{Im}\tau) - \frac{1}{2}(\operatorname{Im} z)(\operatorname{Im}\tau)^{-1}\,{}^t\!(\operatorname{Im} z)\\ &\quad= (\operatorname{Im} z)\,{}^t\!m + \frac{1}{2}m(\operatorname{Im}\tau)\,{}^t\!m\\ &\quad= \operatorname{Im}(z\,{}^t\!m + \frac{1}{2}m\tau\,{}^t\!m).\end{aligned}$$

For the first term, we have

$$h_{\gamma,m,n}(\tau,z) = h_{\gamma,0,0}(\tau,z+m\tau+n) + z\,{}^t\!m + \frac{1}{2}m\tau\,{}^t\!m.$$

Hence we have the expected formula

$$\Psi((\gamma,m,n)\cdot(\tau,z)) = \Psi(\tau,z) + \operatorname{Im} h_{\gamma,m,n}(\tau,z).$$

(3). Let us define a holomorphic function

$$h^\sharp_{\gamma,m,n}(\tau,z) := h_{\gamma,m,n}(\tau,z) - \frac{\sqrt{-1}}{4\pi}\log\det(c_\gamma\tau + d_\gamma).$$

Then for $u = (\gamma_1,m_1,n_1)$ and $v = (\gamma_2,m_2,n_2)$, we see that

$$r(u,v) := h^\sharp_u(v\cdot(\tau,z)) + h^\sharp_v(\tau,z) - h^\sharp_{uv}(\tau,z)$$

is a real number and that the collection $\{r(u,v)\}$ forms a cocycle in $Z^2(\mathrm{Sp}(g,\mathbb{R})\ltimes\mathbb{R}^{2g},\mathbb{R})$ from the calculation in (2) above. We have only to express $r(u,v)$ in terms of $\gamma_1,\gamma_2,m_1,m_2,n_1,n_2$. We have

$$h^\sharp_{\gamma,0,0}(\tau,z) = -\frac{1}{2}z(c_\gamma\tau+d_\gamma)^{-1}c_\gamma\,{}^t\!z, \quad \text{and} \quad h^\sharp_{1,m,n}(\tau,z) = z\,{}^t\!m_1 + \frac{1}{2}m_1 z\,{}^t\!m_1.$$

Let γ_1 and γ_2 be two elements of $\mathrm{Sp}(g,\mathbb{R})$. Then

$$h^\sharp_{\gamma_1,0,0}((\gamma_2,0,0)\cdot(\tau,0)) = h^\sharp_{\gamma_2,0,0}(\tau,0) = h^\sharp_{\gamma_1\gamma_2,0,0}(\tau,0) = 0.$$

Thus $r((\gamma_1,0,0),(\gamma_2,0,0)) = 0$. For γ in $\mathrm{Sp}(g,\mathbb{R})$ and $(m,n)\in\mathbb{R}^{2g}$, we have

$$h^\sharp_{\gamma,0,0}((1,m,n)\cdot(\tau,z)) + h^\sharp_{1,m,n}(\tau,z) = h^\sharp_{\gamma,m,n}(\tau,z).$$

Hence $r((\gamma,0,0),(1,m,n)) = 0$. We set $(m_1,n_1) = (m,n)\cdot\gamma$. Then $(1,m,n)(\gamma,m,n) = (\gamma,m_1,n_1)$ and $m_1\tau + n_1 = m(a_\gamma\tau + b_\gamma) + n(c_\gamma\tau + d_\gamma)$. By using $1_g + c_\gamma\,{}^t\!b_\gamma = d_\gamma\,{}^t\!a_\gamma$, We have

$$\begin{aligned}
&r((1,m,n),(\gamma,0,0)) = h^\sharp_{1,m,n}((\gamma,0,0)\cdot(\tau,0)) + h^\sharp_{\gamma,0,0}(\tau,0) - h^\sharp_{\gamma,m_1,n_1}(\tau,0)\\
&\quad= \frac{1}{2}m(a_\gamma\tau+b_\gamma)(c_\gamma\tau+d_\gamma)^{-1}\,{}^t\!m + \frac{1}{2}(m_1\tau+n_1)(c_\gamma\tau+d_\gamma)^{-1}c_\gamma\,{}^t\!(m_1\tau+n_1) - \frac{1}{2}m_1\tau\,{}^t\!m_1\\
&\quad= \frac{1}{2}m(a_\gamma\tau+b_\gamma)(c_\gamma\tau+d_\gamma)^{-1}(\,{}^t\!m + c_\gamma\,{}^t\!(m_1\tau+n_1)) + \frac{1}{2}nc_\gamma\,{}^t\!(m_1\tau+n_1) - \frac{1}{2}m_1\tau\,{}^t\!m_1\\
&\quad= \frac{1}{2}m(a_\gamma\tau+b_\gamma)(\,{}^t\!a_\gamma\,{}^t\!m + {}^t\!c_\gamma\,{}^t\!n) + \frac{1}{2}nc_\gamma\,{}^t\!(m_1\tau+n_1) - \frac{1}{2}m_1\tau\,{}^t\!m_1\\
&\quad= \frac{1}{2}mb_\gamma\,{}^t\!m_1 + \frac{1}{2}nc_\gamma\,{}^t\!n_1\\
&\quad= \frac{1}{2}ma_\gamma\,{}^t\!b_\gamma\,{}^t\!m + mb_\gamma\,{}^t\!c_\gamma\,{}^t\!n + \frac{1}{2}nc_\gamma\,{}^t\!d_\gamma\,{}^t\!n\\
&\quad= \frac{1}{2}(ma_\gamma + nc_\gamma)\,{}^t\!(mb_\gamma + nd_\gamma).
\end{aligned}$$

Let m_1,n_1,m_2,n_2 be vectors in $\mathbb{R}^g$. Then

$$\begin{aligned}
&r((1,m_1,n_1),(1,m_2,n_2))\\
&\quad= h^\sharp_{1,m_1,n_1}(\tau,m_2\tau+n_2) + h^\sharp_{1,m_2,n_2}(\tau,0) - h^\sharp_{1,m_1+m_2,n_1+n_2}(\tau,0)\\
&\quad= (m_2\tau+n_2)\,{}^t\!m_1 + \frac{1}{2}m_1\tau\,{}^t\!m_1 + \frac{1}{2}m_2\tau\,{}^t\!m_2 - \frac{1}{2}(m_1+m_2)\tau\,{}^t\!(m_1+m_2)\\
&\quad= n_2\,{}^t\!m_1 = m_1\,{}^t\!n_2.
\end{aligned}$$

Therefore we have the following equalities:

$$\begin{aligned}
r((\gamma_1,0,0),(\gamma_2,0,0)) = r((\gamma,0,0),(1,m,n)) &= 0\\
r((1,m,n),(\gamma,0,0)) &= \frac{1}{2}(ma_\gamma + nc_\gamma)\,{}^t\!(mb_\gamma + nd_\gamma)\\
r((1,m_1,n_1),(1,m_2,n_2)) &= m_1\,{}^t\!n_2.
\end{aligned}$$

For u, v, w in $\mathrm{Sp}(g, \mathbb{R}) \ltimes \mathbb{R}^{2g}$, we have

$$r(uv, w) - r(u, vw) = r(v, w) - r(u, v).$$

Since $(\gamma, m, n) = (\gamma, 0, 0)(1, m, n)$, $r((\gamma_1, 0, 0), (\gamma_2, m, n))$ is equal to

$$r((\gamma_1\gamma_2, 0, 0), (1, m, n)) - r((\gamma_2, 0, 0), (1, m, n)) + r((\gamma_1, 0, 0), (\gamma_2, 0, 0)) = 0,$$

for $\gamma_1, \gamma_2 \in \mathrm{Sp}(g, \mathbb{R})$ and $m, n \in \mathbb{R}^g$. Therefore

$$\begin{aligned}
&r((\gamma_1, m_1, n_1), (\gamma_2, m_2, n_2)) \\
&= r((\gamma_1, 0, 0), (1, m_1, n_1)(\gamma_2, m_2, n_2)) + r((1, m_1, n_1), (\gamma_2, m_2, n_2)) \\
&\quad - r((\gamma_1, 0, 0), (1, m_1, n_1)) \\
&= r((1, m_1, n_1)(\gamma_2, 0, 0), (1, m_2, n_2)) - r((\gamma_2, 0, 0), (1, m_2, n_2)) + r((1, m_1, n_1), (\gamma_2, 0, 0)) \\
&= r((\gamma_2, 0, 0), (1, (m_1, n_1) \cdot \gamma_2 + (m_2, n_2))) + r((1, (m_1, n_1)\gamma_2), (1, m_2, n_2)) \\
&\quad - r((\gamma_2, 0, 0), (1, (m_1, n_1) \cdot \gamma)) + r((1, m_1, n_1), (\gamma_2, 0, 0)) \\
&= r((1, (m_1, n_1) \cdot \gamma_2), (1, m_2, n_2)) + r((1, m_1, n_1), (\gamma_2, 0, 0)) \\
&= (m_1 a_{\gamma_2} + n_1 c_{\gamma_2})\, {}^{\mathrm{t}}n_2 + \frac{1}{2}(m_1 a_{\gamma_2} + n_1 c_{\gamma_2})\, {}^{\mathrm{t}}(m_1 b_{\gamma_2} + n_1 d_{\gamma_2}) \\
&= \frac{1}{2}(m_1 a_{\gamma_2} + n_1 c_{\gamma_2})\, {}^{\mathrm{t}}(m_1 b_{\gamma_2} + n_1 d_{\gamma_2} + 2n_2).
\end{aligned}$$

This is just the expected expression.

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