A COHOMOLOGICAL HASSE PRINCIPLE FOR THREE-DIMENSIONAL COMPLETE REGULAR LOCAL RINGS

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Abstract. In this paper, we prove a cohomological Hasse principle for three-dimensional complete regular local rings of positive characteristics with finite residue fields. The main theorem states the exactness of a Hasse principle complex for such a local ring, which at the same time provides a new evidence for the exactness of the Bloch-Ogus complexes.

Contents

1. Introduction

1. Introduction

In this paper, we prove a cohomological Hasse principle for the positive-characteristic three dimensional complete regular local ring with finite residue field, namely the three-variable formal power series ring over a finite field.

The classical Hasse principle is a basic theorem which relates the Brauer group of algebraic number fields, or one-variable function fields over finite fields, with the Brauer groups of their localizations. It was generalized to two-dimensional global fields by K. Kato in [?], as the exactness of a certain complex (he calls Hasse principle complex) defined for each two-dimensional global field. Moreover, he was able to formulate this complex even for arbitrary global fields and conjectured its exactness (called cohomological Hasse primciple for those fields). Until now, works of J. Colliot-Thélène, U. Jannsen or S. Saito have given affirmative answers to this conjecture (cf. [?], [?], [?]). In particular, Colliot-Thélène proved it for arbitrary three-dimensional global fields of positive characteristic (cf. loc.cit.).

On the other hand, consider an arbitrary two-dimensional complete regular local ring R with finite residue field, for example, $R = \mathbb{F}_p[[X, Y]]$. Then, like the case where R is the integer ring of an algebraic number field, there are many height one primes in R. So one may ask whether there holds the local-global relation like the Hasse principle of algebraic number fields or one-variable function fields over finite fields. And it is S. Saito who answered this natural question in [?] by showing the exactness of a certain

complex associated to R which is quite similar to that of one-dimensional global fields (a cohomological Hasse principle for R). This theorem of Saito tells us that even for local rings, there exists a formalism connecting local and global arithmetics like one-dimensioanl global fields.

Then occurs a natural question whether it is possible to generalize this result of Saito to three-dimensional cases which also can be seen as an analogy of a cohomological Hasse principle for two-dimensional global fields proved by Kato (cf. loc.cit.).

In this paper, we show this is indeed possible at least in the positive characteristic cases. Let us state the main reult.

Theorem 3.1. Let A be the three-dimensional complete regular local ring of positive characteristic with finite residue field, namely the three-variable formal power series ring over a finite field. Then for an arbitrary natural number m prime to the characteristic of A, the following Hasse principle complex for A becomes exact:

$$0 \to H^4_{\acute{e}t}(K, \mu_m^{\otimes 3}) \to \bigoplus_{\mathfrak{p} \in P^1_A} H^3_{\acute{e}t}(\kappa(\mathfrak{p}), \mu_m^{\otimes 2}) \to \bigoplus_{\mathfrak{m} \in P^2_A} H^2_{\acute{e}t}(\kappa(\mathfrak{m}), \mu_m) \to \mathbb{Z}/m \to 0 \qquad (3.1),$$

where P^i_A denotes the set of height i prime ideals of A and $\kappa(\mathfrak{p})$, $\kappa(\mathfrak{m})$ denotes the residue fields of $\mathfrak{p}$, $\mathfrak{m}$, respectively. The map in (3.1) is given in the course of the proof in Section 3.

We remark that the degree 'four' of the first cohomology is obtained by adding one to the Krull-dimension of A, and the twist number three of $\mu_m^{\otimes 3}$ also comes from the Krull dimension of A. This is one of the most peculiar formalism in a cohomological Hasse principle, which origins in the classical Hasse principle of the Brauer group of one-dimensional global fields (one can verify this by the isomorphism $\mathrm{Br}(K)_m \cong H^2_{\mathrm{Gal}}(K, \mu_m^{\otimes 1})$ for an arbitrary one-dimensional global field K, where $\mathrm{Br}(K)_m$ denotes the subgroup of $\mathrm{Br}(K)$ consisting of m-torsion elements, and the degree 'two' of the Galois cohomology $H^2_{\mathrm{Gal}}(K, \mu_m^{\otimes 1})$ is obtained by adding one to the Krull dimension of the integer ring $\mathcal{O}_K$.

We now explain briefly how to prove theorem 3.1. First, in Section 2, we review the absolute purity theorem by Fujiwara-Gabber (theorem 2.1) which enables us to interpret each term in the above complex as the local cohomology of certain henselian local rings. And we also review a result of Gabber on the cohomological dimension for fractional fields of complete regular local rings over fields (theorem 2.2). In Section 3, we prove the main result. We consider the coniveau-spectral sequence for A. Then, thanks to theorem 2.1,

we can reduce the exactness of the above Hasse principle complex to the vanishing of each $E_2^{p,q}$-term of the coniveau-spectral sequence $(p+q=4)$. There, theorem 2.2 is fully used and we finally reduce the vanishing of $E_2^{0,4}$, which is the most hard task, to the exactness of the Gersten-Quillen complex of A. Thus, our proof is finished because the exactness of its Gersten-Quillen complex for A is already proved by Quillen (cf. [**?**]).

Finally, we remark the relation of our result with the famous Bloch-Ogus complex(which we do not review here). This complex is believed to be exact for an arbtrary henselian regular local ring over a field, and it is proved only for those rings obtained from the henselization of smooth local rings(roughly, the localization of the affine ring of a smooth variety over a field). The above complex (3.1) is the special case of the Bloch-Ogus complex for A, so our result could be seen as a partial answer to this conjecture for complete regular local rings.

2. The abosute purity theorem by Fujiwara-Gabber

In this section, we recall the absolute purity of Fujiwara and Gabber (cf. [**?**]).

Theorem 2.1. (*Fujiwara-Gabber*) *Let X be an arbitrary noetherian excellent regular scheme over a field k, and Z be its regular closed subscheme over k of codimension c. Then, for an arbitrary natural number m prime to the characteristic k, we have the following canonical isomorphism*:

$$H^i_Z(X_{\acute{e}t}\,,\mu_m^{\otimes j}) \;\cong\; H^{i-2c}_{\acute{e}t}(Z\,,\mu_m^{\otimes(j-c)}). \tag{2.1}$$

Remark. This theorem is proved by O. Gabber and K. Fujiwara. It was conjectured by M. Artin in SGA 4 (tome 3, XIX) about 30 years ago. Fujiwara proved this theorem using his theory of tubular neighborhoods in rigid analytic geometry. For more details, see Fujiwara's paper [Fu], especially Theorem 7.1.1 and sentences just below Corollary 7.1.7.

Next, we explain an another useful result of Gabber on the cohomological dimension for the fractional field of complete regular local ring $k\,[[X_1, X_2, \cdots X_n]]$ $(n \geq 1)$. His proof was lectured at I.H.P (cf. [**?**]) by himself.

Theorem 2.2. (*Gabber*) *Let K_n be the fractional field of a n-dimensional complete regular local ring $k\,[[X_1, X_2, \cdots X_n]]$. Then for an arbitrary prime number $l\,(l \neq p)$, the l-cohomological dimension* $\mathrm{cd}_l\, K_n$ *of K_n satisfies*

$$cd_l\, K_n \;\leq\; cd_l\, k \,+\, n.$$

Proof. We use induction on n. First, we define

$A_{n-1} := k[[X_1, X_2, \dots X_{n-1}]]$ $(n \geq 1)$,

$K_{n-1} :=$ The fractional field of A_{n-1},

$A'_n :=$ the henselization of $A_{n-1}[X_n]$ at the maximal ideal $(X_1, X_2, \cdots, X_{n-1}, X_n)$,

$K'_n :=$ The fractional field of A'_n,

The theorem is clear when $n = 0$.

As a first step, we show

$$\mathrm{cd}_l K'_n \leq \mathrm{cd}_l k + n. \tag{2.2}$$

From the definition of the henselization, there exists a filtering index category Λ and an inductive system of domains B_λ $(\lambda \in \Lambda)$ which is etale over the localization of $A_{n-1}[X_n]$ at $(X_1, ..., X_n)$ such that A'_n can be written as the inductive limit of the form

$$A'_n = \varinjlim_{\lambda \in \Lambda} B_\lambda. \tag{2.3}$$

So, we have

$$K'_n = \varinjlim_{\lambda \in \Lambda} Q(B_\lambda), \tag{2.4}$$

where $Q(B_\lambda)$ denotes the fractional field of B_λ. Since $Q(B_\lambda) = Q(K_{n-1} \otimes_{A_{n-1}} B_\lambda)$ and $Q(K_{n-1} \otimes_{A_{n-1}} B_\lambda) = \varinjlim_{f \neq 0, f \in B_\lambda} K_{n-1} \otimes_{A_{n-1}} B_\lambda[1/f]$, there exist some index category Λ' and index system B'_λ such that each B'_λ is a normal domain and we have

$$K'_n = \varinjlim_{\lambda \in \Lambda'} B'_\lambda. \tag{2.5}$$

Then, we see that $\mathrm{Spec}\, B'_\lambda$ is an affine curve over K_{n-1}.

Next, assume M to be a finite $\mathrm{Gal}(\overline{K'_n}/K'_n)$-module of l-power order. Since we have $\mathrm{Gal}(\overline{K'_n}/K'_n) = \varprojlim \pi_1^{alg}(\mathrm{Spec}\, B'_\lambda)$, it follows that

$$H^i_{\mathrm{Gal}}(K'_n, M) = \varinjlim_{\lambda \in \Lambda} H^i_{\mathrm{Gal}}(\pi_1^{alg}(\mathrm{Spec}\, B'_\lambda), M) = \varinjlim_{\lambda \in \Lambda} H^i_{\acute{e}t}(\mathrm{Spec}\, B'_\lambda, M). \tag{2.6}$$

By putting $\overline{B'_\lambda} = B'_\lambda \otimes_{K_{n-1}} \overline{K_{n-1}}$, we obtain the Hochschild-Serre spectral sequence

$$E_2^{p,q} = H^p_{\mathrm{Gal}}(K_{n-1}, H^q_{\acute{e}t}(\mathrm{Spec}\, \overline{B'_\lambda}, M)) \Longrightarrow H^{p+q}_{\acute{e}t}(\mathrm{Spec}\, B'_\lambda, M). \tag{2.7}$$

Further, the induction assumption together with the weak Lefschetz theorem shows that the $E_2^{p,q}$-term vanishes when $p > \mathrm{cd}_l k + (n-1)$, or $q > 1$. Hence for $i > \mathrm{cd}_l k + n$, we obtain $H^i_{\acute{e}t}(\mathrm{Spec}\, B'_\lambda, M) = 0$, hence $H^i_{\acute{e}t}(K'_n, M) = 0$ by (**??**). This proves $\mathrm{cd}_l K'_n \leq cd_l k + n$.

Next, we show $\mathrm{cd}_l K_n \leq \mathrm{cd}_l k + n$. We see $K_n = \varinjlim_f A_n[1/f]$, where f runs over all non-zero elements of A_n. Put $\widehat{U_f} = \mathrm{Spec}\, A_n[1/f]$. Let M be a finite $\mathrm{Gal}(\overline{K_n}/K_n)$-module of l-power order. Then, M is a $\pi_1(\widehat{U_f})$-module, and we have

$$H^i_{\mathrm{Gal}}(K_n, M) = \varinjlim_f H^i_{\acute{e}t}(\pi_1(\widehat{U_f}), M). \tag{2.8}$$

But for an l-primary torsion module M ($l \neq p = ch\ \widehat{U_f}$), it is easily seen that

$$H^i_{\acute{e}t}(\pi_1(\widehat{U_f}), M) \cong H^i_{\acute{e}t}(\pi_1(\widehat{U_f})^{\mathrm{tame}}, M), \tag{2.9}$$

where $\pi_1(\widehat{U_f})^{\mathrm{tame}} := \varprojlim \mathrm{Aut}\,(V/\widehat{U_f})$, where V runs over all tamely ramified Galois coverings of $\widehat{U_f}$. This is easily seen, for example, by the fact that for a pro-p-subgroup P of the profinite group G, $H^l_{\mathrm{Gal}}(P, M) = 0$ for $l > 0$ by the assumption of M, so the Hochschild-Serre spectral sequence $E_2^{k,l} = H^k_{\mathrm{Gal}}(G/P, H^l_{\mathrm{Gal}}(P, M)) \Longrightarrow H^i_{\mathrm{Gal}}(G, M)$ degenerates giving an isomorphism $H^i_{\mathrm{Gal}}(G, M) \cong H^i_{\mathrm{Gal}}(G/P, M^P)$. Now, we get (**??**) by putting $G = \pi_1(\widehat{U_f})$, $G/P = \pi_1(\widehat{U_f})^{\mathrm{tame}}$ in the just above isomorphism. Thus, we have

$$H^i_{\mathrm{Gal}}(K_n, M) = \varinjlim_f H^i_{\acute{e}t}(\pi_1(\widehat{U_f}), M) \cong \varinjlim_f H^i_{\acute{e}t}(\pi_1(\widehat{U_f})^{\mathrm{tame}}, M). \tag{2.10}$$

Assume $i > \mathrm{cd}_l k + n$, and let $e \in H^i_{\mathrm{Gal}}(K_n, M)$ be an arbitrary element of $H^i_{\mathrm{Gal}}(K_n, M)$. Then from (**??**), e can be represented by an element $\widehat{e_f}$ of $H^i_{\acute{e}t}(\pi_1(\widehat{U_f})^{\mathrm{tame}}, M)$ for some f, and by the Weierstrass Preparation theorem, we may assume $f \in A'_n$. Put $U'_f = \mathrm{Spec}\, A'_n[1/f]$. Now, we have the following result of M. Artin.

Theorem (Artin, [**?**] Th.5.1) Let R be an arbitrary excellent equi-characteristic henselian local ring, $X/\mathrm{Spec}\, R$ proper, $Y \subset X$ a closed subscheme, and $U = X - Y$. Let $\widehat{V}$ be an etale covering of $\widehat{U}$, where $\widehat{U}$ is the fibre product $U \times_{\mathrm{Spec}\, R} \mathrm{Spec}\, \widehat{R}$, where $\widehat{R}$ denotes the completion of R by the maximal ideal of R. Further, assume that X is normal. Then, if the covering $\widehat{V}$ is tamely ramified at points of $\widehat{Y}$ which are of codimension 1 in $\widehat{X}$, then $\widehat{V}$ is induced by an etale covering V of U.

From this theorem, we immediately obtain an isomorphism $\pi_1(\widehat{U_f})^{\mathrm{tame}} \xrightarrow{\sim} \pi_1(U'_f)^{\mathrm{tame}}$. Consequently, we get an isomorphism

$$H^i_{\acute{e}t}(\pi_1(U'_f)^{\mathrm{tame}}, M) \xrightarrow{\sim} H^i_{\acute{e}t}(\pi_1(\widehat{U_f})^{\mathrm{tame}}, M). \tag{2.11}$$

By this isomorphism (**??**), $\widehat{e_f}$ is sent to $e'_f \in H^i_{\acute{e}t}(\pi_1(U'_f)^{\mathrm{tame}}, M)$.

On the other hand, we have

$$H^i_{\mathrm{Gal}}(K'_n, M) = \varinjlim_{f \in A'_n,\, f \neq 0} H^i_{\acute{e}t}(\pi_1(U'_f), M) = \varinjlim_{f \in A'_n,\, f \neq 0} H^i_{\acute{e}t}(\pi_1(U'_f)^{\mathrm{tame}}, M), \tag{2.12}$$

because an arbitrary element of M is annihilated by natural number prime to the characteristic of U'_f. We have proved in the first step that the left hand side is zero. So, there exists some element $g \in A'_n$ such that the image of e'_f by the inflation map $H^i_{\acute{e}t}(\pi(U'_f)^{\mathrm{tame}}, M) \to H^i_{\acute{e}t}(\pi(U'_{fg})^{\mathrm{tame}}, M)$ is zero. It follows that also the image of $\widehat{e_f}$ by the inflation map $H^i_{\acute{e}t}(\pi_1(\widehat{U_f})^{\mathrm{tame}}, M) \to H^i_{\acute{e}t}(\pi_1(\widehat{U_{fg}})^{\mathrm{tame}}, M)$ is zero, because the isomorphism (**??**) commutes with the inflation maps of the etale cohomology in the inductive limits,viz.

$$\begin{array}{ccc} H^i_{\acute{e}t}(\pi_1(U'_f)^{\mathrm{tame}}, M) & \xrightarrow{\sim} & H^i_{\acute{e}t}(\pi_1(\widehat{U_f})^{\mathrm{tame}}, M) \\ \downarrow \text{Inflation} & & \downarrow \text{Inflation} \\ H^i_{\acute{e}t}(\pi_1(U'_{fg})^{\mathrm{tame}}, M) & \xrightarrow{\sim} & H^i_{\acute{e}t}(\pi_1(\widehat{U_{fg}})^{\mathrm{tame}}, M). \end{array} \tag{2.13}$$

Especially by (**??**), e is zero as an element of $H^i_{\mathrm{Gal}}(K_n, M)$, which shows that $H^i_{\mathrm{Gal}}(K_n, M) = 0$. Therefore the desirable result $\mathrm{cd}_l K_n \leq \mathrm{cd}_l k + n$ is proved. □

Remark. Though it seems much more difficult to prove, above theorem **??** would hold even in the mixed characteristic cases. That is, for the quotient field L of a complete regular local ring $\mathbb{Z}_p[[X_1, \ldots X_{n-1}]]$, it would hold that $\mathrm{cd}_l L \leq n + 1$. Indeed, for the case of $n = 2$, i.e. $L = Q(\mathbb{Z}_p[[X]])$, the theorem $\mathrm{cd}_l L \leq 3$ was proved by K. Kato (cf. [**?**]) using his deep calculations of Milnor K-groups for higher local fields and certain results in [**?**]. See also [**?**].

3. Proof of the main theorem

In this section we prove a cohomological Hasse principle for A.

Theorem 3.1. *Let A be the three-dimensional formal power seies ring over finite field. Then for an arbitrary natural number m prime to the characteristic of A, the following Hasse principle complex for A becomes exact:*

$$0 \to H^4_{\acute{e}t}(K, \mu_m^{\otimes 3}) \to \bigoplus_{\mathfrak{p} \in P^1_A} H^3_{\acute{e}t}(\kappa(\mathfrak{p}), \mu_m^{\otimes 2}) \to \bigoplus_{\mathfrak{m} \in P^2_A} H^2_{\acute{e}t}(\kappa(\mathfrak{m}), \mu_m) \to \mathbb{Z}/m \to 0, \tag{3.1}$$

where P^i_A denotes the set of height i prime ideals of A and $\kappa(\mathfrak{p})$, $\kappa(\mathfrak{m})$ denotes the residue fields of $\mathfrak{p}$, $\mathfrak{m}$, respectively.

Proof. For the proof, we use the coniveau-spectral sequence. Let us denote $X := \operatorname{Spec} A$. Consider the following coniveau spectral sequence for X:

$$E_1^{p,q} = \bigoplus_{x \in X^{(p)}} H_x^{p+q}(X_{\text{ét}}, \mu_m^{\otimes 3}) \Rightarrow H_{\text{ét}}^{p+q}(X, \mu_m^{\otimes 3}), \tag{3.2}$$

where $X^{(p)}$ denotes the set of codimension p primes of X. We have an isomorphism $H_x^{p+q}(X_{\text{ét}}, \mu_m^{\otimes 3}) \cong H_x^{p+q}(X_x^h, \mu_m^{\otimes 3})$ by excision, where X_x^h denotes the henselization of X at its prime point x. As X_x^h and its maximal point x satisfy the assumption of theorem 2.1 with $Z = x$ and $X = X_x^h$, we can apply this theorem and find that each component $H_x^{p+q}(X_{\text{ét}}, \mu_m^{\otimes 3})$ in $E_1^{p,q}$-term of (**??**) can be replaced by $H_{\text{ét}}^{q-p}(\kappa(x), \mu_m^{\otimes(3-p)})$. So we can rewrite the above coniveau spectral sequence (**??**) as,

$$E_1^{p,q} = \bigoplus_{x \in X^{(p)}} H_{\text{ét}}^{q-p}(\kappa(x), \mu_m^{\otimes(3-p)}) \Rightarrow H_{\text{ét}}^{p+q}(X, \mu_m^{\otimes 3}). \tag{3.3}$$

We see that each $E_1^{p,q}$-term vanishes if $p > 3$, or $p > q$. Especially, the following complex in $E_1^{p,q}$-terms

$$0 \to E_1^{0,4} \overset{d_1^{0,4}}{\to} E_1^{1,4} \overset{d_1^{1,4}}{\to} E_1^{2,4} \overset{d_1^{2,4}}{\to} E_1^{3,4} \to 0 \tag{3.4}$$

becomes

$$\begin{aligned} 0 \to H_{\text{ét}}^4(K, \mu_m^{\otimes 3}) &\overset{\iota_1}{\to} \bigoplus_{\mathfrak{p} \in X^{(1)}} H_{\text{ét}}^3(\kappa(\mathfrak{p}), \mu_m^{\otimes 2}) \\ &\overset{\iota_2}{\to} \bigoplus_{\mathfrak{m} \in X^{(2)}} H_{\text{ét}}^2(\kappa(\mathfrak{m}), \mu_m) \overset{addition}{\to} (\mathbb{Z}/m) \to 0, \end{aligned}$$

which is nothing but a Hasse principle complex (**??**). So theorem 3.1 follows from the following proposition:

Proposition 3.2. $E_2^{0,4} = E_2^{1,4} = E_2^{2,4} = E_2^{3,4} = 0$.

Proof First, we state a lemma.

Lemma 3.3. *For $p + q > 1$, the convergent term satisfies*
$E^{p+q} = H_{ét}^{p+q}(X, \mu_m^{\otimes 3}) = H_{ét}^{p+q}(\operatorname{Spec} A, \mu_m^{\otimes 3}) = 0$, *hence $E_\infty^{p,q} = 0$ for $p + q > 1$.*

Proof. This follows from an isomorphism $H_{\text{ét}}^{p+q}(\operatorname{Spec} A, \mu_m^{\otimes 3}) \cong H_{\text{ét}}^{p+q}(A/\mathfrak{m}_A, \mu_m^{\otimes 3})$, which is deduced from the comparison theorem of the Henselian local rings (cf. [**?**]), and the latter group is 0 for $p + q > 1$, because the cohomological dimension of the finite field $A/\mathfrak{m}_A$ for torsion sheaves is 1. □

We return to the proof of proposition **??**.

1). First, we prove $E_2^{3,4} = 0$. Just below, we see that both $E_2^{5,3}$ and $E_2^{1,5}$ are zero. For $E_2^{5,3} = 0$, because there is no codimension 5 prime in X. And for the proof of $E_2^{1,5} = 0$, we need the following results:

Theorem 3.4. (*Nagata,* [**?**] *Chap.V, Cor.31.6*) *If R is a complete local integral domain, then R contains a complete regular ring S such that R is a finite S-module and such that $S = I\,[[X_1, \ldots, X_r]]$ with a coefficient ring I of R and analytically independent elements $X_1, \ldots, X_r$.*

Theorem 3.5. (*Serre,* [**?**] *Chap.I, Prop.14*) *Let H be a closed subgroup of the profinite group G. Then, one has $cd_p(H) \leq cd_p(G)$, where p is an arbitrary prime integer.*

It's enogh to prove $E_1^{1,5} = 0$. But $E_1^{1,5} = \bigoplus_{\mathfrak{p} \in X^{(1)}} H^4_{\text{ét}}(\kappa(\mathfrak{p}), \mu_m^{\otimes 2})$ and we will prove each component $H^4_{\text{ét}}(\kappa(x), \mu_m^{\otimes 2})$ in $E_1^{1,5}$ exactly vanishes.

We apply the above theorem **??** by putting $I = \mathbb{F}_p$, $r = 2$ which shows that $\kappa(\mathfrak{p})$ ($\mathfrak{p} \in X^{(1)}$) is a finite extention of $\mathrm{Frac}(\mathbb{F}_p[[X, Y]])$. But we already know $\mathrm{cd}_l \mathrm{Frac}\,(\mathbb{F}_p[[X, Y]]) = 3$ by theorem 2.2, and theorem **??** tells us that $\mathrm{cd}_l\,\kappa(\mathfrak{p}) \leqq \mathrm{cd}_l \mathrm{Frac}\,(\mathbb{F}_p[[X, Y]]) = 3$, because $\mathrm{Gal}\,(\overline{\kappa(\mathfrak{p})}/\kappa(\mathfrak{p}))$ is the closed subgroup of $\mathrm{Gal}\,(\overline{\mathrm{Frac}(\mathbb{F}_p[[X, Y]])}/\mathrm{Frac}\,(\mathbb{F}_p[[X, Y]]))$. Thus, we obtain $H^4_{\text{ét}}(\kappa(\mathfrak{p}), \mu_m^{\otimes 2}) = 0$.

It follows that $E_3^{3,4} = \mathrm{Ker}(E_2^{3,4} \overset{d_2^{3,4}}{\to} E_2^{5,3})/\mathrm{Im}(E_2^{1,5} \overset{d_2^{1,5}}{\to} E_2^{3,4}) = E_2^{3,4}$. But, it's easily seen that $E_3^{3,4} = \ldots = E_\infty^{3,4} = 0$. Hence we get the deasired vanishing of $E_2^{3,4}$.

2). Next, we show $E_2^{2,4} = 0$. This is proved as follows. First, it holds that $E_2^{4,3} = E_2^{0,5} = 0$ which follows from the vanishing of $E_1^{4,3} = E_1^{0,5} = 0$. Of course, $E_1^{4,3} = 0$ follows from the inequality $p = 4 > 3 = q$ and $E_1^{0,5} = 0$ follows again from theorem 2.2. So, we have

$$E_3^{2,4} = \mathrm{Ker}(\mathrm{E}_2^{2,4} \overset{\mathrm{d}_2^{2,4}}{\to} \mathrm{E}_2^{4,3})/\mathrm{Im}(\mathrm{E}_2^{0,5} \overset{\mathrm{d}_2^{0,5}}{\to} \mathrm{E}_2^{2,4}) = \mathrm{E}_2^{2,4}. \tag{3.5}$$

But it follows that $E_3^{2,4} = \cdots = E_\infty^{2,4} = 0$. Hence we are done.

3). Thirdly, we prove $E_2^{1,4} = 0$. We need a lemma.

Lemma 3.6. *We have the vanishing $E_2^{3,3} = 0$.*

Proof. By definition, $E_2^{3,3} = \mathrm{Ker}(E_1^{3,3} \overset{d_1^{3,3}}{\to} E_1^{4,3})/\mathrm{Im}(E_1^{2,3} \overset{d_1^{2,3}}{\to} E_1^{3,3}) = \mathrm{Coker}(E_1^{2,3} \overset{d_1^{2,3}}{\to} E_1^{3,3})$, where the second equality follows from the vanishing $E_1^{4,3} = 0$. But this group is rewritten

as

$$E_2^{3,3} = \mathrm{Coker}(E_1^{2,3} \overset{d_1^{2,3}}{\to} E_1^{3,3}) = \mathrm{Coker}\left(\bigoplus_{\mathfrak{m}\in X^{(2)}} H^1_{et}(\kappa(\mathfrak{m}), \mu_m) \overset{d_1^{2,3}}{\to} \mathbb{Z}/m\right). \tag{3.6}$$

But from the Kummer theory, we can rewrite this group as,

$$E_2^{3,3} = \mathrm{Coker}\left(\bigoplus_{\mathfrak{m}\in X^{(2)}} \kappa(\mathfrak{m})^*/m \overset{d_1^{2,3}}{\to} \mathbb{Z}/m\right). \tag{3.7}$$

But each map $\kappa(\mathfrak{m})^*/m \to \mathbb{Z}/m$ is nothing but the valuation map of one-dimensional complete discrete valuation field $\kappa(\mathfrak{m})$. So if we choose a height two prime $\mathfrak{m}$ such that $A/\mathfrak{m}$ is regular, it becomes surjective, hence $d_1^{2,3}$ is also surjective which proves $E_2^{3,3} = 0$. □

Now we proceed as follows. It's easily seen that $E_3^{1,4} = \ldots = E_\infty^{1,4} = 0$. Further, $E_3^{1,4} = \mathrm{Ker}(E_2^{1,4} \overset{d_2^{1,4}}{\to} E_2^{3,3})$. But lemma **??** shows $E_2^{3,3} = 0$ and it follows immediately that $E_2^{1,4} = 0$.

4). Finally, the proof of proposition **??** is finished by showing $E_2^{0,4} = 0$. For this, we first show that $E_3^{0,4} = 0$. It is easily seen that $E_4^{0,4} = \ldots = E_\infty^{0,4} = 0$. But $E_4^{0,4} = \mathrm{Ker}(E_3^{0,4} \overset{d_3^{0,4}}{\to} E_3^{3,2})$ and $E_3^{3,2} = 0$, which results from $E_1^{3,2} = 0\,(p = 3 > 2 = q)$. Putting all together, we have $E_3^{0,4} = E_4^{0,4} = 0$. On the other hand, $E_3^{0,4} = \mathrm{Ker}(E_2^{0,4} \overset{d_2^{0,4}}{\to} E_2^{2,3})$, so our desirable vanishing $E_2^{0,4} = 0$ follows directly from $E_2^{2,3} = 0$.

Claim 3.7. *We have an equality* $E_2^{2,3} = 0$.

Proof. By definition,

$$E_2^{2,3} = \mathrm{Ker}(E_1^{2,3} \overset{d_1^{2,3}}{\to} E_1^{3,3})/\mathrm{Im}(E_1^{1,3} \overset{d_1^{1,3}}{\to} E_1^{2,3}).$$

So, the exactness of the following sequence furnishes this Claim.

$$E_1^{1,3} \overset{d_1^{1,3}}{\to} E_1^{2,3} \overset{d_1^{2,3}}{\to} E_1^{3,3}. \tag{3.8}$$

But this sequence (**??**) can be rewritten as

$$\bigoplus_{\mathfrak{p}\in X^{(1)}} H^2(\kappa(\mathfrak{p}), \mu_m^{\otimes 2}) \to \bigoplus_{\mathfrak{m}\in X^{(2)}} H^1(\kappa(\mathfrak{m}), \mu_m) \to \mathbb{Z}/m. \tag{3.9}$$

Further by the theorem of Mercurjev-Suslin and the Kummer theory, (**??**) is rewritten as

$$\bigoplus_{\mathfrak{p}\in X^{(1)}} K_2^Q(\kappa(\mathfrak{p}))/m \to \bigoplus_{\mathfrak{m}\in X^{(2)}} K_1^Q(\kappa(\mathfrak{m}))/m \to \mathbb{Z}/m. \tag{3.10}$$

(we used the fact that $K_i^M(k) \cong K_i^Q(k)$ for a field k). So everything is reduced to prove the exactness of the sequence (??). But this is exactly the Gersten-Quillen complex for A which is proved exact by Quillen in [?]. So we finished the proof of claim 3.7, hence propositon 3.2. Thus, the proof of theorem 3.1 is now completely finished. □

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