Automatic Evaluation of the Links–Gould Invariant for all Prime Knots of up to 10 Crossings

David De Wit*

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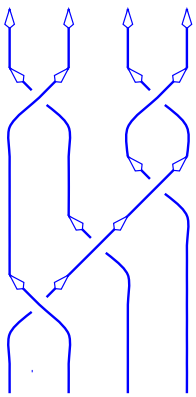

Abstract

This paper describes a method for the automatic evaluation of the Links–Gould two-variable polynomial link invariant (LG) for any link, given only a braid presentation. This method is currently feasible for the evaluation of LG for links for which we have a braid presentation of string index at most 5. Data are presented for the invariant, for all prime knots of up to 10 crossings and various other links. LG distinguishes between these links, and also detects the chirality of those that are chiral. In this sense, it is more sensitive than the well-known two-variable HOMFLY and Kauffman polynomials. When applied to examples which defeat the HOMFLY invariant, interestingly, LG 'almost' fails. The automatic method is in fact applicable to the evaluation of any such state sum invariant for which an appropriate R matrix and cap and cup matrices have been determined.

*RIMS, Kyoto University 606-8502, Japan. ddw@kurims.kyoto-u.ac.jp

1 Background

The Links–Gould two-variable polynomial invariant of oriented links (LG) is based on the $(0,0|\alpha)$ family of 4 dimensional highest weight representations of the quantum superalgebra $U_q[gl(2|1)]$. It was originally described by Jon Links and Mark Gould in [13], although that work did not pursue its evaluation for want of an efficient method. In [5], Louis Kauffman, Jon Links and I described a state model for its evaluation.[1]

In [5], we demonstrated that LG does not detect the noninvertibility or mutation of links. Furthermore, we provided evaluations of it for a small set of sundry interesting links, in particular for the (prime) knots 9_{42} and 10_{48}, these two being infamous as the first chiral knots whose chirality is undetected by the HOMFLY invariant. Whilst it was discovered that LG detected the chirality of these two examples, the method used to evaluate it was labour-intensive, and it was impracticable to evaluate LG for *all* the knots up to 9_{42}. Thus, we did not know whether LG was in fact more sensitive than the HOMFLY invariant as LG might have failed to detect the chirality of some chiral knot before 9_{42}.

Here, we improve upon that situation by describing a more sophisticated method for evaluating LG. This method requires as its input a braid presentation β for the (closed) link $\hat{\beta}$ in question. Using it, we evaluate LG for *all* the (prime) knots of up to 10 crossings.

2 Changes of Variable and Notation

Importantly, we make a change of variable from [5]. In that work, the underlying α-parametric family of representations of $U_q[gl(2|1)]$ led us to an invariant which manifested as an integer-coefficient 'polynomial' in variable q with exponents being integer-linear functions of α. Defining $p \triangleq q^{\alpha}$ ensured that our invariant was a two-variable (Laurent) polynomial in variables q and p (actually q^2 and p^2). We showed that LG would be unchanged under the transformation $\alpha \mapsto -\alpha - 1$ (and hence unable to detect noninvertibility). This meant that it satisfied the symmetry $LG_K(q,p) = LG_K(q,q^{-1}p^{-1})$.

Here, we instead define $p \triangleq q^{\alpha+1/2}$, which changes the symmetry to the simpler $LG_K(q,p) = LG_K(q,p^{-1})$. In these new variables q and p, the representation of the braid generator and the caps and cups take on more symmetric forms than they do using our previous variables. These new variables are thus a more natural choice for our invariant.

[1] In conjunction with this work, the reader should peruse [5], of which a more thorough treatment is found in [4] (available from the author).

The tensor $\psi(\overline{\sigma})$ representing the braid generator in the state model is now:

$$\left[\begin{array}{cccc|cccc|cccc|cccc}
p^{-2}q & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & p^{-1}q^{1/2} & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & p^{-1}q^{1/2} & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & 1 & . & . & . \\
\hline
. & p^{-1}q^{1/2} & . & . & p^{-2}q{-}1 & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & -1 & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & -q & . & . & -q^{1/2}Y & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & pq^{1/2} & . & . \\
\hline
. & . & p^{-1}q^{1/2} & . & . & . & . & . & p^{-2}q{-}1 & . & . & . & . & . & . & . \\
. & . & . & . & . & . & -q & . & . & q^2{-}1 & . & . & q^{3/2}Y & . & . & . \\
. & . & . & . & . & . & . & . & . & . & -1 & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & pq^{1/2} & . \\
\hline
. & . & . & 1 & . & . & -q^{1/2}Y & . & . & q^{3/2}Y & . & . & qY^2 & . & . & . \\
. & . & . & . & . & . & . & pq^{1/2} & . & . & . & . & . & p^2q{-}1 & . & . \\
. & . & . & . & . & . & . & . & . & . & . & pq^{1/2} & . & . & p^2q{-}1 & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & p^2q
\end{array}\right],$$

where we have used the shorthand:

$$Y \triangleq \sqrt{(pq^{1/2} - p^{-1}q^{-1/2})(pq^{-1/2} - p^{-1}q^{1/2})} = \sqrt{p^{+2} + p^{-2} - q^{+1} - q^{-1}};$$

note that this definition of Y differs from that in our previous work. This Y is more symmetric than our previous one – it is invariant under $q \mapsto q^{-1}$ (and hence $p \mapsto p^{-1}$). In our current variables, $\psi(\overline{\sigma}^{-1})$ is obtainable from $\psi(\overline{\sigma})$ by a twisting transformation combined with the mapping $q \mapsto q^{-1}$. Explicitly:

$$\left[\psi(\overline{\sigma}^{-1})\right]_{c\ d}^{a\ b} = [\psi(\overline{\sigma})]_{d\ c}^{b\ a} \quad \text{after mapping} \quad q \mapsto q^{-1}.$$

This transformation was opaque using our previous variables. It effectively illustrates the 'uniqueness' of our R matrix by demonstrating explicitly a unitary transformation relating two candidates. It is in fact also visible in the R matrix used for the bracket polynomial [8, 9], i.e. that R matrix originating in the 2 dimensional vector representation of $U_q[sl(2)]$. In that case, the same twist is used, and the mapping is $A \mapsto A^{-1}$.

To describe the caps ($\Omega^{\pm}$) and cups ($\mho^{\pm}$), we introduce another slight change of notation from [5]; to whit, we exchange Ω^{+} with Ω^{-}. This notation ensures that the superscript "+" (respectively "−") indicates a sector of an anticlockwise (respectively clockwise) arc, and is consistent with the introduction of an additional notation: we shall use $C^{\pm}$ to denote vertically oriented arcs defined by the composition of horizontally oriented arcs. We shall call these new diagram components *left handles*.[2] Our current collection of arcs is shown in Figure 1.

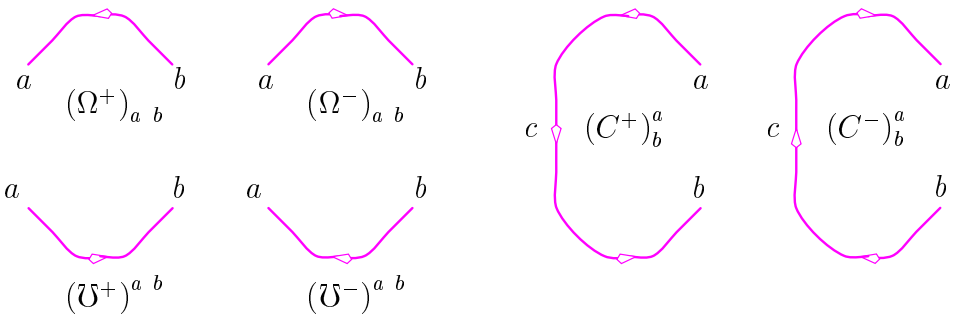

Figure 1: Caps, cups and (left) handles.

The matrices corresponding to the left handles may be described in terms of the caps and cups:

$$\left(C^{\pm}\right)^{a}_{b} = \left(\Omega^{\pm}\right)_{c\ a}\left(\mho^{\pm}\right)^{c\ b}.$$

These definitions are generally applicable to any state model. In [5], we determined appropriate caps and cups for our particular state model. Using our new diagram conventions, in terms of our new variables that information becomes:

$$\begin{aligned}
\Omega^{-} &= I_4 \\
\Omega^{+} &= p^{-2}\ \mathrm{diag}\{q^{+1}, -q^{+1}, -q^{-1}, q^{-1}\} \\
\mho^{+} &= \left(\Omega^{-}\right)^{-1} = I_4 \\
\mho^{-} &= \left(\Omega^{+}\right)^{-1} = p^{+2}\ \mathrm{diag}\{q^{-1}, -q^{-1}, -q^{+1}, q^{+1}\} \\
C^{+} &= p^{-2}\ \mathrm{diag}\{q^{+1}, -q^{+1}, -q^{-1}, q^{-1}\} \\
C^{-} &= p^{+2}\ \mathrm{diag}\{q^{-1}, -q^{-1}, -q^{+1}, q^{+1}\}.
\end{aligned}$$

Again, note that these matrices possess greater symmetries than they do using our previous variables.

Lastly, by inspection, it turns out that our new evaluations of LG are actually Laurent polynomials in q and p^2, so we can introduce the further notational artifice $P \triangleq p^2$.

[2] For our current purposes, we have no need for right handles.

3 The Algorithm

The present algorithm used for evaluating state model invariants abandons the prior drawing of a quasi-Morse diagram for the link (as used in [5]), instead recognising that a closed braid corresponding to the link is already in Morse form. If the braid form is used to represent a link, then the previous method may be simplified to eliminate all the auxiliary abstract tensors, as we need only consider crossings with upwardly-pointing arrows, represented by the tensors σ and σ^{-1}.

We describe the process for a link $K = \hat{\beta}$, corresponding to an n string braid word β (i.e. $\beta \in B_n$). We regard β as an oriented (n, n) tangle, and $\hat{\beta}$ as being obtained by the (vertical) closures of all the strings of β. Recall that for our purposes, we actually require a $(1, 1)$ tangle obtained by closing all but one of the strings of β.

Firstly, we construct Z, a rank $2n$ tensor corresponding to β. Secondly, we construct T, a rank 2 tensor obtained by contracting all but 2 of the indices of Z. In contrast to our previous method, at each stage of the construction this method uses the data contained in σ – the tensors Z and T are concrete, not abstract. The indices of all our tensors run from 1 to M, where M is the dimension of the underlying representation (in our case $M = 4$, but the process is more generally applicable to any state sum invariant). Thus, Z has M^{2n} entries, and T is an $M \times M$ matrix.

3.1 Construction of Z by Accretion

We shall build Z by an m stage accretionary process corresponding to the definition of β as a string of m braid generators. At each stage, the intermediate tensor will also be called Z. Initially, Z corresponds to the trivial (n, n) tangle, so we initialise it to be the identity rank $2n$ tensor $I_M^{\otimes n}$, where I_M is the identity rank 2 tensor (i.e. the usual $M \times M$ identity matrix).

Say that our braid $\beta \in B_n$ may be written as:

$$\beta = \sigma_{j_1}^{i_1} \sigma_{j_2}^{i_2} \cdots \sigma_{j_m}^{i_m}, \qquad i_k = \pm 1, \quad 1 \leqslant j_k < n, \qquad k = 1, \ldots, m,$$

where $\sigma_{j_k}^{\pm 1}$ indicates that letter k in β is a positive (respectively negative) crossing of strings j_k and $j_k + 1$. Where Z corresponds to an (n, n) tangle representing a part $\overline{\beta}$ of β, the accretion of a letter σ_j to $\overline{\beta}$ corresponds to the product of the tensors Z and σ_j according to Figure 2.[3]

Algebraically (Einstein summation convention used throughout):

$$(Z\sigma_j)_{b_1 \cdots b_n}^{a_1 \cdots a_n} = (Z)_{b_1 \cdots b_{j-1}\ c_j\ c_{j+1}\ b_{j+2} \cdots b_n}^{a_1 \cdots a_{j-1}\ a_j\ a_{j+1}\ a_{j+2} \cdots a_n} \cdot (\sigma_j)_{b_j\ b_{j+1}}^{c_j\ c_{j+1}}.$$

[3] These and the following comments of course apply equally to σ^{-1}.

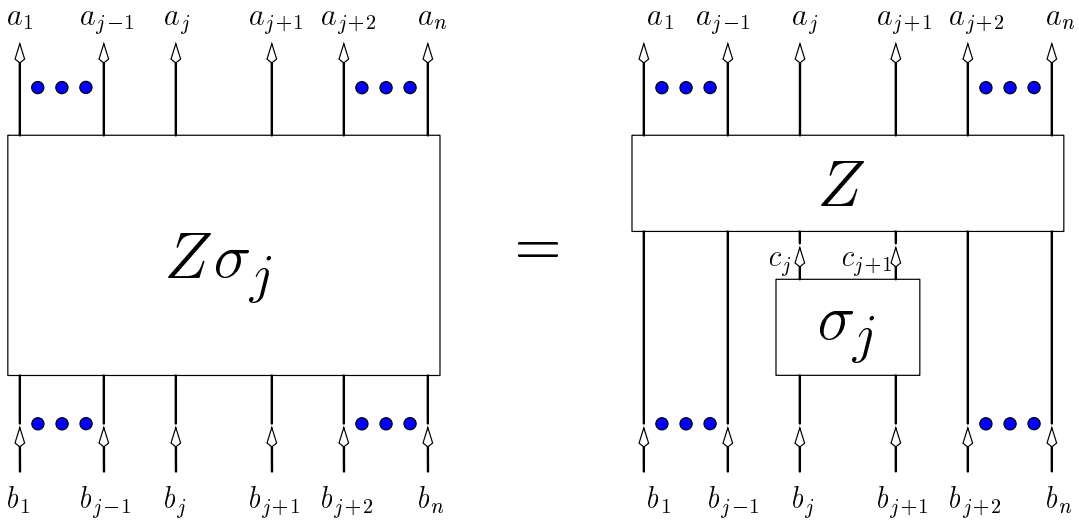

Figure 2: Accreting a braid generator to position j of an (n, n) tangle.

More generally, we may accrete any appropriate object in this fashion, and thus, we may cheat somewhat in the accretionary process: say σ_j actually repeats N times consecutively within β. Having computed the auxiliary rank 4 tensor σ_j^N (described in [5]), we may then accrete σ_j^N to Z in one, rather than N stages, which improves efficiency in a small but practical manner.

3.2 Contraction of Z to T

The tensor T corresponds to the $(1, 1)$ tangle obtained by the closure of all but one of the strings of β – we shall choose the rightmost one for convenience This process is depicted in Figure 3.

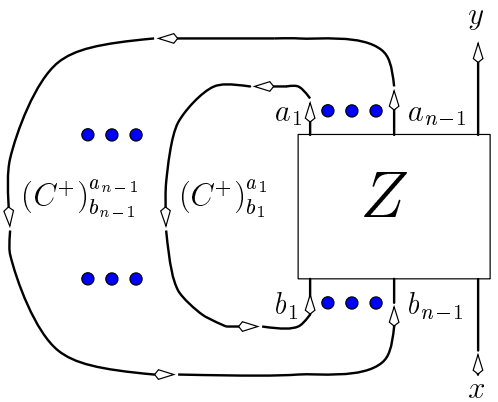

Figure 3: Closure of all but one pair of strings of an (n, n) tangle.

Thus, T is constructed by contraction of all but two of the indices of Z (preferably done 'string-at-a-time'), viz:

$$(T)_x^y \triangleq (Z)_{b_1 \cdots b_{n-1} \; x}^{a_1 \cdots a_{n-1} \; y} \cdot \prod_{j=1}^{n-1} (C^+)_{b_j}^{a_j}.$$

4 Feasibility and Implementation

Where our state model is based on a underlying representation of dimension M, and we are evaluating an invariant based on a braid presentation of n strings, the tensor Z contains M^{2n} components (see Table 1), each of which may become a possibly large polynomial expression (to be fair, it may remain zero throughout). Thus, the algorithm requires storage at least proportional to M^{2n}, which is a serious limitation.

Algebra	$U_q[sl(2)]$	$U_q[gl(2\|1)]$	$U_q[gl(3\|1)]$	$U_q[gl(4\|1)]$
Representation	$(1,0)$	$(0,0\|\alpha)$	$(0,0,0\|\alpha)$	$(0,0,0,0\|\alpha)$
	$M=2$	4	8	16
$n=2$	16	256	4096	6.6×10^4
3	64	4096	2.6×10^5	1.7×10^7
4	256	6.6×10^4	1.7×10^7	·
5	1024	1.0×10^6	·	·
6	4096	1.7×10^7	·	·
...	·	·	·	·
10	1.0×10^6	·	·	·

Table 1: M^{2n} for various state models and braid indices.

Experiments demonstrate that for LG, a current practical limit for M^{2n} is 10^6, thus evaluation of LG is viable for all links where the braid presentation to hand has at most 5 strings. All but five of the prime knots to 10_{166} have braid index (see §5) at most 5; the five outstanding ones have braid index 6. Thus, our algorithm is applicable to almost all of the prime knots to 10_{166}, and LG may be evaluated for the outstanding cases using our previous method (see §6).

We have again implemented the algorithm in the interpreted environment of MATHEMATICA. Conversion to a compiled language would lead to an enormous speedup, but would not enhance the algorithm's applicability to links with braid presentations of more than 5 strings. This algorithm compares poorly in performance with that used by Morton and Short [15] to evaluate the HOMFLY polynomial – their (non-state model) method requires much less storage and time. In compensation, our method described here is simpler to encode.

5 Braid Presentations

To evaluate LG for a particular link K, we require a braid presentation, i.e. a braid word β, such that K is equal to the (vertical) closure $\hat{\beta}$ of β. For computational efficiency, it is preferable that β be of as few strings as possible, and as short as possible. Of these two considerations, the former is critical.

Recall that the *braid index* of a link K is the minimal number of strings of all possible braid presentations β such that $\hat{\beta} = K$. Unfortunately, we do not (as yet!) have an algorithm to determine the braid index of an arbitrary link, much less an algorithm to construct minimal braid presentations [2, 16]. Indeed, such an algorithm would answer Artin's *algebraic link problem*,[4] and the braid word so constructed would itself be a complete invariant for links. Recall that because no such algorithm is known, it is hoped that a polynomial invariant may eventually provide a complete invariant. Indeed, it is entirely possible that no such algorithm *exists*, as [14] describes a finitely-presented group for which it is known that there can be no algorithm to determine whether a word is equal to the unit element.

The construction of braid presentations from links is itself of algorithmic interest. The classical algorithm of Alexander [1] has been used to great effect, but is labour-intensive, and doesn't yield particularly minimal presentations. In 1987, Yamada [18] showed that the braid index of a link is equal to the minimum number of Seifert circles of all braid presentations of that link. This was proven via the demonstration of a braid-generating algorithm, improving the method of Alexander. In 1990, Yamada's algorithm was further improved by Vogel [17].

In 1987, Jones [7, p64] combined the use of the 'Franks–Williams–Morton inequality' together with some explicit braid presentations to demonstrate the braid indices of all (249) prime knots of up to 10 crossings[5] (i.e. up to 10_{166}). Jones went further [7, pp66-71], *manually*[6] applying Alexander's algorithm to generate a set of braid presentations for all prime knots to 10_{166}. These presentations all have string index equal to the braid index. We have used this data for our evaluations of LG, although we have no way of independently confirming that the data does not contain errors originating in the application of Alexander's algorithm.

Manual continuation of the application of Alexander's (or even Vogel's) algorithm to yield braid presentations for the prime knots beyond 10_{166} is an unattractive proposition. Unfortunately, Vogel's algorithm has not yet been implemented on a computer, hence we do not yet have at our disposal a substantial list of braid presentations of prime knots beyond 10_{166}.

[4] Artin stated that the question of the *topological* equivalence (or otherwise) of two arbitrary links could be answered by the solution of the *algebraic* problem of the equivalence of (arbitrary examples of) closed braids associated with the links. Its solution depends on the solution of the *Markov equivalence problem* [2] for closed braids. Open!

[5] Recall that $10_{161} = 10_{162}$. Here, we maintain the artifice that these are distinct.

[6] Personal communication. This must have been a Herculean task!

6 The Knots $10_1, 10_3, 10_{13}, 10_{35}$ and 10_{58}

The five prime knots up to 10_{166} of braid index 6 are $10_1, 10_3, 10_{13}, 10_{35}$ and 10_{58}. Here, we describe quasi-Morse diagrams for these knots, from which we are able to evaluate LG using our previous method, by writing down abstract tensors associated with them. The diagrams (sometimes the reflections!) are taken from [10, pp257-259], and the conventions are the same as those of [5].

10_1 and 10_3: These two knots, presented in Figure 4, are structurally similar. Their abstract tensors are:

$$\begin{aligned}(T_{10_1})^{y}_{x} &\triangleq \left(\sigma^{-1}_{rl}\right)^{a\ y}_{b\ d} \cdot \left(\sigma^{8}_{dudu}\right)^{b\ d}_{c\ x} \cdot \left(C^{-}\right)^{a}_{c} \\ (T_{10_3})^{y}_{x} &\triangleq \left(\sigma^{-4}_{rlrl}\right)^{a\ y}_{b\ d} \cdot \left(\sigma^{6}_{dudu}\right)^{b\ d}_{c\ x} \cdot \left(C^{-}\right)^{a}_{c}.\end{aligned}$$

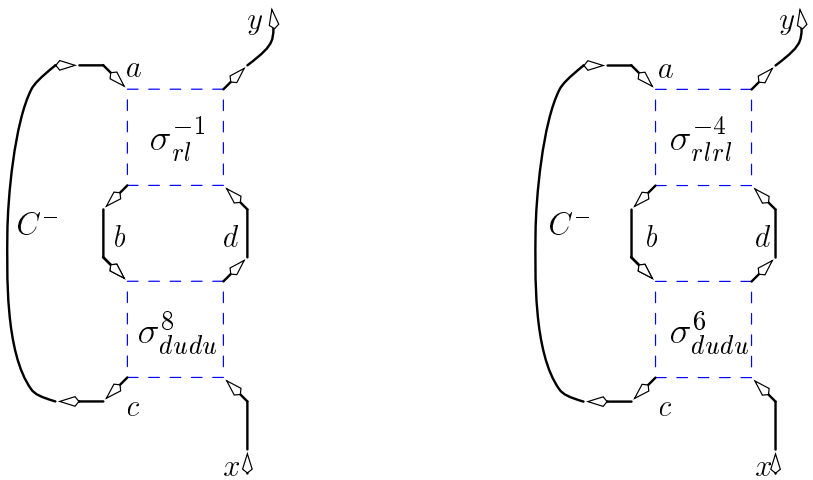

Figure 4: Tangle forms of 10_1 and 10_3.

10_{13} and 10_{35}: These two knots, presented in Figure 5, are also structurally similar. Their abstract tensors are:

$$\begin{aligned}(T_{10_{13}})^{y}_{x} &\triangleq \left(\sigma^{-1}_{rl}\right)^{a\ y}_{b\ j} \cdot \left(\sigma^{-1}_{rl}\right)^{b\ e}_{c\ f} \cdot \left(\sigma^{4}_{dudu}\right)^{c\ f}_{d\ g} \cdot \left(\sigma_{du}\right)^{h\ j}_{i\ x} \cdot \\ &\qquad \left(\Omega^{+}\right)_{e\ h} \cdot \left(\mho^{-}\right)^{g\ i} \cdot \left(C^{-}\right)^{a}_{d} \\ (T_{10_{35}})^{y}_{x} &\triangleq \left(\sigma_{rl}\right)^{a\ y}_{b\ j} \cdot \left(\sigma^{4}_{rlrl}\right)^{b\ e}_{c\ f} \cdot \left(\sigma^{-1}_{du}\right)^{c\ f}_{d\ g} \cdot \left(\sigma^{-1}_{du}\right)^{h\ j}_{i\ x} \cdot \\ &\qquad \left(\Omega^{+}\right)_{e\ h} \cdot \left(\mho^{-}\right)^{g\ i} \cdot \left(C^{-}\right)^{a}_{d}.\end{aligned}$$

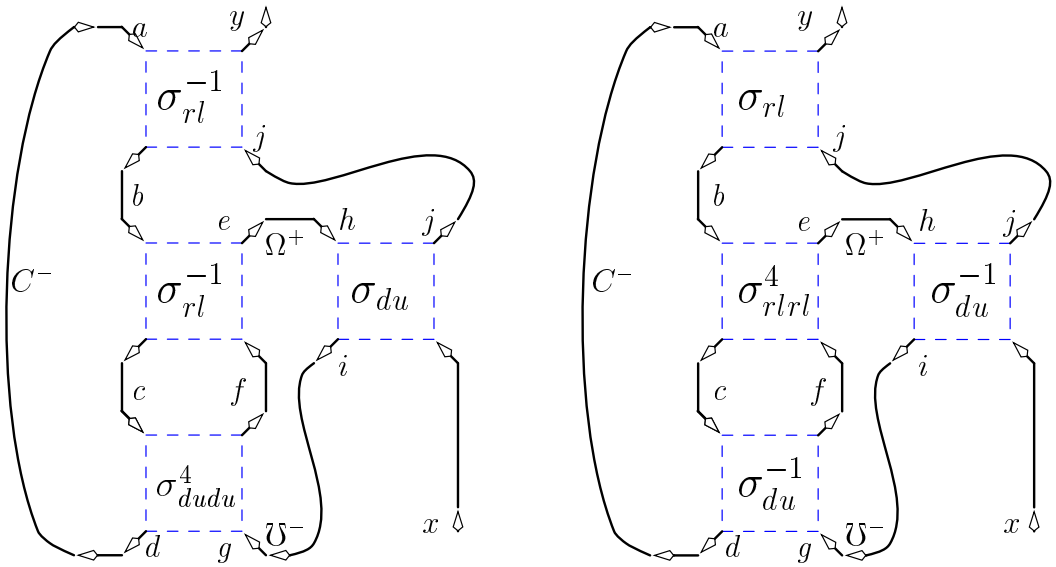

Figure 5: Tangle forms of 10_{13} and 10_{35}.

10_{58}: This knot, depicted in Figure 6, has an elegant symmetry. Its abstract tensor is:

$$(T_{10_{58}})_{x}^{y} \triangleq (\sigma_{du})_{b\ m}^{a\ y} \cdot \left(\sigma_{rl}^{-1}\right)_{c\ f}^{b\ e} \cdot \left(\Omega^{+}\right)_{e\ i} \cdot \left(\mho^{-}\right)^{f\ j} \cdot (\sigma_{du})_{j\ n}^{i\ m} \cdot \left(\sigma_{rl}^{-1}\right)_{d\ h}^{c\ g} \cdot \left(\Omega^{+}\right)_{g\ k} \cdot \left(\mho^{-}\right)^{h\ l} \cdot (\sigma_{du})_{l\ x}^{k\ n} \cdot \left(C^{-}\right)_{d}^{a}.$$

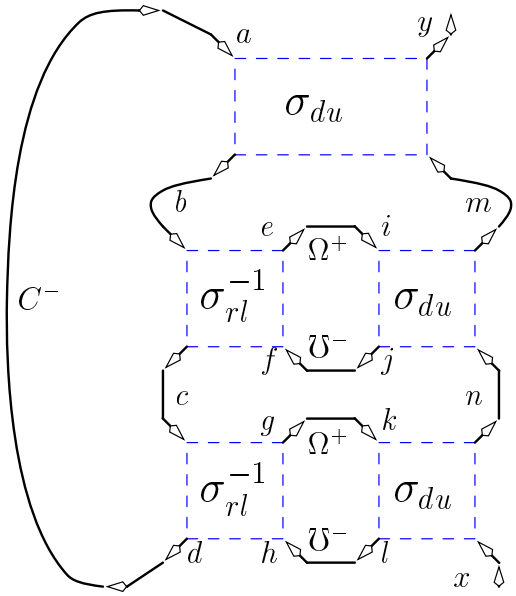

Figure 6: Tangle form of 10_{58}.

7 Behaviour of LG

Evaluations of LG for the prime knots to 10_{166} are presented in the Appendix. To reduce the volume of text, we have utilised the symmetry $LG_K(q,p) = LG_K(q,p^{-1})$ (recall that this indicates that LG is unable to detect inversion), and only printed the (q-polynomial) coefficients of non-negative powers of $P \triangleq p^2$. To illustrate the encoding, consider the entry for 8_{17}:

$$
\begin{aligned}
&LG_{8_{17}} \quad = \\
&4q^{-4} + 68q^{-2} + 139 + 68q^2 + 4q^4, -22q^{-3} - 102q^{-1} - 102q^1 - 22q^3, \\
&3q^{-4} + 40q^{-2} + 82 + 40q^2 + 3q^4, -7q^{-3} - 36q^{-1} - 36q^1 - 7q^3, 7q^{-2} + 18 + 7q^2, \\
&-4q^{-1} - 4q^1, 1.
\end{aligned}
$$

This is intended to be read:

$$
\begin{aligned}
&LG_{8_{17}} \quad = \\
&(4q^{-4} + 68q^{-2} + 139 + 68q^2 + 4q^4) + (P^{-1} + P^1)(-22q^{-3} - 102q^{-1} - 102q^1 - 22q^3) \\
&+(P^{-2} + P^2)(3q^{-4} + 40q^{-2} + 82 + 40q^2 + 3q^4) \\
&+(P^{-3} + P^3)(-7q^{-3} - 36q^{-1} - 36q^1 - 7q^3) + (P^{-4} + P^4)(7q^{-2} + 18 + 7q^2) \\
&+(P^{-5} + P^5)(-4q^{-1} - 4q^1) + (P^{-6} + P^6).
\end{aligned}
$$

The polynomial is invariant under $P \mapsto P^{-1}$, and in this case, is also palindromic in q (i.e. invariant under $q \mapsto q^{-1}$), as 8_{17} is amphichiral. To illustrate the economy and utility of the notation, replacing P with p^2 and reordering, the polynomial takes on the cumbersome traditional form:

$$
\begin{aligned}
&139 + p^{-12} + 18p^{-8} + 82p^{-4} + 82p^4 + 18p^8 + p^{12} + 4q^{-4} + 3p^{-4}q^{-4} + 3p^4q^{-4} \\
&-7p^{-6}q^{-3} - 22p^{-2}q^{-3} - 22p^2q^{-3} - 7p^6q^{-3} + 68q^{-2} + 7p^{-8}q^{-2} + 40p^{-4}q^{-2} \\
&+40p^4q^{-2} + 7p^8q^{-2} - 4p^{-10}q^{-1} - 36p^{-6}q^{-1} - 102p^{-2}q^{-1} - 102p^2q^{-1} - 36p^6q^{-1} \\
&-4p^{10}q^{-1} - 4p^{-10}q^1 - 36p^{-6}q^1 - 102p^{-2}q^1 - 102p^2q^1 - 36p^6q^1 - 4p^{10}q^1 + 68q^2 \\
&+7p^{-8}q^2 + 40p^{-4}q^2 + 40p^4q^2 + 7p^8q^2 - 7p^{-6}q^3 - 22p^{-2}q^3 - 22p^2q^3 - 7p^6q^3 + 4q^4 \\
&+3p^{-4}q^4 + 3p^4q^4.
\end{aligned}
$$

This may be seen to be in agreement with the form presented in the original work if the replacement $p \mapsto q^{1/2}p$ is made, viz, we had the opaque form:

$$
\begin{aligned}
&139 - 7p^{-6} + 40p^{-4} - 102p^{-2} - 102p^2 + 40p^4 - 7p^6 + p^{-12}q^{-6} - 4p^{-10}q^{-6} \\
&+7p^{-8}q^{-6} - 7p^{-6}q^{-6} + 3p^{-4}q^{-6} + 4q^{-4} - 4p^{-10}q^{-4} + 18p^{-8}q^{-4} - 36p^{-6}q^{-4} \\
&+40p^{-4}q^{-4} - 22p^{-2}q^{-4} + 68q^{-2} + 7p^{-8}q^{-2} - 36p^{-6}q^{-2} + 82p^{-4}q^{-2} - 102p^{-2}q^{-2} \\
&-22p^2q^{-2} + 3p^4q^{-2} + 68q^2 + 3p^{-4}q^2 - 22p^{-2}q^2 - 102p^2q^2 + 82p^4q^2 - 36p^6q^2 \\
&+7p^8q^2 + 4q^4 - 22p^2q^4 + 40p^4q^4 - 36p^6q^4 + 18p^8q^4 - 4p^{10}q^4 + 3p^4q^6 - 7p^6q^6 \\
&+7p^8q^6 - 4p^{10}q^6 + p^{12}q^6.
\end{aligned}
$$

7.1 Observations for the Prime Knots to 10_{166}

To evaluate LG for the prime knots to 10_{166}, we have utilised the braid presentations supplied by Jones in [7]. That table of braids also contains evaluations of the Jones polynomial $V(t)$,[7] and we confirm that we have made no transcription errors in obtaining our list of braids by checking that our method yields Jones polynomials in agreement with those of [7]. These comments of course apply to all but the five knots of braid index 6. Recall that the diagrams for those knots were used to generate abstract tensors manually. Again, Jones polynomials that agree with those presented in [7] were obtained, indicating the veracity of our abstract tensors. Confirmation that our algorithm and code are correct is provided by the agreement of our data for LG with that of the subset published in [5]. Thus, our evaluations of LG are correct modulo errors in the original construction of the braids.

Generally, LG for any of the prime knots has the following properties.

- The q-polynomials are nonzero, the only exceptions being those corresponding to $(P^{-5} + P^5)$ for LG of both 10_{154} and 10_{161}. (This fact does not appear to be significant.)
- The q-polynomials corresponding to even (respectively odd) powers of P contain only even (respectively odd) powers of q.
- The coefficients (of the powers of q) within the q-polynomials corresponding to odd (respectively even) powers of P are all negative (respectively positive). This is true for all the knots except the following ten: 8_{19}, 10_{124}, 10_{128}, 10_{132}, 10_{139}, 10_{145}, 10_{152}–10_{154}, and 10_{161}. Of these knots, the coefficients of the q-polynomials of 8_{19} and 10_{124} are all of the same sign within each q-polynomial, whilst those of the other eight sometimes change sign within a q-polynomial, so these latter knots are in a sense, even more unusual. Furthermore:
  - These unusual knots all have braid index either 3 or 4.
  - These unusual knots are all chiral. For the record, the 20 amphichiral knots to 10_{166} are 4_1, 6_3, 8_3, 8_9, 8_{12}, 8_{17}, 8_{18}, 10_{17}, 10_{33}, 10_{37}, 10_{43}, 10_{45}, 10_{79}, 10_{81}, 10_{88}, 10_{99}, 10_{109}, 10_{115}, 10_{118} and 10_{123}.
  - These unusual knots are all invertible except for 10_{153}. For the record, the 36 noninvertible knots to 10_{166} are 8_{17}, 9_{32}, 9_{33}, 10_{67}, 10_{79}–10_{88}, 10_{90}–10_{95}, 10_{98}, 10_{102}, 10_{106}, 10_{107}, 10_{109}, 10_{110}, 10_{115}, 10_{117}–10_{119}, 10_{147}–10_{151} and 10_{153}.
  - It is not yet known whether all, some or none of these unusual knots are nonalternating knots.

[7] Actually the data in [7] is that for a related polynomial $W(t) \triangleq (1 - V(t))/(1 - t)(1 - t^3)$, which has integral coefficients.

Considering the data as a whole, we observe:

- *LG* assigns *distinct* polynomials to each of the (distinct) knots tested. (We also observe that there are no hidden equalities in the data by checking if there are equivalences between any pairs if the interchange $q \mapsto q^{-1}$ is made in *one* of the pair.) This is a significant result, as it is *not* a feature of the HOMFLY polynomial P; indeed [7, 10] to 10_{166} we have four HOMFLY equivalences: $P_{5_1} = P_{10_{132}}$, $P_{8_8} = P_{10_{129}}$, $P_{8_{16}} = P_{10_{156}}$ and $P_{10_{25}} = P_{10_{56}}$. Investigating each of these 'HOMFLY-pairs', we find that whilst *LG* of 5_1 and 10_{132} are quite distinct (recall that *LG* of 10_{132} had unusual coefficients), the other three pairs have Links–Gould invariants that are remarkably similar, in that the coefficients of their q-polynomials are very close (in particular, compare *LG* of 10_{25} with that of 10_{56}). In this sense, *LG only just* distinguishes these pairs.

  (There are also 9 other coincidences of the $V(t)$ amongst the 10 crossing knots [7]. Within each pair, the Alexander–Conway polynomials are distinct, but none of these are also HOMFLY pairs.)

- *LG* correctly detects the chirality of all chiral prime knots tested. This is also a significant result, as it is not true for the HOMFLY polynomial. Indeed, of the prime knots to 10_{166}, the HOMFLY invariant fails to distinguish the chirality of the six knots $9_{42}, 10_{48}, 10_{71}, 10_{91}, 10_{104}$ and 10_{125} [7, 10]. (Recall that in [5], we discovered that *LG* detects the chirality of 9_{42} and 10_{48}.) The *LG* polynomials of these six knots are not palindromic in q, but for four of them ($10_{48}, 10_{71}, 10_{91}$ and 10_{104}) are curiously 'almost palindromic', in that the q-polynomials are almost palindromic. In this sense, *LG only just* succeeds in detecting the chirality of these knots.

7.2 *LG* of a Set of Divers Links

It is of interest to evaluate *LG* for multicomponent links, to see if the resultant polynomials have the same properties as those of the prime knots. Here, we present evaluations of *LG* for a small set of divers links, mostly taken from [6, pp108-110] (same notation). The list includes several links of 2 and 3 components.

4^2_{1a} and 4^2_{1b} are different orientations of the same two-component link 4^2_1; the latter may be obtained from the former by reversing the orientation of only *one* of its components – they *are* distinct oriented links [6, p110]. The notation "JE" means "Jones' Example". "KT" is the Kinoshita–Terasaka Knot [11] and "C" is the Conway Knot [3].

We have used our method to evaluate $V(t)$ for these links. That our $V(t)$ are in agreement with [6] gives us confidence that our evaluations of *LG* are correct modulo errors in the source braids.

$LG_{2_1^2} = -1 - q^2, q^1$

$LG_{4_{1a}^2} = -1 - 2q^2 - q^4, q^1 + 2q^3, -q^2 - q^4, q^3$

$LG_{4_{1b}^2} = -3 - 4q^2 - q^4, 3q^1 + q^3$

$LG_{5_1^2} = -8q^{-2} - 10 - 2q^2, 2q^{-3} + 9q^{-1} + 4q^1, -3q^{-2} - 3, q^{-1}$

$LG_{6_1^2} = -1 - 2q^2 - 2q^4 - q^6, q^1 + 2q^3 + 2q^5, -q^2 - 2q^4 - q^6, q^3 + 2q^5, -q^4 - q^6, q^5$

$LG_{6_2^2} = -3 - 12q^2 - 10q^4 - q^6, 5q^1 + 12q^3 + 4q^5, -5q^2 - 6q^4 - q^6, 3q^3 + q^5$

$LG_{6_3^2} = -4q^{-2} - 23 - 22q^2 - 3q^4, 8q^{-1} + 20q^1 + 7q^3, -5 - 5q^2, q^1$

$LG_{6_1^3} = 2q^{-2} + 13 + 18q^2 + 3q^4, -5q^{-1} - 15q^1 - 10q^3, 5 + 10q^2 + 2q^4, -3q^1 - 3q^3, q^2$

$LG_{6_2^3} = 16q^{-2} + 38 + 16q^2, -3q^{-3} - 25q^{-1} - 25q^1 - 3q^3, 6q^{-2} + 16 + 6q^2, -4q^{-1} - 4q^1, 1$

$LG_{6_3^3} = 1 + 2q^2 + q^4, -q^1 - q^3, q^4, -q^3 - q^5, q^4$

$LG_{JE_{1,1}} = -3 - 4q^2 - q^4, 3q^1 + q^3$

$LG_{JE_{1,2}} = -1 - 2q^2 - q^4, q^{-1} + q^1 + q^3, -q^{-2} - 1, q^{-1}$

$LG_{JE_{2,1}} = LG_{JE_{2,2}} =$
$-1 - 11q^2 - 32q^4 - 24q^6 - 2q^8, 3q^1 + 19q^3 + 30q^5 + 9q^7, -5q^2 - 19q^4 - 15q^6 - q^8, 5q^3 + 11q^5 + 3q^7,$
$-3q^4 - 3q^6, q^5$

$LG_{KT} = LG_C =$
$10q^{-4} + 6q^{-2} - 23 - 8q^2 + 14q^4 + 2q^6, -2q^{-5} - 11q^{-3} + 9q^{-1} + 18q^1 - 7q^3 - 7q^5,$
$3q^{-4} + 2q^{-2} - 9 - 3q^2 + 6q^4 + q^6, -q^{-3} + q^{-1} + 2q^1 - q^3 - q^5$

- As for the prime knots, the q-polynomials corresponding to even (respectively odd) powers of P contain only even (respectively odd) powers of q. Also, the coefficients (of the powers of q) within each q-polynomial are generally of consistent sign.
- Only $LG_{6_2^3}$ is palindromic, and this is so as only 6_2^3 is amphichiral.
- $LG_{KT} = LG_C$, as shown in [5].
- The two links $JE_{1,1}$ and $JE_{1,2}$ have homeomorphic complements, but are distinguished by LG (and for that matter, also by $V(t)$).
- Jones provides braid presentations $\sigma_1^2\sigma_2^3\sigma_3^3$ for $JE_{2,1}$ and $\sigma_1^3\sigma_2^2\sigma_3^3$ for $JE_{2,2}$, so these links are actually conjugate (the conjugacy of a similar pair is demonstrated in [2]). Thus, any invariant should take on the same value for both of them, and this is true for both $V(t)$ and LG.

7.3 LG of the $(3, 5, 7)$ Pretzel Knot

For good measure, we include the evaluation of the $(3, 5, 7)$ pretzel (for comparison with our previously published result [5]):

$LG_{(3,5,7)} = 57 + 308q^2 + 468q^4 + 464q^6 + 338q^8 + 174q^{10} + 56q^{12} + 8q^{14},$
$-113q^1 - 279q^3 - 329q^5 - 279q^7 - 172q^9 - 71q^{11} - 16q^{13} - q^{15},$
$57q^2 + 84q^4 + 83q^6 + 60q^8 + 30q^{10} + 9q^{12} + q^{14}.$

8 Summary

Significantly:

- LG allocates *distinct* polynomials to each of the (249) prime knots to 10_{166}.
- Furthermore, it detects the chirality of those that are chiral.

In both these features, LG is more sensitive than both the well-known two-variable HOMFLY and Kauffman invariants.

We do not yet have a counterexample to show that it does not always detect the chirality of chiral links, nor do we have a pair of distinct prime knots to which LG allocates the same polynomial, or a nontrivial prime knot K such that $LG_K = LG_{0_1}$. The evidence that it almost fails to discriminate in some cases where the HOMFLY polynomial fails (§7) suggests that such counterexamples exist. Investigation of the prime knots beyond 10_{166} might expose such examples, but this would require braid presentations of these knots (§5), and such are not currently to hand.

The method used to evaluate LG is generally applicable to any state sum invariant once appropriate R matrices and caps and cups have been determined. Stay tuned for future experiments with other such invariants – specifically ones based on $(\dot{0}_m|\dot{\alpha}_n)$ representations of $U_q[gl(m|n)]$.

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A LG of the Prime Knots to 10_{166}

$LG_{3_1} = 1 + 2q^2, -q^1 - q^3, q^2$

$LG_{4_1} = 2q^{-2} + 7 + 2q^2, -3q^{-1} - 3q^1, 1$

$LG_{5_1} = 1 + 2q^2 + 2q^4, -q^1 - 2q^3 - q^5, q^2 + 2q^4, -q^3 - q^5, q^4$

$LG_{5_2} = 3 + 10q^2 + 4q^4, -5q^1 - 6q^3 - q^5, 3q^2 + q^4$

$LG_{6_1} = 4q^{-2} + 17 + 10q^2 + 2q^4, -7q^{-1} - 10q^1 - 3q^3, 3 + q^2$

$LG_{6_2} = 2q^{-2} + 11 + 14q^2 + 2q^4, -4q^{-1} - 12q^1 - 8q^3, 4 + 9q^2 + 2q^4, -3q^1 - 3q^3, q^2$

$LG_{6_3} = 10q^{-2} + 25 + 10q^2, -2q^{-3} - 16q^{-1} - 16q^1 - 2q^3, 4q^{-2} + 11 + 4q^2, -3q^{-1} - 3q^1, 1$

$LG_{7_1} = 1 + 2q^2 + 2q^4 + 2q^6, -q^1 - 2q^3 - 2q^5 - q^7, q^2 + 2q^4 + 2q^6, -q^3 - 2q^5 - q^7, q^4 + 2q^6, -q^5 - q^7, q^6$

$LG_{7_2} = 5 + 20q^2 + 14q^4 + 4q^6, -9q^1 - 14q^3 - 6q^5 - q^7, 5q^2 + 3q^4 + q^6$

$LG_{7_3} = 3 + 12q^2 + 16q^4 + 4q^6, -5q^1 - 14q^3 - 10q^5 - q^7, 5q^2 + 12q^4 + 4q^6, -5q^3 - 6q^5 - q^7, 3q^4 + q^6$

$LG_{7_4} = 9 + 38q^2 + 28q^4 + 6q^6, -17q^1 - 27q^3 - 11q^5 - q^7, 9q^2 + 6q^4 + q^6$

$LG_{7_5} = 3 + 20q^2 + 32q^4 + 10q^6, -7q^1 - 26q^3 - 21q^5 - 2q^7, 9q^2 + 20q^4 + 7q^6, -7q^3 - 8q^5 - q^7, 3q^4 + q^6$

$LG_{7_6} = 4q^{-2} + 31 + 50q^2 + 16q^4, -10q^{-1} - 37q^1 - 30q^3 - 3q^5, 10 + 22q^2 + 7q^4, -5q^1 - 5q^3, q^2$

$LG_{7_7} = 24q^{-2} + 67 + 38q^2 + 4q^4, -4q^{-3} - 39q^{-1} - 46q^1 - 11q^3, 8q^{-2} + 24 + 11q^2, -5q^{-1} - 5q^1, 1$

$LG_{8_1} = 6q^{-2} + 27 + 22q^2 + 10q^4 + 2q^6, -11q^{-1} - 18q^1 - 10q^3 - 3q^5, 5 + 3q^2 + q^4$

$LG_{8_2} = 2q^{-2} + 11 + 18q^2 + 14q^4 + 2q^6, -4q^{-1} - 13q^1 - 17q^3 - 8q^5, 4 + 13q^2 + 14q^4 + 2q^6, -4q^1 - 12q^3 - 8q^5,$
$4q^2 + 9q^4 + 2q^6, -3q^3 - 3q^5, q^4$

$LG_{8_3} = 4q^{-4} + 26q^{-2} + 53 + 26q^2 + 4q^4, -7q^{-3} - 29q^{-1} - 29q^1 - 7q^3, 3q^{-2} + 10 + 3q^2$

$LG_{8_4} = 2q^{-4} + 14q^{-2} + 33 + 30q^2 + 4q^4, -4q^{-3} - 19q^{-1} - 31q^1 - 16q^3, 4q^{-2} + 16 + 21q^2 + 4q^4,$
$-3q^{-1} - 10q^1 - 7q^3, 1 + 3q^2$

$LG_{8_5} = 2q^{-2} + 15 + 30q^2 + 26q^4 + 4q^6, -5q^{-1} - 21q^1 - 30q^3 - 14q^5, 7 + 22q^2 + 22q^4 + 3q^6, -7q^1 - 17q^3 - 10q^5,$
$5q^2 + 10q^4 + 2q^6, -3q^3 - 3q^5, q^4$

$LG_{8_6} = 4q^{-2} + 33 + 62q^2 + 28q^4 + 2q^6, -10q^{-1} - 44q^1 - 44q^3 - 10q^5, 12 + 33q^2 + 17q^4 + 2q^6,$
$-9q^1 - 12q^3 - 3q^5, 3q^2 + q^4$

$LG_{8_7} = 10q^{-2} + 35 + 40q^2 + 10q^4, -2q^{-3} - 18q^{-1} - 39q^1 - 25q^3 - 2q^5, 4q^{-2} + 20 + 31q^2 + 10q^4,$
$-4q^{-1} - 18q^1 - 16q^3 - 2q^5, 4 + 11q^2 + 4q^4, -3q^1 - 3q^3, q^2$

$LG_{8_8} = 22q^{-2} + 73 + 54q^2 + 12q^4, -4q^{-3} - 42q^{-1} - 60q^1 - 24q^3 - 2q^5, 10q^{-2} + 35 + 23q^2 + 4q^4,$
$-9q^{-1} - 12q^1 - 3q^3, 3 + q^2$

$LG_{8_9} = 2q^{-4} + 28q^{-2} + 59 + 28q^2 + 2q^4, -10q^{-3} - 43q^{-1} - 43q^1 - 10q^3, 2q^{-4} + 18q^{-2} + 37 + 18q^2 + 2q^4,$
$-4q^{-3} - 18q^{-1} - 18q^1 - 4q^3, 4q^{-2} + 11 + 4q^2, -3q^{-1} - 3q^1, 1$

$LG_{8_{10}} = 12q^{-2} + 51 + 62q^2 + 16q^4, -2q^{-3} - 25q^{-1} - 58q^1 - 38q^3 - 3q^5, 5q^{-2} + 29 + 43q^2 + 13q^4,$
$-6q^{-1} - 23q^1 - 19q^3 - 2q^5, 5 + 12q^2 + 4q^4, -3q^1 - 3q^3, q^2$

$LG_{8_{11}} = 6q^{-2} + 51 + 90q^2 + 38q^4 + 2q^6, -16q^{-1} - 65q^1 - 61q^3 - 12q^5, 18 + 44q^2 + 21q^4 + 2q^6,$
$-11q^1 - 14q^3 - 3q^5, 3q^2 + q^4$

$LG_{8_{12}} = 6q^{-4} + 64q^{-2} + 129 + 64q^2 + 6q^4, -17q^{-3} - 81q^{-1} - 81q^1 - 17q^3, 17q^{-2} + 41 + 17q^2, -7q^{-1} - 7q^1, 1$

$LG_{8_{13}} = 34q^{-2} + 105 + 74q^2 + 14q^4, -6q^{-3} - 61q^{-1} - 83q^1 - 30q^3 - 2q^5, 14q^{-2} + 46 + 29q^2 + 4q^4,$
$-11q^{-1} - 14q^1 - 3q^3, 3 + q^2$

$LG_{8_{14}} = 6q^{-2} + 63 + 124q^2 + 60q^4 + 4q^6, -18q^{-1} - 85q^1 - 86q^3 - 19q^5, 22 + 56q^2 + 28q^4 + 2q^6,$
$-13q^1 - 16q^3 - 3q^5, 3q^2 + q^4$

$LG_{8_{15}} = 7 + 68q^2 + 128q^4 + 60q^6 + 4q^8, -21q^1 - 93q^3 - 91q^5 - 19q^7, 28q^2 + 67q^4 + 33q^6 + 2q^8,$
$-19q^3 - 24q^5 - 5q^7, 6q^4 + 3q^6$

$LG_{8_{16}} = 20q^{-2} + 89 + 106q^2 + 28q^4, -3q^{-3} - 43q^{-1} - 100q^1 - 65q^3 - 5q^5, 8q^{-2} + 50 + 72q^2 + 22q^4,$
$-10q^{-1} - 38q^1 - 31q^3 - 3q^5, 8 + 18q^2 + 6q^4, -4q^1 - 4q^3, q^2$

$LG_{8_{17}} = 4q^{-4} + 68q^{-2} + 139 + 68q^2 + 4q^4, -22q^{-3} - 102q^{-1} - 102q^1 - 22q^3, 3q^{-4} + 40q^{-2} + 82 + 40q^2 + 3q^4,$
$-7q^{-3} - 36q^{-1} - 36q^1 - 7q^3, 7q^{-2} + 18 + 7q^2, -4q^{-1} - 4q^1, 1$

$LG_{8_{18}} = 6q^{-4} + 102q^{-2} + 205 + 102q^2 + 6q^4, -33q^{-3} - 152q^{-1} - 152q^1 - 33q^3,$
$4q^{-4} + 60q^{-2} + 122 + 60q^2 + 4q^4, -10q^{-3} - 53q^{-1} - 53q^1 - 10q^3, 10q^{-2} + 25 + 10q^2, -5q^{-1} - 5q^1, 1$

$LG_{8_{19}} = 1 + 2q^2 + 2q^4, -q^1 - q^3, -q^4 - q^6, q^3 + q^5, q^6, -q^5 - q^7, q^6$

$LG_{8_{20}} = 7 + 8q^2 + 4q^4, -3q^{-1} - 7q^1 - 5q^3 - q^5, q^{-2} + 5 + 3q^2 + q^4, -2q^{-1} - 2q^1, 1$

$LG_{8_{21}} = 9 + 30q^2 + 18q^4 + 2q^6, -q^{-1} - 17q^1 - 23q^3 - 7q^5, 3 + 14q^2 + 8q^4 + q^6, -3q^1 - 4q^3 - q^5, q^2$

$$LG_{9_1} = 1 + 2q^2 + 2q^4 + 2q^6 + 2q^8, -q^1 - 2q^3 - 2q^5 - 2q^7 - q^9, q^2 + 2q^4 + 2q^6 + 2q^8, -q^3 - 2q^5 - 2q^7 - q^9,$$
$$q^4 + 2q^6 + 2q^8, -q^5 - 2q^7 - q^9, q^6 + 2q^8, -q^7 - q^9, q^8$$
$$LG_{9_2} = 7 + 30q^2 + 26q^4 + 14q^6 + 4q^8, -13q^1 - 22q^3 - 14q^5 - 6q^7 - q^9, 7q^2 + 5q^4 + 3q^6 + q^8$$
$$LG_{9_3} = 3 + 12q^2 + 18q^4 + 16q^6 + 4q^8, -5q^1 - 14q^3 - 18q^5 - 10q^7 - q^9, 5q^2 + 14q^4 + 16q^6 + 4q^8,$$
$$-5q^3 - 14q^5 - 10q^7 - q^9, 5q^4 + 12q^6 + 4q^8, -5q^5 - 6q^7 - q^9, 3q^6 + q^8$$
$$LG_{9_4} = 5 + 24q^2 + 40q^4 + 20q^6 + 4q^8, -9q^1 - 30q^3 - 30q^5 - 10q^7 - q^9, 9q^2 + 26q^4 + 16q^6 + 4q^8,$$
$$-9q^3 - 14q^5 - 6q^7 - q^9, 5q^4 + 3q^6 + q^8$$
$$LG_{9_5} = 15 + 70q^2 + 70q^4 + 32q^6 + 6q^8, -29q^1 - 55q^3 - 36q^5 - 11q^7 - q^9, 15q^2 + 14q^4 + 6q^6 + q^8$$
$$LG_{9_6} = 3 + 20q^2 + 42q^4 + 40q^6 + 10q^8, -7q^1 - 28q^3 - 44q^5 - 25q^7 - 2q^9, 9q^2 + 30q^4 + 36q^6 + 10q^8,$$
$$-9q^3 - 28q^5 - 21q^7 - 2q^9, 9q^4 + 20q^6 + 7q^8, -7q^5 - 8q^7 - q^9, 3q^6 + q^8$$
$$LG_{9_7} = 5 + 42q^2 + 88q^4 + 52q^6 + 10q^8, -13q^1 - 60q^3 - 69q^5 - 24q^7 - 2q^9, 17q^2 + 48q^4 + 31q^6 + 7q^8,$$
$$-13q^3 - 20q^5 - 8q^7 - q^9, 5q^4 + 3q^6 + q^8$$
$$LG_{9_8} = 4q^{-4} + 34q^{-2} + 91 + 100q^2 + 28q^4, -10q^{-3} - 48q^{-1} - 89q^1 - 56q^3 - 5q^5, 10q^{-2} + 37 + 48q^2 + 13q^4,$$
$$-5q^{-1} - 16q^1 - 11q^3, 1 + 3q^2$$
$$LG_{9_9} = 3 + 22q^2 + 58q^4 + 62q^6 + 16q^8, -7q^1 - 36q^3 - 64q^5 - 38q^7 - 3q^9, 11q^2 + 42q^4 + 50q^6 + 13q^8,$$
$$-13q^3 - 36q^5 - 25q^7 - 2q^9, 11q^4 + 22q^6 + 7q^8, -7q^5 - 8q^7 - q^9, 3q^6 + q^8$$
$$LG_{9_{10}} = 9 + 62q^2 + 110q^4 + 54q^6 + 6q^8, -21q^1 - 82q^3 - 82q^5 - 22q^7 - q^9, 25q^2 + 66q^4 + 39q^6 + 6q^8,$$
$$-21q^3 - 31q^5 - 11q^7 - q^9, 9q^4 + 6q^6 + q^8$$
$$LG_{9_{11}} = 4q^{-2} + 33 + 76q^2 + 70q^4 + 16q^6, -10q^{-1} - 47q^1 - 76q^3 - 42q^5 - 3q^7, 12 + 47q^2 + 58q^4 + 16q^6,$$
$$-12q^1 - 39q^3 - 30q^5 - 3q^7, 10q^2 + 22q^4 + 7q^6, -5q^3 - 5q^5, q^4$$
$$LG_{9_{12}} = 8q^{-2} + 75 + 152q^2 + 88q^4 + 16q^6, -22q^{-1} - 99q^1 - 110q^3 - 36q^5 - 3q^7, 24 + 64q^2 + 38q^4 + 7q^6,$$
$$-13q^1 - 18q^3 - 5q^5, 3q^2 + q^4$$
$$LG_{9_{13}} = 9 + 72q^2 + 142q^4 + 80q^6 + 12q^8, -23q^1 - 100q^3 - 110q^5 - 35q^7 - 2q^9, 29q^2 + 80q^4 + 51q^6 + 9q^8,$$
$$-23q^3 - 35q^5 - 13q^7 - q^9, 9q^4 + 6q^6 + q^8$$
$$LG_{9_{14}} = 48q^{-2} + 161 + 138q^2 + 44q^4 + 4q^6, -8q^{-3} - 87q^{-1} - 134q^1 - 66q^3 - 11q^5, 18q^{-2} + 64 + 48q^2 + 11q^4,$$
$$-13q^{-1} - 18q^1 - 5q^3, 3 + q^2$$
$$LG_{9_{15}} = 8q^{-2} + 89 + 196q^2 + 120q^4 + 20q^6, -24q^{-1} - 123q^1 - 143q^3 - 47q^5 - 3q^7, 28 + 78q^2 + 47q^4 + 7q^6,$$
$$-15q^1 - 20q^3 - 5q^5, 3q^2 + q^4$$
$$LG_{9_{16}} = 3 + 32q^2 + 98q^4 + 106q^6 + 28q^8, -9q^1 - 57q^3 - 108q^5 - 65q^7 - 5q^9, 16q^2 + 67q^4 + 81q^6 + 22q^8,$$
$$-19q^3 - 55q^5 - 39q^7 - 3q^9, 16q^4 + 31q^6 + 10q^8, -9q^5 - 10q^7 - q^9, 3q^6 + q^8$$
$$LG_{9_{17}} = 4q^{-4} + 44q^{-2} + 115 + 108q^2 + 24q^4, -12q^{-3} - 67q^{-1} - 115q^1 - 64q^3 - 4q^5, 16q^{-2} + 66 + 83q^2 + 24q^4,$$
$$-15q^{-1} - 50q^1 - 39q^3 - 4q^5, 11 + 24q^2 + 8q^4, -5q^1 - 5q^3, q^2$$
$$LG_{9_{18}} = 9 + 90q^2 + 184q^4 + 104q^6 + 14q^8, -27q^1 - 128q^3 - 141q^5 - 42q^7 - 2q^9, 37q^2 + 98q^4 + 60q^6 + 9q^8,$$
$$-27q^3 - 39q^5 - 13q^7 - q^9, 9q^4 + 6q^6 + q^8$$
$$LG_{9_{19}} = 8q^{-4} + 96q^{-2} + 221 + 144q^2 + 28q^4, -25q^{-3} - 133q^{-1} - 161q^1 - 57q^3 - 4q^5, 29q^{-2} + 81 + 50q^2 + 8q^4,$$
$$-15q^{-1} - 20q^1 - 5q^3, 3 + q^2$$
$$LG_{9_{20}} = 4q^{-2} + 43 + 124q^2 + 130q^4 + 34q^6, -12q^{-1} - 71q^1 - 131q^3 - 78q^5 - 6q^7, 18 + 75q^2 + 91q^4 + 25q^6,$$
$$-18q^1 - 53q^3 - 38q^5 - 3q^7, 12q^2 + 24q^4 + 7q^6, -5q^3 - 5q^5, q^4$$
$$LG_{9_{21}} = 12q^{-2} + 119 + 244q^2 + 142q^4 + 22q^6, -34q^{-1} - 156q^1 - 172q^3 - 53q^5 - 3q^7, 36 + 93q^2 + 53q^4 + 7q^6,$$
$$-17q^1 - 22q^3 - 5q^5, 3q^2 + q^4$$
$$LG_{9_{22}} = 4q^{-4} + 50q^{-2} + 143 + 142q^2 + 34q^4, -13q^{-3} - 81q^{-1} - 146q^1 - 84q^3 - 6q^5, 19q^{-2} + 81 + 101q^2 + 29q^4,$$
$$-18q^{-1} - 57q^1 - 43q^3 - 4q^5, 12 + 25q^2 + 8q^4, -5q^1 - 5q^3, q^2$$
$$LG_{9_{23}} = 9 + 102q^2 + 226q^4 + 140q^6 + 22q^8, -29q^1 - 150q^3 - 177q^5 - 59q^7 - 3q^9, 41q^2 + 114q^4 + 74q^6 + 12q^8,$$
$$-29q^3 - 43q^5 - 15q^7 - q^9, 9q^4 + 6q^6 + q^8$$
$$LG_{9_{24}} = 4q^{-4} + 82q^{-2} + 193 + 122q^2 + 20q^4, -26q^{-3} - 134q^{-1} - 156q^1 - 51q^3 - 3q^5,$$
$$4q^{-4} + 53q^{-2} + 117 + 67q^2 + 9q^4, -10q^{-3} - 51q^{-1} - 53q^1 - 12q^3, 10q^{-2} + 25 + 10q^2, -5q^{-1} - 5q^1, 1$$
$$LG_{9_{25}} = 10q^{-2} + 119 + 268q^2 + 170q^4 + 28q^6, -32q^{-1} - 170q^1 - 204q^3 - 70q^5 - 4q^7, 41 + 117q^2 + 76q^4 + 12q^6,$$
$$-25q^1 - 36q^3 - 11q^5, 6q^2 + 3q^4$$
$$LG_{9_{26}} = 28q^{-2} + 143 + 206q^2 + 82q^4 + 4q^6, -4q^{-3} - 62q^{-1} - 173q^1 - 141q^3 - 26q^5,$$
$$10q^{-2} + 74 + 128q^2 + 57q^4 + 4q^6, -12q^{-1} - 57q^1 - 56q^3 - 11q^5, 10 + 26q^2 + 11q^4, -5q^1 - 5q^3, q^2$$
$$LG_{9_{27}} = 4q^{-4} + 94q^{-2} + 235 + 158q^2 + 28q^4, -28q^{-3} - 157q^{-1} - 193q^1 - 68q^3 - 4q^5,$$
$$4q^{-4} + 58q^{-2} + 136 + 83q^2 + 12q^4, -10q^{-3} - 55q^{-1} - 60q^1 - 15q^3, 10q^{-2} + 26 + 11q^2, -5q^{-1} - 5q^1, 1$$

$LG_{9_{28}} = 30q^{-2} + 171 + 254q^2 + 104q^4 + 6q^6, -4q^{-3} - 73q^{-1} - 210q^1 - 172q^3 - 31q^5,$
$12q^{-2} + 91 + 150q^2 + 62q^4 + 3q^6, -16q^{-1} - 66q^1 - 59q^3 - 9q^5, 12 + 27q^2 + 10q^4, -5q^1 - 5q^3, q^2$

$LG_{9_{29}} = 6q^{-4} + 76q^{-2} + 217 + 214q^2 + 52q^4, -20q^{-3} - 123q^{-1} - 216q^1 - 122q^3 - 9q^5,$
$29q^{-2} + 116 + 136q^2 + 37q^4, -25q^{-1} - 71q^1 - 50q^3 - 4q^5, 14 + 27q^2 + 8q^4, -5q^1 - 5q^3, q^2$

$LG_{9_{30}} = 6q^{-4} + 120q^{-2} + 287 + 186q^2 + 30q^4, -35q^{-3} - 192q^{-1} - 230q^1 - 77q^3 - 4q^5,$
$4q^{-4} + 67q^{-2} + 158 + 96q^2 + 13q^4, -10q^{-3} - 60q^{-1} - 67q^1 - 17q^3, 10q^{-2} + 27 + 12q^2, -5q^{-1} - 5q^1, 1$

$LG_{9_{31}} = 32q^{-2} + 197 + 308q^2 + 134q^4 + 8q^6, -4q^{-3} - 80q^{-1} - 246q^1 - 211q^3 - 41q^5,$
$12q^{-2} + 100 + 173q^2 + 76q^4 + 4q^6, -16q^{-1} - 71q^1 - 66q^3 - 11q^5, 12 + 28q^2 + 11q^4, -5q^1 - 5q^3, q^2$

$LG_{9_{32}} = 46q^{-2} + 241 + 334q^2 + 128q^4 + 6q^6, -6q^{-3} - 106q^{-1} - 284q^1 - 222q^3 - 38q^5,$
$17q^{-2} + 124 + 199q^2 + 83q^4 + 5q^6, -21q^{-1} - 87q^1 - 80q^3 - 14q^5, 15 + 35q^2 + 14q^4, -6q^1 - 6q^3, q^2$

$LG_{9_{33}} = 8q^{-4} + 162q^{-2} + 377 + 242q^2 + 38q^4, -49q^{-3} - 257q^{-1} - 302q^1 - 99q^3 - 5q^5,$
$6q^{-4} + 94q^{-2} + 211 + 125q^2 + 16q^4, -15q^{-3} - 82q^{-1} - 88q^1 - 21q^3, 14q^{-2} + 35 + 15q^2,$
$-6q^{-1} - 6q^1, 1$

$LG_{9_{34}} = 12q^{-4} + 216q^{-2} + 503 + 330q^2 + 54q^4, -63q^{-3} - 334q^{-1} - 400q^1 - 136q^3 - 7q^5,$
$6q^{-4} + 112q^{-2} + 262 + 162q^2 + 22q^4, -15q^{-3} - 92q^{-1} - 104q^1 - 27q^3, 14q^{-2} + 37 + 17q^2,$
$-6q^{-1} - 6q^1, 1$

$LG_{9_{35}} = 19 + 92q^2 + 100q^4 + 48q^6 + 8q^8, -37q^1 - 75q^3 - 53q^5 - 16q^7 - q^9, 19q^2 + 20q^4 + 9q^6 + q^8$

$LG_{9_{36}} = 4q^{-2} + 41 + 100q^2 + 94q^4 + 22q^6, -12q^{-1} - 62q^1 - 101q^3 - 55q^5 - 4q^7, 17 + 62q^2 + 72q^4 + 19q^6,$
$-16q^1 - 46q^3 - 33q^5 - 3q^7, 11q^2 + 23q^4 + 7q^6, -5q^3 - 5q^5, q^4$

$LG_{9_{37}} = 10q^{-4} + 120q^{-2} + 273 + 176q^2 + 32q^4, -31q^{-3} - 164q^{-1} - 196q^1 - 67q^3 - 4q^5,$
$35q^{-2} + 96 + 58q^2 + 8q^4, -17q^{-1} - 22q^1 - 5q^3, 3 + q^2$

$LG_{9_{38}} = 17 + 178q^2 + 364q^4 + 214q^6 + 30q^8, -53q^1 - 249q^3 - 279q^5 - 87q^7 - 4q^9, 70q^2 + 183q^4 + 116q^6 + 17q^8,$
$-47q^3 - 69q^5 - 23q^7 - q^9, 14q^4 + 10q^6 + q^8$

$LG_{9_{39}} = 16q^{-2} + 177 + 382q^2 + 238q^4 + 38q^6, -49q^{-1} - 242q^1 - 282q^3 - 94q^5 - 5q^7, 57 + 154q^2 + 97q^4 + 14q^6,$
$-30q^1 - 42q^3 - 12q^5, 6q^2 + 3q^4$

$LG_{9_{40}} = 72q^{-2} + 399 + 566q^2 + 228q^4 + 12q^6, -9q^{-3} - 172q^{-1} - 471q^1 - 375q^3 - 67q^5,$
$27q^{-2} + 200 + 316q^2 + 132q^4 + 7q^6, -33q^{-1} - 130q^1 - 116q^3 - 19q^5, 21 + 46q^2 + 18q^4, -7q^1 - 7q^3, q^2$

$LG_{9_{41}} = 72q^{-2} + 261 + 246q^2 + 82q^4 + 6q^6, -12q^{-3} - 138q^{-1} - 232q^1 - 126q^3 - 20q^5,$
$29q^{-2} + 111 + 94q^2 + 24q^4, -23q^{-1} - 36q^1 - 13q^3, 6 + 3q^2$

$LG_{9_{42}} = 2q^{-4} + 6q^{-2} + 3, -3q^{-3} - 3q^{-1} - q^1 - q^3, q^{-2} + 1 + 3q^2 + q^4, -2q^1 - 2q^3, q^2$

$LG_{9_{43}} = 2q^{-2} + 11 + 12q^2 + 4q^4, -4q^{-1} - 10q^1 - 7q^3 - q^5, 3 + 6q^2 + 5q^4, -2q^1 - 7q^3 - 5q^5, 3q^2 + 8q^4 + 2q^6,$
$-3q^3 - 3q^5, q^4$

$LG_{9_{44}} = 8q^{-2} + 35 + 30q^2 + 10q^4, -q^{-3} - 17q^{-1} - 29q^1 - 15q^3 - 2q^5, 3q^{-2} + 14 + 10q^2 + 3q^4,$
$-3q^{-1} - 4q^1 - q^3, 1$

$LG_{9_{45}} = 19 + 70q^2 + 54q^4 + 12q^6, -2q^{-1} - 36q^1 - 56q^3 - 24q^5 - 2q^7, 5 + 26q^2 + 19q^4 + 4q^6, -4q^1 - 6q^3 - 2q^5, q^2$

$LG_{9_{46}} = 2q^{-2} + 11 + 10q^2 + 8q^4 + 2q^6, -4q^{-1} - 7q^1 - 6q^3 - 3q^5, 2 + q^2 + q^4$

$LG_{9_{47}} = 18q^{-2} + 57 + 50q^2 + 6q^4, -3q^{-3} - 31q^{-1} - 55q^1 - 27q^3, 6q^{-2} + 30 + 41q^2 + 11q^4,$
$-6q^{-1} - 26q^1 - 23q^3 - 3q^5, 6 + 16q^2 + 6q^4, -4q^1 - 4q^3, q^2$

$LG_{9_{48}} = 8q^{-2} + 63 + 106q^2 + 42q^4 + 2q^6, -20q^{-1} - 74q^1 - 64q^3 - 10q^5, 18 + 39q^2 + 14q^4, -7q^1 - 7q^3, q^2$

$LG_{9_{49}} = 7 + 44q^2 + 66q^4 + 22q^6, -16q^1 - 54q^3 - 44q^5 - 6q^7, 18q^2 + 41q^4 + 18q^6 + q^8, -14q^3 - 18q^5 - 4q^7,$
$6q^4 + 3q^6$

$$
\begin{aligned}
LG_{10_1} &= 8q^{-2} + 37 + 34q^2 + 22q^4 + 10q^6 + 2q^8, -15q^{-1} - 26q^1 - 18q^3 - 10q^5 - 3q^7, 7 + 5q^2 + 3q^4 + q^6 \\
LG_{10_2} &= 2q^{-2} + 11 + 18q^2 + 18q^4 + 14q^6 + 2q^8, -4q^{-1} - 13q^1 - 18q^3 - 17q^5 - 8q^7, \\
&\quad 4 + 13q^2 + 18q^4 + 14q^6 + 2q^8, -4q^1 - 13q^3 - 17q^5 - 8q^7, 4q^2 + 13q^4 + 14q^6 + 2q^8, -4q^3 - 12q^5 - 8q^7, \\
&\quad 4q^4 + 9q^6 + 2q^8, -3q^5 - 3q^7, q^6 \\
LG_{10_3} &= 6q^{-4} + 42q^{-2} + 95 + 68q^2 + 26q^4 + 4q^6, -11q^{-3} - 49q^{-1} - 60q^1 - 29q^3 - 7q^5, 5q^{-2} + 18 + 10q^2 + 3q^4 \\
LG_{10_4} &= 6q^{-4} + 46q^{-2} + 59 + 38q^2 + 14q^4 + 2q^6, -24q^{-3} - 51q^{-1} - 42q^1 - 19q^3 - 4q^5, \\
&\quad 6q^{-4} + 33q^{-2} + 32 + 16q^2 + 4q^4, -11q^{-3} - 18q^{-1} - 10q^1 - 3q^3, 5q^{-2} + 3 + q^2 \\
LG_{10_5} &= 10q^{-2} + 35 + 50q^2 + 40q^4 + 10q^6, -2q^{-3} - 18q^{-1} - 41q^1 - 48q^3 - 25q^5 - 2q^7, \\
&\quad 4q^{-2} + 20 + 41q^2 + 40q^4 + 10q^6, -4q^{-1} - 20q^1 - 39q^3 - 25q^5 - 2q^7, 4 + 20q^2 + 31q^4 + 10q^6, \\
&\quad -4q^1 - 18q^3 - 16q^5 - 2q^7, 4q^2 + 11q^4 + 4q^6, -3q^3 - 3q^5, q^4 \\
LG_{10_6} &= 4q^{-2} + 33 + 78q^2 + 82q^4 + 28q^6 + 2q^8, -10q^{-1} - 47q^1 - 83q^3 - 56q^5 - 10q^7, \\
&\quad 12 + 49q^2 + 70q^4 + 28q^6 + 2q^8, -12q^1 - 46q^3 - 44q^5 - 10q^7, 12q^2 + 33q^4 + 17q^6 + 2q^8, \\
&\quad -9q^3 - 12q^5 - 3q^7, 3q^4 + q^6 \\
LG_{10_7} &= 10q^{-2} + 95 + 202q^2 + 138q^4 + 38q^6 + 2q^8, -28q^{-1} - 129q^1 - 158q^3 - 69q^5 - 12q^7, \\
&\quad 32 + 90q^2 + 66q^4 + 21q^6 + 2q^8, -19q^1 - 30q^3 - 14q^5 - 3q^7, 5q^2 + 3q^4 + q^6 \\
LG_{10_8} &= 2q^{-4} + 14q^{-2} + 37 + 46q^2 + 30q^4 + 4q^6, -4q^{-3} - 20q^{-1} - 40q^1 - 40q^3 - 16q^5, \\
&\quad 4q^{-2} + 20 + 37q^2 + 30q^4 + 4q^6, -4q^{-1} - 19q^1 - 31q^3 - 16q^5, 4 + 16q^2 + 21q^4 + 4q^6, \\
&\quad -3q^1 - 10q^3 - 7q^5, q^2 + 3q^4 \\
LG_{10_9} &= 2q^{-4} + 28q^{-2} + 75 + 82q^2 + 28q^4 + 2q^6, -10q^{-3} - 46q^{-1} - 82q^1 - 56q^3 - 10q^5, \\
&\quad 2q^{-4} + 18q^{-2} + 52 + 69q^2 + 28q^4 + 2q^6, -4q^{-3} - 20q^{-1} - 49q^1 - 43q^3 - 10q^5, \\
&\quad 4q^{-2} + 37q^2 + 20 + 18q^4 + 2q^6, -4q^{-1} - 18q^1 - 18q^3 - 4q^5, 4 + 11q^2 + 4q^4, -3q^1 - 3q^3, q^2 \\
LG_{10_{10}} &= 58q^{-2} + 201 + 194q^2 + 82q^4 + 14q^6, -10q^{-3} - 109q^{-1} - 180q^1 - 109q^3 - 30q^5 - 2q^7, \\
&\quad 24q^{-2} + 88 + 78q^2 + 29q^4 + 4q^6, -19q^{-1} - 30q^1 - 14q^3 - 3q^5, 5 + 3q^2 + q^4 \\
LG_{10_{11}} &= 4q^{-4} + 42q^{-2} + 143 + 182q^2 + 68q^4 + 4q^6, -10q^{-3} - 69q^{-1} - 155q^1 - 118q^3 - 22q^5, \\
&\quad 12q^{-2} + 64 + 102q^2 + 43q^4 + 4q^6, -9q^{-1} - 37q^1 - 35q^3 - 7q^5, 3 + 10q^2 + 3q^4 \\
LG_{10_{12}} &= 22q^{-2} + 107 + 172q^2 + 88q^4 + 12q^6, -4q^{-3} - 48q^{-1} - 139q^1 - 132q^3 - 39q^5 - 2q^7, \\
&\quad 10q^{-2} + 62 + 119q^2 + 69q^4 + 12q^6, -12q^{-1} - 58q^1 - 68q^3 - 24q^5 - 2q^7, 12 + 37q^2 + 23q^4 + 4q^6, \\
&\quad -9q^1 - 12q^3 - 3q^5, 3q^2 + q^4 \\
LG_{10_{13}} &= 12q^{-4} + 148q^{-2} + 361 + 272q^2 + 76q^4 + 6q^6, -37q^{-3} - 206q^{-1} - 271q^1 - 119q^3 - 17q^5, \\
&\quad 41q^{-2} + 120 + 83q^2 + 17q^4, -19q^{-1} - 26q^1 - 7q^3, 3 + q^2 \\
LG_{10_{14}} &= 6q^{-2} + 67 + 202q^2 + 232q^4 + 82q^6 + 4q^8, -18q^{-1} - 111q^1 - 224q^3 - 157q^5 - 26q^7, \\
&\quad 26 + 123q^2 + 176q^4 + 71q^6 + 4q^8, -28q^1 - 103q^3 - 94q^5 - 19q^7, 24q^2 + 58q^4 + 28q^6 + 2q^8, \\
&\quad -13q^3 - 16q^5 - 3q^7, 3q^4 + q^6 \\
LG_{10_{15}} &= 12q^{-4} + 64q^{-2} + 123 + 102q^2 + 22q^4, -2q^{-5} - 26q^{-3} - 86q^{-1} - 117q^1 - 59q^3 - 4q^5, \\
&\quad 4q^{-4} + 32q^{-2} + 82 + 85q^2 + 22q^4, -4q^{-3} - 30q^{-1} - 64q^1 - 42q^3 - 4q^5, 4q^{-2} + 23 + 35q^2 + 10q^4, \\
&\quad -3q^{-1} - 12q^1 - 9q^3, 1 + 3q^2 \\
LG_{10_{16}} &= 6q^{-4} + 60q^{-2} + 181 + 216q^2 + 78q^4 + 4q^6, -16q^{-3} - 92q^{-1} - 188q^1 - 136q^3 - 24q^5, \\
&\quad 18q^{-2} + 79 + 116q^2 + 47q^4 + 4q^6, -11q^{-1} - 41q^1 - 37q^3 - 7q^5, 3 + 10q^2 + 3q^4 \\
LG_{10_{17}} &= 10q^{-4} + 60q^{-2} + 109 + 60q^2 + 10q^4, -2q^{-5} - 28q^{-3} - 86q^{-1} - 86q^1 - 28q^3 - 2q^5, \\
&\quad 10q^{-4} + 46q^{-2} + 79 + 46q^2 + 10q^4, -2q^{-5} - 18q^{-3} - 49q^{-1} - 49q^1 - 18q^3 - 2q^5, \\
&\quad 4q^{-4} + 20q^{-2} + 37 + 20q^2 + 4q^4, -4q^{-3} - 18q^{-1} - 18q^1 - 4q^3, 4q^{-2} + 11 + 4q^2, -3q^{-1} - 3q^1, 1 \\
LG_{10_{18}} &= 6q^{-4} + 72q^{-2} + 247 + 322q^2 + 130q^4 + 8q^6, -18q^{-3} - 118q^{-1} - 265q^1 - 204q^3 - 39q^5, \\
&\quad 22q^{-2} + 103 + 155q^2 + 64q^4 + 4q^6, -13q^{-1} - 49q^1 - 43q^3 - 7q^5, 3 + 10q^2 + 3q^4 \\
LG_{10_{19}} &= 14q^{-4} + 86q^{-2} + 181 + 154q^2 + 34q^4, -2q^{-5} - 32q^{-3} - 120q^{-1} - 174q^1 - 90q^3 - 6q^5, \\
&\quad 4q^{-4} + 40q^{-2} + 116 + 125q^2 + 34q^4, -4q^{-3} - 38q^{-1} - 89q^1 - 61q^3 - 6q^5, 4q^{-2} + 29 + 46q^2 + 14q^4, \\
&\quad -3q^{-1} - 14q^1 - 11q^3, 1 + 3q^2 \\
LG_{10_{20}} &= 6q^{-2} + 55 + 122q^2 + 88q^4 + 28q^6 + 2q^8, -16q^{-1} - 78q^1 - 100q^3 - 48q^5 - 10q^7, \\
&\quad 20 + 61q^2 + 47q^4 + 17q^6 + 2q^8, -15q^1 - 24q^3 - 12q^5 - 3q^7, 5q^2 + 3q^4 + q^6 \\
LG_{10_{21}} &= 6q^{-2} + 53 + 126q^2 + 124q^4 + 38q^6 + 2q^8, -16q^{-1} - 77q^1 - 130q^3 - 81q^5 - 12q^7, \\
&\quad 20 + 79q^2 + 104q^4 + 38q^6 + 2q^8, -20q^1 - 69q^3 - 61q^5 - 12q^7, 18q^2 + 44q^4 + 21q^6 + 2q^8, \\
&\quad -11q^3 - 14q^5 - 3q^7, 3q^4 + q^6
\end{aligned}
$$

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\begin{aligned}
LG_{10_{22}} &= 4q^{-4}+68q^{-2}+185+148q^{2}+42q^{4}+2q^{6}, -22q^{-3}-121q^{-1}-168q^{1}-81q^{3}-12q^{5},\\
&\quad 4q^{-4}+48q^{-2}+123+93q^{2}+26q^{4}+2q^{6}, -10q^{-3}-56q^{-1}-72q^{1}-30q^{3}-4q^{5},\\
&\quad 12q^{-2}+37+23q^{2}+4q^{4}, -9q^{-1}-12q^{1}-3q^{3}, 3+q^{2}\\
LG_{10_{23}} &= 38q^{-2}+193+290q^{2}+134q^{4}+14q^{6}, -6q^{-3}-86q^{-1}-241q^{1}-212q^{3}-53q^{5}-2q^{7},\\
&\quad 16q^{-2}+108+192q^{2}+101q^{4}+14q^{6}, -20q^{-1}-91q^{1}-99q^{3}-30q^{5}-2q^{7}, 18+50q^{2}+29q^{4}+4q^{6},\\
&\quad -11q^{1}-14q^{3}-3q^{5}, 3q^{2}+q^{4}\\
LG_{10_{24}} &= 12q^{-2}+141+334q^{2}+240q^{4}+56q^{6}+2q^{8}, -38q^{-1}-207q^{1}-270q^{3}-115q^{5}-14q^{7},\\
&\quad 50+152q^{2}+115q^{4}+29q^{6}+2q^{8}, -33q^{1}-53q^{3}-23q^{5}-3q^{7}, 9q^{2}+6q^{4}+q^{6}\\
LG_{10_{25}} &= 6q^{-2}+79+274q^{2}+332q^{4}+120q^{6}+6q^{8}, -20q^{-1}-145q^{1}-311q^{3}-221q^{5}-35q^{7},\\
&\quad 34+167q^{2}+231q^{4}+88q^{6}+4q^{8}, -38q^{1}-127q^{3}-108q^{5}-19q^{7}, 28q^{2}+62q^{4}+28q^{6}+2q^{8},\\
&\quad -13q^{3}-16q^{5}-3q^{7}, 3q^{4}+q^{6}\\
LG_{10_{26}} &= 6q^{-4}+122q^{-2}+313+234q^{2}+56q^{4}+2q^{6}, -38q^{-3}-209q^{-1}-274q^{1}-117q^{3}-14q^{5},\\
&\quad 6q^{-4}+82q^{-2}+198+139q^{2}+32q^{4}+2q^{6}, -16q^{-3}-87q^{-1}-105q^{1}-38q^{3}-4q^{5},\\
&\quad 18q^{-2}+50+29q^{2}+4q^{4}, -11q^{-1}-14q^{1}-3q^{3}, 3+q^{2}\\
LG_{10_{27}} &= 42q^{-2}+259+442q^{2}+236q^{4}+30q^{6}, -6q^{-3}-106q^{-1}-345q^{1}-338q^{3}-97q^{5}-4q^{7},\\
&\quad 18q^{-2}+146+280q^{2}+158q^{4}+22q^{6}, -26q^{-1}-125q^{1}-138q^{3}-41q^{5}-2q^{7}, 24+64q^{2}+36q^{4}+4q^{6},\\
&\quad -13q^{1}-16q^{3}-3q^{5}, 3q^{2}+q^{4}\\
LG_{10_{28}} &= 70q^{-2}+263+270q^{2}+112q^{4}+16q^{6}, -12q^{-3}-141q^{-1}-249q^{1}-156q^{3}-38q^{5}-2q^{7},\\
&\quad 32q^{-2}+126+118q^{2}+41q^{4}+4q^{6}, -29q^{-1}-49q^{1}-23q^{3}-3q^{5}, 9+6q^{2}+q^{4}\\
LG_{10_{29}} &= 6q^{-4}+84q^{-2}+283+338q^{2}+122q^{4}+6q^{6}, -20q^{-3}-145q^{-1}-309q^{1}-222q^{3}-38q^{5},\\
&\quad 30q^{-2}+150+218q^{2}+89q^{4}+6q^{6}, -29q^{-1}-105q^{1}-93q^{3}-17q^{5}, 19+43q^{2}+17q^{4}, -7q^{1}-7q^{3}, q^{2}\\
LG_{10_{30}} &= 18q^{-2}+217+516q^{2}+384q^{4}+96q^{6}+4q^{8}, -58q^{-1}-312q^{1}-412q^{3}-181q^{5}-23q^{7},\\
&\quad 72+213q^{2}+162q^{4}+40q^{6}+2q^{8}, -41q^{1}-65q^{3}-27q^{5}-3q^{7}, 9q^{2}+6q^{4}+q^{6}\\
LG_{10_{31}} &= 30q^{-4}+200q^{-2}+373+222q^{2}+40q^{4}, -4q^{-5}-74q^{-3}-257q^{-1}-270q^{1}-89q^{3}-6q^{5},\\
&\quad 10q^{-4}+83q^{-2}+165+92q^{2}+14q^{4}, -9q^{-3}-45q^{-1}-47q^{1}-11q^{3}, 3q^{-2}+10+3q^{2}\\
LG_{10_{32}} &= 6q^{-4}+142q^{-2}+391+318q^{2}+86q^{4}+4q^{6}, -42q^{-3}-255q^{-1}-357q^{1}-166q^{3}-22q^{5},\\
&\quad 6q^{-4}+98q^{-2}+252+187q^{2}+44q^{4}+2q^{6}, -18q^{-3}-109q^{-1}-136q^{1}-49q^{3}-4q^{5},\\
&\quad 22q^{-2}+62+36q^{2}+4q^{4}, -13q^{-1}-16q^{1}-3q^{3}, 3+q^{2}\\
LG_{10_{33}} &= 46q^{-4}+286q^{-2}+505+286q^{2}+46q^{4}, -6q^{-5}-107q^{-3}-351q^{-1}-351q^{1}-107q^{3}-6q^{5},\\
&\quad 14q^{-4}+110q^{-2}+208+110q^{2}+14q^{4}, -11q^{-3}-53q^{-1}-53q^{1}-11q^{3}, 3q^{-2}+10+3q^{2}\\
LG_{10_{34}} &= 34q^{-2}+123+122q^{2}+58q^{4}+12q^{6}, -6q^{-3}-68q^{-1}-114q^{1}-74q^{3}-24q^{5}-2q^{7},\\
&\quad 16q^{-2}+61+55q^{2}+23q^{4}+4q^{6}, -15q^{-1}-24q^{1}-12q^{3}-3q^{5}, 5+3q^{2}+q^{4}\\
LG_{10_{35}} &= 10q^{-4}+122q^{-2}+299+230q^{2}+70q^{4}+6q^{6}, -31q^{-3}-171q^{-1}-228q^{1}-105q^{3}-17q^{5},\\
&\quad 35q^{-2}+103+73q^{2}+17q^{4}, -17q^{-1}-24q^{1}-7q^{3}, 3+q^{2}\\
LG_{10_{36}} &= 10q^{-2}+121+292q^{2}+224q^{4}+66q^{6}+4q^{8}, -32q^{-1}-175q^{1}-235q^{3}-111q^{5}-19q^{7},\\
&\quad 40+120q^{2}+93q^{4}+28q^{6}+2q^{8}, -23q^{1}-36q^{3}-16q^{5}-3q^{7}, 5q^{2}+3q^{4}+q^{6}\\
LG_{10_{37}} &= 28q^{-4}+178q^{-2}+319+178q^{2}+28q^{4}, -4q^{-5}-68q^{-3}-227q^{-1}-227q^{1}-68q^{3}-4q^{5},\\
&\quad 10q^{-4}+77q^{-2}+147+77q^{2}+10q^{4}, -9q^{-3}-43q^{-1}-43q^{1}-9q^{3}, 3q^{-2}+10+3q^{2}\\
LG_{10_{38}} &= 12q^{-2}+153+380q^{2}+294q^{4}+80q^{6}+4q^{8}, -40q^{-1}-229q^{1}-314q^{3}-146q^{5}-21q^{7},\\
&\quad 54+168q^{2}+133q^{4}+36q^{6}+2q^{8}, -35q^{1}-57q^{3}-25q^{5}-3q^{7}, 9q^{2}+6q^{4}+q^{6}\\
LG_{10_{39}} &= 6q^{-2}+77+242q^{2}+276q^{4}+94q^{6}+4q^{8}, -20q^{-1}-133q^{1}-267q^{3}-182q^{5}-28q^{7},\\
&\quad 32+147q^{2}+201q^{4}+77q^{6}+4q^{8}, -34q^{1}-115q^{3}-100q^{5}-19q^{7}, 26q^{2}+60q^{4}+28q^{6}+2q^{8},\\
&\quad -13q^{3}-16q^{5}-3q^{7}, 3q^{4}+q^{6}\\
LG_{10_{40}} &= 46q^{-2}+301+510q^{2}+266q^{4}+32q^{6}, -6q^{-3}-122q^{-1}-398q^{1}-383q^{3}-105q^{5}-4q^{7},\\
&\quad 20q^{-2}+168+313q^{2}+170q^{4}+22q^{6}, -30q^{-1}-137q^{1}-146q^{3}-41q^{5}-2q^{7}, 26+66q^{2}+36q^{4}+4q^{6},\\
&\quad -13q^{1}-16q^{3}-3q^{5}, 3q^{2}+q^{4}\\
LG_{10_{41}} &= 8q^{-4}+387+116q^{-2}+448q^{2}+154q^{4}+6q^{6}, -28q^{-3}-200q^{-1}-411q^{1}-283q^{3}-44q^{5},\\
&\quad 42q^{-2}+197+269q^{2}+103q^{4}+6q^{6}, -37q^{-1}-123q^{1}-103q^{3}-17q^{5}, 21+45q^{2}+17q^{4}, -7q^{1}-7q^{3}, q^{2}\\
LG_{10_{42}} &= 60q^{-4}+394q^{-2}+679+368q^{2}+50q^{4}, -8q^{-5}-157q^{-3}-507q^{-1}-490q^{1}-138q^{3}-6q^{5},\\
&\quad 26q^{-4}+198q^{-2}+352+181q^{2}+20q^{4}, -33q^{-3}-133q^{-1}-127q^{1}-27q^{3}, 21q^{-2}+47+19q^{2},\\
&\quad -7q^{-1}-7q^{1}, 1
\end{aligned}
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\begin{aligned}
LG_{10_{43}} &= 44q^{-4}+296q^{-2}+527+296q^{2}+44q^{4}, -6q^{-5}-118q^{-3}-393q^{-1}-393q^{1}-118q^{3}-6q^{5},\\
&\quad 20q^{-4}+157q^{-2}+291+157q^{2}+20q^{4}, -27q^{-3}-115q^{-1}-115q^{1}-27q^{3}, 19q^{-2}+45+19q^{2},\\
&\quad -7q^{-1}-7q^{1}, 1\\
LG_{10_{44}} &= 8q^{-4}+130q^{-2}+477+594q^{2}+226q^{4}+12q^{6}, -30q^{-3}-236q^{-1}-520q^{1}-379q^{3}-65q^{5},\\
&\quad 48q^{-2}+237+330q^{2}+128q^{4}+6q^{6}, -43q^{-1}-141q^{1}-115q^{3}-17q^{5}, 23+47q^{2}+17q^{4}, -7q^{1}-7q^{3}, q^{2}\\
LG_{10_{45}} &= 68q^{-4}+478q^{-2}+851+478q^{2}+68q^{4}, -8q^{-5}-181q^{-3}-616q^{-1}-616q^{1}-181q^{3}-8q^{5},\\
&\quad 26q^{-4}+224q^{-2}+417+224q^{2}+26q^{4}, -33q^{-3}-145q^{-1}-145q^{1}-33q^{3}, 21q^{-2}+49+21q^{2},\\
&\quad -7q^{-1}-7q^{1}, 1\\
LG_{10_{46}} &= 2q^{-2}+15+34q^{2}+42q^{4}+30q^{6}+4q^{8}, -5q^{-1}-22q^{1}-39q^{3}-38q^{5}-16q^{7},\\
&\quad 7+26q^{2}+38q^{4}+28q^{6}+4q^{8}, -8q^{1}-26q^{3}-32q^{5}-14q^{7}, 8q^{2}+23q^{4}+22q^{6}+3q^{8},\\
&\quad -7q^{3}-17q^{5}-10q^{7}, 5q^{4}+10q^{6}+2q^{8}, -3q^{5}-3q^{7}, q^{6}\\
LG_{10_{47}} &= 12q^{-2}+53+88q^{2}+70q^{4}+16q^{6}, -2q^{-3}-25q^{-1}-68q^{1}-84q^{3}-42q^{5}-3q^{7},\\
&\quad 5q^{-2}+31+69q^{2}+66q^{4}+16q^{6}, -6q^{-1}-32q^{1}-61q^{3}-38q^{5}-3q^{7}, 6+30q^{2}+43q^{4}+13q^{6},\\
&\quad -6q^{1}-23q^{3}-19q^{5}-2q^{7}, 5q^{2}+12q^{4}+4q^{6}, -3q^{3}-3q^{5}, q^{4}\\
LG_{10_{48}} &= 16q^{-4}+94q^{-2}+165+88q^{2}+12q^{4}, -3q^{-5}-43q^{-3}-132q^{-1}-129q^{1}-39q^{3}-2q^{5},\\
&\quad 13q^{-4}+67q^{-2}+119+68q^{2}+12q^{4}, -2q^{-5}-22q^{-3}-69q^{-1}-72q^{1}-25q^{3}-2q^{5},\\
&\quad 4q^{-4}+24q^{-2}+50+29q^{2}+5q^{4}, -4q^{-3}-21q^{-1}-23q^{1}-6q^{3}, 4q^{-2}+12+5q^{2}, -3q^{-1}-3q^{1}, 1\\
LG_{10_{49}} &= 7+72q^{2}+206q^{4}+232q^{6}+82q^{8}+4q^{10}, -21q^{1}-119q^{3}-229q^{5}-157q^{7}-26q^{9},\\
&\quad 32q^{2}+135q^{4}+182q^{6}+71q^{8}+4q^{10}, -35q^{3}-116q^{5}-100q^{7}-19q^{9}, 31q^{4}+70q^{6}+33q^{8}+2q^{10},\\
&\quad -19q^{5}-24q^{7}-5q^{9}, 6q^{6}+3q^{8}\\
LG_{10_{50}} &= 6q^{-2}+67+182q^{2}+194q^{4}+64q^{6}+4q^{8}, -19q^{-1}-108q^{1}-195q^{3}-126q^{5}-20q^{7},\\
&\quad 29+115q^{2}+147q^{4}+53q^{6}+3q^{8}, -29q^{1}-89q^{3}-74q^{5}-14q^{7}, 21q^{2}+48q^{4}+22q^{6}+2q^{8},\\
&\quad -11q^{3}-14q^{5}-3q^{7}, 3q^{4}+q^{6}\\
LG_{10_{51}} &= 44q^{-2}+255+402q^{2}+194q^{4}+22q^{6}, -6q^{-3}-109q^{-1}-323q^{1}-292q^{3}-75q^{5}-3q^{7},\\
&\quad 19q^{-2}+140+247q^{2}+128q^{4}+17q^{6}, -26q^{-1}-110q^{1}-115q^{3}-33q^{5}-2q^{7}, 21+54q^{2}+30q^{4}+4q^{6},\\
&\quad -11q^{1}-14q^{3}-3q^{5}, 3q^{2}+q^{4}\\
LG_{10_{52}} &= 16q^{-4}+114q^{-2}+259+228q^{2}+52q^{4}, -2q^{-5}-41q^{-3}-168q^{-1}-250q^{1}-130q^{3}-9q^{5},\\
&\quad 5q^{-4}+56q^{-2}+161+166q^{2}+43q^{4}, -6q^{-3}-51q^{-1}-109q^{1}-70q^{3}-6q^{5}, 5q^{-2}+33+49q^{2}+14q^{4},\\
&\quad -3q^{-1}-14q^{1}-11q^{3}, 1+3q^{2}\\
LG_{10_{53}} &= 21+246q^{2}+568q^{4}+410q^{6}+96q^{8}+4q^{10}, -69q^{1}-361q^{3}-466q^{5}-197q^{7}-23q^{9},\\
&\quad 94q^{2}+272q^{4}+207q^{6}+49q^{8}+2q^{10}, -63q^{3}-103q^{5}-45q^{7}-5q^{9}, 18q^{4}+15q^{6}+3q^{8}\\
LG_{10_{54}} &= 14q^{-4}+80q^{-2}+153+126q^{2}+28q^{4}, -2q^{-5}-33q^{-3}-108q^{-1}-144q^{1}-72q^{3}-5q^{5},\\
&\quad 5q^{-4}+42q^{-2}+101+99q^{2}+25q^{4}, -6q^{-3}-37q^{-1}-72q^{1}-45q^{3}-4q^{5}, 5q^{-2}+25+36q^{2}+10q^{4},\\
&\quad -3q^{-1}-12q^{1}-9q^{3}, 1+3q^{2}\\
LG_{10_{55}} &= 13+158q^{2}+388q^{4}+298q^{6}+80q^{8}+4q^{10}, -43q^{1}-239q^{3}-326q^{5}-151q^{7}-21q^{9},\\
&\quad 60q^{2}+184q^{4}+148q^{6}+41q^{8}+2q^{10}, -41q^{3}-70q^{5}-34q^{7}-5q^{9}, 12q^{4}+10q^{6}+3q^{8}\\
LG_{10_{56}} &= 6q^{-2}+83+276q^{2}+328q^{4}+118q^{6}+6q^{8}, -21q^{-1}-148q^{1}-309q^{3}-218q^{5}-36q^{7},\\
&\quad 35+165q^{2}+229q^{4}+90q^{6}+5q^{8}, -37q^{1}-125q^{3}-109q^{5}-21q^{7}, 27q^{2}+62q^{4}+29q^{6}+2q^{8},\\
&\quad -13q^{3}-16q^{5}-3q^{7}, 3q^{4}+q^{6}\\
LG_{10_{57}} &= 48q^{-2}+331+580q^{2}+314q^{4}+40q^{6}, -6q^{-3}-131q^{-1}-443q^{1}-438q^{3}-125q^{5}-5q^{7},\\
&\quad 21q^{-2}+182+345q^{2}+191q^{4}+25q^{6}, -32q^{-1}-146q^{1}-156q^{3}-44q^{5}-2q^{7}, 27+68q^{2}+37q^{4}+4q^{6},\\
&\quad -13q^{1}-16q^{3}-3q^{5}, 3q^{2}+q^{4}\\
LG_{10_{58}} &= 14q^{-4}+192q^{-2}+505+416q^{2}+124q^{4}+8q^{6}, -47q^{-3}-283q^{-1}-405q^{1}-197q^{3}-28q^{5},\\
&\quad 58q^{-2}+183+143q^{2}+34q^{4}, -31q^{-1}-48q^{1}-17q^{3}, 6+3q^{2}\\
LG_{10_{59}} &= 8q^{-4}+128q^{-2}+439+514q^{2}+180q^{4}+8q^{6}, -30q^{-3}-225q^{-1}-466q^{1}-322q^{3}-51q^{5},\\
&\quad 47q^{-2}+220+297q^{2}+112q^{4}+6q^{6}, -41q^{-1}-132q^{1}-108q^{3}-17q^{5}, 22+46q^{2}+17q^{4}, -7q^{1}-7q^{3}, q^{2}\\
LG_{10_{60}} &= 12q^{-4}+274q^{-2}+725+574q^{2}+148q^{4}+8q^{6}, -76q^{-3}-453q^{-1}-619q^{1}-274q^{3}-32q^{5},\\
&\quad 8q^{-4}+147q^{-2}+373+266q^{2}+52q^{4}, -20q^{-3}-124q^{-1}-149q^{1}-45q^{3}, 18q^{-2}+48+23q^{2},\\
&\quad -7q^{-1}-7q^{1}, 1\\
LG_{10_{61}} &= 2q^{-4}+18q^{-2}+49+62q^{2}+42q^{4}+6q^{6}, -5q^{-3}-28q^{-1}-55q^{1}-54q^{3}-22q^{5},\\
&\quad 7q^{-2}+30+50q^{2}+38q^{4}+5q^{6}, -7q^{-1}-26q^{1}-37q^{3}-18q^{5}, 5+18q^{2}+22q^{4}+4q^{6},\\
&\quad -3q^{1}-10q^{3}-7q^{5}, q^{2}+3q^{4}
\end{aligned}
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$$LG_{10_{62}} = 12q^{-2} + 61 + 112q^{2} + 94q^{4} + 22q^{6}, -2q^{-3} - 27q^{-1} - 84q^{1} - 110q^{3} - 55q^{5} - 4q^{7},$$
$$5q^{-2} + 38 + 88q^{2} + 82q^{4} + 19q^{6}, -7q^{-1} - 41q^{1} - 73q^{3} - 42q^{5} - 3q^{7}, 8 + 35q^{2} + 46q^{4} + 13q^{6},$$
$$-7q^{1} - 24q^{3} - 19q^{5} - 2q^{7}, 5q^{2} + 12q^{4} + 4q^{6}, -3q^{3} - 3q^{5}, q^{4}$$
$$LG_{10_{63}} = 13 + 144q^{2} + 332q^{4} + 244q^{6} + 66q^{8} + 4q^{10}, -41q^{1} - 212q^{3} - 276q^{5} - 124q^{7} - 19q^{9},$$
$$56q^{2} + 164q^{4} + 128q^{6} + 36q^{8} + 2q^{10}, -39q^{3} - 65q^{5} - 31q^{7} - 5q^{9}, 12q^{4} + 10q^{6} + 3q^{8}$$
$$LG_{10_{64}} = 2q^{-4} + 42q^{-2} + 141 + 166q^{2} + 58q^{4} + 4q^{6}, -12q^{-3} - 80q^{-1} - 160q^{1} - 111q^{3} - 19q^{5},$$
$$2q^{-4} + 27q^{-2} + 97 + 127q^{2} + 48q^{4} + 3q^{6}, -5q^{-3} - 36q^{-1} - 85q^{1} - 67q^{3} - 13q^{5},$$
$$7q^{-2} + 34 + 53q^{2} + 22q^{4} + 2q^{6}, -7q^{-1} - 24q^{1} - 21q^{3} - 4q^{5}, 5 + 12q^{2} + 4q^{4}, -3q^{1} - 3q^{3}, q^{2}$$
$$LG_{10_{65}} = 40q^{-2} + 217 + 342q^{2} + 168q^{4} + 20q^{6}, -6q^{-3} - 94q^{-1} - 276q^{1} - 253q^{3} - 68q^{5} - 3q^{7},$$
$$17q^{-2} + 120 + 218q^{2} + 118q^{4} + 17q^{6}, -22q^{-1} - 99q^{1} - 108q^{3} - 33q^{5} - 2q^{7}, 19 + 52q^{2} + 30q^{4} + 4q^{6},$$
$$-11q^{1} - 14q^{3} - 3q^{5}, 3q^{2} + q^{4}$$
$$LG_{10_{66}} = 7 + 98q^{2} + 348q^{4} + 430q^{6} + 162q^{8} + 8q^{10}, -25q^{1} - 185q^{3} - 401q^{5} - 290q^{7} - 49q^{9},$$
$$46q^{2} + 220q^{4} + 303q^{6} + 119q^{8} + 6q^{10}, -54q^{3} - 175q^{5} - 147q^{7} - 26q^{9}, 43q^{4} + 91q^{6} + 41q^{8} + 2q^{10},$$
$$-22q^{5} - 27q^{7} - 5q^{9}, 6q^{6} + 3q^{8}$$
$$LG_{10_{67}} = 14q^{-2} + 177 + 442q^{2} + 344q^{4} + 92q^{6} + 4q^{8}, -46q^{-1} - 263q^{1} - 363q^{3} - 169q^{5} - 23q^{7},$$
$$60 + 188q^{2} + 150q^{4} + 40q^{6} + 2q^{8}, -37q^{1} - 61q^{3} - 27q^{5} - 3q^{7}, 9q^{2} + 6q^{4} + q^{6}$$
$$LG_{10_{68}} = 82q^{-2} + 309 + 322q^{2} + 134q^{4} + 18q^{6}, -14q^{-3} - 163q^{-1} - 292q^{1} - 185q^{3} - 44q^{5} - 2q^{7},$$
$$36q^{-2} + 142 + 135q^{2} + 47q^{4} + 4q^{6}, -31q^{-1} - 53q^{1} - 25q^{3} - 3q^{5}, 9 + 6q^{2} + q^{4}$$
$$LG_{10_{69}} = 72q^{-2} + 471 + 798q^{2} + 426q^{4} + 56q^{6}, -8q^{-3} - 182q^{-1} - 594q^{1} - 574q^{3} - 161q^{5} - 7q^{7},$$
$$24q^{-2} + 218 + 406q^{2} + 216q^{4} + 25q^{6}, -30q^{-1} - 142q^{1} - 146q^{3} - 34q^{5}, 20 + 49q^{2} + 22q^{4}, -7q^{1} - 7q^{3}, q^{2}$$
$$LG_{10_{70}} = 6q^{-4} + 92q^{-2} + 329 + 398q^{2} + 142q^{4} + 6q^{6}, -21q^{-3} - 165q^{-1} - 360q^{1} - 258q^{3} - 42q^{5},$$
$$33q^{-2} + 171 + 246q^{2} + 98q^{4} + 6q^{6}, -32q^{-1} - 114q^{1} - 99q^{3} - 17q^{5}, 20 + 44q^{2} + 17q^{4}, -7q^{1} - 7q^{3}, q^{2}$$
$$LG_{10_{71}} = 48q^{-4} + 340q^{-2} + 605 + 334q^{2} + 46q^{4}, -6q^{-5} - 131q^{-3} - 448q^{-1} - 446q^{1} - 129q^{3} - 6q^{5},$$
$$20q^{-4} + 172q^{-2} + 323 + 174q^{2} + 21q^{4}, -27q^{-3} - 122q^{-1} - 124q^{1} - 29q^{3}, 19q^{-2} + 46 + 20q^{2},$$
$$-7q^{-1} - 7q^{1}, 1$$
$$LG_{10_{72}} = 6q^{-2} + 93 + 344q^{2} + 430q^{4} + 162q^{6} + 8q^{8}, -22q^{-1} - 176q^{1} - 395q^{3} - 290q^{5} - 49q^{7},$$
$$39 + 204q^{2} + 294q^{4} + 119q^{6} + 6q^{8}, -44q^{1} - 156q^{3} - 138q^{5} - 26q^{7}, 33q^{2} + 75q^{4} + 35q^{6} + 2q^{8},$$
$$-15q^{3} - 18q^{5} - 3q^{7}, 3q^{4} + q^{6}$$
$$LG_{10_{73}} = 68q^{-2} + 429 + 712q^{2} + 372q^{4} + 48q^{6}, -8q^{-3} - 170q^{-1} - 539q^{1} - 511q^{3} - 140q^{5} - 6q^{7},$$
$$24q^{-2} + 205 + 373q^{2} + 194q^{4} + 22q^{6}, -30q^{-1} - 136q^{1} - 137q^{3} - 31q^{5}, 20 + 48q^{2} + 21q^{4}, -7q^{1} - 7q^{3}, q^{2}$$
$$LG_{10_{74}} = 18q^{-2} + 203 + 460q^{2} + 320q^{4} + 70q^{6} + 2q^{8}, -56q^{-1} - 286q^{1} - 360q^{3} - 146q^{5} - 16q^{7},$$
$$68 + 195q^{2} + 142q^{4} + 33q^{6} + 2q^{8}, -39q^{1} - 61q^{3} - 25q^{5} - 3q^{7}, 9q^{2} + 6q^{4} + q^{6}$$
$$LG_{10_{75}} = 8q^{-4} + 232q^{-2} + 641 + 522q^{2} + 140q^{4} + 8q^{6}, -64q^{-3} - 399q^{-1} - 558q^{1} - 254q^{3} - 31q^{5},$$
$$8q^{-4} + 134q^{-2} + 341 + 245q^{2} + 49q^{4}, -20q^{-3} - 118q^{-1} - 140q^{1} - 42q^{3}, 18q^{-2} + 47 + 22q^{2},$$
$$-7q^{-1} - 7q^{1}, 1$$
$$LG_{10_{76}} = 4q^{-2} + 53 + 198q^{2} + 258q^{4} + 100q^{6} + 6q^{8}, -13q^{-1} - 101q^{1} - 234q^{3} - 177q^{5} - 31q^{7},$$
$$23 + 122q^{2} + 180q^{4} + 73q^{6} + 4q^{8}, -27q^{1} - 97q^{3} - 87q^{5} - 17q^{7}, 21q^{2} + 50q^{4} + 24q^{6} + 2q^{8},$$
$$-11q^{3} - 14q^{5} - 3q^{7}, 3q^{4} + q^{6}$$
$$LG_{10_{77}} = 28q^{-2} + 189 + 346q^{2} + 196q^{4} + 28q^{6}, -4q^{-3} - 75q^{-1} - 262q^{1} - 271q^{3} - 84q^{5} - 4q^{7},$$
$$13q^{-2} + 109 + 219q^{2} + 129q^{4} + 20q^{6}, -20q^{-1} - 97q^{1} - 110q^{3} - 35q^{5} - 2q^{7}, 19 + 52q^{2} + 30q^{4} + 4q^{6},$$
$$-11q^{1} - 14q^{3} - 3q^{5}, 3q^{2} + q^{4}$$
$$LG_{10_{78}} = 8q^{-2} + 109 + 356q^{2} + 420q^{4} + 152q^{6} + 8q^{8}, -28q^{-1} - 189q^{1} - 384q^{3} - 266q^{5} - 43q^{7},$$
$$44 + 192q^{2} + 251q^{4} + 91q^{6} + 4q^{8}, -40q^{1} - 120q^{3} - 93q^{5} - 13q^{7}, 22q^{2} + 44q^{4} + 15q^{6}, -7q^{3} - 7q^{5}, q^{4}$$
$$LG_{10_{79}} = 20q^{-4} + 154q^{-2} + 283 + 154q^{2} + 20q^{4}, -3q^{-5} - 64q^{-3} - 221q^{-1} - 221q^{1} - 64q^{3} - 3q^{5},$$
$$16q^{-4} + 107q^{-2} + 194 + 107q^{2} + 16q^{4}, -2q^{-5} - 32q^{-3} - 107q^{-1} - 107q^{1} - 32q^{3} - 2q^{5},$$
$$5q^{-4} + 36q^{-2} + 69 + 36q^{2} + 5q^{4}, -6q^{-3} - 27q^{-1} - 27q^{1} - 6q^{3}, 5q^{-2} + 13 + 5q^{2}, -3q^{-1} - 3q^{1}, 1$$
$$LG_{10_{80}} = 7 + 96q^{2} + 316q^{4} + 366q^{6} + 128q^{8} + 6q^{10}, -25q^{1} - 175q^{3} - 354q^{5} - 242q^{7} - 38q^{9},$$
$$45q^{2} + 203q^{4} + 268q^{6} + 100q^{8} + 5q^{10}, -52q^{3} - 163q^{5} - 134q^{7} - 23q^{9}, 42q^{4} + 88q^{6} + 39q^{8} + 2q^{10},$$
$$-22q^{5} - 27q^{7} - 5q^{9}, 6q^{6} + 3q^{8}$$
$$LG_{10_{81}} = 56q^{-4} + 408q^{-2} + 731 + 408q^{2} + 56q^{4}, -7q^{-5} - 158q^{-3} - 543q^{-1} - 543q^{1} - 158q^{3} - 7q^{5},$$
$$25q^{-4} + 213q^{-2} + 396 + 213q^{2} + 25q^{4}, -35q^{-3} - 152q^{-1} - 152q^{1} - 35q^{3}, 24q^{-2} + 56 + 24q^{2},$$
$$-8q^{-1} - 8q^{1}, 1$$

$$
\begin{aligned}
LG_{10_{82}} &= 4q^{-4}+72q^{-2}+217+240q^{2}+82q^{4}+4q^{6}, -22q^{-3}-127q^{-1}-240q^{1}-161q^{3}-26q^{5},\\
&\quad 3q^{-4}+43q^{-2}+148+191q^{2}+75q^{4}+4q^{6}, -7q^{-3}-51q^{-1}-131q^{1}-109q^{3}-22q^{5},\\
&\quad 8q^{-2}+49+86q^{2}+40q^{4}+3q^{6}, -8q^{-1}-37q^{1}-36q^{3}-7q^{5}, 7+18q^{2}+7q^{4}, -4q^{1}-4q^{3}, q^{2}\\
LG_{10_{83}} &= 12q^{-4}+250q^{-2}+641+492q^{2}+118q^{4}+4q^{6}, -74q^{-3}-420q^{-1}-562q^{1}-244q^{3}-28q^{5},\\
&\quad 9q^{-4}+153q^{-2}+381+278q^{2}+63q^{4}+3q^{6}, -25q^{-3}-150q^{-1}-189q^{1}-71q^{3}-7q^{5},\\
&\quad 27q^{-2}+76+47q^{2}+7q^{4}, -14q^{-1}-18q^{1}-4q^{3}, 3+q^{2}\\
LG_{10_{84}} &= 56q^{-2}+391+700q^{2}+394q^{4}+54q^{6}, -7q^{-3}-153q^{-1}-528q^{1}-538q^{3}-163q^{5}-7q^{7},\\
&\quad 24q^{-2}+213+416q^{2}+242q^{4}+35q^{6}, -36q^{-1}-171q^{1}-191q^{3}-59q^{5}-3q^{7}, 30+79q^{2}+46q^{4}+6q^{6},\\
&\quad -14q^{1}-18q^{3}-4q^{5}, 3q^{2}+q^{4}\\
LG_{10_{85}} &= 20q^{-2}+101+182q^{2}+146q^{4}+34q^{6}, -3q^{-3}-45q^{-1}-137q^{1}-175q^{3}-86q^{5}-6q^{7},\\
&\quad 8q^{-2}+61+142q^{2}+130q^{4}+31q^{6}, -11q^{-1}-65q^{1}-118q^{3}-69q^{5}-5q^{7}, 12+57q^{2}+75q^{4}+22q^{6},\\
&\quad -11q^{1}-39q^{3}-31q^{5}-3q^{7}, 8q^{2}+18q^{4}+6q^{6}, -4q^{3}-4q^{5}, q^{4}\\
LG_{10_{86}} &= 68q^{-2}+395+618q^{2}+304q^{4}+36q^{6}, -9q^{-3}-168q^{-1}-497q^{1}-453q^{3}-120q^{5}-5q^{7},\\
&\quad 28q^{-2}+214+379q^{2}+202q^{4}+28q^{6}, -38q^{-1}-165q^{1}-176q^{3}-52q^{5}-3q^{7}, 30+77q^{2}+44q^{4}+6q^{6},\\
&\quad -14q^{1}-18q^{3}-4q^{5}, 3q^{2}+q^{4}\\
LG_{10_{87}} &= 8q^{-4}+188q^{-2}+541+468q^{2}+136q^{4}+6q^{6}, -54q^{-3}-341q^{-1}-505q^{1}-253q^{3}-35q^{5},\\
&\quad 7q^{-4}+124q^{-2}+337+269q^{2}+70q^{4}+3q^{6}, -21q^{-3}-135q^{-1}-180q^{1}-73q^{3}-7q^{5},\\
&\quad 25q^{-2}+74+47q^{2}+7q^{4}, -14q^{-1}-18q^{1}-4q^{3}, 3+q^{2}\\
LG_{10_{88}} &= 86q^{-4}+618q^{-2}+1099+618q^{2}+86q^{4}, -10q^{-5}-233q^{-3}-797q^{-1}-797q^{1}-233q^{3}-10q^{5},\\
&\quad 33q^{-4}+290q^{-2}+538+290q^{2}+33q^{4}, -42q^{-3}-185q^{-1}-185q^{1}-42q^{3}, 26q^{-2}+60+26q^{2},\\
&\quad -8q^{-1}-8q^{1}, 1\\
LG_{10_{89}} &= 104q^{-2}+641+1032q^{2}+528q^{4}+66q^{6}, -12q^{-3}-257q^{-1}-786q^{1}-727q^{3}-194q^{5}-8q^{7},\\
&\quad 36q^{-2}+298+523q^{2}+266q^{4}+29q^{6}, -43q^{-1}-185q^{1}-182q^{3}-40q^{5}, 26+60q^{2}+26q^{4}, -8q^{1}-8q^{3}, q^{2}\\
LG_{10_{90}} &= 10q^{-4}+198q^{-2}+519+408q^{2}+104q^{4}+4q^{6}, -58q^{-3}-335q^{-1}-459q^{1}-208q^{3}-26q^{5},\\
&\quad 7q^{-4}+119q^{-2}+307+232q^{2}+57q^{4}+3q^{6}, -19q^{-3}-119q^{-1}-156q^{1}-63q^{3}-7q^{5},\\
&\quad 21q^{-2}+63+41q^{2}+7q^{4}, -12q^{-1}-16q^{1}-4q^{3}, 3+q^{2}\\
LG_{10_{91}} &= 32q^{-4}+218q^{-2}+387+212q^{2}+28q^{4}, -5q^{-5}-95q^{-3}-308q^{-1}-305q^{1}-91q^{3}-4q^{5},\\
&\quad 25q^{-4}+154q^{-2}+274+155q^{2}+24q^{4}, -3q^{-5}-47q^{-3}-155q^{-1}-158q^{1}-50q^{3}-3q^{5},\\
&\quad 7q^{-4}+52q^{-2}+103+57q^{2}+8q^{4}, -8q^{-3}-40q^{-1}-42q^{1}-10q^{3}, 7q^{-2}+19+8q^{2}, -4q^{-1}-4q^{1}, 1\\
LG_{10_{92}} &= 10q^{-2}+157+552q^{2}+660q^{4}+242q^{6}+12q^{8}, -38q^{-1}-289q^{1}-611q^{3}-431q^{5}-71q^{7},\\
&\quad 66+315q^{2}+426q^{4}+165q^{6}+8q^{8}, -68q^{1}-218q^{3}-182q^{5}-32q^{7}, 44q^{2}+93q^{4}+41q^{6}+2q^{8},\\
&\quad -17q^{3}-20q^{5}-3q^{7}, 3q^{4}+q^{6}\\
LG_{10_{93}} &= 24q^{-4}+160q^{-2}+341+286q^{2}+64q^{4}, -3q^{-5}-60q^{-3}-228q^{-1}-320q^{1}-160q^{3}-11q^{5},\\
&\quad 8q^{-4}+80q^{-2}+211+206q^{2}+52q^{4}, -10q^{-3}-70q^{-1}-137q^{1}-84q^{3}-7q^{5}, 8q^{-2}+42+58q^{2}+16q^{4},\\
&\quad -4q^{-1}-16q^{1}-12q^{3}, 1+3q^{2}\\
LG_{10_{94}} &= 4q^{-4}+86q^{-2}+283+322q^{2}+112q^{4}+6q^{6}, -24q^{-3}-161q^{-1}-317q^{1}-215q^{3}-35q^{5},\\
&\quad 3q^{-4}+52q^{-2}+193+248q^{2}+95q^{4}+5q^{6}, -8q^{-3}-67q^{-1}-167q^{1}-133q^{3}-25q^{5},\\
&\quad 11q^{-2}+63+102q^{2}+44q^{4}+3q^{6}, -11q^{-1}-43q^{1}-39q^{3}-7q^{5}, 8+19q^{2}+7q^{4}, -4q^{1}-4q^{3}, q^{2}\\
LG_{10_{95}} &= 78q^{-2}+479+778q^{2}+398q^{4}+48q^{6}, -10q^{-3}-198q^{-1}-610q^{1}-570q^{3}-154q^{5}-6q^{7},\\
&\quad 32q^{-2}+254+454q^{2}+241q^{4}+30q^{6}, -44q^{-1}-190q^{1}-197q^{3}-53q^{5}-2q^{7}, 34+83q^{2}+44q^{4}+4q^{6},\\
&\quad -15q^{1}-18q^{3}-3q^{5}, 3q^{2}+q^{4}\\
LG_{10_{96}} &= 18q^{-4}+346q^{-2}+895+706q^{2}+182q^{4}+10q^{6}, -95q^{-3}-553q^{-1}-753q^{1}-334q^{3}-39q^{5},\\
&\quad 8q^{-4}+169q^{-2}+436+315q^{2}+62q^{4}, -20q^{-3}-135q^{-1}-167q^{1}-52q^{3}, 18q^{-2}+50+25q^{2},\\
&\quad -7q^{-1}-7q^{1}, 1\\
LG_{10_{97}} &= 26q^{-2}+343+864q^{2}+686q^{4}+180q^{6}+8q^{8}, -88q^{-1}-507q^{1}-707q^{3}-329q^{5}-41q^{7},\\
&\quad 113+350q^{2}+281q^{4}+68q^{6}+2q^{8}, -65q^{1}-107q^{3}-45q^{5}-3q^{7}, 14q^{2}+10q^{4}+q^{6}\\
LG_{10_{98}} &= 10q^{-2}+141+462q^{2}+534q^{4}+190q^{6}+10q^{8}, -36q^{-1}-250q^{1}-503q^{3}-344q^{5}-55q^{7},\\
&\quad 60+267q^{2}+348q^{4}+129q^{6}+6q^{8}, -60q^{1}-183q^{3}-148q^{5}-25q^{7}, 38q^{2}+79q^{4}+34q^{6}+2q^{8},\\
&\quad -15q^{3}-18q^{5}-3q^{7}, 3q^{4}+q^{6}\\
LG_{10_{99}} &= 36q^{-4}+272q^{-2}+491+272q^{2}+36q^{4}, -5q^{-5}-115q^{-3}-388q^{-1}-388q^{1}-115q^{3}-5q^{5},\\
&\quad 28q^{-4}+192q^{-2}+344+192q^{2}+28q^{4}, -3q^{-5}-57q^{-3}-192q^{-1}-192q^{1}-57q^{3}-3q^{5},\\
&\quad 8q^{-4}+64q^{-2}+122+64q^{2}+8q^{4}, -10q^{-3}-46q^{-1}-46q^{1}-10q^{3}, 8q^{-2}+20+8q^{2}, -4q^{-1}-4q^{1}, 1
\end{aligned}
$$

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\begin{aligned}
LG_{10_{100}} &= 22q^{-2}+129+252q^{2}+204q^{4}+46q^{6},-3q^{-3}-54q^{-1}-184q^{1}-242q^{3}-117q^{5}-8q^{7},\\
&\quad 9q^{-2}+79+191q^{2}+173q^{4}+40q^{6},-14q^{-1}-86q^{1}-152q^{3}-86q^{5}-6q^{7},16+72q^{2}+90q^{4}+25q^{6},\\
&\quad -14q^{1}-45q^{3}-34q^{5}-3q^{7},9q^{2}+19q^{4}+6q^{6},-4q^{3}-4q^{5},q^{4}\\
LG_{10_{101}} &= 25+308q^{2}+752q^{4}+582q^{6}+148q^{8}+6q^{10},-84q^{1}-464q^{3}-636q^{5}-292q^{7}-36q^{9},\\
&\quad 117q^{2}+358q^{4}+293q^{6}+76q^{8}+3q^{10},-80q^{3}-139q^{5}-67q^{7}-8q^{9},23q^{4}+21q^{6}+5q^{8}\\
LG_{10_{102}} &= 8q^{-4}+166q^{-2}+447+364q^{2}+100q^{4}+4q^{6},-50q^{-3}-288q^{-1}-404q^{1}-192q^{3}-26q^{5},\\
&\quad 7q^{-4}+107q^{-2}+274+210q^{2}+55q^{4}+3q^{6},-19q^{-3}-111q^{-1}-144q^{1}-59q^{3}-7q^{5},\\
&\quad 21q^{-2}+61+39q^{2}+7q^{4},-12q^{-1}-16q^{1}-4q^{3},3+q^{2}\\
LG_{10_{103}} &= 52q^{-2}+309+502q^{2}+258q^{4}+34q^{6},-7q^{-3}-130q^{-1}-396q^{1}-374q^{3}-106q^{5}-5q^{7},\\
&\quad 22q^{-2}+168+307q^{2}+170q^{4}+26q^{6},-30q^{-1}-132q^{1}-145q^{3}-46q^{5}-3q^{7},\\
&\quad 24+64q^{2}+38q^{4}+6q^{6},-12q^{1}-16q^{3}-4q^{5},3q^{2}+q^{4}\\
LG_{10_{104}} &= 30q^{-4}+244q^{-2}+449+250q^{2}+34q^{4},-4q^{-5}-101q^{-3}-351q^{-1}-354q^{1}-105q^{3}-5q^{5},\\
&\quad 25q^{-4}+173q^{-2}+309+172q^{2}+26q^{4},-3q^{-5}-53q^{-3}-173q^{-1}-170q^{1}-50q^{3}-3q^{5},\\
&\quad 8q^{-4}+60q^{-2}+109+55q^{2}+7q^{4},-10q^{-3}-43q^{-1}-41q^{1}-8q^{3},8q^{-2}+19+7q^{2},-4q^{-1}-4q^{1},1\\
LG_{10_{105}} &= 10q^{-4}+174q^{-2}+645+794q^{2}+300q^{4}+16q^{6},-39q^{-3}-319q^{-1}-697q^{1}-503q^{3}-86q^{5},\\
&\quad 64q^{-2}+317+435q^{2}+168q^{4}+8q^{6},-57q^{-1}-183q^{1}-148q^{3}-22q^{5},29+58q^{2}+21q^{4},-8q^{1}-8q^{3},q^{2}\\
LG_{10_{106}} &= 4q^{-4}+94q^{-2}+327+376q^{2}+128q^{4}+6q^{6},-25q^{-3}-182q^{-1}-367q^{1}-248q^{3}-38q^{5},\\
&\quad 3q^{-4}+56q^{-2}+219+281q^{2}+105q^{4}+5q^{6},-8q^{-3}-73q^{-1}-184q^{1}-145q^{3}-26q^{5},\\
&\quad 11q^{-2}+67+108q^{2}+46q^{4}+3q^{6},-11q^{-1}-44q^{1}-40q^{3}-7q^{5},8+19q^{2}+7q^{4},-4q^{1}-4q^{3},q^{2}\\
LG_{10_{107}} &= 78q^{-4}+524q^{-2}+901+490q^{2}+66q^{4},-10q^{-5}-207q^{-3}-672q^{-1}-651q^{1}-184q^{3}-8q^{5},\\
&\quad 33q^{-4}+260q^{-2}+463+241q^{2}+27q^{4},-42q^{-3}-171q^{-1}-165q^{1}-36q^{3},26q^{-2}+58+24q^{2},\\
&\quad -8q^{-1}-8q^{1},1\\
LG_{10_{108}} &= 24q^{-4}+148q^{-2}+293+236q^{2}+52q^{4},-3q^{-5}-58q^{-3}-202q^{-1}-271q^{1}-133q^{3}-9q^{5},\\
&\quad 8q^{-4}+74q^{-2}+185+179q^{2}+46q^{4},-10q^{-3}-64q^{-1}-125q^{1}-78q^{3}-7q^{5},8q^{-2}+40+56q^{2}+16q^{4},\\
&\quad -4q^{-1}-16q^{1}-12q^{3},1+3q^{2}\\
LG_{10_{109}} &= 38q^{-4}+308q^{-2}+561+308q^{2}+38q^{4},-5q^{-5}-126q^{-3}-440q^{-1}-440q^{1}-126q^{3}-5q^{5},\\
&\quad 29q^{-4}+212q^{-2}+383+212q^{2}+29q^{4},-3q^{-5}-60q^{-3}-208q^{-1}-208q^{1}-60q^{3}-3q^{5},\\
&\quad 8q^{-4}+67q^{-2}+128+67q^{2}+8q^{4},-10q^{-3}-47q^{-1}-47q^{1}-10q^{3},8q^{-2}+20+8q^{2},-4q^{-1}-4q^{1},1\\
LG_{10_{110}} &= 10q^{-4}+158q^{-2}+537+622q^{2}+218q^{4}+10q^{6},-37q^{-3}-277q^{-1}-568q^{1}-391q^{3}-63q^{5},\\
&\quad 58q^{-2}+271+364q^{2}+139q^{4}+8q^{6},-51q^{-1}-163q^{1}-134q^{3}-22q^{5},27+56q^{2}+21q^{4},-8q^{1}-8q^{3},q^{2}\\
LG_{10_{111}} &= 10q^{-2}+137+416q^{2}+460q^{4}+158q^{6}+8q^{8},-36q^{-1}-233q^{1}-445q^{3}-295q^{5}-47q^{7},\\
&\quad 58+243q^{2}+312q^{4}+116q^{6}+6q^{8},-56q^{1}-170q^{3}-139q^{5}-25q^{7},36q^{2}+77q^{4}+34q^{6}+2q^{8},\\
&\quad -15q^{3}-18q^{5}-3q^{7},3q^{4}+q^{6}\\
LG_{10_{112}} &= 6q^{-4}+130q^{-2}+431+488q^{2}+170q^{4}+8q^{6},-36q^{-3}-245q^{-1}-482q^{1}-325q^{3}-52q^{5},\\
&\quad 4q^{-4}+78q^{-2}+294+373q^{2}+143q^{4}+7q^{6},-11q^{-3}-100q^{-1}-251q^{1}-199q^{3}-37q^{5},\\
&\quad 15q^{-2}+93+151q^{2}+66q^{4}+4q^{6},-15q^{-1}-62q^{1}-57q^{3}-10q^{5},11+26q^{2}+10q^{4},-5q^{1}-5q^{3},q^{2}\\
LG_{10_{113}} &= 94q^{-2}+663+1178q^{2}+666q^{4}+90q^{6},-11q^{-3}-257q^{-1}-884q^{1}-898q^{3}-271q^{5}-11q^{7},\\
&\quad 38q^{-2}+349+675q^{2}+391q^{4}+53q^{6},-56q^{-1}-266q^{1}-293q^{3}-86q^{5}-3q^{7},44+112q^{2}+63q^{4}+6q^{6},\\
&\quad -18q^{1}-22q^{3}-4q^{5},3q^{2}+q^{4}\\
LG_{10_{114}} &= 14q^{-4}+288q^{-2}+753+602q^{2}+156q^{4}+6q^{6},-85q^{-3}-488q^{-1}-673q^{1}-309q^{3}-39q^{5},\\
&\quad 10q^{-4}+176q^{-2}+449+342q^{2}+84q^{4}+4q^{6},-28q^{-3}-173q^{-1}-226q^{1}-91q^{3}-10q^{5},\\
&\quad 30q^{-2}+87+57q^{2}+10q^{4},-15q^{-1}-20q^{1}-5q^{3},3+q^{2}\\
LG_{10_{115}} &= 100q^{-4}+712q^{-2}+1261+712q^{2}+100q^{4},-12q^{-5}-272q^{-3}-921q^{-1}-921q^{1}-272q^{3}-12q^{5},\\
&\quad 40q^{-4}+342q^{-2}+630+342q^{2}+40q^{4},-51q^{-3}-220q^{-1}-220q^{1}-51q^{3},31q^{-2}+71+31q^{2},\\
&\quad -9q^{-1}-9q^{1},1\\
LG_{10_{116}} &= 6q^{-4}+150q^{-2}+525+602q^{2}+210q^{4}+10q^{6},-39q^{-3}-293q^{-1}-590q^{1}-399q^{3}-63q^{5},\\
&\quad 4q^{-4}+90q^{-2}+356+451q^{2}+170q^{4}+8q^{6},-12q^{-3}-120q^{-1}-299q^{1}-232q^{3}-41q^{5},\\
&\quad 18q^{-2}+110+172q^{2}+72q^{4}+4q^{6},-18q^{-1}-69q^{1}-61q^{3}-10q^{5},12+27q^{2}+10q^{4},-5q^{1}-5q^{3},q^{2}\\
LG_{10_{117}} &= 90q^{-2}+593+1014q^{2}+552q^{4}+72q^{6},-11q^{-3}-236q^{-1}-772q^{1}-759q^{3}-221q^{5}-9q^{7},\\
&\quad 36q^{-2}+308+578q^{2}+326q^{4}+44q^{6},-50q^{-1}-229q^{1}-249q^{3}-73q^{5}-3q^{7},38+97q^{2}+55q^{4}+6q^{6},\\
&\quad -16q^{1}-20q^{3}-4q^{5},3q^{2}+q^{4}
\end{aligned}
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\begin{aligned}
LG_{10_{118}} &= 52q^{-4}+392q^{-2}+703+392q^{2}+52q^{4}, -7q^{-5}-166q^{-3}-557q^{-1}-557q^{1}-166q^{3}-7q^{5},\\
&\quad 40q^{-4}+277q^{-2}+493+277q^{2}+40q^{4}, -4q^{-5}-82q^{-3}-276q^{-1}-276q^{1}-82q^{3}-4q^{5},\\
&\quad 11q^{-4}+92q^{-2}+174+92q^{2}+11q^{4}, -14q^{-3}-65q^{-1}-65q^{1}-14q^{3}, 11q^{-2}+27+11q^{2},\\
&\quad -5q^{-1}-5q^{1}, 1\\
LG_{10_{119}} &= 16q^{-4}+342q^{-2}+917+748q^{2}+196q^{4}+8q^{6}, -98q^{-3}-583q^{-1}-819q^{1}-380q^{3}-46q^{5},\\
&\quad 11q^{-4}+203q^{-2}+528+403q^{2}+93q^{4}+3q^{6}, -31q^{-3}-196q^{-1}-254q^{1}-96q^{3}-7q^{5},\\
&\quad 33q^{-2}+94+58q^{2}+7q^{4}, -16q^{-1}-20q^{1}-4q^{3}, 3+q^{2}\\
LG_{10_{120}} &= 41+508q^{2}+1192q^{4}+890q^{6}+210q^{8}+8q^{10}, -140q^{1}-744q^{3}-983q^{5}-426q^{7}-47q^{9},\\
&\quad 189q^{2}+548q^{4}+429q^{6}+99q^{8}+3q^{10}, -120q^{3}-200q^{5}-88q^{7}-8q^{9}, 31q^{4}+28q^{6}+5q^{8}\\
LG_{10_{121}} &= 124q^{-2}+783+1278q^{2}+666q^{4}+82q^{6}, -15q^{-3}-319q^{-1}-991q^{1}-935q^{3}-258q^{5}-10q^{7},\\
&\quad 49q^{-2}+402+719q^{2}+388q^{4}+49q^{6}, -66q^{-1}-285q^{1}-298q^{3}-82q^{5}-3q^{7}, 47+114q^{2}+62q^{4}+6q^{6},\\
&\quad -18q^{1}-22q^{3}-4q^{5}, 3q^{2}+q^{4}\\
LG_{10_{122}} &= 14q^{-4}+336q^{-2}+951+820q^{2}+232q^{4}+10q^{6}, -96q^{-3}-596q^{-1}-877q^{1}-434q^{3}-57q^{5},\\
&\quad 12q^{-4}+211q^{-2}+564+450q^{2}+113q^{4}+4q^{6}, -34q^{-3}-212q^{-1}-282q^{1}-114q^{3}-10q^{5},\\
&\quad 36q^{-2}+104+67q^{2}+10q^{4}, -17q^{-1}-22q^{1}-5q^{3}, 3+q^{2}\\
LG_{10_{123}} &= 80q^{-4}+622q^{-2}+1113+622q^{2}+80q^{4}, -10q^{-5}-260q^{-3}-882q^{-1}-882q^{1}-260q^{3}-10q^{5},\\
&\quad 59q^{-4}+435q^{-2}+776+435q^{2}+59q^{4}, -5q^{-5}-124q^{-3}-429q^{-1}-429q^{1}-124q^{3}-5q^{5},\\
&\quad 15q^{-4}+139q^{-2}+263+139q^{2}+15q^{4}, -20q^{-3}-94q^{-1}-94q^{1}-20q^{3}, 15q^{-2}+36+15q^{2},\\
&\quad -6q^{-1}-6q^{1}, 1\\
LG_{10_{124}} &= 1+2q^{2}+2q^{4}+2q^{6}, -q^{1}-q^{3}-q^{5}-q^{7}, -q^{4}, q^{3}+2q^{5}+q^{7}, -q^{4}-2q^{6}-q^{8}, q^{5}+q^{7}, q^{8}, -q^{7}-q^{9}, q^{8}\\
LG_{10_{125}} &= 4q^{-4}+8q^{-2}+5+2q^{2}, -q^{-5}-5q^{-3}-6q^{-1}-3q^{1}-q^{3}, q^{-4}+3q^{-2}+4+4q^{2}, -2q^{-1}-5q^{1}-3q^{3},\\
&\quad 2+5q^{2}+q^{4}, -2q^{1}-2q^{3}, q^{2}\\
LG_{10_{126}} &= 11+32q^{2}+20q^{4}+4q^{6}, -3q^{-1}-20q^{1}-26q^{3}-10q^{5}-q^{7}, q^{-2}+7+19q^{2}+13q^{4}+4q^{6},\\
&\quad -2q^{-1}-8q^{1}-10q^{3}-5q^{5}-q^{7}, 2+6q^{2}+3q^{4}+q^{6}, -2q^{1}-2q^{3}, q^{2}\\
LG_{10_{127}} &= 9+46q^{2}+70q^{4}+28q^{6}+2q^{8}, -q^{-1}-20q^{1}-59q^{3}-50q^{5}-10q^{7}, 3+26q^{2}+50q^{4}+23q^{6}+2q^{8},\\
&\quad -4q^{1}-24q^{3}-27q^{5}-7q^{7}, 4q^{2}+15q^{4}+8q^{6}+q^{8}, -3q^{3}-4q^{5}-q^{7}, q^{4}\\
LG_{10_{128}} &= 3+12q^{2}+12q^{4}+2q^{6}, -5q^{1}-9q^{3}-3q^{5}+q^{7}, 2q^{2}-2q^{6}-q^{8}, q^{3}-q^{5}-2q^{7}, 2q^{4}+8q^{6}+3q^{8},\\
&\quad -5q^{5}-6q^{7}-q^{9}, 3q^{6}+q^{8}\\
LG_{10_{129}} &= 20q^{-2}+69+56q^{2}+16q^{4}, -5q^{-3}-39q^{-1}-58q^{1}-27q^{3}-3q^{5}, q^{-4}+12q^{-2}+33+21q^{2}+5q^{4},\\
&\quad -2q^{-3}-10q^{-1}-10q^{1}-2q^{3}, q^{-2}+3\\
LG_{10_{130}} &= 13+24q^{2}+22q^{4}+6q^{6}, -7q^{-1}-18q^{1}-20q^{3}-10q^{5}-q^{7}, 3q^{-2}+14+12q^{2}+6q^{4}+q^{6},\\
&\quad -6q^{-1}-8q^{1}-2q^{3}, 3+q^{2}\\
LG_{10_{131}} &= 25+102q^{2}+96q^{4}+32q^{6}+2q^{8}, -3q^{-1}-52q^{1}-92q^{3}-52q^{5}-9q^{7}, 9+45q^{2}+40q^{4}+13q^{6}+q^{8},\\
&\quad -9q^{1}-15q^{3}-7q^{5}-q^{7}, 3q^{2}+q^{4}\\
LG_{10_{132}} &= -1-2q^{2}+4q^{4}+4q^{6}, -q^{-1}+2q^{1}-4q^{5}-q^{7}, q^{-2}+2-q^{2}+q^{6}, -2q^{-1}-q^{1}+q^{3}, 1\\
LG_{10_{133}} &= 9+36q^{2}+36q^{4}+18q^{6}+2q^{8}, -q^{-1}-18q^{1}-32q^{3}-22q^{5}-7q^{7}, 3+15q^{2}+13q^{4}+7q^{6}+q^{8},\\
&\quad -3q^{1}-4q^{3}-2q^{5}-q^{7}, q^{2}\\
LG_{10_{134}} &= 3+20q^{2}+34q^{4}+22q^{6}+2q^{8}, -7q^{1}-25q^{3}-29q^{5}-11q^{7}, 8q^{2}+22q^{4}+22q^{6}+4q^{8},\\
&\quad -7q^{3}-21q^{5}-15q^{7}-q^{9}, 8q^{4}+18q^{6}+6q^{8}, -7q^{5}-8q^{7}-q^{9}, 3q^{6}+q^{8}\\
LG_{10_{135}} &= 2q^{-4}+60q^{-2}+159+108q^{2}+20q^{4}, -15q^{-3}-99q^{-1}-126q^{1}-45q^{3}-3q^{5},\\
&\quad q^{-4}+28q^{-2}+76+47q^{2}+7q^{4}, -2q^{-3}-21q^{-1}-25q^{1}-6q^{3}, q^{-2}+6+2q^{2}\\
LG_{10_{136}} &= 4q^{-4}+18q^{-2}+19+14q^{2}+4q^{4}, -8q^{-3}-15q^{-1}-15q^{1}-9q^{3}-q^{5}, 5q^{-2}+8+10q^{2}+3q^{4},\\
&\quad -q^{-1}-4q^{1}-3q^{3}, q^{2}\\
LG_{10_{137}} &= 16q^{-2}+69+72q^{2}+34q^{4}+4q^{6}, -2q^{-3}-32q^{-1}-60q^{1}-40q^{3}-10q^{5}, 5q^{-2}+24+21q^{2}+8q^{4},\\
&\quad -4q^{-1}-6q^{1}-2q^{3}, 1\\
LG_{10_{138}} &= 4q^{-4}+44q^{-2}+99+74q^{2}+8q^{4}, 12q^{-3}+62q^{-1}+89q^{1}+39q^{3}, 15q^{-2}+55+64q^{2}+16q^{4},\\
&\quad 13q^{-1}+43q^{1}+34q^{3}+4q^{5}, 10+23q^{2}+8q^{4}, 5q^{1}+5q^{3}, q^{2}\\
LG_{10_{139}} &= 1+2q^{2}+4q^{4}+10q^{6}+4q^{8}, -q^{1}-q^{3}-5q^{5}-6q^{7}-q^{9}, -2q^{4}+q^{6}+q^{8}, 2q^{3}+5q^{5}+3q^{7},\\
&\quad -3q^{4}-5q^{6}-2q^{8}, 2q^{5}+2q^{7}, q^{8}, -q^{7}-q^{9}, q^{8}\\
LG_{10_{140}} &= 3+4q^{2}+8q^{4}+4q^{6}, -2q^{-1}-3q^{1}-5q^{3}-5q^{5}-q^{7}, q^{-2}+4+2q^{2}+2q^{4}+q^{6}, -2q^{-1}-2q^{1}, 1\\
LG_{10_{141}} &= 2q^{-2}+23+32q^{2}+18q^{4}+2q^{6}, -12q^{-1}-28q^{1}-23q^{3}-7q^{5}, 5q^{-2}+21+19q^{2}+8q^{4}+q^{6},\\
&\quad -q^{-3}-12q^{-1}-15q^{1}-5q^{3}-q^{5}, 3q^{-2}+10+4q^{2}, -3q^{-1}-3q^{1}, 1
\end{aligned}
$$

$$
\begin{aligned}
LG_{10_{142}} &= 3+12q^2+14q^4+6q^6, -5q^1-12q^3-9q^5-2q^7, 4q^2+8q^4+6q^6, -3q^3-8q^5-5q^7, 4q^4+10q^6+3q^8,\\
&\quad -5q^5-6q^7-q^9, 3q^6+q^8\\
LG_{10_{143}} &= 21+64q^2+46q^4+10q^6, -5q^{-1}-40q^1-56q^3-23q^5-2q^7, q^{-2}+14+40q^2+28q^4+7q^6,\\
&\quad -3q^{-1}-17q^1-21q^3-8q^5-q^7, 4+11q^2+5q^4+q^6, -3q^1-3q^3, q^2\\
LG_{10_{144}} &= 14q^{-2}+103+180q^2+84q^4+6q^6, -q^{-3}-36q^{-1}-132q^1-124q^3-27q^5,\\
&\quad 3q^{-2}+42+88q^2+42q^4+3q^6, -3q^{-1}-25q^1-27q^3-5q^5, 1+6q^2+2q^4\\
LG_{10_{145}} &= 3+4q^2-2q^4+4q^6+4q^8, -3q^1+2q^3+2q^5-4q^7-q^9, -q^2-4q^4-q^6+q^8, q^3+q^5, q^4\\
LG_{10_{146}} &= 2q^{-4}+46q^{-2}+133+100q^2+24q^4, -10q^{-3}-75q^{-1}-106q^1-45q^3-4q^5, 16q^{-2}+54+38q^2+8q^4,\\
&\quad -11q^{-1}-16q^1-5q^3, 3+q^2\\
LG_{10_{147}} &= 4q^{-4}+32q^{-2}+71+66q^2+14q^4, -10q^{-3}-42q^{-1}-65q^1-35q^3-2q^5, 10q^{-2}+31+36q^2+8q^4,\\
&\quad -5q^{-1}-14q^1-9q^3, 1+3q^2\\
LG_{10_{148}} &= 29+92q^2+66q^4+12q^6, -6q^{-1}-56q^1-79q^3-31q^5-2q^7, q^{-2}+17+53q^2+38q^4+8q^6,\\
&\quad -3q^{-1}-20q^1-26q^3-10q^5-q^7, 4+12q^2+6q^4+q^6, -3q^1-3q^3, q^2\\
LG_{10_{149}} &= 13+94q^2+156q^4+68q^6+4q^8, -q^{-1}-36q^1-126q^3-113q^5-22q^7, 4+52q^2+102q^4+48q^6+3q^8,\\
&\quad -7q^1-45q^3-49q^5-11q^7, 7q^2+23q^4+12q^6+q^8, -4q^3-5q^5-q^7, q^4\\
LG_{10_{150}} &= 4q^{-2}+33+62q^2+48q^4+8q^6, -11q^{-1}-44q^1-58q^3-26q^5-q^7, 14+41q^2+41q^4+8q^6,\\
&\quad -12q^1-30q^3-19q^5-q^7, 8q^2+16q^4+4q^6, -4q^3-4q^5, q^4\\
LG_{10_{151}} &= 28q^{-2}+135+178q^2+60q^4+2q^6, -4q^{-3}-62q^{-1}-155q^1-112q^3-15q^5,\\
&\quad 11q^{-2}+70+105q^2+37q^4+q^6, -13q^{-1}-48q^1-40q^3-5q^5, 9+20q^2+7q^4, -4q^1-4q^3, q^2\\
LG_{10_{152}} &= 1+2q^2+10q^4+30q^6+18q^8+2q^{10}, -q^1-q^3-16q^5-23q^7-7q^9, -q^2-2q^4+10q^6+8q^8+q^{10},\\
&\quad 4q^3+6q^5+q^7-q^9, -5q^4-8q^6-4q^8, 4q^5+5q^7+q^9, -q^6, -q^7-q^9, q^8\\
LG_{10_{153}} &= 4q^{-4}+4q^{-2}+1+2q^2+4q^4, -q^{-5}-4q^{-3}+2q^1-2q^3-q^5, q^{-4}-q^{-2}-4-3q^2,\\
&\quad q^{-3}+4q^{-1}+3q^1, -2q^{-2}-2+2q^2+q^4, q^{-1}-q^1-2q^3, q^2\\
LG_{10_{154}} &= 1+6q^2+26q^4+30q^6+18q^8+2q^{10}, -q^1-9q^3-20q^5-19q^7-7q^9, -q^2+2q^6+6q^8+q^{10},\\
&\quad 4q^3+8q^5+3q^7-q^9, -3q^4-4q^6-q^8, 0, q^6\\
LG_{10_{155}} &= 4q^{-2}+39+50q^2+24q^4+2q^6, -19q^{-1}-45q^1-34q^3-8q^5, 6q^{-2}+30+29q^2+11q^4+q^6,\\
&\quad -q^{-3}-14q^{-1}-20q^1-8q^3-q^5, 3q^{-2}+11+5q^2, -3q^{-1}-3q^1, 1\\
LG_{10_{156}} &= 24q^{-2}+91+102q^2+26q^4, -4q^{-3}-46q^{-1}-98q^1-62q^3-6q^5, 9q^{-2}+48+70q^2+24q^4+q^6,\\
&\quad -9q^{-1}-36q^1-32q^3-5q^5, 7+18q^2+7q^4, -4q^1-4q^3, q^2\\
LG_{10_{157}} &= 17+134q^2+226q^4+102q^6+6q^8, -q^{-1}-50q^1-181q^3-165q^5-33q^7,\\
&\quad 5+74q^2+146q^4+70q^6+4q^8, -10q^1-64q^3-69q^5-15q^7, 10q^2+31q^4+16q^6+q^8,\\
&\quad -5q^3-6q^5-q^7, q^4\\
LG_{10_{158}} &= 6q^{-4}+100q^{-2}+217+122q^2+14q^4, -31q^{-3}-150q^{-1}-162q^1-44q^3-q^5,\\
&\quad 4q^{-4}+54q^{-2}+115+61q^2+6q^4, -9q^{-3}-45q^{-1}-46q^1-10q^3, 8q^{-2}+20+8q^2, -4q^{-1}-4q^1, 1\\
LG_{10_{159}} &= 45+144q^2+108q^4+20q^6, -9q^{-1}-88q^1-127q^3-51q^5-3q^7, q^{-2}+27+85q^2+62q^4+12q^6,\\
&\quad -4q^{-1}-32q^1-42q^3-15q^5-q^7, 6+18q^2+9q^4+q^6, -4q^1-4q^3, q^2\\
LG_{10_{160}} &= 4q^{-2}+25+32q^2+14q^4, -9q^{-1}-26q^1-23q^3-6q^5, 8+19q^2+18q^4+3q^6, -6q^1-18q^3-13q^5-q^7,\\
&\quad 6q^2+14q^4+4q^6, -4q^3-4q^5, q^4\\
LG_{10_{161}} &= 1+6q^4+8q^6+4q^8, -q^3-5q^5-5q^7-q^9, -q^2+q^8, q^3+3q^5+2q^7, -q^4-2q^6-q^8, 0, q^6\\
LG_{10_{163}} &= 14q^{-2}+87+138q^2+58q^4+4q^6, -q^{-3}-33q^{-1}-106q^1-93q^3-19q^5,\\
&\quad 3q^{-2}+36+72q^2+33q^4+3q^6, -3q^{-1}-22q^1-24q^3-5q^5, 1+6q^2+2q^4\\
LG_{10_{164}} &= 42q^{-2}+195+246q^2+80q^4+2q^6, -6q^{-3}-91q^{-1}-219q^1-154q^3-20q^5,\\
&\quad 16q^{-2}+100+147q^2+53q^4+2q^6, -18q^{-1}-67q^1-57q^3-8q^5, 12+27q^2+10q^4, -5q^1-5q^3, q^2\\
LG_{10_{165}} &= 4q^{-4}+94q^{-2}+245+172q^2+34q^4, -23q^{-3}-148q^{-1}-192q^1-72q^3-5q^5,\\
&\quad q^{-4}+38q^{-2}+105+68q^2+11q^4, -2q^{-3}-25q^{-1}-31q^1-8q^3, q^{-2}+6+2q^2\\
LG_{10_{166}} &= 41+166q^2+164q^4+58q^6+4q^8, -5q^{-1}-82q^1-149q^3-88q^5-16q^7, 13+65q^2+61q^4+20q^6+q^8,\\
&\quad -11q^1-19q^3-9q^5-q^7, 3q^2+q^4
\end{aligned}
$$

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