# ARITHMETIC MIXED HODGE STRUCTURES

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We give a formalism of arithmetic mixed Hodge structures which are recently studied by M. Green [12], [13] and M. Asakura [1], and are named by them. This notion became necessary for us to describe the image of the Abel-Jacobi map for a generic hypersurface (inspired by previous work of M. Green [11] and C. Voisin [25]), and also to prove the following variant of results of D. Mumford [17] and A. Roitman [18] suggested by S. Bloch in Exercise of Appendix to Lecture 1 of [6].

**0.1. Proposition.** *If there is a morphism of complex varieties $S \to X$ inducing a surjective morphism $\mathrm{CH}_0(S)_{\mathbb{Q}} \to \mathrm{CH}_0(X)_{\mathbb{Q}}$ where $X$ is smooth proper, then $\Gamma(X,\Omega_X^j) = 0$ for $j > \dim S$.*

Actually, it turns out that the usual Hodge theory [9] is enough for this. See (4.4) below. But the attempt led us to the following formulation.

For a subfield $A$ of $\mathbb{R}$, and a subfield $k$ of $\mathbb{C}$, let $\mathrm{MHS}(A)_{\langle k\rangle}$ be the category of $k$-finite mixed $A$-Hodge structures, which is by definition the inductive limit of the category of mixed $A$-Hodge Modules on $S = \operatorname{Spec} R$ with $R$ running over finitely generated $k$-subalgebras of $\mathbb{C}$. See (2.2). It has a natural functor $\iota$ to $\mathrm{MHS}(A)$ the category of graded-polarizable mixed $A$-Hodge structures in the usual sense [9], and has the constant object $A_{\langle k\rangle}$ as the limit of the constant variation of Hodge structure of type $(0,0)$ on smooth $S$. For a complex variety $X$, we can define the cohomology $H^j(X, A_{\langle k\rangle})$ in $\mathrm{MHS}(A)_{\langle k\rangle}$ in a compatible way with the functor $\iota$.

**0.2. Theorem.** *Let $X$ be a smooth proper complex variety of dimension $n$. Assume $k$ has finite transcendental degree. If $\Gamma(X,\Omega_X^j) \neq 0$, then $\mathrm{Ext}^j(A_{\langle k\rangle}, H^{2n-j}(X, A_{\langle k\rangle}(n))) \neq 0$, where $\mathrm{Ext}^j$ is taken in $\mathrm{MHS}(A)_{\langle k\rangle}$. See (4.3).*

As to the image of the Abel-Jacobi map, let $A = \mathbb{Q}$, and consider a cycle map of $\mathrm{CH}^p(X)_{\mathbb{Q}}$ to

$$H^{2p}_{\mathcal{D}}(X, \mathbb{Q}_{\langle k\rangle}(p)) := \mathrm{Ext}^{2p}(\mathbb{Q}_{\langle k\rangle}, (a_X)_* a_X^* \mathbb{Q}_{\langle k\rangle}(p)),$$

where $\mathrm{Ext}^{2p}$ is taken in the derived category of $\mathrm{MHS}(A)_{\langle k\rangle}$, and $a_X : X \to pt$ is the structure morphism. We have a natural morphism of $H^{2p}_{\mathcal{D}}(X, \mathbb{Q}_{\langle k\rangle}(p))$ to the usual Deligne

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cohomology $H^{2p}_{\mathcal{D}}(X,\mathbb{Q}(p))$, and its image is called the $k$-finite part of the Deligne cohomology, and is denoted by $H^{2p}_{\mathcal{D}}(X,\mathbb{Q}(p))^{\langle k\rangle}$. The $k$-finite part $J^p(X)^{\langle k\rangle}_{\mathbb{Q}}$ of the Griffiths intermediate Jacobian tensored with $\mathbb{Q}$ is defined by intersecting it with $H^{2p}_{\mathcal{D}}(X,\mathbb{Q}(p))^{\langle k\rangle}$. We have the Abel-Jacobi map of $\mathrm{CH}^p(X)^{(0)}_{\mathbb{Q}}$ (which is the kernel of the cycle map of $\mathrm{CH}^p(X)_{\mathbb{Q}}$ to $H^{2p}(X,\mathbb{Q}(p))$) to $J^p(X)^{\langle k\rangle}_{\mathbb{Q}}$. Then by an argument similar to [22], [23], we can show (see (4.2)) :

**0.3. Proposition.** *Let $X$ and $k$ be as in (0.2). If $H^{2p-1}(X,\mathbb{Q}_{\langle k\rangle})$ is global section-free (see (2.2)), and the Hodge conjecture holds for any smooth projective complex varieties defined over $k$, then the cycle map $cl:\mathrm{CH}^p(X)_{\mathbb{Q}}\to H^{2p}_{\mathcal{D}}(X,\mathbb{Q}(p))^{\langle k\rangle}$ and the Abel-Jacobi map $\mathrm{CH}^p(X)^{(0)}_{\mathbb{Q}}\to J^p(X)^{\langle k\rangle}_{\mathbb{Q}}$ are surjective.*

The first assumption is satisfied if $X$ is a generic hypersurface, or more generally, if $X$ is a generic hypersurface of a smooth projective variety $Y$ defined over a subfield of $k$ such that $H^{2p-1}(Y,\mathbb{Q})=0$. If the first assumption of (0.3) is not satisfied, the cycle map $cl:\mathrm{CH}^p(X)_{\mathbb{Q}}\to H^{2p}_{\mathcal{D}}(X,\mathbb{Q}(p))^{\langle k\rangle}$ does not seem to be surjective, because $\mathrm{Ext}^1$ in $\mathrm{MHS}(k,A)$ is too big. See (1.4).

For the injectivity of the cycle map, let $Y$ be a smooth variety over an algebraic number field, and consider the Abel-Jacobi map

$$\mathrm{CH}^p(Y)^{(0)}_{\mathbb{Q}}\to J^p(Y/k)_{\mathbb{Q}}:=\mathrm{Ext}^1(\mathbb{Q}_k,H^{2p-1}(Y/k,\mathbb{Q}(p)))$$

where $k$ is the maximal algebraic number field contained in the function field of $Y$, and $\mathbb{Q}_k, H^{2p-1}(Y/k,\mathbb{Q}(p))$ are defined in $\mathrm{MHS}(k,A)$, the category of mixed $\mathbb{Q}$-Hodge structures whose $\mathbb{C}$-part is defined over $k$. See (1.3).

**0.4. Proposition.** *Let $X,k$ be as in (0.2), and assume $k$ is a number field. Then the cycle map $\mathrm{CH}^p(X)_{\mathbb{Q}}\to H^{2p}_{\mathcal{D}}(X,\mathbb{Q}_{\langle k\rangle}(p))$ is injective if the above Abel-Jacobi map is injective for any smooth variety $Y$ over a number field. See (4.5).*

The assumption would be reduced to the injectivity of the Abel-Jacobi map for any smooth projective varieties over number fields if the Hodge conjecture holds for any $p-1$ codimensional cycles on smooth projective varieties defined over number fields (e.g. if $p=2$).

In §1, we introduce the notion of mixed Hodge structure whose $\mathbb{C}$-part is defined over $k$, and calculate the extension group. In §2, we study the category of $k$-finite mixed Hodge Modules, and define the cycle map in §3. The proofs of the above assertions are given in §4.

## 1. Mixed Hodge Modules defined over a subfield

**1.1.** Let $A$ be a subfield of $\mathbb{R}$, and $k$ a subfield of $\mathbb{C}$. (The reader may assume $k$ is algebraically closed, because an $l$-adic sheaf [5] is not used in this note.) For a variety $X$ over $k$, let $\mathrm{MHM}(X/k,A)$ denote the category of mixed Hodge Modules [20] (see also

Remark below) on $X_{\mathbb{C}}$ ($:= X \otimes_k \mathbb{C}$) whose underlying filtered $\mathcal{D}$-Module $(M, F; W)$ on $X_{\mathbb{C}}$ is defined over $k$ (i.e., a filtered $\mathcal{D}$-Module $(M_k, F; W)$ on $X/k$ together with an isomorphism $(M_k, F; W) \otimes_k \mathbb{C} = (M, F; W)$ is given). We also assume that polarizations on the graded pieces of the weight filtration are defined over $k$. See (1.8, ii) in [21].

*Remarks.* (i) If $X$ is a smooth complex variety, a mixed $A$-Hodge Module consists of $((M, F; W), (K, W), \alpha)$ where $(M, F)$ is a filtered regular holonomic $\mathcal{D}_X$-Module, $K$ is a perverse sheaf with $A$-coefficients on $X^{\mathrm{an}}$, $W$ is a finite increasing filtration on $M, K$ (called the weight filtration), and $\alpha : \mathrm{DR}(M, W) = (K, W) \otimes_A \mathbb{C}$ is an isomorphism of filtered perverse sheaves with $\mathbb{C}$-coefficients on $X^{\mathrm{an}}$. They satisfy several good conditions. See [20].

(ii) If $X$ is a singular complex variety, the underlying filtered $\mathcal{D}$-Module of $a$ mixed $A$-Hodge Module is defined by using closed embeddings $U \to V$ where $U$ are open subvariety of $X$ and $V$ are smooth. See for example [19, 2.1.20].

**1.2. Theorem.** *The category* $\mathrm{MHM}(X/k, A)$ *is an abelian category, and every morphism is strictly compatible with* $(F, W)$. *The bounded derived categories* $D^b\mathrm{MHM}(X/k, A)$ *are stable by standard functors like* $f_*, f_!, f^*, f^!$, *etc.*

*Proof.* The first assertion follows from the same argument as in [19, 5.1.14]. Then the second follows from [21].

*Remark.* For the construction of the direct images $f_*, f_!$, it is enough to construct the cohomological direct images $H^i f_*, H^i f_!$ (thanks to the strictness of $(F, W)$). Indeed, if $X$ is quasi-projective, $f_*, f_!$ are defined by taking two sets of open coverings of $X$ associated with general hyperplane sections, and using the combination of co-Cech and Cech complexes together with Artin's vanishing theorem for the (perverse) cohomological direct images by an affine morphism. See [4], [5].

For the pull-back by a closed embedding $i : X \to Y$, it is enough to show an equivalence of categories

$$i_* : D^b\mathrm{MHM}(X/k, A) \to D^b_X(\mathrm{MHM}(Y/k, A),$$

where the target is the full subcategory of $D^b(\mathrm{MHM}(Y/k, A)$ defined by the condition : $\operatorname{supp} H^j M \subset X$ for any $j$. This is reduced to the case where $X$ is a divisor defined by a function $g$. Then the assertion is verified by using the functor $\xi_g$ in [20, 2.22]. This is inspired by [4]. (The paper quoted in loc. cit. was not available at that time). A similar functor which is denoted by $\Xi_g$, and corresponds to $\xi_g j_*$ (where $j : Y \backslash X \to Y$ is the inclusion morphism) is used in [4]. It is not clear whether we can construct this functor in the category of mixed Hodge Modules by using projective limit and inductive limit as in the paper quoted in loc. cit.

**1.3.** Let $\mathrm{MHS}(k, A) = \mathrm{MHM}(\operatorname{Spec} k/k, A)$ (where we put $F^p = F_{-p}$). This is the category of graded-polarizable mixed $A$-Hodge structures [9] whose $\mathbb{C}$-part is defined over $k$ (and polarizations are assumed to be defined over $k$ as above). An object $H$ of $\mathrm{MHS}(k, A)$ consists of $((H_k, F; W), (H_A, W), (H_{\mathbb{C}}, W); \alpha_k, \alpha_A)$, where $H_\Lambda$ is a filtered (or bifiltered) $\Lambda$-vector spaces for $\Lambda = k, A, \mathbb{C}$ with comparison isomorphisms $\alpha_k : H_k \otimes_k \mathbb{C} \to H_{\mathbb{C}}$, $\alpha_A : H_A \otimes_A \mathbb{C} \to H_{\mathbb{C}}$ compatible with the filtration $W$.

For $j \in \mathbb{Z}$, we define $A_k(j) \in \mathrm{MHS}(k,A)$ by $H_k = k$ and $H_A = (2\pi i)^j A \subset H_{\mathbb{C}} = \mathbb{C}$, where $\alpha_k$ and $\alpha_A$ are natural isomorphisms, and $\mathrm{Gr}_F^i = 0$ for $i \neq -j$ and $\mathrm{Gr}_i^W = 0$ for $i \neq -2j$. Let $A_k = A_k(0)$.

For a $k$-variety $X$ with structure morphism $a_{X/k} : X \to \mathrm{Spec}\, k$, let

$$A_{X/k} = a_{X/k}^* A_k, \quad H^j(X/k, A) = H^j(a_{X/k})_* A_{X/k}. \tag{1.3.1}$$

Let $H_i \in \mathrm{MHS}(k,A)$ for $i = 1,2$. Put $H = \mathcal{H}om(H_1, H_2) \in \mathrm{MHS}(k,A)$. We define

$$C(H)^0 = \mathrm{Ker}(F^0 W_0 H_k \oplus W_0 H_A \to \mathrm{Gr}_0^W H_{\mathbb{C}}), \quad C(H)^1 = W_{-1} H_{\mathbb{C}},$$

so that we get a complex $C(H) = [C(H)^0 \to C(H)^1]$.

**1.4. Lemma.** *We have natural isomorphisms*

$$H^0(C(H)) = \mathrm{Hom}(H_1, H_2), \;\; H^1(C(H)) = \mathrm{Ext}^1(H_1, H_2),$$

*where* Hom *and* $\mathrm{Ext}^1$ *are taken in* $\mathrm{MHS}(k,A)$.

*Proof.* The assertion on $H^0(C(H))$ is clear. For $H^1(C(H))$, we have $\mathrm{Ext}^1(H_1,H_2) = \mathrm{Ext}^1(A_k, H)$, and the assertion is reduced to the case $H_1 = A_k$ and $H_2 = H$. We may assume further $H = W_0 H$ since $\mathrm{Ext}^1(A_k, H/W_0 H) = 0$ (using the splitting by the weight filtration ).

Let $\mathrm{MHS}(k,A)'$ denote the category of mixed $A$-Hodge structures whose $\mathbb{C}$-part is defined over $k$ as above (but the graded pieces of the weight filtration are not necessarily polarizable). Then by the same argument as in [8], we get an isomorphism

$$'\mathrm{Ext}^1(A_k, H) = H_{\mathbb{C}}/(F^0 H_k + H_A), \tag{1.4.1}$$

where $'\mathrm{Ext}^1$ is taken in $\mathrm{MHS}(k,A)'$. Using the semisimplicity of the graded pieces of the weight filtration, we see that $\mathrm{Ext}^1(A_k, H)$ is identified with

$$\mathrm{Ker}('\mathrm{Ext}^1(A_k, H) \to '\mathrm{Ext}^1(A_k, \mathrm{Gr}_0^W H)),$$

which is isomorphic to $\mathrm{Im}('\mathrm{Ext}^1(A_k, W_{-1}H) \to '\mathrm{Ext}^1(A_k, H))$. So the assertion follows.

**1.5. Corollary.** *For* $i > 1$, $\mathrm{Ext}^i(H_1, H_2) = 0$.

*Proof.* It is enough to show that $H^1(C(H))$ is a right exact functor. But this is clear by definition of $C(H)^1$.

## 2. k-finite Mixed Hodge Modules

**2.1.** *Mixed Hodge Modules.* Let $A, k$ be as above. For a complex variety $X$, let $R$ be a finitely generated $k$-subalgebra of $\mathbb{C}$ such that $X$ is defined over $R$, i.e., there is an $R$-scheme $X_R$ with an isomorphism $X_R \otimes_R \mathbb{C} = X$. Then, for a finitely generated

smooth $k$-subalgebra $R'$ of $\mathbb{C}$ (i.e. $\operatorname{Spec} R'$ is smooth over $k$) containing $R$, we put $X_{R'} = X_R \otimes_R R', S' = \operatorname{Spec} R', d_{R'} = \dim_k S'$, and $k_{R'} = \overline{k} \cap k(R')$ where $k(R')$ is the fractional field of $R'$. For $R', R''$ as above with $R' \subset R''$, we have a natural morphism by the pull-back

$$\mathrm{MHM}(X_{R'}/k_{R'}, A)[-d_{R'}] \to \mathrm{MHM}(X_{R''}/k_{R''}, A)[-d_{R''}], \tag{2.1.1}$$

where $\mathrm{MHM}(X_{R'}/k_{R'}, A)[-d_{R'}]$ denotes the category of mixed Hodge Modules shifted in the derived category. Since $X$ is identified with the (general) closed fiber of $X_{R'} \otimes_{k_{R'}} \mathbb{C}$ over the closed point corresponding to the inclusion $R' \subset \mathbb{C}$, we have also a natural functor

$$\mathrm{MHM}(X_{R'}/k_{R'}, A)[-d_{R'}] \to \mathrm{MHM}(X, A) \tag{2.1.2}$$

compatible with the above functor. We define

$$\mathrm{MHM}(X, A)_{\langle k \rangle} = \text{ind } \lim_{R'} \mathrm{MHM}(X_{R'}/k_{R'}, A)[-d_{R'}]. \tag{2.1.3}$$

This is independent of the choice of $R$ and $X_R$. An object of this category is represented by an object of $\mathrm{MHM}(X_{R'}/k, A)[-d_{R'}]$ for some $R'$, and is called a $k$-finite mixed $A$-Hodge Module. By (2.1.2) we have a natural functor

$$\iota : \mathrm{MHM}(X, A)_{\langle k \rangle} \to \mathrm{MHM}(X, A). \tag{2.1.4}$$

We have cohomological functors to $\mathrm{MHM}(X_{R'}/k_{R'}, A)[-d_{R'}], \mathrm{MHM}(X, A)_{\langle k \rangle}, \mathrm{MHM}(X, A)$ from their derived categories, and they are compatible with the functors (2.1.1), (2.1.2) and (2.1.4).

**2.2.** *Mixed Hodge structures.* With the above notation, let

$$\mathrm{MHS}(A)_{\langle k \rangle} = \mathrm{MHM}(\operatorname{Spec} \mathbb{C}, A)_{\langle k \rangle}. \tag{2.2.1}$$

This is the category of $k$-finite mixed $A$-Hodge structures. It has a natural functor

$$\iota : \mathrm{MHS}(A)_{\langle k \rangle} \to \mathrm{MHS}(A), \tag{2.2.2}$$

where $\mathrm{MHS}(A)$ is the category of graded-polarizable mixed $A$-Hodge structures in the usual sense [9].

For a finitely generated smooth $k$-algebra $R$ of $\mathbb{C}$, put $S = \operatorname{Spec} R$ and $S_{\mathbb{C}} = \operatorname{Spec} R \otimes_{k_R} \mathbb{C}$. Let $\mathrm{VMHS}(S/k_R, A)^{\mathrm{ad}}$ be the category of admissible variations of mixed $A$-Hodge structures on $S_{\mathbb{C}}^{\mathrm{an}}$ (see [15], [24]) whose $\mathbb{C}$-part is defined over $S$ by a bifiltered $\mathcal{O}_S$-Module $(M_S; F, W)$ with an integrable connection. We assume as before that polarization on the graded pieces of the weight filtration are defined over $S$ as pairings of $\mathcal{O}_S$-Modules compatible with connection. Then we have by [20, 3.27]

$$\mathrm{MHS}(A)_{\langle k \rangle} = \text{ind } \lim_R \mathrm{VMHS}(\operatorname{Spec} R/k_R, A)^{\mathrm{ad}}, \tag{2.2.3}$$

where $R$ runs over finitely generated smooth $k$-subalgebras of $\mathbb{C}$.

For $j \in \mathbb{Z}$, we have $A_{\langle k\rangle}(j) \in \mathrm{MHS}(A)_{\langle k\rangle}$ which is represented by a constant variation of Hodge structure of type $(-j,-j)$ on $S = \mathrm{Spec}\, R$. More precisely, this is represented by

$$A_{S/k_R}(j) = ((\mathcal{O}_S; F, W), ((2\pi i)^j A_{S_{\mathbb{C}}}, W), (\mathbb{C}_{S_{\mathbb{C}}}, W), \alpha_k, \alpha_A),$$

where $\mathcal{O}_S$ is the structure sheaf with natural connection $d$, $(2\pi i)^j A_{S_{\mathbb{C}}}$ and $\mathbb{C}_{S_{\mathbb{C}}}$ are constant local systems on $S_{\mathbb{C}}^{\mathrm{an}}$ with stalks $(2\pi i)^j A (\subset \mathbb{C})$ and $\mathbb{C}$,

$$\alpha_k : \mathrm{Ker}\, d^{\mathrm{an}} \to \mathbb{C}_{S_{\mathbb{C}}}, \quad \alpha_A : (2\pi i)^j A_{S_{\mathbb{C}}} \otimes_A \mathbb{C} \to \mathbb{C}_{S_{\mathbb{C}}}$$

are natural isomorphisms, and the filtrations $F, W$ are defined by the conditions : $\mathrm{Gr}_F^i = 0$ for $i \neq -j$, $\mathrm{Gr}_i^W = 0$ for $i \neq -2j$.

We say that $H \in \mathrm{MHS}(A)_{\langle k\rangle}$ is global section-free, if for any finitely generated smooth $k$-subalgebra $R$ of $\mathbb{C}$ such that $H$ is represented by an admissible variation of mixed Hodge structure on $S := \mathrm{Spec}\, R$, the underlying local system on $S_{\mathbb{C}}$ has no nonzero global section.

**2.3.** *Remarks.* (i) When $k = \overline{\mathbb{Q}}$, $\mathrm{MHS}(A)_{\langle k\rangle}$ is called the category of arithmetic mixed Hodge structures according to M. Asakura [1], and is closely related to recent work of M. Green [12], [13] (see also [26]).

(ii) If $H \in \mathrm{MHS}(A)$ belongs to the image of the functor $\iota$ in (2.2.2), then the underlying $\mathbb{C}$-part $H_{\mathbb{C}}$ has naturally an integrable connection $\nabla : H_{\mathbb{C}} \to H_{\mathbb{C}} \otimes_{\mathbb{C}} \Omega^1_{\mathbb{C}/\overline{k}}$ which is the limit of the connection $M_R \to M_R \otimes_R \Omega^1_{R/k_R}$, where $M_R = \Gamma(S, M_S)$.

So we can also consider the category $\mathrm{MHS}(A)_{\nabla/\overline{k}}$ of mixed Hodge structures whose $\mathbb{C}$-part has an integrable connection $\nabla$ over $\overline{k}$. Then we can show that $\mathrm{MHS}(A)_{\langle k\rangle}$ is a full subcategory of $\mathrm{MHS}(A)_{\nabla/\overline{k}}$. Indeed, a morphism $A \to \iota(H)$ in $\mathrm{MHS}(A)_{\nabla/\overline{k}}$ defines a section of $\mathrm{Ker}\,\nabla$ in $M_R$ for $R$ sufficiently large, and the corresponding global section of $H_{\mathbb{C}}$ on $S_{\mathbb{C}}$ belongs to the $A$-local system $H_A$ at the point corresponding to the inclusion $R \to \mathbb{C}$. See also [1], [12], [13].

**2.4. Proposition.** *We have equivalences of categories*

$$\mathrm{MHM}(X, A)_{\langle k\rangle} \to \mathrm{MHM}(X, A)_{\langle \overline{k}\rangle}, \quad \mathrm{MHS}(A)_{\langle k\rangle} \to \mathrm{MHS}(A)_{\langle \overline{k}\rangle}.$$

*induced by* $\mathrm{MHM}(X_{R'}/k_{R'}, A) \to \mathrm{MHM}(X_{R'} \otimes_{k_{R'}} \overline{k}/\overline{k}, A)$.

*Proof.* It is enough to verify that a filtered $\mathcal{D}$-Module on $X_{R'} \otimes_{k_{R'}} \overline{k}/\overline{k}$ is defined on $X_{R'} \otimes_{k_{R'}} k'/k'$ for a sufficiently large algebraic extension $k'$ of $k$ in $\overline{k}$ (and a similar assertion holds for morphisms).

**2.5. Theorem.** *The category* $\mathrm{MHM}(X, A)_{\langle k\rangle}$ *is an abelian category, and the bounded derived categories* $D^b\mathrm{MHM}(X, A)_{\langle k\rangle}$ *are stable by standard functors like* $f_*, f_!, f^*, f^!$, *etc. in a compatible way with the functor* $\iota$.

*Proof.* This follows from (1.2).

## 3. Cycle Maps

**3.1.** For a complex variety $X$ with structure morphism $a_X : X \to \mathbb{C}$, we define

$$A_{X,\langle k\rangle}(j) = a_X^* A_{\langle k\rangle}(j), \quad H^i(X, A_{\langle k\rangle}(j)) = H^i(a_X)_* A_{X,\langle k\rangle}(j).$$

These are compatible with the functor $\iota$.

If $X$ is proper smooth and $A = \mathbb{Q}$, let

$$H^{p,p}(X, \mathbb{Q}(p))^{\langle k\rangle} = \mathrm{Hom}(\mathbb{Q}_{\langle k\rangle}, H^{2p}(X, \mathbb{Q}_{\langle k\rangle}(p))).$$

This is called the group of $k$-finite (or $k$-horizontal) Hodge cycles of codimension $p$.

**3.2.** *Remarks.* (i) By Remark (ii) of (2.3), $H^{p,p}(X, \mathbb{Q}(p))^{\langle k\rangle}$ is identified with the space of $\mathbb{Q}$-Hodge cycles (in the usual sense) which are annihilated by the connection over $\overline{k}$.

(ii) When $A = \mathbb{Q}$, the Hodge conjecture implies that every $\mathbb{Q}$-Hodge cycle in the usual sense is a $\overline{\mathbb{Q}}$-finite (or $\overline{\mathbb{Q}}$-horizontal) Hodge cycle. It may be more natural to restrict the Hodge conjecture to $\overline{\mathbb{Q}}$-finite (or $\overline{\mathbb{Q}}$-horizontal) Hodge cycles.

**3.3.** *Deligne cohomology.* Let $X$ be a proper smooth complex variety, and $H^i_{\mathcal{D}}(X, \mathbb{Q}(j))$ the $\mathbb{Q}$-Deligne cohomology in the usual sense [10]. By [2] it is isomorphic to

$$\mathrm{Ext}^i(\mathbb{Q}^H, (a_X)_* \mathbb{Q}^H_X(j)) = \mathrm{Ext}^i(\mathbb{Q}^H_X, \mathbb{Q}^H_X(j)), \tag{3.3.1}$$

where $\mathbb{Q}^H$ and $\mathbb{Q}^H_X$ belong to $D^b\mathrm{MHS}(\mathbb{Q})$ and $D^b\mathrm{MHM}(X, \mathbb{Q})$ respectively (see [20]), and the isomorphism follows from the adjoint relation between the direct image and the pull-back. We define $H^i_{\mathcal{D}}(X, \mathbb{Q}_{\langle k\rangle}(j))$ as

$$\mathrm{Ext}^i(\mathbb{Q}_{\langle k\rangle}, (a_X)_* \mathbb{Q}_{X,\langle k\rangle}(j)) = \mathrm{Ext}^i(\mathbb{Q}_{X,\langle k\rangle}, \mathbb{Q}_{X,\langle k\rangle}(j)). \tag{3.3.2}$$

This is the limit of $H^{2p}_{\mathcal{D}}(X_R/k_R, \mathbb{Q}_k(p))$ which is defined as

$$\mathrm{Ext}^i(\mathbb{Q}_k, (a_{X_R/k})_* \mathbb{Q}_{X_R/k}(j)) = \mathrm{Ext}^i(\mathbb{Q}_{X_R/k}, \mathbb{Q}_{X_R/k}(j)). \tag{3.3.3}$$

See (1.3). By (2.1.4) we have a natural morphism $H^i_{\mathcal{D}}(X, \mathbb{Q}_{\langle k\rangle}(j)) \to H^i_{\mathcal{D}}(X, \mathbb{Q}(j))$. Its image is denoted by $H^i_{\mathcal{D}}(X, \mathbb{Q}(j))^{\langle k\rangle}$, and is called the $k$-finite part of the Deligne cohomology. We have the decreasing filtration $L$ on $H^i_{\mathcal{D}}(X, \mathbb{Q}_{\langle k\rangle}(j))$ induced by the truncation $\tau$ on $(a_X)_* \mathbb{Q}_{X,\langle k\rangle}(j)$ so that

$$\mathrm{Gr}^r_L H^i_{\mathcal{D}}(X, \mathbb{Q}_{\langle k\rangle}(j)) = \mathrm{Ext}^r(\mathbb{Q}_{\langle k\rangle}, H^{i-r}(a_X)_* \mathbb{Q}_{X,\langle k\rangle}(j)). \tag{3.3.4}$$

See [21, (8.1)]. It may be expected that this filtration induces the conjectural filtration of Beilinson [3] and Bloch [6] on the Chow group. See also [16].

For $(i,j)=(2p,p)$, we have a morphism of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & J^p(X)_{\mathbb{Q}}^{\langle k\rangle} & \longrightarrow & H_{\mathcal{D}}^{2p}(X,\mathbb{Q}(p))^{\langle k\rangle} & \longrightarrow & H^{p,p}(X,\mathbb{Q}(p))^{\langle k\rangle} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & J^p(X)_{\mathbb{Q}} & \longrightarrow & H_{\mathcal{D}}^{2p}(X,\mathbb{Q}(p)) & \longrightarrow & H^{p,p}(X,\mathbb{Q}(p)) & \longrightarrow & 0 \end{array} \tag{3.3.5}$$

where $J^p(X)_{\mathbb{Q}}$ is the Griffiths intermediate Jacobian [14] tensored by $\mathbb{Q}$, and $J^p(X)_{\mathbb{Q}}^{\langle k\rangle}$ is defined as

$$\mathrm{Im}(\mathrm{Ext}^1(\mathbb{Q}_{\langle k\rangle},H^{2p-1}(X,\mathbb{Q}_{\langle k\rangle}(p))) \to \mathrm{Ext}^1(\mathbb{Q}^H,H^{2p-1}(X,\mathbb{Q}(p)))),$$

using $J^p(X)_{\mathbb{Q}}=\mathrm{Ext}^1(\mathbb{Q}^H,H^{2p-1}(X,\mathbb{Q}(p)))$.

**3.4.** *Cycle map.* For a smooth proper complex variety $X$, let $\mathrm{CH}^p(X)_{\mathbb{Q}}$ be the Chow group of codimension $p$ cycles on $X$ tensored with $\mathbb{Q}$. By [20], [21] we have a cycle map

$$cl : \mathrm{CH}^p(X)_{\mathbb{Q}} \to H_{\mathcal{D}}^{2p}(X,\mathbb{Q}_{\langle k\rangle}(p)). \tag{3.4.1}$$

This is the limit of the cycle map

$$cl : \mathrm{CH}^p(X_R)_{\mathbb{Q}} \to H_{\mathcal{D}}^{2p}(X_R/k_R,\mathbb{Q}_k(p)). \tag{3.4.2}$$

Let $\mathrm{CH}^p(X)_{\mathbb{Q}}^{(0)}$ be the kernel of $cl : \mathrm{CH}^p(X)_{\mathbb{Q}} \to H^{2p}(X,\mathbb{Q}(p))$, and $\mathrm{CH}^p(X)_{\mathbb{Q}/\,\mathrm{hom}}$ its image so that we get a short exact sequence

$$0 \to \mathrm{CH}^p(X)_{\mathbb{Q}}^{(0)} \to \mathrm{CH}^p(X)_{\mathbb{Q}} \to \mathrm{CH}^p(X)_{\mathbb{Q}/\,\mathrm{hom}} \to 0. \tag{3.4.3}$$

Then the cycle map induces a morphism of this exact sequence to the first row of the commutative diagram (3.3.5).

*Remark.* We can define a cycle map from Bloch's higher Chow group $\mathrm{CH}^p(X,n)_{\mathbb{Q}}$ [7] to $\mathrm{Ext}^{2p-n}(\mathbb{Q}_{X,\langle k\rangle},\mathbb{Q}_{X,\langle k\rangle}(p))$. See [21, (8.3)].

## 4. Calculation of Extensions

**4.1. Proposition.** *Let $S$ be a $k$-variety with structure morphism $a_{S/k} : S \to \mathrm{Spec}\,k$. Then for $M \in \mathrm{MHM}(S/k,A)$ we have a natural short exact sequence*

$$0 \to \mathrm{Ext}^1(A_k,H^{j-1}(a_{S/k})_*M) \to \mathrm{Ext}^j(A_{S/k},M) \to \mathrm{Hom}(A_k,H^j(a_{S/k})_*M) \to 0.$$

*Proof.* This follows from the adjoint relation for the direct image and the pull-back by $a_{S/k}$ together with (1.5).

**4.2.** *Proof of* (0.3). It is enough to show the surjectivity of

$$\mathrm{CH}^p(X)_{\mathbb{Q}} \to H^{p,p}(X,\mathbb{Q}(p))^{\langle k\rangle}, \quad \mathrm{CH}^p(X)^{(0)}_{\mathbb{Q}} \to J^p(X)^{\langle k\rangle}_{\mathbb{Q}}.$$

Let $R$ be a finitely generated smooth $k$-subalgebra of $\mathbb{C}$ with a smooth proper morphism $\pi : X_R \to S := \operatorname{Spec} R$ such that $X_R \otimes_R \mathbb{C} = X$. Put $A = \mathbb{Q}$. Then $H^i(X, \mathbb{Q}_{\langle k\rangle}(j))$ is represented by a variation of Hodge structure $H^i\pi_* A_{X_R/k_R}(j)$ on $S$. For $i = 0, 1$, we have by (4.1)

$$\mathrm{Ext}^i(A_{S/k_R}, H^{2p-i}\pi_* A_{X_R/k_R}(p)) = \mathrm{Hom}(A_{k_R}, H^i(a_{S/k_R})_* H^{2p-i}\pi_* A_{X_R/k_R}(p)),$$

because $H^0(a_{S/k_R})_* H^{2p-1}\pi_* A_{X_R/k_R}(p)) = 0$ by assumption. Since we have a noncanonical isomorphism (see [21, (6.10)]) :

$$\pi_* A_{X_R/k_R} = \bigoplus_i (H^i\pi_* A_{X_R/k_R})[-i] \quad \text{in } D^b\mathrm{MHM}(S/k_R, A), \tag{4.2.1}$$

we can identify $H^i(a_{S/k_R})_* H^{2p-i}\pi_* A_{X_R/k_R}(p)$ noncanonically with a subobject (and canonically with a subquotient) of $H^{2p}(X_R/k_R, A(p))$. So the assertion is reduced to the Hodge conjecture for a smooth compactification of $X_R \otimes_{k_R} \mathbb{C}$, and is further reduced to the Hodge conjecture for a smooth projective variety due to Chow's lemma.

**4.3.** *Proof of* (0.2). Let $R$ be a finitely generated smooth $k$-subalgebra of $\mathbb{C}$ with a smooth proper morphism $\pi : X_R \to S := \operatorname{Spec} R$ such that $X_R \otimes_R \mathbb{C} = X$. We may assume $X$ connected, and $k$ is algebraically closed by (2.4). For a finitely generated smooth $k$-subalgebra $R'$ of $\mathbb{C}$ containing $R$, let $S' = \operatorname{Spec} R'$, $X_{R'} = X_R \otimes_R R'$. Then $H^{2n-j}(X, A_{\langle k\rangle}(n))$ is represented by $H^{2n-j}\pi_* A_{X_{R'}/k}$ (where the morphism $X_{R'} \to S'$ is also denoted by $\pi$), and

$$\mathrm{Ext}^j(A_{\langle k\rangle}, H^{2n-j}(X, A_{\langle k\rangle}(n)) = \operatorname{ind\,lim}_{R'} \mathrm{Ext}^j(A_{S'/k}, H^{2n-j}\pi_* A_{X_{R'}/k}(n)).$$

Let $\pi' : S' \to S$ denote the canonical morphism. Then it is enough to show

$$\operatorname{ind\,lim}_{R'} \mathrm{Hom}(A_{S/k}, H^j\pi'_* H^{2n-j}\pi_* A_{X_{R'}/k}(n)) \neq 0,$$

by using the canonical surjective morphism

$$\mathrm{Ext}^j(A_{S'/k}, H^{2n-j}\pi_* A_{X_{R'}/k}(n)) \to \mathrm{Hom}(A_{S/k}, H^j\pi'_* H^{2n-j}\pi_* A_{X_{R'}/k}(n))$$

induced by $\tau_{\le 0}(\pi'_* H^{2n-j}\pi_* A_{X_{R'}/k}(n)[j]) \to H^j\pi'_* H^{2n-j}\pi_* A_{X_{R'}/k}(n)$ (because this morphism is compatible with the transition morphism of the inductive limit).

Now let $R'$ be the affine ring of an affine open subvariety $U$ of $X_R$ (i.e. $U = \operatorname{Spec} R$), and choose an embedding $R' \to \mathbb{C}$ over $R$. We may assume that $H^j\pi'_* A_{U/k}$ is a variation of Hodge structure (shrinking $S$ if necessary), where $\pi'$ is the restriction of $\pi$ to $U$. Then

$$H^j\pi'_* H^{2n-j}\pi_* A_{X_{R'}/k}(n) = H^j\pi'_*\pi'^* H^{2n-j}\pi_* A_{X_R/k}(n)$$
$$= H^j\pi'_* A_{U/k} \otimes H^{2n-j}\pi_* A_{X_R/k}(n) = \mathcal{H}om(H^{2n-j}\pi'_! A_{U/k}, H^{2n-j}\pi_* A_{X_R/k}).$$

We have a canonical element $\xi$ in

$$\mathrm{Hom}(A_{S/k}, \mathcal{H}om(H^{2n-j}\pi'_!A_{U/k}, H^{2n-j}\pi_*A_{X_R/k}) = \mathrm{Hom}(H^{2n-j}\pi'_!A_{U/k}, H^{2n-j}\pi_*A_{X_R/k}),$$

corresponding to the canonical morphism $H^{2n-j}\pi'_!A_{U/k} \to H^{2n-j}\pi_*A_{X_R/k}$. This is nonzero for any $U \subset X_R$ because $\Gamma(X, \Omega_X^j) \neq 0$. (Indeed $H^{2n-j}(X, A) \to H^{2n-1}(Z, A)$ is not injective for any proper subvariety $Z$ of $X$ by using the level of Hodge structure.)

Then we have to show the nonvanishing of the pull-back of $\xi$ to

$$\mathrm{Hom}(A_{S/k}, H^j\pi''_*\pi''^*H^{2n-j}\pi_*A_{X_R/k}(n))$$

for any sufficiently large finitely generated smooth $k$-subalgebra $R''$ of $\mathbb{C}$ containing $R'$, where $\pi'' : S'' := \mathrm{Spec}\, R'' \to S$ denotes the canonical morphism. But this can be done by taking a multivalued section of $S'' \to U$ where we may assume that the image of the multivalued section is proper and étale over $U$ by shrinking $U$.

**4.4.** *Proof of* (0.1). We may assume $X$ connected. Let $k$ be a subfield of $\mathbb{C}$ with finite transcendental degree such that $S, X$ and the morphism $S \to X$ are defined over $k$, i.e., there exist $k$-varieties $S_k, X_k$ with a morphism $S_k \to X_k$ which gives $S \to X$ by the base change. Let $Z$ be the image of $S$ in $X$, and similarly for $Z_k$. Let $K$ be the function field of $X_k$, and choose an embedding $K \to \mathbb{C}$. Let $\xi$ denote the zero cycle of $X_K := X_k \otimes_k K$ defined by the diagonal of $X_k \times_k X_k$. By assumption its image in $\mathrm{CH}_0(X)_{\mathbb{Q}}$ is represented by a cycle supported in $Z$. Then this holds also in $\mathrm{CH}_0(X_K)_{\mathbb{Q}}$. (Indeed, it is easy to show it for $\mathrm{CH}_0(X_{K'})_{\mathbb{Q}}$ where $K'$ is a finite Galois extension of $K$, and then we can use the trace morphism.) See [7].

This assertion means that the middle dimensional cycle defined by the diagonal of $X_k \times_k X_k$ is represented in $\mathrm{CH}^n(X_k \times_k X_k)_{\mathbb{Q}}$ by a cycle supported in $X_k \times_k Z_k \cup Y_k \times_k X_k$ where $Y_k$ is a proper closed subvariety of $X_k$. Taking the base change by $k \to \mathbb{C}$, the same assertion holds for $X \times X$.

So it is enough to show that the image of the morphism induced by the correspondence

$$\zeta_* : H^{2n-j}(X, \mathbb{C}) \to H^{2n-j}(X, \mathbb{C})$$

has level $\leq i$ for $j > i$, if the support of the cycle $\zeta$ is contained in $Y \times X$ or $X \times Z$, where $Y$ is a proper closed subvariety of $X$ and $Z$ is a closed subvariety with pure dimension $i$. But the morphism $\zeta_*$ factors through $H^{2n-j}(Y, \mathbb{C})$ in the first case, and the image of $\zeta_*$ is contained in the image of the Gysin morphism of $H^{2i-j}(Z, \mathbb{C})$ in the second case.

**4.5.** *Proof of* (0.5). With the notation of (4.2), we have

$$\mathrm{Ext}^{2p}(A_{S/k_R}, \pi_*A_{X_R/k_R}(p)) = \mathrm{Ext}^{2p}(A_{k_R}, (a_{X_R/k_R})_*A_{X_R/k_R}(p)),$$

and $H^{2p}_{\mathcal{D}}(X, \mathbb{Q}_{\langle k\rangle}(p))$ is equal to the limit of $\mathrm{Ext}^{2p}(A_{k_R}, (a_{X_R/k_R})_*A_{X_R/k_R}(p))$ over $R$. Here we put $A = \mathbb{Q}$. So the assertion is reduced to the injectivity of the cycle map

$$\mathrm{CH}^p(X_R)_{\mathbb{Q}} \to \mathrm{Ext}^{2p}(A_{k_R}, (a_{X_R/k_R})_*A_{X_R/k_R}(p)).$$

See (3.3.3) and (3.4.2). Then it follows from the assumption by (4.1).

*Remark.* Let $X$ be a smooth proper variety over $k$, and $Y$ a divisor with normal crossings on $X$ such that the irreducible components $Y_i$ are smooth. Let $U = X \backslash Y$. Then we have an exact sequence

$$\begin{aligned}\mathrm{Hom}(A_k(-p), \mathrm{Gr}_{2p}^W H^{2p-1}(U/k, A)) \to \mathrm{Ext}^1(A_k(-p), \mathrm{Gr}_{2p-1}^W H^{2p-1}(U/k, A)) \\ \to \mathrm{Ext}^1(A_k(-p), H^{2p-1}(U/k, A)),\end{aligned}$$

where $W$ is the weight filtration so that $\mathrm{Gr}_{2p-1}^W H^{2p-1}(U/k, A)$ and $\mathrm{Gr}_{2p}^W H^{2p-1}(U/k, A)$ are isomorphic to a quotient of $H^{2p-1}(X/k, A)$ and a subquotient of $\bigoplus_i H^{2p-2}(Y_i/k, A(-1))$ respectively. See [9]. The first morphism is a part of the commutative diagram

$$\begin{array}{ccc} (\bigoplus_i \mathrm{CH}^{p-1}(Y_i)_{\mathbb{Q}})^{(0)} & \longrightarrow & \mathrm{CH}^p(X)_{\mathbb{Q}}^{(0)} \\ \downarrow & & \downarrow \\ \mathrm{Hom}(A_k(-p), \mathrm{Gr}_{2p}^W H^{2p-1}(U/k, A)) & \longrightarrow & \mathrm{Ext}^1(A_k(-p), \mathrm{Gr}_{2p-1}^W H^{2p-1}(U/k, A)) \end{array}$$

where $(\bigoplus_i \mathrm{CH}^{p-1}(Y_i)_{\mathbb{Q}})^{(0)} = \mathrm{Ker}((\bigoplus_i \mathrm{CH}^{p-1}(Y_i)_{\mathbb{Q}}) \to H^{2p}(X_{\mathbb{C}}, \mathbb{C}))$, and the vertical morphisms are cycle maps. If the Hodge conjecture for codimension $p-1$ cycles on $Y_i$ is true, the left vertical morphism would be surjective. This implies the remark after (0.4).

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