A Continuous Euler Transformation and its Application to the Fourier Transform of a Slowly Decaying Function

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Abstract

The Euler transformation is a linear sequence transformation to accelerate the convergence of an alternating series. The sequence of weights of the transformation is extended to a continuous weight function which can accelerate Fourier type integrals including Hankel transforms with a slowly convergent integrand. We show that the continuous weight function can also be used to compute the Fourier transform of a slowly decaying function using FFT.

Keywords: Continuous Euler transformation; Computation of the Fourier transform of a slowly decaying function; Fourier type integrals having a Bessel function in its integrand

1 Introduction

The Euler transformation is a linear sequence transformation to accelerate the convergence of alternating series. Now, an alternating series

$$S = \sum_{n=0}^{\infty} (-1)^n f(n) = \sum_{n=0}^{\infty} e^{i\pi n} f(n) \tag{1.1}$$

is a discrete approximation to a Fourier type integral

$$I = \int_0^{\infty} e^{i\pi x} f(x)\, dx \tag{1.2}$$

with a mesh size 1. So we may expect that a continuous version of the Euler transformation, if any, will accelerate the convergence of Fourier type integrals.

In this paper, we derive a continuous weight function of the Euler transformation and define a continuous Euler transformation for Fourier type integrals. We next apply it to the numerical evaluation of the Fourier transform of a slowly decaying function.

2 Continuous weight function of the Euler transformation

The Euler transformation of an alternating series

$$S = \sum_{n=0}^{\infty} (-1)^n a_n \tag{2.1}$$

is defined by

$$S_{\rm Euler} = \frac{1}{2} \sum_{n=0}^{\infty} (-\frac{1}{2}\Delta)^n a_0 \quad (\Delta a_n \equiv a_{n+1} - a_n). \tag{2.2}$$

In actual calculation, the summation (2.2) must be truncated at the N-th term as

$$\begin{aligned} S_{\rm Euler}^{(N)} &= \frac{1}{2} \sum_{n=0}^{N-1} (-\frac{1}{2}\Delta)^n a_0 \\ &= \sum_{m=0}^{N-1} w_m^{(N)} (-1)^m a_m \ , \end{aligned} \tag{2.3}$$

where $w_m^{(N)}$ are the weights

$$w_m^{(N)} = \sum_{n=m+1}^{N} \frac{1}{2^N} \binom{N}{n} \tag{2.4}$$

of the linear sequence transformation [4].

We first consider the behavior of $w_m^{(N)}$ as $N \to \infty$. Since the weights $w_m^{(N)}$ are the probability of the binomial distribution, we can use the central limit theorem to obtain

$$\begin{aligned} w_m^{(N)} &\sim \int_{m+1}^{\infty} \frac{1}{\sqrt{2\pi \cdot N/4}} e^{-(x-N/2)^2/(2\cdot N/4)}\, dx\,, \quad N \to \infty \\ &\sim \frac{1}{\sqrt{\pi}} \int_{m/\sqrt{N/2}-\sqrt{N/2}}^{\infty} e^{-t^2}\, dt\,, \quad N \to \infty. \end{aligned} \tag{2.5}$$

Thus

$$w_m^{(N)} \sim \frac{1}{2}\mathrm{erfc}(m/\sqrt{N/2} - \sqrt{N/2})\,, \quad N \to \infty\,, \tag{2.6}$$

where $\mathrm{erfc}(x)$ is the error function that is defined by

$$\mathrm{erfc}(x) \equiv \frac{2}{\sqrt{\pi}} \int_x^{\infty} e^{-t^2}\, dt.$$

Motivated by (2.6) we introduce a continuous weight function

$$w(x;p,q) = \frac{1}{2}\mathrm{erfc}(x/p - q) \tag{2.7}$$

with parameters $p,q > 0$ depending on N [2]. In Fig. 1 we show $w_m^{(N)}$ and $w(x;p,q)$ with $N = 16$, $p = q = \sqrt{N/2}$.

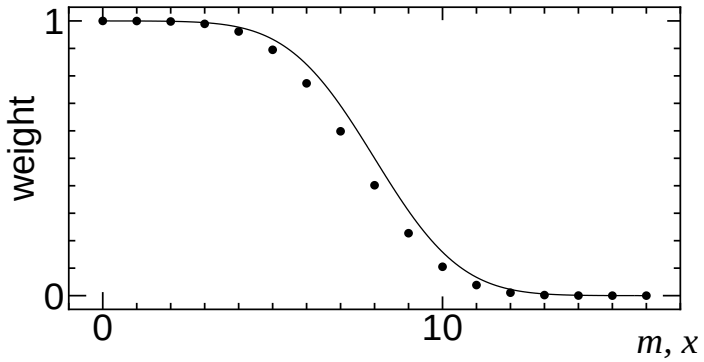


Fig. 1: The discrete weights $w_m^{(N)}$ and the continuous weight function $w(x;p,q)$ ($N = 16$, $p = q = (N/2)^{1/2}$)

We next show that a sequence transformation using the continuous weight function $w(x;p,q)$ accelerates the convergence of alternating series. We introduce the notation

$$S^{(N)} = \sum_{n=0}^{N-1} (-1)^n f(n)\,, \tag{2.8}$$

$$S_w^{(N)} = \sum_{n=0}^{N-1} w(n;p,q)(-1)^n f(n) \tag{2.9}$$

and $S^{(\infty)} = \lim_{N\to\infty} S^{(N)}$, $S_w^{(\infty)} = \lim_{N\to\infty} S_w^{(N)}$.

Theorem 1 We assume the following:

1. $w(x;p,q) = \frac{1}{2}\mathrm{erfc}(x/p - q)$ with $p,q > 0$.
2. $f(z)$ is regular in a domain $|\arg(z+1/2)| \le \delta$ with $0 < \delta < \pi/2$, and $|f(z)| \le M$ in the domain and
$$\lim_{R\to\infty} \max_{|\theta|\le\delta} |f(R - 1/2 + iR\tan\theta)| = 0.$$

Then, for arbitrary α such that $\alpha \le \tan\delta$, $0 < \alpha < 1$ we have

$$|S^{(\infty)} - S_w^{(\infty)}| < \frac{M\sqrt{1+\alpha^2}}{1 - e^{-\pi\alpha/2}} \left(\frac{\sqrt{\pi}p}{\sqrt{1-\alpha^2}} e^{(q'-\pi\alpha p/2)^2/(1-\alpha^2)} + \frac{2}{\pi\alpha} e^{(q'-\pi\alpha p)q'} \right) e^{-q'^2} \tag{2.10}$$

with $q' = q + 1/(2p)$.

Proof. By the residue theorem, we have

$$\begin{aligned} S^{(N)} - S_w^{(N)} &= \sum_{n=0}^{N-1} (-1)^n (1 - w(n;p,q)) f(n) \\ &= \frac{1}{2\pi i} \int_C \frac{\pi}{\sin\pi z} (1 - w(z;p,q)) f(z)\, dz \end{aligned} \tag{2.11}$$

with the contour of integration C illustrated in Fig. 2.

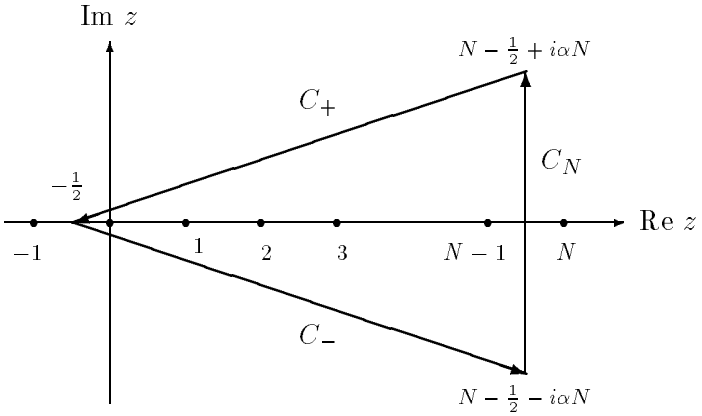


Fig. 2: The contour of integration C

Hence

$$|S^{(N)} - S_w^{(N)}| \le \frac{1}{2\pi} \int_C \left| \frac{\pi}{\sin\pi z} \right| \cdot |1 - w(z;p,q)| \cdot |f(z)| \cdot |dz|. \tag{2.12}$$

To estimate (2.12), we consider the behavior of the error function on the complex plane. From the definition of the error function, we have

$$\begin{aligned} 1 - \frac{1}{2}\mathrm{erfc}(z) &= 1 - \frac{1}{\sqrt{\pi}} \int_0^\infty e^{-(t+z)^2}\, dt \\ &= \frac{1}{\sqrt{\pi}} \int_{-\infty}^0 e^{-(t+z)^2}\, dt \end{aligned} \tag{2.13}$$

and

$$\left|1-\frac{1}{2}\mathrm{erfc}(z)\right| \le \begin{cases} 1+\dfrac{1}{\sqrt{\pi}}\displaystyle\int_0^\infty |e^{-t^2-z^2}|\,dt = 1+\frac{1}{2}|e^{-z^2}|\,, & \mathrm{Re}\ z \ge 0 \\ \dfrac{1}{\sqrt{\pi}}\displaystyle\int_{-\infty}^0 |e^{-t^2-z^2}|\,dt = \frac{1}{2}|e^{-z^2}| \qquad\quad , & \mathrm{Re}\ z < 0. \end{cases} \tag{2.14}$$

By (2.14), the integral on C_N is estimated as

$$\begin{aligned} &\frac{1}{2\pi}\int_{C_N}\left|\frac{\pi}{\sin\pi z}\right|\cdot|1-w(z;p,q)|\cdot|f(z)|\cdot|dz| \\ &\quad< \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\pi}{\cosh\pi y}dy\cdot\left(1+\frac{1}{2}e^{-((N-1/2)/p-q)^2+(N\alpha/p)^2}\right) \\ &\qquad\qquad \cdot\max_{|y|\le\alpha}|f(N-1/2+iNy)| \\ &\quad\to\ 0\,, \quad N\to\infty. \end{aligned} \tag{2.15}$$

For the integral on C_+, putting $z = t-1/2+i\alpha t$, we obtain

$$\begin{aligned} \left|\frac{\pi}{\sin\pi z}\right| &= \frac{2\pi}{|1-e^{2\pi iz}|}|e^{\pi iz}| \\ &\le \frac{2\pi}{1+e^{-2\pi\alpha t}\cos 2\pi t}e^{-\pi\alpha t} \\ &< \frac{2\pi}{1-e^{-\pi\alpha/2}}e^{-\pi\alpha t} \end{aligned} \tag{2.16}$$

and

$$\begin{aligned} |1-w(z;p,q)| &= \left|1-\frac{1}{2}\mathrm{erfc}((t-1/2)/p-q+i\alpha t/p)\right| \\ &\le \begin{cases} 1+\dfrac{1}{2}e^{-(t/p-q')^2+(\alpha t/p)^2}\,, & t\ge pq' \\ \dfrac{1}{2}e^{-(t/p-q')^2+(\alpha t/p)^2} \qquad , & t<pq'. \end{cases} \end{aligned} \tag{2.17}$$

Hence, the integral on C_+ is estimated as

$$\begin{aligned} &\frac{1}{2\pi}\int_{C_+}\left|\frac{\pi}{\sin\pi z}\right|\cdot|1-w(z;p,q)|\cdot|f(z)|\cdot|dz| \\ &\quad< \frac{1}{2\pi}\frac{2\pi}{1-e^{-\pi\alpha/2}}\left(\int_0^\infty e^{-\pi\alpha t}\frac{1}{2}e^{-(t/p-q')^2+(\alpha t/p)^2}M\sqrt{1+\alpha^2}\,dt\right. \\ &\qquad\qquad \left.+\int_{pq'}^\infty e^{-\pi\alpha t}M\sqrt{1+\alpha^2}\,dt\right) \\ &\quad< \frac{M\sqrt{1+\alpha^2}}{1-e^{-\pi\alpha/2}}\left(\frac{\sqrt{\pi}p}{2\sqrt{1-\alpha^2}}e^{(q'-\pi\alpha p/2)^2/(1-\alpha^2)}+\frac{1}{\pi\alpha}e^{(q'-\pi\alpha p)q'}\right)e^{-q'^2}. \end{aligned} \tag{2.18}$$

The integral on C_- is bounded similarly.

Thus we obtain

$$\begin{aligned} |S^{(\infty)}-S_w^{(\infty)}| &= \lim_{N\to\infty}|S^{(N)}-S_w^{(N)}| \\ &< \frac{M\sqrt{1+\alpha^2}}{1-e^{-\pi\alpha/2}}\left(\frac{\sqrt{\pi}p}{\sqrt{1-\alpha^2}}e^{(q'-\pi\alpha p/2)^2/(1-\alpha^2)}+\frac{2}{\pi\alpha}e^{(q'-\pi\alpha p)q'}\right)e^{-q'^2}. \end{aligned} \tag{2.19}$$

□

By Theorem 1, if we choose p, q and α such that

$$q' = q + \frac{1}{2p} = \frac{\pi\alpha}{2} p \,, \tag{2.20}$$

then we have

$$|S^{(\infty)} - S_w^{(\infty)}| < \frac{M\sqrt{1+\alpha^2}}{1 - e^{-\pi\alpha/2}} \left(\frac{\sqrt{\pi} p}{\sqrt{1-\alpha^2}} + \frac{2}{\pi\alpha} \right) e^{-q^2}. \tag{2.21}$$

Hence, under the relation (2.20), the error $|S^{(\infty)} - S_w^{(\infty)}|$ decays exponentially as $q \to \infty$.

Next, we estimate the truncated summation $S_w^{(N)}$ to obtain the total error of the sequence transformation using the continuous weight function. Since $w(n; p, q) \le \frac{1}{2}\exp(-(n/p - q)^2)$ for large n, the truncation error is estimated as

$$\begin{aligned} |S_w^{(\infty)} - S_w^{(N)}| &\le \sum_{n=N}^{\infty} |w(n;p,q)(-1)^n f(n)| \\ &\le \frac{M}{2} \sum_{n=N}^{\infty} \exp(-(n/p - q)^2) \\ &< \frac{M}{2} \int_{N-1}^{\infty} \exp(-(x/p - q)^2)\, dx \\ &= \frac{\sqrt{\pi} M p}{2} w(N-1; p, q) \\ &\le \frac{\sqrt{\pi} M p}{4} \exp(-((N-1)/p - q)^2). \end{aligned} \tag{2.22}$$

Hence, if we choose

$$N = \lfloor 2pq + 1 \rfloor = \lfloor \frac{4}{\pi\alpha} q'^2 \rfloor \,, \tag{2.23}$$

then

$$|S_w^{(\infty)} - S_w^{(N)}| < \frac{\sqrt{\pi} M p}{4} e^{-q^2} \,, \tag{2.24}$$

which is of the same order as (2.21) as $q \to \infty$.

Corollary 1 Let p, q and N be defined as in (2.20) and (2.23). Then

$$\begin{aligned} |S^{(\infty)} - S_w^{(N)}| &< \frac{M\sqrt{1+\alpha^2}}{1 - e^{-\pi\alpha/2}} \left(\frac{\sqrt{\pi} p}{\sqrt{1-\alpha^2}} + \frac{2}{\pi\alpha} \right) e^{-q^2} + \frac{\sqrt{\pi} M p}{4} e^{-q^2} \\ &= O(\alpha^{-2}(1-\alpha)^{-1/2} q e^{-q^2}) \\ &= O(\alpha^{-3/2}(1-\alpha)^{-1/2}\sqrt{N} e^{-\pi\alpha N/4}) \,, \quad N \to \infty. \end{aligned} \tag{2.25}$$

Corollary 1 means that $S_w^{(N)}$ approaches $S^{(\infty)}$ exponentially as $N \to \infty$ even if the decay of $f(n)$ is very slow.

Example 1 The example we evaluate is

$$S = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} = \log 2. \tag{2.26}$$

We applied

1. Direct summation :

$$S^{(N)} = \sum_{n=0}^{N-1} (-1)^n a_n$$

2. The Euler transformation :

$$S^{(N)}_{\text{Euler}} = \frac{1}{2} \sum_{n=0}^{N-1} (-\frac{1}{2}\Delta)^n a_0$$

3. The sequence transformation using $w(n;p,q) = \frac{1}{2}\text{erfc}(n/p - q)$:

$$S^{(N)}_w = \sum_{n=0}^{N-1} w(n;p,q)(-1)^n a_n$$

to the series (2.26) and calculated the summation. The numerical errors are shown in Table 1.

Table 1: Approximation to the alternating series (2.26)

	Method	N	Absolute error
1	Direct summation	21	2.3×10^{-2}
2	Euler transformation	21	2.1×10^{-8}
3	$w(n;p,q) = \frac{1}{2}\text{erfc}(n/p - q)$	21	1.6×10^{-9}

On the basis of Theorem 1, the parameters p, q were taken such that $N = 2pq$, $q = \pi p/2$. The result shows that the sequence transformation using the continuous weight function $w(n;p,q)$ accelerates the convergence of the alternating series as well as the Euler transformation.

3 Continuous Euler transformation for Fourier type integrals

We define a continuous Euler transformation for the following integral

$$I = \int_0^\infty f(x) e^{i\omega x}\, dx \tag{3.1}$$

by

$$I^{(L)}_w = \int_0^L w(x;p,q) f(x) e^{i\omega x}\, dx\ , \tag{3.2}$$

where $w(x;p,q)$ is the continuous weight function that is defined by (2.7) and p and q depend on L and ω.

We can show that the continuous Euler transformation accelerates the convergence of Fourier type integrals including Hankel transforms.

Theorem 2 We assume the following:

1. $w(x;p,q) = \frac{1}{2}\text{erfc}(x/p - q)$ with $p, q > 0$.
2. $\omega > 0$, r is real, $f(z)$ and $E(z;\omega)$ are regular in a domain $0 \le \arg(z - r) \le \delta$ with $0 < \delta < \pi/2$, $|f(z)| \le M_1$, $|E(z;\omega)| \le M_2 |e^{i\omega z}|$ in the domain and

$$\lim_{R\to\infty} \max_{0\le\theta\le\delta} |f(R + r + iR\tan\theta)| = 0.$$

Then, for arbitrary α such that $\alpha \le \tan\delta$, $0 < \alpha < 1$ we have

$$\left| \int_r^\infty f(x)E(x;\omega)\,dx - \int_r^\infty w(x;p,q)f(x)E(x;\omega)\,dx \right| \le M_1 M_2 \sqrt{1+\alpha^2} \left(\frac{\sqrt{\pi}p}{2\sqrt{1-\alpha^2}} e^{(q'-\omega\alpha p/2)^2/(1-\alpha^2)} + \frac{1}{\omega\alpha} e^{(q'-\omega\alpha p)q'} \right) e^{-q'^2} \tag{3.3}$$

with $q' = q - r/p$.

Proof. Let $R > 0$ and consider the integral

$$\begin{aligned} \Delta I_w^{(R)} &= \int_r^{r+R} f(x)E(x;\omega)\,dx - \int_r^{r+R} w(x;p,q)f(x)E(x;\omega)\,dx \\ &= \int_r^{r+R} (1 - w(x;p,q))f(x)E(x;\omega)\,dx. \end{aligned} \tag{3.4}$$

From the regularity of $f(z)$ and $E(z;\omega)$, the path of the integral can be moved from the real axis to C_+ and C_R, as illustrated in Fig. 3.

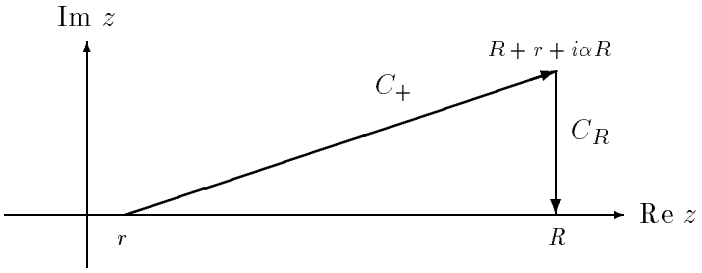


Fig. 3: The contour of integration $C_+ + C_R$

Then, we have

$$\begin{aligned} |\Delta I_w^{(R)}| &= \left| \int_{C_+ + C_R} (1 - w(z;p,q))f(z)E(z;\omega)\,dz \right| \\ &\le \int_{C_+ + C_R} |1 - w(z;p,q)| \cdot |f(z)| \cdot |E(z;\omega)| \cdot |dz| \end{aligned} \tag{3.5}$$

to get the result by estimating the integral as in the proof of Theorem 1. □

By using Theorem 2 with $r = 0$, $E(z;\omega) = \exp(i\omega z)$, we can estimate the error of the Fourier type integral

$$I_w = \int_0^\infty w(x;p,q)f(x)e^{i\omega x}\,dx. \tag{3.6}$$

If we choose p, q and α such that

$$q = \frac{\omega\alpha}{2}p\,, \tag{3.7}$$

then we have

$$|I - I_w| < M_1 \sqrt{1+\alpha^2} \left(\frac{\sqrt{\pi}p}{2\sqrt{1-\alpha^2}} + \frac{1}{\omega\alpha} \right) e^{-q^2}. \tag{3.8}$$

We next consider the truncated integral $I_w^{(L)}$. If we take

$$L = 2pq = \frac{4}{\omega\alpha}q^2\,, \tag{3.9}$$

then the truncation error becomes

$$\begin{aligned} |I_w - I_w^{(L)}| &\le \int_L^\infty |w(x;p,q)f(x)e^{i\omega x}|\,dx \\ &\le \frac{M_1}{2}\int_{2pq}^\infty \exp(-(x/p-q)^2)\,dx \\ &\le \frac{\sqrt{\pi}M_1 p}{4}e^{-q^2} \end{aligned} \tag{3.10}$$

which is of the same order as (3.8) as $q \to \infty$.

Corollary 2 Let p, q and L be defined as in (3.7), (3.9). Then

$$\begin{aligned} |I - I_w^{(L)}| &\le |I - I_w| + |I_w - I_w^{(L)}| \\ &< M_1\left(\frac{\sqrt{\pi}\sqrt{1+\alpha^2}p}{2\sqrt{1-\alpha^2}} + \frac{\sqrt{\pi}p}{4} + \frac{\sqrt{1+\alpha^2}}{\omega\alpha}\right)e^{-q^2} \\ &= O((\omega\alpha)^{-1/2}(1-\alpha)^{-1/2}\sqrt{L}e^{-\omega\alpha L/4}), \quad L \to \infty. \end{aligned} \tag{3.11}$$

Corollary 2 means that $I_w^{(L)}$ approaches I exponentially as $L \to \infty$ even if the decay of the integrand $f(x)$ is very slow.

In actual computation, the parameter q should be taken so that e^{-q^2} is smaller than the relative error requested. The parameter p and the interval L can be determined once α is given.

By Theorem 2, we can apply the continuous Euler transformation not only to Fourier type integrals but to the integrals whose integrand has a Bessel function. Let $E(z;\omega) = \sqrt{\omega z}H_\nu^1(\omega z)$, where $H_\nu^1(x) = J_\nu(x) + iY_\nu(x)$ is the Hankel function. Then the continuous Euler transformation for the integrals having a Bessel function in its integrand can be estimated in a similar way as Fourier type integrals.

Example 2 The integrals we evaluate are

$$I_1 = \int_0^\infty \frac{\cos x}{\sqrt{1+x^2}}\,dx = K_0(1), \tag{3.12}$$

$$I_2 = \int_0^\infty \frac{\sin x}{\log(1+x)}\,dx = 2.0410186151477098866\cdots, \tag{3.13}$$

$$I_3 = \int_0^\infty \log(1+x^2)J_1(x)\,dx = 2K_0(1), \tag{3.14}$$

where $J_\nu(x)$ is the Bessel function of the first kind and $K_\nu(x)$ is the modified Bessel function of the second kind. We took $q = p/2 = 4$, $L = 2pq$ so that the error factor e^{-q^2} becomes about 10^{-7}, and calculated the continuous Euler transformation (3.2) using 60 point Legendre-Gauss quadrature. The result is shown in Table 2.

Table 2: Approximation to I_1, I_2, I_3

Transformation	Error of I_1	Error of I_2	Error of I_3
Truncation with no weight function	1.4×10^{-2}	9.5×10^{-2}	7.7×10^{-1}
Continuous Euler transformation	2.1×10^{-9}	4.8×10^{-8}	9.7×10^{-8}

From the result, we can see that the continuous Euler transformation works well for Fourier type integrals with a slowly decaying function. Moreover it works well also for integrals with a Bessel function in its integrand.

4 Application to the Fourier transform of slowly decaying function

In this section, we apply the continuous Euler transformation to numerical evaluation of the Fourier transform using FFT. We are to compute the Fourier transform

$$F(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{-i\omega x}\,dx\ , \tag{4.1}$$

where $f(x)$ is a function slowly decaying as $|x| \to \infty$. We first extend the continuous Euler transformation to $(-\infty, \infty)$ and apply it to (4.1) to obtain

$$F_w^{(L)}(\omega) = \frac{1}{2\pi}\int_{-L_-}^{L_+} w(|x|;p,q)f(x)e^{-i\omega x}\,dx. \tag{4.2}$$

We next approximate it by the discrete summation with mesh size h to obtain

$$F_w^{(N,h)}(\omega) = \frac{h}{2\pi}\sum_{n=-N_-}^{N_+} w(|nh|;p,q)f(nh)e^{-i\omega nh}\ , \tag{4.3}$$

where $N_\pm = \lfloor L_\pm/h \rfloor$.

To keep the precision of the continuous Euler transformation, we should make p, q depend on $L_\pm$ and ω. Here we fix the parameters p, q with a view to using FFT. For $L_+ = Nh/2 - h$, $L_- = Nh/2$ and $\omega = 2\pi k/(Nh)$, (4.3) becomes

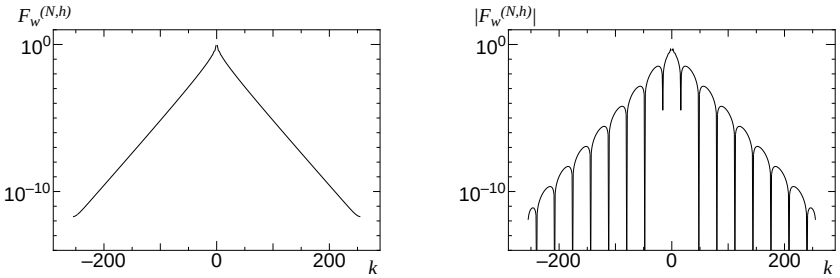
$$F_w^{(N,h)}(\frac{2\pi}{Nh}k) = \frac{h}{2\pi}\sum_{n=-N/2}^{N/2-1} w(|nh|;p,q)f(nh)e^{-2\pi ink/N}\ , \tag{4.4}$$

which can be computed by FFT.

Example 3 The functions to transform are

$$\begin{aligned} f_1(x) &= \frac{1}{\sqrt{1+x^2}}\ , \quad (F_1(\omega) = \frac{1}{\pi}K_0(|\omega|))\ , \\ f_2(x) &= \frac{x^3}{4+x^4}\ , \quad (F_2(\omega) = i\frac{-\mathrm{sgn}\ \omega}{2}e^{-|\omega|}\cos\omega). \end{aligned}$$

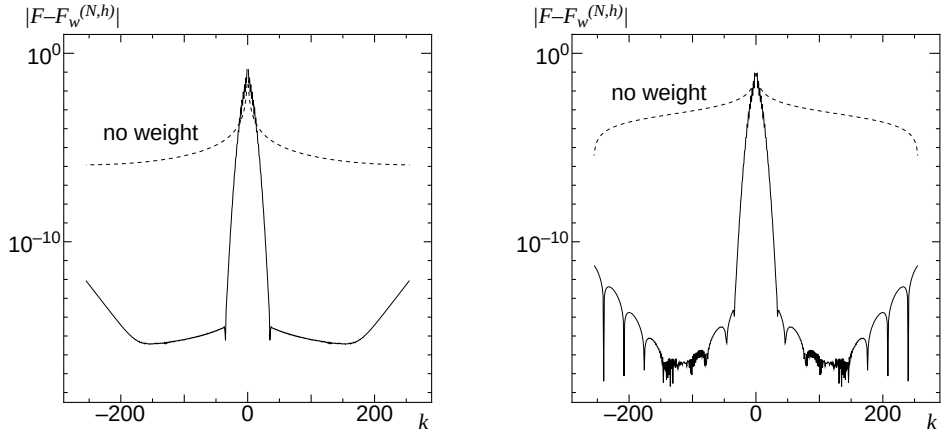
The result of (4.4) is shown in Fig. 4, and the error of (4.4) is shown in Fig. 5.



(a) $f(x) = \frac{1}{\sqrt{1+x^2}}$ (b) $f(x) = \frac{x^3}{4+x^4}$

Fig. 4: Approximations to the Fourier transform: $F_w^{(N,h)}(\frac{2\pi}{Nh}k)$, $N = 512$, $h = 0.125$

We chose $N = 512$, $h = 0.125$ and computed in double precision that has 53 bit accuracy. Truncation error of the continuous Euler transformation was set to 10^{-12} and the parameters were



(a) $f(x) = \frac{1}{\sqrt{1+x^2}}$ (b) $f(x) = \frac{x^3}{4+x^4}$

Fig. 5: Absolute error of the Fourier transform: $F_w^{(N,h)}(\frac{2\pi}{Nh}k)$, $N = 512$, $h = 0.125$

chosen such that $\exp(-q^2) = 10^{-12}$, $L_- = L_+ + h = Nh/2 = 2pq$. The dotted line in Fig. 5 is the error of the direct transform that does not use $w(x; p, q)$.

As seen from Fig. 5, the error of the transform increases exponentially for large $|\omega|$. Also the error becomes very large in the interval $|k| < 35$ ($|\omega| < 3.44$).

In Theorem 2, the right hand side of (3.3) becomes large if $q' - \omega\alpha p/2 \neq 0$. Since α must be $0 < \alpha < 1$, the error bound of $F(\omega) - F_w(\omega)$ becomes large if $q - |\omega|p/2 > 0$, i.e. $|\omega| < 2q/p = 3.45$, where

$$F_w(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} w(|x|; p, q) f(x) e^{-i\omega x}\, dx. \tag{4.5}$$

The truncation error $F_w(\omega) - F_w^{(L)}(\omega)$ is bounded as (3.10) and its bound is about $\exp(-q^2) = 10^{-12}$. The error of the discrete approximation $F_w^{(L)}(\omega) - F_w^{(N,h)}(\omega)$ can be estimated as $O(\exp(|\omega| - 2\pi/h) + \exp(-q^2))$ as $-|\omega| + 2\pi/h, q \to \infty$ by using the theorem on the discrete Fourier transform [3, chapter 3.3]. The approximation $F_w^{(N,h)}(\omega)$ includes the above errors and these estimations are consistent with the numerical result.

References

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