AN ESTIMATION ON THE NUMBER OF RATIONAL VALUES RELATED TO G-FUNCTIONS.

Makoto NAGATA

Research Institute for Mathematical Sciences, Kyoto University

Introduction

Let K be an algebraic number field of finite degree. We recall the definition of G-function [1].

Definition.

We call $f(z)$ *G-function* if it is satisfying the following conditions:

(i) $f(z)$ has a Taylor expansion at the origin in $K[[z]]$; $f(z) = \sum_{i=0}^{\infty} a_i z^i \in K[[z]]$.

(ii) There exists a constant c_0 for $i = 0, 1, 2, \ldots$ such that the maximal absolute values of a_i and its conjugates $\leq c_0^i$, and such that the common denominator ($\in \mathbb{N}$) of $a_0, \ldots, a_i \leq c_0^i$.

(iii) $f(z)$ satisfies a linear differential equation whose coefficients are in $K(z)$.

The logarithm $\log(1-z) = -\sum_{i=1}^{\infty} z^i/i$, polylogarithms $L_k(z) = \sum_{i=1}^{\infty} z^i/i^k$ with $k \in \mathbb{N}$, Gauss' hypergeometric series ${}_2F_1(a,b,c;z) = \sum_{i=0}^{\infty}((a)_i(b)_i/(c)_i i!)z^i$ with $a, b, c \in \mathbb{Q}$, and algebraic functions $\in K[[z]]$ over $\mathbb{Q}$ are G-functions. See [1].

In this paper, our motivation is the following question:

Question.

How many rational values of G-functions in an interval exist?

Here, an interval means an interval on $\mathbb{Q}$, which is not so short, for example, $[0,1] \cap \mathbb{Q}$.

Since algebraic functions are G-functions, the number of rational values of G-functions in an interval is not bounded in general. So our question expects some estimations on the number.

We know that there are some results on arithmetic properties of G-functions ([2], [3], [5], etc.) Unfortunately, it seems there is a gap between known results and our question.

The explanation of the gap will be given later, we first state our results. Following Theorem 1 and Theorem 7 are our main results, Corollary 2 and Corollary 4 are estimations for our question.

Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX

Let q be a natural number. We put

$$U_r(q) := \{\zeta \in \mathbb{Q} \ ; \ |\zeta| < r \text{ and } q\zeta \in \mathbb{Z}\}.$$

Especially, elements in $U_r(q)$ do not need to be irreducible fractions.

Now we introduce a terminology: *projectively $\mathbb{Q}$-rational.*

Let $y = y(z)$ be a vector of holomorphic functions, and suppose that y is convergent at $\zeta \in \mathbb{C}$.

We say y is *projectively $\mathbb{Q}$-rational at ζ* if all ratios of coordinates of $y(\zeta)$ are rational numbers.

In particular, if all coordinates of $y(\zeta)$ are rational numbers, y is projectively $\mathbb{Q}$-rational at ζ.

Theorem 1.

Let m, n be natural numbers with $n \geq m \geq 6$, $A \in M_n(\mathbb{Q}(z))$. We suppose that y is a non-trivial vector solution of the differential equation

$$(\frac{d}{dz} - A)y = 0$$

and its coordinates are holomorphic at the origin with radius of convergence $R > 0$. Moreover we assume that $(d/dz - A)$ is a G-operator and that at least m coordinates of y are linearly independent over $\mathbb{Q}(z)$. Fix a real number r with $0 < r < R$ such that there exist no poles and no zeros of A on the circle $\{\zeta \in \mathbb{C} \ ; \ |\zeta| = r\}$. Then there exist effective positive constants c_1 and c_2, depending only on A, m, n, r and y, satisfying the following:

For a natural number $q > c_1$, if there exists distinct l numbers in $U_r(q)$, $\zeta_0, \zeta_1, \dots,$ ζ_{l-1} except zeros and poles of A, at which y is projectively $\mathbb{Q}$-rational with

$$|\zeta_i - \zeta_j| \leq q^{-35/6m} \quad (i, j = 0, 1, \dots, l-1),$$

then $l < c_2$. (Here c_1, c_2 are independent of q and ζ_i.)

Since an interval with its length $2r$ is covered by $[2rq^{35/6m}] + 1$ intervals with their length $q^{-35/6m}$, we obtain the following, which is a nontrivial estimation $35/6m < 1$ for our question.

Corollary 2.

Under the assumptions in Theorem 1, we have

$$\overline{\lim_{q \to \infty}} \frac{\log \#\{\zeta \in U_r(q) \ ; y \text{ is projectively } \mathbb{Q}\text{-rational at } \zeta\}}{\log q} \leq \frac{35}{6m}.$$

Combining Corollary 2 with Theorem III of [3], we obtain the following.

Example 3.

Let f be a Gauss' hypergeometric series with rational parameters. If 1, f, f', f^2, f'^2, $f'f$ are linearly independent over $\mathbb{Q}(z)$, then for any real number r smaller than the radius of convergence of f, we have

$$\overline{\lim_{q \to \infty}} \frac{\log \#\{\zeta \in U_r(q) \ ; \ f(\zeta), f'(\zeta) \in \mathbb{Q}\}}{\log q} \leq \frac{35}{36}.$$

Now let f satisfy a linear differential equation of order n with coefficients in $\mathbb{Q}(z)$.

By considering $y = (1, f, f', f^{(2)}, \dots, f^{(n-1)}, f^2, ff', ff^{(2)}, \dots, f^3, \dots)$, we obtain the following since Corollary 2 is independent result of the *size* of A. (See in [1] for the definition of *size*.)

Corollary 4.

Let f satisfy a linear differential equation of order n with coefficients in $\mathbb{Q}(z)$.

We suppose that f is a G-function, holomorphic on a disk at origin with radius $R > 0$ and f is algebraically independent over $\mathbb{Q}(z)$. Then for any $r < R$, we have

$$\overline{\lim_{q\to\infty}} \frac{\log \#\{\zeta \in U_r(q) \ ; \ f^{(i)}(\zeta) \in \mathbb{Q}, \ i = 0, \dots, n-1\}}{\log q} = 0.$$

Example 5.

For $r < 1$, we have

$$\overline{\lim_{q\to\infty}} \frac{\log \#\{\zeta \in U_r(q) \ ; \ \log(1-\zeta) \in \mathbb{Q}\}}{\log q} = 0.$$

Above Example 5 is obvious by Lindemann Theorem, but our proof of this does not require properties of the exponential function e^z.

Remark 6.

Let f be a G-function and satisfy a linear differential equation of order n with coefficients in $\mathbb{Q}(z)$. If

$$\overline{\lim_{q\to\infty}} \frac{\log \#\{\zeta \in U_r(q) \ ; \ f^{(i)}(\zeta) \in \mathbb{Q}, \ i = 0, \dots, n-1\}}{\log q} > 0,$$

then f is an algebraic function (over $\mathbb{Q}$) by Corollary 4. Thus after confirming f is an algebraic function over $\mathbb{Q}$, then using the theory of algebraic curve in [9]*, pp.133, we obtain the following:*

If

$$\overline{\lim_{q\to\infty}} \frac{\log \#\{\zeta \in U_r(q) \ ; \ f(\zeta) \in \mathbb{Q}\}}{\log q} > \frac{2}{m},$$

then f is an algebraic function of genus 0 and its degree $< m$.

For the case of several variables, we obtain the following result.

The definition of G-operator on several variables will be given later. We write $z = (z_0, z_1, \dots, z_{\mu-1})$ and $\mathbf{i} = (i_0, i_1, \dots, i_{\mu-1}) \in (\mathbb{Z}_{\geq 0})^\mu$.

Theorem 7.

Let μ, m, n be natural numbers with $n \geq m \geq (1+\sqrt{2}\mu)(\sqrt{2}+\mu)^\mu$, $A_\lambda \in M_n(\mathbb{Q}(z))$ for $\lambda = 0, 1, \dots, \mu-1$.

We suppose that y is a non-trivial vector solution of the system of differential equations

$$(\text{eq.1}) \qquad (\frac{\partial}{\partial z_\lambda} - A_\lambda)y = 0 \qquad (\lambda = 0, 1, \dots, \mu-1)$$

and its coordinates are holomorphic on a polydisk at the origin with radius $R > 0$.

Moreover we assume that the connection (eq.1) *is a G-operator and that at least m coordinates of y are linearly independent over $\mathbb{Q}(z)$.*

For each $\lambda = 0, 1, \dots, \mu - 1$, fix a real number r_λ with $0 < r_\lambda < R$ such that there exist no poles and no zeros of A_λ on the torus $\prod_{\tau=0}^{\mu-1}\{\zeta_\tau \in \mathbb{C} \ ; \ |\zeta_\tau| = r_\tau\}$ and fix a real number ϵ with

$$\epsilon > (\sqrt{2} + \mu)(\frac{1 + \sqrt{2}\mu}{m})^{1/\mu}.$$

Then there exists effective positive constants c_3 and c_4, depending only on μ, m, n, A_λ, y, r_λ and ϵ, satisfying the following:

For a natural number $q > c_3$, let

$$S_\lambda := \{\zeta_i \in U_{r_\lambda}(q) \ ; \ i = 0, 1, \dots, l-1, \ |\zeta_i - \zeta_j| \leq q^{-\epsilon}\} \qquad (\lambda = 0, 1, \dots, l-1),$$

and assume that any of $\zeta_{\mathbf{i}} \in S_0 \times S_1 \times \cdots \times S_{\mu-1}$ is neither zero nor pole of A_λ for $\lambda = 0, 1, \dots, \mu - 1$.

If there exists $\{\alpha_{\mathbf{i}}\} \subset \mathbb{C} \setminus \{0\}$ for $\zeta_{\mathbf{i}} \in S_0 \times S_1 \times \cdots \times S_{\mu-1}$ such that

$$\alpha_{\mathbf{i}} y(\zeta_{\mathbf{i}}) \in \mathbb{Q}^n,$$

then $l < c_4$. (Here c_3, c_4 are independent of q, ζ_i and $\alpha_{\mathbf{i}}$.)

Theorem 1 is the case of $\mu = 1, \epsilon = 35/6m$ in Theorem 7.

We here explain the gap between known arithmetic properties of G-functions and our question. In order to explain this, we quote [4], pp.14. Almost all known results on arithmetic properties of G-functions seem some variants of the following:

> $\cdots$, The explanation lies in the fact that in the G-function method, in order to show that the values $f(\xi)$, of a G-function $f(z)$ at algebraic $\xi \neq 0$ satisfying a linear differential equation of order q, in not algebraic of degree D (irrational if $D = 1$), it is necessary to assume
>
> $$|\xi| < \exp(-C(D)(\log H(\xi))^{q/(q+1)}) \tag{0.1}$$
>
> for some (not effective!) $C(D) > 0$. $\cdots$

The main obstacle is the fact that $C(D)$ is not effective.

By (0.1), the above is a result on a very short interval depending on $C(D)$. In their proofs, one may find that the constant $C(D)$ depends on the coefficients of the Taylor expansion of $f(z)$ at the origin, that is, $C(D)$ depends on a point. Therefore $C(D)$ is not effective for a whole interval.

In the same paper, Chudnovsky states other kinds of results, which are concerning irrational (and transcendental) measures of values of G-function without the assumption (0.1). Although his results do not need the assumption (0.1), they need the same assumptions of so-called Schneider-Lang theorem. See [4].

Schneider-Lang theorem is a result concerning the number of algebraic values of meromorphic functions. The original [7] and its generalization [12] are results about meromorphic functions defined on a Riemann surface whose complement in a compact Riemann surface is a finite set. They need also an assumption that the functions are

algebraically independent. For further details, one can refer [7], [12], [6], [13]. Now we recall examples of G-functions. One finds immediately that the assumptions of Schneider-Lang theorem are too strong for our question.

These facts may show that there is a gap between known results and our question.

Our results in this paper, briefly speaking, are variants of Schneider-Lang theorem. Our method depends on some characteristic properties of G-operator (its definition in below), however it is an analogue of the method of the original Schneider-Lang theorem [7].

Here we explain the difference between the method of the original Schneider-Lang theorem and ours. The original method is, as the first step, to consider an auxiliary function, and the second, to take r, the radius of convergence of the function, to the infinity. Our method is, as the first, similar to the original one, but, the second, to take q, q is in $U_r(q)$, to a very large number.

Our method seems classical, however, without the strong assumptions of Schneider-Lang theorem, it deduces some estimations on the number of rational values of G-functions. Although this method also forces a restricted result on a very short interval like as Theorem 1, it is independent of the position of the interval. Therefore we obtain above Corollary 2, it is free from the non-effective constant $C(D)$ in (0.1).

The rest of this paper is dedicated to the proof of Theorem 7. Before the proof, we consider some properties of G-operators. After some preparations, we prove Theorem 7.

We note that our proof of Theorem 7 does not use an assumption that all coordinates of y are G-functions. It is enough to assume that y is a solution of a G-operator.

Notations

Let $z = (z_0, z_1, \dots, z_{\mu-1})$. For $\mathbf{k} = (k_0, k_1, \dots, k_{\mu-1}) \in (\mathbb{Z}_{\geq 0})^{\mu}$, we use the following notations:

$$\mathbf{k}! := \prod_{\lambda=0}^{\mu-1} k_\lambda !, \qquad ||\mathbf{k}|| := \sum_{\lambda=0}^{\mu-1} k_\lambda ,$$

$$z^{\mathbf{k}} := \prod_{\lambda=0}^{\mu-1} z_\lambda^{k_\lambda}, \qquad (\frac{\partial}{\partial z})^{\mathbf{k}} := \prod_{\lambda=0}^{\mu-1} (\frac{\partial}{\partial z_\lambda})^{k_\lambda},$$

$$A_\lambda^{(k_\lambda)} := (\frac{\partial}{\partial z_\lambda} + {}^t A_\lambda)^{k_\lambda}, \qquad A^{(\mathbf{k})} := A_0^{(k_0)} A_1^{(k_1)} \cdots A_{\mu-1}^{(k_{\mu-1})}.$$

Then $A_\lambda^{(n+1)} = A_\lambda^{(1)} A_\lambda^{(n)}$ and if $A_i^{(1)} A_j^{(1)} = A_j^{(1)} A_i^{(1)}$ (i.e., if the connection $(\partial/\partial_\lambda - A_\lambda)$ is integrable), one finds that $A^{(\mathbf{k}+\mathbf{k}')} = A^{(\mathbf{k})} A^{(\mathbf{k}')}$.

Let $\mathbf{e}_\lambda$ be the element in $(\mathbb{Z}_{\geq 0})^{\mu}$ with the λ-th coordinate is 1, the rest are 0, and let $\mathbf{1} = (1, 1, \dots, 1)$, $L\mathbf{1} = (L, L, \dots, L)$ for $L \in \mathbb{Z}_{\geq 0}$. We write that $\mathbf{k} \leq \mathbf{k}'$ if $k_\lambda \leq k'_\lambda$ for each $\lambda = 0, 1, \dots, \mu - 1$.

G-OPERATORS

We here recall G-operator (or (G,C)-property, G-connection) on several variables. See [1], [3], etc.

Let I be the identity matrix in $M_n(\mathbb{Q}[z])$.

Definition 2.1.

Let $d \in \mathbb{Z}[z]$ be the common denominator such that $dA_\lambda \in M_n(\mathbb{Z}[z])$ for $\lambda = 0, 1, \ldots, \mu-1$.

We call a connection $(\partial/\partial z_\lambda - A_\lambda)$ $(\lambda = 0, 1, \ldots, \mu-1)$ *G-operator* if it it integrable and if there exists a constant C_0 independent of i satisfying the following:

There exists a sequence $\{e_i\} \subset \mathbb{N}$ with $e_i \leq C_0^i$ such that

$$\frac{e_i d^{||\mathbf{k}||}}{\mathbf{k}!} A^{(\mathbf{k})} I \in M_n(\mathbb{Z}[z])$$

for any $\mathbf{k}$ with $||\mathbf{k}|| \leq i$, and $i = 0, 1, \ldots$.

Proposition 2.2. *(See* [3], [8]*)*

Let B be any element in $M_n(C^\infty)$. For any $k_\lambda \in \mathbb{Z}_{\geq 0}$, we have

$$\frac{1}{k_\lambda!} A_\lambda^{(k_\lambda)} B = \sum_{i+j=k_\lambda} \frac{1}{i!}(A_\lambda^{(i)} I)\frac{1}{j!}(\frac{\partial}{\partial z_\lambda})^j B. \tag{2.2.1}$$

Proof.

Induction on k_λ.

The case of $k_\lambda = 0$, (2.2.1) is true.

For any B, $C \in M_n(C^\infty)$, one finds that

$$\begin{aligned} A_\lambda^{(1)}(B \cdot C) &= (\frac{\partial}{\partial z_\lambda} + {}^t A_\lambda)(B \cdot C) = (\frac{\partial}{\partial z_\lambda} + {}^t A_\lambda)B \cdot C + B \cdot \frac{\partial}{\partial z_\lambda} C \\ &= (A_\lambda^{(1)} B) \cdot C + B \cdot (\frac{\partial}{\partial z_\lambda} C). \end{aligned} \tag{2.2.2}$$

We assume that (2.2.1) is valid for k_λ. We shall prove that for $k_\lambda + 1$.

By (2.2.2), we have

$$\begin{aligned} &\frac{1}{(k_\lambda+1)!} A_\lambda^{(k_\lambda+1)} B = \frac{1}{k_\lambda+1} A_\lambda^{(1)} \frac{1}{k_\lambda!} A_\lambda^{(k_\lambda)} B \\ &= \frac{1}{k_\lambda+1} A_\lambda^{(1)} \sum_{i+j=k_\lambda} \frac{1}{i!}(A_\lambda^{(i)} I)\frac{1}{j!}(\frac{\partial}{\partial z_\lambda})^j B \\ &= \frac{1}{k_\lambda+1} \sum_{i+j=k_\lambda} \frac{1}{i!}(A_\lambda^{(1)} A_\lambda^{(i)} I)\frac{1}{j!}(\frac{\partial}{\partial z_\lambda})^j B + \frac{1}{i!}(A_\lambda^{(i)} I)\frac{1}{j!}(\frac{\partial}{\partial z_\lambda})^{j+1} B \\ &= \frac{1}{k_\lambda+1} \sum_{i+j=k_\lambda} \frac{1}{i!}(A_\lambda^{(i+1)} I)\frac{1}{j!}(\frac{\partial}{\partial z_\lambda})^j B + \frac{1}{i!}(A_\lambda^{(i)} I)\frac{1}{j!}(\frac{\partial}{\partial z_\lambda})^{j+1} B. \end{aligned} \tag{2.2.3}$$

For elements of a $\mathbb{Q}$-algebra, a_i, b_j, one finds in general that

$$(2.2.4) \qquad (k_\lambda+1)\sum_{i+j=k_\lambda+1}\frac{a_i}{i!}\frac{b_j}{j!}=\sum_{i+j=k_\lambda}(\frac{a_{i+1}}{i!}\frac{b_j}{j!}+\frac{a_i}{i!}\frac{b_{j+1}}{j!}).$$

Then we have

$$(2.2.3)=\sum_{i+j=k_\lambda+1}\frac{1}{i!}(A_\lambda^{(i)}I)\frac{1}{j!}(\frac{\partial}{\partial z_\lambda})^jB. \quad \square$$

Lemma 2.3.
Let B be any element in $M_n(C^\infty)$. Then for any $\mathbf{k}\in(\mathbb{Z}_{\geq 0})^\mu$, we have

$$(2.3.1) \qquad \frac{1}{\mathbf{k}!}A^{(\mathbf{k})}B=\sum_{\mathbf{i}+\mathbf{j}=\mathbf{k}}\frac{1}{\mathbf{i}!}(A^{(\mathbf{i})}I)\frac{1}{\mathbf{j}!}(\frac{\partial}{\partial z})^{\mathbf{j}}B.$$

Proof.
We show (2.3.1) by induction.
The case of any $\mathbf{k}=(0,\dots,0,k_{\mu-1})$ is true by Proposition 2.2.
We assume that the case of any $\mathbf{k}=(0,\dots,0,0,k_{\lambda+1},\dots,k_{\mu-1})$ is true for (2.3.1). For a $k_\lambda\geq 0$, we also assume that the case of $\mathbf{k}=(0,\dots,0,k_\lambda,k_{\lambda+1},\dots,k_{\mu-1})$ is valid for (2.3.1). Now we shall show that the case of $\mathbf{k}=(0,\dots,0,k_\lambda,k_{\lambda+1},\dots,k_{\mu-1})+\mathbf{e}_\lambda$ is valid.
We write

$$\alpha_{\mathbf{s}}=\frac{1}{\mathbf{s}!}A^{(\mathbf{s})}I, \qquad \beta_{\mathbf{t}}=\frac{1}{\mathbf{t}!}(\frac{\partial}{\partial z})^{\mathbf{t}}B.$$

Let $\mathbf{k}'=\mathbf{k}-(k_\lambda+1)\mathbf{e}_\lambda$. By our assumptions, we have

$$\frac{1}{(\mathbf{k}-\mathbf{e}_\lambda)!}A^{(\mathbf{k}-\mathbf{e}_\lambda)}B=\sum_{\mathbf{s}+\mathbf{t}=\mathbf{k}',i+j=k_\lambda}\frac{1}{i!}A_\lambda^{(i)}\alpha_{\mathbf{s}}\frac{1}{j!}(\frac{\partial}{\partial z_\lambda})^j\beta_{\mathbf{t}}.$$

Then by (2.2.2) in Proposition 2.2, we have

$$\begin{aligned}
&\frac{1}{\mathbf{k}!}A^{(\mathbf{k})}B=\frac{1}{k_\lambda+1}A_\lambda^{(1)}\frac{1}{(\mathbf{k}-\mathbf{e}_\lambda)!}A^{(\mathbf{k}-\mathbf{e}_\lambda)}B\\
=&\frac{1}{k_\lambda+1}A_\lambda^{(1)}\sum_{\substack{\mathbf{s}+\mathbf{t}=\mathbf{k}',\\ i+j=k_\lambda}}\frac{1}{i!}A_\lambda^{(i)}\alpha_{\mathbf{s}}\frac{1}{j!}(\frac{\partial}{\partial z_\lambda})^j\beta_{\mathbf{t}}\\
=&\frac{1}{k_\lambda+1}\sum_{\substack{\mathbf{s}+\mathbf{t}=\mathbf{k}',\\ i+j=k_\lambda}}\frac{1}{i!}A_\lambda^{(1)}A_\lambda^{(i)}\alpha_{\mathbf{s}}\frac{1}{j!}(\frac{\partial}{\partial z_\lambda})^j\beta_{\mathbf{t}}+\frac{1}{i!}A_\lambda^{(i)}\alpha_{\mathbf{s}}\frac{1}{j!}(\frac{\partial}{\partial z_\lambda})^{j+1}\beta_{\mathbf{t}}\\
=&\frac{1}{k_\lambda+1}\sum_{\substack{\mathbf{s}+\mathbf{t}=\mathbf{k}',\\ i+j=k_\lambda}}\frac{1}{i!}A_\lambda^{(i+1)}\alpha_{\mathbf{s}}\frac{1}{j!}(\frac{\partial}{\partial z_\lambda})^j\beta_{\mathbf{t}}+\frac{1}{i!}A_\lambda^{(i)}\alpha_{\mathbf{s}}\frac{1}{j!}(\frac{\partial}{\partial z_\lambda})^{j+1}\beta_{\mathbf{t}}.
\end{aligned}$$

By same argument in Proposition 2.2, we conclude that

$$= \sum_{\substack{\mathbf{s}+\mathbf{t}=\mathbf{k}',\\ i+j=k_\lambda+1}} \frac{1}{i!} A_\lambda^{(i)} \alpha_{\mathbf{s}} \frac{1}{j!} (\frac{\partial}{\partial z_\lambda})^j \beta_{\mathbf{t}}$$

$$= \sum_{\mathbf{i}+\mathbf{j}=\mathbf{k}} \frac{1}{\mathbf{i}!} (A^{(\mathbf{i})} I) \frac{1}{\mathbf{j}} (\frac{\partial}{\partial z})^{\mathbf{j}} B. \quad \square$$

We use notations $\deg_\lambda$ (resp. deg) means the degree in z_λ (resp. the total degree), $H(\cdot)$ means that $H(\sum_{\mathbf{j}} a_{\mathbf{j}} z^{\mathbf{j}}) := \max_{\mathbf{j}} |a_{\mathbf{j}}|$.

We put

$$\kappa_1 := \max(\deg d, \max_{0 \le \lambda \le \mu-1} \deg dA_\lambda),$$

$$\kappa_2 := (2\mu+n)(\kappa_1+1)^2 \max(H(d), \max_{0 \le \lambda \le \mu-1} H(dA_\lambda)).$$

Proposition 2.4.

For any $\mathbf{k}$, *we have*

$$\deg(\frac{d^{||\mathbf{k}||}}{\mathbf{k}!} A^{(\mathbf{k})} I) \le \kappa_1 ||\mathbf{k}||, \tag{2.4.1}$$

$$H(\frac{d^{||\mathbf{k}||}}{\mathbf{k}!} A^{(\mathbf{k})} I) \le \kappa_2^{||\mathbf{k}||}. \tag{2.4.2}$$

Proof.

First of all, for $a, b \in \mathbb{Q}[z]$, we note that

$$H(ab) \le (\min(\deg a, \deg b)+1) H(a) H(b). \tag{2.4.3}$$

We show Proposition 2.4 by induction.

In the case of $\mathbf{k} = (0, 0, \dots, 0)$, (2.4.1) and (2.4.2) are valid.

Now we assume that the both are valid for $\mathbf{k} = (k_0, k_1, \dots, k_{\mu-1})$ with

$$k_0 = k_1 = \cdots = k_{\lambda-1} = k_\lambda + 1 = k_{\lambda+1} + 1 = \cdots = k_{\mu-1} + 1 \tag{2.4.4}$$

for some $\lambda \in \{0, 1, \dots, \mu-1\}$.

Since

$$A_\lambda^{(1)} (\frac{d^{||\mathbf{k}||}}{\mathbf{k}!} A^{(\mathbf{k})} I) = \frac{d^{||\mathbf{k}||}}{\mathbf{k}!} A^{(\mathbf{k}+\mathbf{e}_\lambda)} I + ||\mathbf{k}|| (\frac{\partial}{\partial z_\lambda} d) \frac{d^{||\mathbf{k}||-1}}{\mathbf{k}!} A^{(\mathbf{k})} I,$$

we have

$$\frac{d^{||\mathbf{k}+\mathbf{e}_\lambda||}}{(\mathbf{k}+\mathbf{e}_\lambda)!} A^{(\mathbf{k}+\mathbf{e}_\lambda)} I = \frac{d}{k_\lambda+1} A_\lambda^{(1)} (\frac{d^{||\mathbf{k}||}}{\mathbf{k}!} A^{(\mathbf{k})} I) - \frac{||\mathbf{k}||}{k_\lambda+1} (\frac{\partial}{\partial z_\lambda} d) \frac{d^{||\mathbf{k}||}}{\mathbf{k}!} A^{(\mathbf{k})} I. \tag{2.4.5}$$

Then (2.4.5) implies that

$$\begin{aligned}\deg \frac{d^{||\mathbf{k}+\mathbf{e}_\lambda||}}{(\mathbf{k}+\mathbf{e}_\lambda)!}A^{(\mathbf{k}+\mathbf{e}_\lambda)}I \leq & \max(\deg d+\kappa_1||\mathbf{k}||, \deg dA_\lambda+\kappa_1||\mathbf{k}||)\\ \leq & \kappa_1||\mathbf{k}+\mathbf{e}_\lambda||\end{aligned}$$

and that

$$\begin{aligned}&H(\frac{d^{||\mathbf{k}+\mathbf{e}_\lambda||}}{(\mathbf{k}+\mathbf{e}_\lambda)!}A^{(\mathbf{k}+\mathbf{e}_\lambda)}I) \leq \frac{1}{k_\lambda+1}(\deg d+1)H(d)(\kappa_1||\mathbf{k}||\kappa_2^{||\mathbf{k}||})\\ &+\frac{1}{k_\lambda+1}(n(\deg(dA_\lambda)+1)H(dA_\lambda))\kappa_2^{||\mathbf{k}||}+\frac{||\mathbf{k}||}{k_\lambda+1}(\deg d+1)^2H(d)\kappa_2^{||\mathbf{k}||}\\ \leq&(\mu\kappa_1(\deg d+1)H(d)+(\deg(dA_\lambda)+1)nH(dA_\lambda)+\mu(\deg d+1)^2H(d))\kappa_2^{||\mathbf{k}||}\\ \leq&\kappa_2^{||\mathbf{k}+\mathbf{e}_\lambda||}\end{aligned}$$

since $||\mathbf{k}||/(k_\lambda+1) \leq \mu$ by (2.4.4). Therefore (2.4.1) and (2.4.2) are valid for

$$k_0 = k_1 = \cdots = k_{\lambda-1} = k_\lambda = k_{\lambda+1}+1 = \cdots = k_{\mu-1}+1.$$

Thus if the both (2.4.1) and (2.4.2) are true for $\mathbf{k} = (k_0, k_0, \ldots, k_0)$, then the case of $\mathbf{k} = (k_0+1, k_0, \ldots, k_0)$ and $(k_0+1, k_0+1, \ldots, k_0), \ldots, (k_0+1, k_0+1, \ldots, k_0+1)$ are also true. □

Lemma 2.5.

We assume that N is sufficiently large. Let $B \in M_n(\mathbb{Z}[z])$ with $\deg_\lambda B < N$.

If (eq.1) *is a G-operator, then there exists a constant C_1 depending only on i, κ_1 and κ_2, such that for any $||\mathbf{k}|| \leq i$ and $i = 0, 1, \ldots$*

$$\frac{e_i d^{||\mathbf{k}||}}{\mathbf{k}!}A^{(\mathbf{k})}B \in M_n(\mathbb{Z}[z]), \tag{2.5.1}$$

$$H(\frac{e_i d^{||\mathbf{k}||}}{\mathbf{k}!}A^{(\mathbf{k})}B) < C_1^{i+N}H(B) \tag{2.5.2}$$

and

$$\deg \frac{e_i d^{||\mathbf{k}||}}{\mathbf{k}!}A^{(\mathbf{k})}B < C_1 i + \mu N, \tag{2.5.3}$$

where C_1 is a positive constant independent of i, N and B.

Proof.

Since

$$\frac{1}{\mathbf{j}!}(\frac{\partial}{\partial z})^{\mathbf{j}}B \in M_n(\mathbb{Z}[z]), \tag{2.5.4}$$

by Lemma 2.3 and Definition 2.1, we obtain (2.5.1).

By Lemma 2.3 and Proposition 2.4, and by (2.4.3) in Proposition 2.4, we obtain

$$\begin{aligned}\deg \frac{e_i d^{||\mathbf{k}||}}{\mathbf{k}!} A^{(\mathbf{k})} B &\leq \kappa_1 i + \mu N \\ &< C_1 i + \mu N.\end{aligned}$$

Since the number of terms in the summation in (2.3.1) $\leq (||\mathbf{k}|| + 1)^\nu$ and since $H((\partial/\partial z)^{\mathbf{k}} z^{\mathbf{k}'}) \leq 2^{||\mathbf{k}'||}$, we obtain

$$\begin{aligned}H(\frac{e_i d^{||\mathbf{k}||}}{\mathbf{k}!} A^{(\mathbf{k})} B) &\leq C_0^i (||\mathbf{k}|| + 1)^\mu \deg(d^{||\mathbf{k}||} + 1)(\deg d + 1)^{||\mathbf{k}||} \\ &\quad \times H(d)^{||\mathbf{k}||} n \kappa_1 ||\mathbf{k}|| \kappa_2^{||\mathbf{k}||} 2^{\mu N} H(B) \\ &< C_1^{i+N} H(B).\end{aligned}$$

Here C_0 is in Definition 2.1 and one may put $C_1 > \max(\kappa_1, C_0 e^{2\mu}(\kappa_1 + 1)(\kappa_1 + 1)\kappa_2 n \kappa_1, (2e)^\mu)$ by $(i+1)^\mu \leq e^\mu e^{\mu i}$. $\square$

Some lemmas

Let $y = (y_0, y_1, \dots, y_{n-1})$ be a nontrivial holomorphic vector solution of (eq.1). Suppose that $y_0, y_1, \dots, y_{m-1}$ be linearly independent over $\mathbb{Q}(z)$.

Let N be a sufficiently large natural number (in particular $N \gg q$). For $a = {}^t(a_0, a_1, \dots, a_{m-1}, 0, \dots, 0) \in (\mathbb{Q}[z])^n$ with $\deg_{z_\lambda} a < N$, we put

$$\psi := {}^t a\, y \quad \text{and} \quad \psi^{(\mathbf{k})} := (\frac{\partial}{\partial z})^{\mathbf{k}} \psi.$$

From now, $C_2, C_3, \dots$ are positive constants independent of $\mathbf{k}, N$ and q. We note here we assume that N is sufficiently larger than q.

Let δ be a positive real number.

Lemma 2.6.

We assume that N is sufficiently large. Let L satisfy $L^\mu < (\delta m/((1+\delta) l^\mu)) N^\mu$. We suppose that (eq.1) *is a G-operator. Then there exists nontrivial $a = {}^t(a_0, a_1, \dots, a_{m-1}, 0, \dots, 0) \in (\mathbb{Z}[z])^n$ such that*

$$H(a) < C_3^N q^{2\delta\mu(C_1 L + N)}$$

with

$$\psi^{(\mathbf{k})}(\zeta_{\mathbf{j}}) = 0 \quad \text{for } \mathbf{j} \leq (l-1)\mathbf{1} \text{ and for any } \mathbf{k} \leq (L-1)\mathbf{1}.$$

Proof.

Let $\{a_{i,\mathbf{t}}\}_{i=0,1,\dots,m-1}^{||\mathbf{t}|| \leq (N-1)\mathbf{1}}$ be mN^μ unknowns and put $a_i = \sum_{||\mathbf{t}|| \leq (N-1)\mathbf{1}} a_{i,\mathbf{t}} z^{\mathbf{t}}$.

Since

$$\begin{aligned}\psi^{(\mathbf{e}_\lambda)} &= \frac{\partial}{\partial z_\lambda}\psi = \frac{\partial}{\partial z_\lambda}\, {}^t a\, y \\ &= (\frac{\partial}{\partial z_\lambda}\, {}^t a)\, y + {}^t a\, \frac{\partial}{\partial z_\lambda} y = (\frac{\partial}{\partial z_\lambda}\, {}^t a)\, y + {}^t a\, A_\lambda y \\ &= {}^t((\frac{\partial}{\partial z_\lambda} + {}^t A_\lambda)\, a) y = {}^t(\, A^{(\mathbf{e}_\lambda)} a) y,\end{aligned}$$

and since (eq.1) is integrable, i.e., $A^{(\mathbf{k}+\mathbf{k}')} = A^{(\mathbf{k})}A^{(\mathbf{k}')}$, we find that

$$\psi^{(\mathbf{k})} = {}^t(A^{(\mathbf{k})}\, a)y. \tag{2.6.1}$$

Let $\zeta_{\mathbf{j}} = p_{\mathbf{j}}/q, \alpha_{\mathbf{j}} y(\zeta_{\mathbf{j}}) = P_{\mathbf{j}}/Q$, $(p, Q \in \mathbb{N})$ and $p_{\mathbf{j}} \in (\mathbb{Z})^{\mu}$, $P_{\mathbf{j}} \in (\mathbb{Z})^n$.
For any $||\mathbf{k}|| \leq (L-1)\mathbf{1}$, it is sufficient to find $\{a_{i,\mathbf{t}}\}$ with

$$\begin{aligned} &\frac{e_{\mu L} d(\zeta_{\mathbf{j}})^{||\mathbf{k}||}}{\mathbf{k}!}\, {}^t(A^{(\mathbf{k})}\, a)_{|z=\zeta_{\mathbf{j}}} \alpha_{\mathbf{j}} y(\zeta_{\mathbf{j}}) Q \\ &= Q \frac{e_{\mu L} d(p_{\mathbf{j}}/q)^{||\mathbf{k}||}}{\mathbf{k}!}\, {}^t(A^{(\mathbf{k})}\, a)_{|z=p_{\mathbf{j}}/q} P_{\mathbf{j}} = 0. \end{aligned} \tag{2.6.2}$$

By Lemma 2.3, Lemma 2.5 and (2.5.4),

$$q^{C_1||\mathbf{k}||+\mu N} Q \frac{e_{\mu L} d(p_{\mathbf{j}}/q)^{||\mathbf{k}||}}{\mathbf{k}!}\, {}^t(A^{(\mathbf{k})}\, a)_{|z=p_{\mathbf{j}}/q} P_{\mathbf{j}} = 0 \tag{2.6.3}$$

is a linear combination of $\{a_{i,\mathbf{t}}\}$ over $\mathbb{Z}$.

Let $P'_{\mathbf{j}} = \max|\text{coordinates of } P_{\mathbf{j}}|$ and $p'_{\mathbf{j}} = \max|\text{coordinates of } p_{\mathbf{j}}|$. Since $\max|p_{\mathbf{j}}| \leq rq$, i.e., $p'_{\mathbf{j}} \leq rq$, and by (2.5.2) , the maximal modulo of coefficients of $\{a_{i,\mathbf{t}}\}$ in the left side of (2.6.3) is less than

$$\gamma_1^{L+N} P'_{\mathbf{j}} Q (q^2)^{C_1 \mu L + \mu N},$$

where $\gamma_1 = \max(n\mu^{\mu} e^{\mu} (C_1 \mu e + 1) C_1^{\mu} 2^{\mu}, \mu^{2\mu})$.

The other hands, the number of linear equations in (2.6.3) is $(lL)^{\mu}$.

Therefore by the below Siegel's Lemma, we conclude that there exists a non-trivial system of solutions of (2.6.3) such that

$$\begin{aligned} \max|a_{i,\mathbf{t}}| &\leq 2(2mN^{\mu} \gamma_1^{L+N} P'_{\mathbf{j}} Q (q^2)^{C_1 \mu L + \mu N})^{l^{\mu} L^{\mu}/(mN^{\mu} - l^{\mu} L^{\mu})} \\ &\leq C_3^N q^{2\delta\mu(C_1 L + N)}, \end{aligned}$$

where $C_3 = 2(2m)^{\delta} e^{\delta\mu} \gamma_1^{m+1} e^{\delta} + 1$ with $N > \log(1, QP'_i)$. $\square$

Siegel's lemma. *(Lemma 1 in [7])*
Let $\sum_{t=1}^{n} a_{s,t} x_t = 0$, $(j = 1, 2, \ldots, r)$ be a system of linear equations with integer coefficients $a_{i,j}$, and $n > r$. Let A be a number ≥ 1 such that $|a_{i,j}| \leq A$ for all i, j. Then there exists an integral, non-trivial solution with

$$|x_j| \leq 2(2nA)^{r/(n-r)}.$$

Lemma 2.7.
We assume that N is sufficiently large. Let a is as in Lemma 2.6. For $\mathbf{j} \leq (l-1)\mathbf{1}$ and for any $\mathbf{k} \in (\mathbb{Z}_{\geq 0})^{\mu}$, we have

$$|\frac{1}{\mathbf{k}!} \psi^{(\mathbf{k})}(\zeta_{\mathbf{j}})| \geq C_5^{-(||\mathbf{k}||+N)} q^{-(C_1||\mathbf{k}||+\mu N)},$$

if it is not vanished.

Proof.

From (2.6.1) and Lemma 2.5, we find that

(2.7.1)

$$q^{C_1||\mathbf{k}||+\mu N}Q\frac{\alpha_{\mathbf{j}}e_{||\mathbf{k}||}d(\zeta_{\mathbf{j}})^{||\mathbf{k}||}}{\mathbf{k}!}\psi^{(\mathbf{k})}(\zeta_{\mathbf{j}}) = q^{C_1||\mathbf{k}||+\mu N}Q\frac{e_{||\mathbf{k}||}d(\zeta_{\mathbf{j}})^{||\mathbf{k}||}}{\mathbf{k}!}\,{}^t(A^{(\mathbf{k})}\,a)_{|z=\zeta_{\mathbf{j}}}P_{\mathbf{j}}$$

is an integer. Let $C_5 > (\alpha_{\mathbf{j}}Q)^{1/N}(\min d(\zeta_{\mathbf{j}}))^{-1}C_1$. Since (2.7.1) is an integer, we obtain

$$\begin{aligned}|\frac{1}{\mathbf{k}!}\psi^{(\mathbf{k})}(\zeta_{\mathbf{j}})| &\geq (\alpha_{\mathbf{j}}Qq^{C_1||\mathbf{k}||+\mu N}e_{||\mathbf{k}||}d(\zeta_{\mathbf{j}})^{||\mathbf{k}||})^{-1}\\ &\geq C_5^{-(||\mathbf{k}||+N)}q^{-(C_1||\mathbf{k}||+\mu N)}\end{aligned}$$

if (2.7.1) is not zero. Here we put $C_5 = e(\min d(\zeta_{\mathbf{j}}))^{-1}C_1$ with $N \geq \log(\alpha_{\mathbf{j}}Q)$. □

Let $L_\nu \in \mathbb{Z}_{\geq 0}$ for $\nu \in \{0,1,\dots,\mu-1\}$. The parameters $\{L_\nu\}$ in the following Lemma 2.8 will be used in our proof of Theorem 7.

Lemma 2.8.

Let $\nu \in \{0,1,\dots,\mu-1\}$, $k_\lambda \in \mathbb{Z}_{\geq 0}, \lambda \neq \nu$ be fixed. For any $\mathbf{k} = (k_0,k_1,\dots,k_\nu,\dots,k_{\mu-1})$ with $k_\nu < L_\nu$, and for any $\mathbf{j}$, we suppose that

$$\frac{1}{\mathbf{k}!}\psi^{(\mathbf{k})}(\zeta_{\mathbf{j}}) = 0.$$

Then under the assumptions in Theorem 7, we have

$$|\frac{1}{(\mathbf{k}+\mathbf{e}_\nu)!}\psi^{(\mathbf{k}+\mathbf{e}_\nu)}(\zeta_{\mathbf{j}})| \leq C_7^{||\mathbf{k}||+N}q^{2\delta\mu(C_1L+N)-\epsilon(l-1)k_\nu}.$$

Proof.

We fix $\mathbf{j}$ and put

$$z_{\mathbf{j},\nu} = (\zeta_{\mathbf{j},0},\zeta_{\mathbf{j},1},\dots,\zeta_{\mathbf{j},\nu-1},z_\nu,\zeta_{\mathbf{j},\nu+1},\dots,\zeta_{\mathbf{j},\mu-1}).$$

We put l points as $\{\zeta_0,\zeta_1,\dots,\zeta_{l-1}\} = \{\zeta_{\mathbf{j}'}\ ;\ \zeta_{\mathbf{j}',\lambda} = \zeta_{\mathbf{j},\lambda} \text{ for any } \lambda \neq \nu\}$. We put $\zeta_i = (\zeta_{i,0},\zeta_{i,1},\dots,\zeta_{i,\mu-1})$.

We consider the followings for each $i \in \{0,1,\dots,l-1\}$.

Put

$$\phi_{\mathbf{j},\nu,i}(\zeta_\nu) = \prod_{j=0,j\neq i}^{l-1}(z_\nu - \zeta_{j,\nu}).$$

By the assumption, we have for any $k'_\nu \leq k_\nu$ and for $j = 0,1,\dots,l-1$,

$$(\frac{\partial}{\partial z_\nu})^{k'_\nu}\psi^{(\mathbf{k}-k'_\nu\mathbf{e}_\nu)}(\zeta_j) = 0. \tag{2.8.1}$$

Since $\psi^{(\mathbf{k}-k'_\nu\mathbf{e}_\nu)}(z_{\mathbf{j},\nu})$ is holomorphic on one variable z_ν, one finds that $\psi^{(\mathbf{k}-k_\nu\mathbf{e}_\nu)}(z_{\mathbf{j},\nu})/(\phi_{\mathbf{j},\nu,i}(z_\nu))^{k_\nu}$ is also holomorphic on the disk with radius $< R$.

By the Cauchy's integral theorem, we have

$$\begin{aligned}&\frac{1}{(k_\nu+1)!}(\frac{\partial}{\partial z_\nu})^{k_\nu+1}\frac{1}{(\mathbf{k}-k_\nu\mathbf{e}_\nu)!}\frac{\psi^{(\mathbf{k}-k_\nu\mathbf{e}_\nu)}(\zeta_i)}{(\phi_{\mathbf{j},\nu,i}(\zeta_i))^{k_\nu}}\\ &=\frac{1}{2\pi\sqrt{-1}}\int_{T_\nu}\frac{1}{(\mathbf{k}-k_\nu\mathbf{e}_\nu)!}\frac{\psi^{(\mathbf{k}-k_\nu\mathbf{e}_\nu)}(z_{\mathbf{j},\nu})dz_\nu}{(z_\nu-\zeta_{i,\nu})^{k\nu+2}(\phi_{\mathbf{j},\nu,i}(z_\nu))^{k_\nu}},\end{aligned}\tag{2.8.2}$$

where $T_\nu=\{z_\nu\in\mathbb{C}\ ;\ |z_\nu|=r'_\nu, r_\nu<r'_\nu<R\}$. The left side of (2.8.2) is, by (2.8.1),

$$\frac{1}{(\mathbf{k}+\mathbf{e}_\nu)!}\frac{\psi^{(\mathbf{k}+\mathbf{e}_\nu)}(\zeta_i)}{(\phi_{\mathbf{j},\nu,i}(\zeta_i))^{k_\nu}}.\tag{2.8.3}$$

By Lemma 2.5 and by (2.6.1), we have

$$\max_{z_\nu\in T_\nu}|\frac{1}{(\mathbf{k}-k_\nu\mathbf{e}_\nu)!}\frac{\psi^{(\mathbf{k}-k_\nu\mathbf{e}_\nu)}(z_{\mathbf{j},\nu})}{(z_\nu-\zeta_{i,\nu})^{k\nu+2}(\phi_{\mathbf{j},\nu,i}(z_\nu))^{k_\nu}}|\le\gamma_2^{||\mathbf{k}||+\mu N}H(a)\max_{z_\nu\in T_\nu}|y(z_{\mathbf{j},\nu})|,$$

where

$$\gamma_2=n(C_1e+e^\mu+1)C_1\max(1,r'_\nu)^{C_1}(r'_\nu-r_\nu)^{-l}(\min_{z_\nu\in T_\nu}|d(z_{\mathbf{j},\nu})|)^{-1}+1.$$

Therefore by Lemma 2.6 , by (2.5.3) and since $|\zeta_{j,\nu}-\zeta_{i,\nu}|\le q^{-\epsilon}$, we conclude that

$$\begin{aligned}|\frac{1}{(\mathbf{k}+\mathbf{e}_\nu)!}\psi^{(\mathbf{k}+\mathbf{e}_\nu)}(\zeta_i)|&\le\gamma_3^{||\mathbf{k}||+\mu N}H(a)\max_{z_\nu\in T_\nu}|y(z_{\mathbf{j},\nu})||\phi_{\mathbf{j},\nu,i}(\zeta_i)|^{k_\nu}\\ &\le\gamma_4^{||\mathbf{k}||+\mu N}C_3^{||\mathbf{k}||+\mu N}q^{2\delta\mu(C_1L+N)-\epsilon(l-1)k_\nu},\end{aligned}$$

where $\gamma_3=r'_\nu\gamma_2$ and $\gamma_4=e\gamma_3$ and C_3 is in Lemma 2.6. For $N>\log\max_{z_\nu\in T_\nu}|y(z_{\mathbf{j},\nu})|$ for any $\mathbf{j}$ and ν, we put $C_7=(\gamma_4C_3)^\mu$. □

Proof of Theorem 7

Proof.

From Lemma 2.6, the case of $L_\nu\le L$ for $\nu=0,1,\dots,\mu-1$ of the assumptions in Lemma 2.8 hold.

Now we assume that the assumptions in Lemma 2.8 for a case of $L_\nu\ge L$ are valid. Put

$$\begin{cases}L_\lambda=L_\nu+1 & (\lambda<\nu),\\ L_\lambda=L_\nu & (\lambda>\nu).\end{cases}\tag{3.1.1}$$

In any above case, if

$$|\frac{1}{(\mathbf{k}+\mathbf{e}_\nu)!}\psi^{(\mathbf{k}+\mathbf{e}_\nu)}(\zeta_{\mathbf{j}})|<C_5^{-(||\mathbf{k}+\mathbf{e}_\nu||+N)}q^{-(C_1||\mathbf{k}+\mathbf{e}_\nu||+\mu N)}\tag{3.1.2}$$

for $k_\lambda=L_\lambda$ $(\lambda=0,1,\dots,\mu-1)$, then one finds that

$$\frac{1}{(\mathbf{k}+\mathbf{e}_\nu)!}\psi^{(\mathbf{k}+\mathbf{e}_\nu)}(\zeta_{\mathbf{j}})=0$$

by Lemma 2.7. After counting up L_ν (i.e., $L_\nu \Leftarrow L_\nu + 1$), one finds that the assumptions in Lemma 2.8, that is, the case of

$$\begin{cases} L_\lambda = L_\nu & (\lambda < \nu), \\ L_\lambda = L_\nu - 1 & (\lambda > \nu), \end{cases}$$

hold.

By induction (this is the induction on ν and L_ν. If $\nu < \mu - 1$, then count up L_ν and ν and repeat the above argument. Otherwise, put $\nu = 0$ and repeat the above argument), we have $\psi^{\mathbf{k}}(\zeta_{\mathbf{j}}) = 0$ for any $\mathbf{k} \in (\mathbb{Z}_{\geq 0})^\mu$, that is $\psi = 0$. Therefore $y_0, \ldots, y_{m-1}$ are linearly dependent over $\mathbb{Q}(z)$. This is a contradiction. So we shall find a sufficient condition of the validity for (3.1.2).

From Lemma 2.8, it is sufficient for (3.1.2) to show that

$$C_7^{||\mathbf{k}||+N} q^{2\delta\mu(C_1 L+N)-\epsilon(l-1)k_\nu} < C_5^{-(||\mathbf{k}||+1+N)} q^{-(C_1||\mathbf{k}||+1+\mu N)}. \tag{3.1.3}$$

Since $\mu k_\nu + \mu - 1 \geq ||\mathbf{k}|| \geq \mu k_\nu$ and by (3.1.1), it is enough to show that

$$(\epsilon(l-1) - \mu C_1)k_\nu - 2\delta\mu C_1 L - \mu(1+2\delta)N - 1 > (\mu k_\nu + \mu + N)\frac{\log C_7 - \log C_5}{\log q}.$$

Let $x = k_\nu / L$, $(x \geq 1)$. The last inequality is implied by

$$(\epsilon(l-1) - \mu C_1)x - 2\delta\mu C_1 - \mu(1+2\delta)\frac{N+1}{L} > (\mu x + \frac{N+\mu}{L})\frac{\log C_7 - \log C_5}{\log q} \tag{3.1.4}$$

for large N. Now we put

$$L = [(\frac{\delta m}{1+\delta})^{1/\mu}\frac{N}{l}]$$

and $\theta = (1 + 1/L)((1+\delta)/(\delta m))^{1/\mu}$ (i.e., $\theta l \geq N/L$).

Let $\delta = \sqrt{2}/(2\mu)$. (So we have $\mu(1+2\delta)\theta \to (\sqrt{2}+\mu)((1+\sqrt{2}\mu)/m)^{1/\mu}$ as $L \to \infty$.) and put

$$\epsilon_1 = \epsilon - (\sqrt{2} + \mu)(\frac{1+\sqrt{2}\mu}{m})^{1/\mu} \quad (> 0.)$$

Let $\gamma_5 = (\log C_7 - \log C_5)/\log q$. By (3.1.4), we have

$$(\epsilon - \frac{\mu(C_1 - \gamma_5)}{l-1})x > \mu(1+2\delta)\theta + \gamma_5\theta + \frac{\mu(1+2\delta)(\theta + 1/L) + 2\delta C_1\mu + \gamma_5\theta + \gamma_5\mu/L}{l-1}.$$

Let l and q be any integers such that

$$\begin{aligned} &\frac{\epsilon_1}{5} \geq \frac{\mu(C_1 - \gamma_5)}{l-1}, \\ &\frac{\epsilon_1}{5} \geq \frac{\mu(1+2\delta)(\theta + 1/L) + 2\delta C_1\mu + \gamma_5\theta + \gamma_5\mu/L}{l-1}, \\ &\gamma_5 < \frac{\epsilon_1}{2\theta}. \end{aligned} \tag{3.1.5}$$

It is easy to see that, if q is large, there exists a constant c_4' such that the system of inequalities (3.1.5) holds for any $l > c_4'$.

Now we find that (3.1.4) is implied by

$$(\epsilon - \frac{\epsilon_1}{5})x > \mu(1+2\delta)\theta + \frac{\epsilon_1}{2} + \frac{\epsilon_1}{5} \quad (x \geq 1).$$

Therefore

$$\epsilon > \mu(1+2\delta)\theta + \frac{9\epsilon_1}{10}$$

implies (3.1.4). This means that if q are large, and if $l > c_4'$, then (3.1.2) holds.

Therefore we find the constants c_3, c_4 such that for any $q > c_3$, it is nessesary that $l < c_4$, where c_3, c_4 are given in (3.1.5). $\square$

References

1. Y. André, *G-functions and Geometry*, Max-Planck-Institut, Bonn, 1989.
2. E. Bombieri, *On G-functions*, Recent progress in analytic number theory **2** (1981), Academic Press, New York, 1 – 67.
3. D. V. Chudnovsky and G. V. Chudnovsky., *Applications of Padé approximations to diophantine inequalities in values of G-functions.*, Lect. Notes in Math. **1135** (1985), Springer-Verlag, Berlin, 9–51.
4. G. V. Chudnovsky., *Measures of irrationality, transcendence and algebraic independence, Recent progress*, London Math. Soc. Lecture Note **56** (1982), Cambridge Univ. Press, Cambridge-New York, 11–82.
5. A. I. Galočhkin, *Estimates from below of polynomials in the values of analytic functions of a certain class*, Math. USSR Sbornik **24** (1974), 385 – 407, Original article in Math. Sbornik **95 (137)** (1974), 396 – 417.
6. F. Gramain, M .Mignotte, M .Waldschmidt, *Valeurs algébriques de fonctions analytiques.*, Acta Arith. **47** (1986), 97–121.
7. S. Lang, *Introduction to transcendental numbers*, ddison-Wesley Publishing Co., Massachusetts, Palo Alto, London, Ontario, 1966.
8. M. Nagata, *A generalization of the sizes of differential equations and its applications to G-function theory*, Preprint Math. Tokyo Institute of Technology (1994).
9. J. P. Serre, *Lectures on the Mordell-Weil Theorem*, Max-Planck-Institut, Bonn, 1989.
10. A. B. Shidlovskii, *Transcendental Numbers*, Walter de Gruyter, Berlin, New York, 1989.
11. C. L. Siegel, *Über einige Anwendungen diophantischer Approximationen*, Abh. Preuss. Akad. Wiss., Phys. Math. Kl. nr.1 (1929).
12. I. Wakabayashi, *Algebraic values of meromorphic functions on Riemann surfaces*, J. Number Theory **25** ((1987), no. 2,), 220–229.
13. ———, *Algebraic values of functions on the unit disk*, Prospects of mathematical science (Tokyo, 1986), (1988), World Sci. Publishing, Singapore, 235–266.

Oiwake-cho, Sakyo-ku, Kyoto, 606-8502, Japan
E-mail address: mnagata@kurims.kyoto-u.ac.jp