COMBINATORICS OF THE $\widehat{sl}_2$ SPACES OF COINVARIANTS

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ABSTRACT. We consider two types of quotients of the integrable modules of $\widehat{sl}_2$. These spaces of coinvariants have dimensions described in terms of the Verlinde algebra of level-k. We describe monomial bases for the spaces of coinvariants, which leads to a fermionic description of these spaces. For $k = 1$, we give the explicit formulas for the characters. We also present recursion relations satisfied by the characters and the monomial bases.

1. INTRODUCTION

Let $\widehat{sl}_2$ be the central extension of the current algebra

$$\widehat{sl}_2 \quad = \quad sl_2 \otimes \mathbb{C}[t, t^{-1}]] \oplus \mathbb{C}c \oplus \mathbb{C}d. \tag{1}$$

Here, by $\mathbb{C}[t, t^{-1}]]$ we mean the completion of the ring of Laurent polynomials with respect to t^{-1}. Let $\hat{\mathfrak{n}}_- \subset \widehat{sl}_2$ be the subalgebra generated by the annihilation operators, $sl_2 \otimes t^{-1}\mathbb{C}[[t^{-1}]] \oplus e \otimes 1$. Consider an integrable representation L of $\widehat{sl}_2$, and a subalgebra $\mathfrak{b} \subset \widehat{sl}_2$ such that

$$\dim(\hat{\mathfrak{n}}_- \cap \mathfrak{b}) < \infty \text{ and } \dim(\widehat{sl}_2/(\hat{\mathfrak{n}}_- + \mathfrak{b})) < \infty. \tag{2}$$

Introduce the space of coinvariants, $I(L, \mathfrak{b}) = L/(\mathfrak{b}L)$. It is known that this space is finite-dimensional.

Spaces of coinvariants appear naturally in representation theory and in number theory, as spaces of automorphic forms. In conformal field theory, spaces of coinvariants are studied in connection with modular functors, spaces of conformal blocks, etc. (see [F],[S],[TUY]).

The first natural question is how to find the dimension of $I(L, \mathfrak{b})$. For general $\mathfrak{b}$, the dimension is not yet known, but in some cases, Verlinde's rule gives us an answer.

Verlinde's rule [V] was formulated for the subalgebra $\mathfrak{b}$ associated with a compact algebraic curve and a point on it. Let $\mathcal{E}$ be an algebraic curve, $P \in \mathcal{E}$ a point and ξ a rank-2 SL_2-bundle on $\mathcal{E}$. Fix a local coordinate t^{-1} and a trivialization of the bundle in a small vicinity of P. These data determine a subalgebra $\mathfrak{b}(\mathcal{E}, \xi) \subset sl_2 \otimes \mathbb{C}[t, t^{-1}]]$. Namely, using the local coordinate and the trivialization of the bundle, we can identify the algebra $\widehat{sl}_2$ with the algebra of infinitesimal automorphisms (i.e., the sl_2 action induced from the SL_2 action) of the bundle ξ restricted to the punctured vicinity of P. Let $\mathfrak{b}(\mathcal{E}, \xi)$ be the algebra of infinitesimal automorphhisms of the bundle ξ restricted to the domain $\mathcal{E} \setminus P$. Clearly, we have the natural inclusion $\mathfrak{b}(\mathcal{E}, \xi) \hookrightarrow \widehat{sl}_2$. Classically, Verlinde's rule gives the dimension of the space of coinvariants with respect to this subalgebra by reducing the calculation to inductive steps governed by the Verlinde algebra.

Let us consider a somewhat more general situation. Let $(z_1, \dots, z_s; w_1, \dots w_r)$ be a set of distinct points, not equal to P. At each point w_i, fix a line l_i in the 2-dimensional fiber $\xi|_{w_i}$. Now define a subalgebra $\mathfrak{b}(\mathcal{E}, \xi; z_1, \dots, z_s; w_1, \dots w_r) \subset \mathfrak{b}(\mathcal{E}, \xi)$ of infinitesimal automorphism g such that $g(z_i) = 0$ and $g(w_i)|_{l_i} = 0$ for all i. In the next section we formulate Verlinde's rule in the case when $\mathcal{E} \cong \mathbb{C}P^1$, ξ is the trivial bundle and either s or r is equal to zero.

We described a family of subalgebras in $\widehat{sl}_2$ depending on some geometric data (curve, bundle, points, lines). It is known that the spaces $I(L, \mathfrak{b}(\mathcal{E}, \xi; z_1, \dots, z_s; w_1, \dots w_r))$ form a vector bundle over the moduli space of these data. Actually this bundle is equipped with a projective connection and many other interesting structures [TUY].

Consider the compactification of the moduli space. Above, we assumed that all the points z_i and w_i are distinct. The simplest "strata at infinity" arise if some points collide. With any point of these strata, we can associate a subalgebra in $\widehat{sl}_2$, so we can consider the space of coinvariants, i.e., the quotient space with respect to this subalgebra. In general, we know only that the dimension of this space is still finite and not less than the dimension at a generic point of the moduli space.

In our special case ($\mathcal{E} = \mathbb{C}P^1$, and ξ is trivial), we conjecture that the spaces of coinvariants form a vector bundle over the configuration space of points on $\mathbb{C}P^1$. This means that the algebra $\mathfrak{b}(\mathcal{E}, \xi; z_1, \dots, z_s; w_1, \dots w_r)$ is well-defined for all values of z_i, w_i, and the dimension of the space of coinvariants does not depend on these values. In this paper we prove this fact for $r = 0$ (Section 3) and $s = 0$, $k = 1$ (Sections 4, 5).

Let d be the grading operator, $d(g \otimes t^n) = n \cdot g \otimes t^n$. We can decompose the irreducible representation L into the sum of d-graded components $L(0)$, $L(1), \dots$. Denote by $L_{\leq j}$ the sum $L(0) \oplus \dots \oplus L(j)$. We have a filtration $\{L_{\leq j}\}$ on L. For any subalgebra $\mathfrak{b}$, consider the induced filtration on the space of coinvariants $I(L, \mathfrak{b})$. Namely, let $I_{\leq j}$ be the image of $L_{\leq j}$ in $I(L, \mathfrak{b})$. Let $c_j = \dim(I_{\leq j}(L, \mathfrak{b})/I_{\leq j-1}(L, \mathfrak{b}))$. We call the polynomial $\sum c_j q^j$ the q-dimension of the space $I(L, \mathfrak{b})$. In fact, using the bi–grading of L we can obtain polynomial in two variables. Such polynomials are certain "finitizations" of the character of the integrable representation L.

In this paper we partially answer the following questions for the special cases described above (see [FF], [FL] for more about the same subject).

1) How to find explicitly the set of graded (or bi–graded) vectors in L such that their images form a basis in $I(L, \mathfrak{b})$?

2) How to find explicit formulas for q-dimensions? There are two types of such formulas: "bosonic" and "fermionic" ([KKMM]). In this text we present only fermionic formulas for $k = 1$.

3) How to find a q-version of Verlinde's rule which gives a recursive description of the spaces of coinvariants?

Unfortunately, in general, it is not known how to investigate the filtration on the space $I(L, \mathfrak{b})$ explicitly. Let us describe our strategy in the special situation considered in this paper.

Our subalgebra $\mathfrak{b}$ is a member of the family of Lie subalgebras depending on geometric data $(z_1, \dots, z_s; w_1, \dots, w_r)$, which constitute a configuration space $\mathcal{C}$. We consider the limit to the special point $(0, \dots, 0; 0, \dots, 0) \in \mathcal{C}$. At this point the subalgebra $\mathfrak{b}^0$ and therefore the quotient space $I(L, \mathfrak{b}^0)$ is bi–graded. We find that the dimension does not change: $\dim I(L, \mathfrak{b}) = \dim I(L, \mathfrak{b}^0)$. This implies that $\operatorname{gr} I(L, \mathfrak{b})$ is canonically isomorphic to $I(L, \mathfrak{b}^0)$ as bi–graded spaces. In other words, we can, in principle, construct a basis of $\operatorname{gr} I(L, \mathfrak{b})$ which remains valid as a basis in the limit.

In the case $r = 0$, it is enough to work with the subalgebra in this limit to obtain the answers to the above questions. In the case $s = 0$ we need to deform not only the subalgebra $\mathfrak{b}$ in $\widehat{sl}_2$ but also $\widehat{sl}_2$ itself in $\widehat{sl}_3$, as explained below.

Consider the Lie algebra sl_3 and its irreducible representation $\mathbb{C}^3$. Let us decompose the space $\mathbb{C}^3$ into the sum of two subspaces V_1 and V_2, where $\dim V_1 = 1$. Then the subalgebra $sl_2 \subset sl_3$ consists of operators which annihilate V_1 and map V_2 to itself.

Consider the deformation of subspaces $V_1(\varepsilon)$, $V_2(\varepsilon)$, $\varepsilon \in \mathbb{C}$ such that $V_1(1) = V_1$, $V_2(1) = V_2$, for any $\varepsilon \neq 0$ we have $V_1(\varepsilon) + V_2(\varepsilon) = \mathbb{C}^3$ and for $\varepsilon = 0$ we have $V_1(0) \subset V_2(0)$. Let

$\mathfrak{a}(\varepsilon) \subset sl_3 (\varepsilon \neq 0)$ be the subalgebra of operators which annihilates $V_1(\varepsilon)$ and maps $V_2(\varepsilon)$ to $V_2(\varepsilon)$. Clearly, $\mathfrak{a}(\varepsilon)$ is isomorphic to sl_2 if $\varepsilon \neq 0$, and

$$\mathfrak{a}(0) \stackrel{\text{def}}{=} \lim_{\varepsilon \to 0} \mathfrak{a}(\varepsilon)$$

is the maximal nilpotent subalgebra in sl_3, so it is isomorphic to the 3-dimensional Heisenberg algebra.

Let L be an integrable representation of $\widehat{sl}_2 \simeq \widehat{\mathfrak{a}}(\varepsilon)$. We can deform it inside an integrable representation of $\widehat{sl}_3$, and consider the limit to a representation of $\widehat{\mathfrak{a}}(0)$. It is known ([FS],[St]) that the q-dimension of the limit $L(0)$ coincides with that of L.

We can also deform a graded subalgebra $\mathfrak{b}^0 \subset \widehat{sl}_2$ and obtain a deformed space of coinvariants $I(L(0), \mathfrak{b}^0(0))$ in the limit $\varepsilon \to 0$. We will prove for the case $k = 1$ that the dimension of the space of coinvariants does not change in this limit.

Since $\widehat{\mathfrak{a}}(0)$ is an infinite dimensional Heisenberg algebra, we can explicitly describe the dual spaces to its representation spaces in terms of symmetric functions. This gives an explicit model for the dual spaces for the coinvariants, and we obtain fermionic character formulas for the $k = 1$ coinvariants.

The plan of the paper is as follows. In Section 2 we formulate the problem. We use a simpler definition of $\widehat{sl}_2$, i.e., using the ring of Laurant polynomials. In Section 3 we give a monomial basis of the space of coinvariants in the case $r = 0$ with arbitrary level k. In Sections 4 and 5 we study the case $s = 0$, and give monomial bases for the space of coinvariants with $k = 1$.

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2. Spaces of coinvariants

Let $\{e, f, h\}$ be the standard basis of sl_2,

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The algebra $\widehat{sl}_2 = sl_2 \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}d$ is generated by $\{x_i = x \otimes t^i, c, d \mid x \in sl_2, i \in \mathbb{Z}\}$, with c the central element and d the degree operator, $[d, x_i] = ix_i$. We will also use the generating current notation, $e(z) = \sum_{i \in \mathbb{Z}} e_i z^i$, etc..

Consider the Verma module $M_{k,l} = U(\widehat{sl}_2)v_{k,l}$, with $k \in \mathbb{Z}_{>0}$ and $0 \leqslant l \leqslant k$, generated by the highest weight vector $v_{k,l}$ satisfying

$$\begin{aligned} &cv_{k,l} = kv_{k,l}, \quad h_0 v_{k,l} = lv_{k,l}, \quad dv_{k,l} = 0, \\ &e_i v_{k,l} = 0 \ (i \leqslant 0), \quad h_i v_{k,l} = f_i v_{k,l} = 0 \ (i < 0). \end{aligned} \tag{3}$$

The irreducible highest weight module $L_{k,l}$ is the quotient of $M_{k,l}$ by the relations

$$e(z)^{k+1} = 0 \quad (\text{or equivalently } f(z)^{k+1} = 0).$$

Note that, in particular,

$$e_1^{k-l+1} v_{k,l} = f_0^{l+1} v_{k,l} = 0.$$

Given $\mathcal{Z} = (z_1, \ldots, z_n) \in \mathbb{C}^N$, z_i not necessarily distinct, we define two types of subalgebras of $sl_2 \otimes \mathbb{C}[t]$, and their corresponding spaces of coinvariants. The first subalgebra

is

$$sl_2(\mathcal{Z};0) \quad = \quad sl_2 \otimes (t-z_1)\cdots(t-z_N)\mathbb{C}[t],$$

which reduces to 0 at each point $t = z_i$. (Here the symbol 0 means the trivial subalgebra of $sl_2 \otimes \mathbb{C}[t]$.) The corresponding space of coinvariants is defined by

$$L^{(N)}_{k,l}(\mathcal{Z};0) \quad = \quad L_{k,l}/sl_2(\mathcal{Z};0)L_{k,l}. \tag{4}$$

The second subalgebra is

$$sl_2(\mathcal{Z};\mathfrak{n}) \quad = \quad \mathbb{C}e \otimes \mathbb{C}[t] \oplus (\mathbb{C}h \oplus \mathbb{C}f) \otimes (t-z_1)\cdots(t-z_N)\mathbb{C}[t].$$

This subalgebra reduces to $\mathfrak{n} = \mathbb{C}e \subset sl_2$ at $t = z_i$. The corresponding space of coinvariants is defined by

$$L^{(N)}_{k,l}(\mathcal{Z};\mathfrak{n}) \quad = \quad L_{k,l}/sl_2(\mathcal{Z};\mathfrak{n})L_{k,l}. \tag{5}$$

In each case, the spaces of coinvariants are finite-dimensional, and their dimensions are given in terms of the Verlinde algebra. Let us recall its definition.

Let π_l be the irreducible $(l+1)$-dimensional representation of sl_2. The level-k Verlinde algebra associated to sl_2, which we denote by $\mathcal{V}_k$, is an algebra over $\mathbb{C}$, with generators $\{\pi_l;\ 0 \leqslant l \leqslant k\}$ and multiplication

$$\pi_l \cdot \pi_{l'} = \sum_{\substack{i=|l-l'| \\ i+l-l' : \text{even}}}^{\min(2k-l-l',l+l')} \pi_i. \tag{6}$$

Define the positive integers $d^{(N)}_{k,l}(0)$ and $d^{(N)}_{k,l}(\mathfrak{n})$ by

$$\begin{aligned}(\pi_0 + 2\pi_1 + \cdots + (k+1)\pi_k)^N \quad &= \quad \sum_l d^{(N)}_{k,l}(0)\pi_l, \\ (\pi_0 + \pi_1 + \cdots + \pi_k)^N \quad &= \quad \sum_l d^{(N)}_{k,l}(\mathfrak{n})\pi_l.\end{aligned}$$

We have

Theorem 2.1.1. *If the points $z_1, \ldots, z_N$ are distinct, then*

$$\begin{aligned}\dim L^{(N)}_{k,l}(\mathcal{Z};0) \quad &= \quad d^{(N)}_{k,l}(0), \\ \dim L^{(N)}_{k,l}(\mathcal{Z};\mathfrak{n}) \quad &= \quad d^{(N)}_{k,l}(\mathfrak{n}).\end{aligned}$$

Example: When $k = 1$,

$$d_{1,l}(0) = \frac{3^N + (-1)^{N+l}}{2}, \qquad d_{1,l}(\mathfrak{n}) = 2^{N-1}, \quad l = 0, 1.$$

We also define a special case of the coinvariants (4), (5), at $\mathcal{Z} = (0, \ldots, 0)$:

$$\begin{aligned}L^{(N)}_{k,l}(0) &\stackrel{\text{def}}{=} L^{(N)}_{k,l}/sl_2(0)L^{(N)}_{k,l}, \\ L^{(N)}_{k,l}(\mathfrak{n}) &\stackrel{\text{def}}{=} L^{(N)}_{k,l}/sl_2(\mathfrak{n})L^{(N)}_{k,l}.\end{aligned}$$

where

$$\begin{aligned}sl_2(0) &= \lim_{z_1,\ldots,z_N \to 0} sl_2(\mathcal{Z};0), \\ sl_2(\mathfrak{n}) &= \lim_{z_1,\ldots,z_N \to 0} sl_2(\mathcal{Z};\mathfrak{n}).\end{aligned}$$

These spaces inherit a bi-grading with respect to (h_0, d) from $L_{k,l}$. We define the character of any of these bi-graded space V by

$$\mathrm{ch}_{q,z} V \quad = \quad \mathrm{trace}_V \left(q^d z^{h_0} \right).$$

3. The space of coinvariants $L_{k,l}^{(N)}(\mathcal{Z};0)$.

In this section we consider the space of coinvariants $L_{k,l}^{(N)}(\mathcal{Z};0)$ (see (4)) where $z_1, \ldots, z_N$ are not necessarily distinct points in $\mathbb{C}$. We introduce a monomial basis for this space, and derive a recursion relation for the characters and dimensions of these coinvariants. We prove that at the special point $\mathcal{Z} = 0$, the dimension of the coinvariants is equal to the Verlinde number as in the case of generic $\mathcal{Z}$.

3.1. **Monomial basis.** Here we propose a monomial basis of $L_{k,l}^{(N)}(\mathcal{Z};0)$, and of the whole representation $L_{k,l}$.

Both the space of coinvariants $L_{k,l}^{(N)}(\mathcal{Z};0)$ and the $\widehat{sl}_2$-module $L_{k,l}$ are quotients of the Verma module $M_{k,l} = U(\hat{\mathfrak{n}}_+)v_{k,l}$, where $\hat{\mathfrak{n}}_+ = sl_2 \otimes t\mathbb{C}[t] \oplus f_0$ is the nilpotent part of the Borel subalgebra of $\widehat{sl}_2$. Define an ordering of the basis of $\hat{\mathfrak{n}}_+$ in the following way:

$$f_0 < e_1 < h_1 < f_1 < e_2 < h_2 < f_2 < \cdots .$$

Definition: A monomial m in $U(\hat{\mathfrak{n}}_+)$ is called *normally ordered* (or just simply *ordered*) if it is of the form

$$f_{n-1}^{a_{n-1}} h_{n-1}^{b_{n-1}} e_{n-1}^{c_{n-1}} f_{n-2}^{a_{n-2}} \cdots e_2^{c_2} f_1^{a_1} h_1^{b_1} e_1^{c_1} f_0^{a_0}. \tag{7}$$

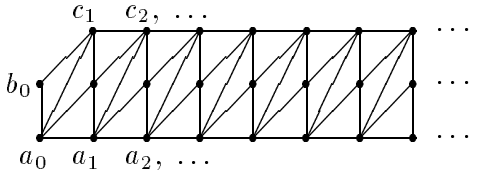
By the PBW Theorem, normally ordered monomials form a basis of $U(\hat{\mathfrak{n}}_+)$.

In the following, we will not make a notational distinction between the highest weight vector $v_{k,l} \in M_{k,l}$ and its image in its quotients, e.g., $L_{k,l}$. By abuse of language, we will also call a vector of the form $m \cdot v_{k,l}$ (m a monomial in $U(\hat{\mathfrak{n}}_+)$) a monomial.

Definition: Let k and l be fixed integers. We call an ordered monomial of the form (7) an *ehf–monomial* if it satisfies the following conditions:

(i) $a_i + a_{i+1} + b_{i+1} \leqslant k$ for all $i \geqslant 0$,
(ii) $a_i + b_{i+1} + c_{i+1} \leqslant k$ for all $i \geqslant 0$,
(iii) $a_i + b_i + c_{i+1} \leqslant k$ for all $i > 0$,
(iv) $b_i + c_i + c_{i+1} \leqslant k$ for all $i > 0$,
(v) $a_0 \leqslant l$, $c_1 \leqslant k - l$.

Remark: Conditions (i)–(v) can be expressed by the following picture:



We formally introduce the variable h_0 and consider monomials

$$f_{n-1}^{a_{n-1}} \cdots h_1^{b_1} e_1^{c_1} f_0^{a_0} h_0^{b_0},$$

with $b_0 = l - a_0$ such that the sum of exponents over any triangle (corresponding to the conditions (i)–(iv)) is less than or equal to k.

Definition: An ehf–monomial has *length* N if $a_i = b_i = c_i = 0$ for $i \geqslant N$.

Our purpose is to prove that the set of ehf–monomials forms a basis in $L_{k,l}$, and that the set of ehf–monomials of length N forms a basis in $L^{(N)}_{k,l}(\mathcal{Z};0)$.

Define a linear isomorphism from the polynomial ring $R = \mathbb{C}[f_0, e_1, h_1, f_1, \dots]$ to $U(\hat{\mathfrak{n}}_+)$, $m \mapsto : m :$, such that the inverse image of an ordered monomial $m \in U(\hat{\mathfrak{n}}_+)$ is $m \in R$. Namely, for a given element in R, we specify by (7) how to order the non-commuting product in $U(\hat{\mathfrak{n}}_+)$. The existence of such isomorphism is a consequence of the PBW theorem.

We often abuse the notation m to mean both an ordered monomial of $U(\hat{\mathfrak{n}}_+)$ and the corresponding element in R.

Next, we define a complete lexicographic ordering on the set of normally ordered monomials. (Note the difference between "lexicographic ordering" and "normally ordered monomials.") We define the homogeneous degree $\operatorname{Deg} m$ of an ordered monomial m:

$$\operatorname{Deg} m \quad = \quad \sum_i (a_i + b_i + c_i).$$

If $\operatorname{Deg} m > \operatorname{Deg} m'$, we say that $m > m'$. If $\operatorname{Deg} m = \operatorname{Deg} m'$, we compare the exponents a_0, c_1, b_1, a_1, etc., in this order. Namely, we say $m > m'$ if the value of a_0 in m is *smaller* than that in m'; if these values are equal, then we compare c_1 next, and so on.

We define the dominant monomial of an element $u \in U(\hat{\mathfrak{n}}_+)$ to be the largest monomial in the expression of u written as a linear combination of ordered monomials. We note that if m_i $(i = 1,2)$ is the dominant monomial of u_i, then the dominant monomial of the product $u_1 u_2$ is $: m_1 m_2 :$.

We will now consider the list of elements in $U(\hat{\mathfrak{n}}_+)$ which are trivial in $L_{k,l} = U(\hat{\mathfrak{n}}_+)v_{k,l}$. We will show that the dominant monomial of each element in this list breaks one or more of the conditions (i)–(v). The remaining monomials, which are not dominant monomials of these trivial elements, span the space $L_{k,l}$.

Lemma 3.1.1. *Let*

$$\psi_{r,s} = (\operatorname{ad} e_0)^s \sum_{\substack{i_1,\dots,i_{k+1} \geqslant 0 \\ i_1+\dots+i_{k+1}=r}} f_{i_1} f_{i_2} \dots f_{i_{k+1}} \in U(\hat{\mathfrak{n}}_+).$$

Then $\psi_{r,s} v_{k,l} = 0$ *in* $L_{k,l}$.

Proof. Since $f(z)^{k+1} = 0$ in $L_{k,l}$, all d–graded elements of the form

$$\psi_r = \sum_{i_1+\dots+i_{k+1}=r} f_{i_1} f_{i_2} \dots f_{i_{k+1}}$$

act trivially in $L_{k,l}$. When acting on the highest weight vector $v_{k,l}$,

$$\psi_r v_{k,l} = \sum_{\substack{i_1,\dots,i_{k+1} \geqslant 0 \\ i_1+\dots+i_{k+1}=r}} f_{i_1} f_{i_2} \dots f_{i_{k+1}} v_{k,l} = 0 \in L_{k,l}.$$

In addition, since $e_0 v_{k,l} = 0$,

$$\psi_{r,s} v_{k,l} = (\operatorname{ad} e_0)^s (\psi_r) v_{k,l} = 0.$$

□

Theorem 3.1.2. *The set of* ehf*–monomials spans* $L_{k,l}$.

Proof.

Consider the set of elements in $U(\hat{\mathfrak{n}}_+)$,

$$m \cdot \psi_{r,s} \quad (2k+2 \geqslant s \geqslant 0,\ r \geqslant s/2) \tag{8}$$

and

$$m \cdot f_0^{l+1}, m \cdot e_1^{k-l+1}, \tag{9}$$

where m is an ordered monomial.

Due to Lemma 3.1.1 and the condition (3), when applied to the highest weight vector $v_{k,l}$, these are trivial in $L_{k,l}$. Let us show that an ordered monomial is the dominant monomial of an element in this set if and only if it violates one or more of the conditions (i)–(v).

The ordered monomials $: m \cdot f_0^{l+1} :$ and $: m \cdot e_1^{k-l+1} :$ are the dominant monomials of $m \cdot f_0^{l+1}$ and $m \cdot e_1^{k-l+1}$, respectively, and an ordered monomial violates the condition (v) if and only if it can be represented in one of these forms.

Consider $\psi_{r,s}$ with $r = \alpha(k+1)+\beta$ and $0 \leqslant \beta \leqslant k$. Their dominant monomials violate one of the conditions (i)–(iv), as shown in the following table:

Case	Dominant monomial of $\psi_{r,s}$	Violates condition
$s \leqslant \beta$	$f_{\alpha+1}^{\beta-s} h_{\alpha+1}^{s} f_{\alpha}^{k+1-\beta}$	(i)
$\beta \leqslant s \leqslant 2\beta$	$h_{\alpha+1}^{2\beta-s} e_{\alpha+1}^{s-\beta} f_{\alpha}^{k+1-\beta}$	(ii)
$2\beta \leqslant s \leqslant k+1+\beta$	$e_{\alpha+1}^{\beta} f_{\alpha}^{k+1+\beta-s} h_{\alpha}^{s-2\beta}$	(iii)
$k+1+\beta \leqslant s$	$e_{\alpha+1}^{\beta} h_{\alpha}^{2k+2-s} e_{\alpha}^{s-(k+1)-\beta}$	(iv)

Given any ordered monomial m, the dominant term in $m \cdot \psi_{r,s}$ also violates the conditions as listed above, hence it is not an ehf-monomial.

On the other hand, suppose that an ordered monomial m violates one of the conditions (i)–(iv). It can decomposed in R as $m = m_1 \cdot m_2$, in such a way that $\mathrm{Deg}\ : m_2 := k+1$, and $: m_2 :$ violates the same condition as m. Then, using this table, there exist r and s such that m_2 is the dominant monomial of $\psi_{r,s}$. Since m is the dominant monomial of $: m_1 : \psi_{r,s}$, any monomial which violates any of the conditions (i)–(iv) is of the form $m \cdot \psi_{r,s}$.

We have shown that the set of ehf-monomials, which satisfy conditions (i)–(v) span the space $L_{k,l}$. □

Theorem 3.1.3. *The set of ehf–monomials of length N spans $L_{k,l}^{(N)}(\mathcal{Z};0)$.*

Proof. The proof is similar to Theorem 3.1.2. Recall that $L_{k,l}^{(N)}(\mathcal{Z};0)$ is a quotient of $L_{k,l}$, and therefore of $M_{k,l} = U(\hat{\mathfrak{n}}_+)v_{k,l}$. For any $g \in sl_2$, $i \geqslant 0$, define $g^{(i)} = g \otimes t^i(t-z_1)\dots(t-z_N) \in U(\hat{\mathfrak{n}}_+)$. By definition it is trivial in $L_{k,l}^{(N)}(\mathcal{Z};0)$. If $g = e$, h or f, then the dominant monomial of $g^{(i)}$ is g_{i+N}.

Consider the set of elements

$$\{e^{(i)} \cdot m, h^{(i)} \cdot m, f^{(i)} \cdot m\} \in U(\hat{\mathfrak{n}}_+),$$

together with the elements (8), (9). Clearly, their image in $L_{k,l}^{(N)}(\mathcal{Z};0)$ is zero. The dominant monomials of $g^{(i)} \cdot m$ $(g = e,h,f; i \geqslant 0)$ are precisely the monomials of length larger than N. Therefore, our assertion follows. □

3.2. **Dimension of the coinvariants.** We have shown that the set of ehf–monomials of length N spans the space of coinvariants $L^{(N)}_{k,l}(\mathcal{Z};0)$. In order to prove that they are indeed linearly independent, we compare the dimension of $L^{(N)}_{k,l}(\mathcal{Z};0)$ with the number of all ehf–monomials of length N.

If $\mathcal{Z}=(z_1,\dots,z_N)$ are all distinct complex numbers, the dimension of $L^{(N)}_{k,l}(\mathcal{Z};0)$ is given by the Verlinde numbers $d^{(N)}_{k,l}(0)$ (see Theorem 2.1.1). These satisfy a recursion relation.

Lemma 3.2.1. *The Verlinde numbers $d^{(N)}_{k,l}(0)$ satisfy the recursion relation*

$$d^{(N+1)}_{k,l}(0) = \sum_{l'} \mathcal{D}^{l'}_{l}\, d^{(N)}_{k,l'}(0), \quad N\geqslant 1; \tag{10}$$

$$d^{(1)}_{k,l}(0) = l+1, \tag{11}$$

where

$$\mathcal{D}^{l'}_{l} = (\min(l,k-l')+1)\cdot(\min(l',k-l)+1). \tag{12}$$

Proof. The equation (10) follows from (6) and the fact that $\mathcal{D}^{i}_{j}$ is the sum of the dimensions of the representations appearing in the right hand side of this formula. □

In other words, we have the matrix $\mathcal{D}=\left(\mathcal{D}^{l'}_{l}\right)$ and vectors $\mathbf{d}^{(N)}_{k}=\left(d^{(N)}_{k,l}(0)\right)$ such that $\mathbf{d}^{(N+1)}_{k}=\mathcal{D}\cdot\mathbf{d}^{(N)}_{k}$, or, equivalently, $\mathbf{d}^{(N)}_{k}=\mathcal{D}^{N-1}\cdot\mathbf{d}^{(1)}_{k}$.

Let $p(k,l,N)$ be the number of ehf–monomials of length N. In order to prove that these monomials are linearly independent, we must show that $d^{(N)}_{k,l}(0)=p(k,l,N)$. The proof is based on the following technical lemma for reduction of the size of matrices.

Lemma 3.2.2. *Let $A=(A^{\alpha}_{\alpha'})$ be an $n\times n$ matrix, and v, φ vectors of length n. Suppose that for some indices $\alpha_0,\alpha_1,\dots,\alpha_s$, $\varphi_{\alpha_i}=\varphi_{\alpha_j}$ and $A^{\alpha}_{\alpha_i}=A^{\alpha}_{\alpha_j}$ for any i,j,α. Then for any m we have*

$$\sum_{i,j=1,\dots,n} \varphi_i\,(A^m)^i_j\,v^j = \sum_{i,j=1,\dots,n-s} \tilde{\varphi}_i\left(\tilde{A}^m\right)^i_j\,\tilde{v}^j,$$

where $\tilde{\varphi}$ is obtained from φ by deleting φ_{α_i} for $i\neq 0$, $\tilde{v}$ from v by deleting v^{α_i} for $i\neq 0$ and replacing v^{α_0} with $\sum_{i\geqslant 0} v^{\alpha_i}$, and $\tilde{A}$ from A by deleting the rows and columns with indices α_i for $i\neq 0$ and replacing $A^{\alpha_0}_{\alpha'}$ with $\sum_{i\geqslant 0} A^{\alpha_i}_{\alpha'}$.

Sometimes we will use this lemma for the case $\varphi_i=\tilde{\varphi}_i=1$ for all i. In this case we omit writing φ_i or $\tilde{\varphi}_i$.

Proposition 3.2.3. $d^{(N)}_{k,l}(0)=p(k,l,N)$.

Proof. Let $p^{a,b,c}(k,l,N)$ be the number of ehf–monomials of length N with $a_{N-1}=a$, $b_{N-1}=b$, $c_{N-1}=c$. Conditions (i)–(v) imply that

$$p^{a,b,c}(k,l,N+1) = \sum_{\substack{a',b',c' \\ a'+a+b\leqslant k,\ a'+b+c\leqslant k \\ a'+b'+c\leqslant k,\ b'+c'+c\leqslant k}} p^{a',b',c'}(k,l,N). \tag{13}$$

Consider the matrix $\mathcal{P}=(\mathcal{P}^{a,b,c}_{a',b',c'})$ given by

$$\mathcal{P}^{a,b,c}_{a',b',c'} = \begin{cases} 1, & a'+a+b\leqslant k, a'+b+c\leqslant k, a'+b'+c\leqslant k, b'+c'+c\leqslant k; \\ 0 & \text{otherwise.}\end{cases}$$

Note that this matrix does not depend on l. Clearly, $p(k,l,N) = \sum_{a,b,c} p^{a,b,c}(k,l,N)$.

¿From (13) we have

$$p(k,l,N) = \sum_{\substack{a,b,c\\ a',b',c'}} \left(\mathcal{P}^{N-1}\right)^{a,b,c}_{a',b',c'} p^{a',b',c'}(k,l,1), \qquad N > 1, \tag{14}$$

with

$$p^{a,b,c}(k,l,1) = \begin{cases} 1, & c = 0,\ a+b = l; \\ 0 & \text{otherwise.} \end{cases}$$

Let us simplify this formula using Lemma 3.2.2. Note that $\mathcal{P}^{a,b,c}_{a',b',c'}$ depends only on a, b, c, a' and $m' = \max(a'+b', b'+c')$. So we can apply Lemma 3.2.2 to all the multi–indices a,b,c with the same a and m in (14). We obtain

$$p(k,l,N) = \sum_{\substack{0\leqslant a\leqslant m\leqslant k\\ 0\leqslant a'\leqslant m'\leqslant k}} \left(\widetilde{\mathcal{P}}^{N-1}\right)^{a,m}_{a',m'} \widetilde{p}^{a',m'}(k,l,1), \tag{15}$$

$$\widetilde{\mathcal{P}}^{a,m}_{a',m'} = \begin{cases} 0, & a'+m > k; \\ \min(k-m', m)+1 & \text{otherwise,} \end{cases}$$

$$\widetilde{p}^{a',m'} = \begin{cases} 1, & m' = l; \\ 0 & \text{otherwise.} \end{cases}$$

For example, if $k = 1$, we have

$$\mathcal{P} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}; \quad \tilde{\mathcal{P}} = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \\ 2 & 1 & 0 \end{pmatrix}; \quad D = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}.$$

See also (17) (with $q = z = 1$) and the matrix $\tilde{\mathcal{P}}$ in the proof.

Note that $\widetilde{\mathcal{P}}^{a,m}_{a',m'}$ and $\widetilde{p}^{a,m}$ do not depend on a, so we can transpose (15) and then apply Lemma 3.2.2 again. The reduced matrix is in fact equal to the matrix $\mathcal{D}$ of (12), and the reduced column is nothing but (11):

$$p(k,l,N) = \sum_{0\leqslant m'\leqslant k} \left(\mathcal{D}^{N-1}\right)^{l}_{m'} \cdot (m'+1) = d^{(N)}_{k,l}(0).$$

□

Theorem 3.2.4.

(i) *The set of ehf–monomials forms a basis of $L_{k,l}$.*

(ii) *The set of efh–monomials of length N forms a basis of $L^{(N)}_{k,l}(\mathcal{Z};0)$ for an arbitrary set of points $\mathcal{Z}$.*

Proof. Let us begin with (ii). If $\mathcal{Z}$ consists of distinct complex numbers then (ii) follows from Theorem 3.1.3 and Proposition 3.2.3. Arguments from deformation theory show that for any (not necessarily distinct) $\mathcal{Z}$ we have $\dim L^{(N)}_{k,l}(0) \geqslant \dim L^{(N)}_{k,l}(\mathcal{Z};0) \geqslant d^{(N)}_{k,l}(0)$. On the other hand, Theorem 3.1.3 for $\mathcal{Z} = (0,\dots,0)$ shows that $\dim L^{(N)}_{k,l}(0) \leqslant p(k,l,N)$. Therefore, by using Proposition 3.2.3, we have the equality of dimensions for any $\mathcal{Z}$. The statement (ii) follows.

The statement (i) follows from Theorem 3.1.2 and (ii); we need only to prove that ehf–monomials are linearly independent. Suppose that some linear combination of $m_1, \dots, m_s$ is trivial in $L_{k,l}$. Let N be the maximal length of monomials m_i. Then, this linear combination is trivial in $L^{(N)}_{k,l}(\mathcal{Z};0)$, which contradicts (ii). $\square$

3.3. **Recursion relations for characters of the coinvariants.** As an application of Theorem 3.2.4 we can write a recursion relation for $\mathrm{ch}_{q,z} L^{(N)}_{k,l}(0)$:

Corollary 3.3.1.

$$\mathrm{ch}_{q,z} L^{(N)}_{k,l}(0) = \sum_{0 \leqslant a \leqslant l} z^{l-2a} \left(\mathcal{P}^{(N)}\right)^{0,0}_{a,l}(q,z),$$

where

$$\mathcal{P}^{(N)}(q,z) = \tilde{\mathcal{P}}(q^N,z) \cdot \tilde{\mathcal{P}}(q^{N-1},z) \cdots \tilde{\mathcal{P}}(q^2,z) \cdot \tilde{\mathcal{P}}(q,z),$$

and the matrix $\tilde{\mathcal{P}}$ is given as follows:

$$\tilde{\mathcal{P}}^{a,m}_{a',m'}(q,z) = 0 \quad \text{if } a' + m > k;$$

otherwise,

$$\tilde{\mathcal{P}}^{a,m}_{a',m'}(q,z) = \begin{cases} q^m z^{-2a} \cdot \frac{1-(qz^2)^{k-m'+1}}{1-qz^2}, & k - m' \leqslant a; \\ q^m z^{-2a} \left(\frac{1-(qz^2)^a}{1-qz^2} + (qz^2)^a \cdot \frac{1-z^{2(k-m'-a+1)}}{1-z^2} \right), & a \leqslant k - m' \leqslant m; \\ q^m z^{-2a} \left(\frac{1-(qz^2)^a}{1-qz^2} + (qz^2)^a \cdot \frac{1-z^{2(m-a+1)}}{1-z^2} \right), & m \leqslant k - m'. \end{cases} \tag{16}$$

Proof. Each monomial (7) contributes $z^{2\alpha} q^\beta$ to the character where $\alpha = \sum_i (c_i - a_i)$, $\beta = \sum_i i \cdot (a_i + b_i + c_i)$; there is also a contribution from $v_{k,l}$, which is equal to z^l.

Let $p^{a,b,c}(k,l,N;q,z)$ be the partial character of the linear span of ehf–monomials of length N with $a_{N-1} = a$, $b_{N-1} = b$, $c_{N-1} = c$. Then we have

$$p^{a,b,c}(k,l,N+1;q,z) = \sum_{\substack{a',b',c' \\ a'+a+b \leqslant k,\ a'+b+c \leqslant k \\ a'+b'+c \leqslant k,\ b'+c'+c \leqslant k}} q^{N(a+b+c)} z^{2(c-a)} p^{a',b',c'}(k,l,N;q,z),$$

and therefore

$$\mathrm{ch}_{q,z} L^N_{k,l}(0) = \sum_{\substack{a,b,c \\ a',b',c'}} \left(\mathcal{P}(q^{N-1},z) \cdot \mathcal{P}(q^{N-2},z) \cdots \mathcal{P}(q,z)\right)^{a,b,c}_{a',b',c'} \cdot p^{a',b',c'}(k,l,1;q,z),$$

where

$$\mathcal{P}^{a,b,c}_{a',b',c'}(q,z) = \begin{cases} q^{(a+b+c)} z^{2(c-a)}, & a' + a + b \leqslant k,\ a' + b + c \leqslant k,\ a' + b' + c \leqslant k,\ b' + c' + c \leqslant k; \\ 0 & \text{otherwise.} \end{cases}$$

The coefficients $\mathcal{P}^{a,b,c}_{a',b',c'}(q,z)$ depend only on a, b, c, a' and $m' = \max(a' + b', b' + c')$. Therefore, we can apply Lemma 3.2.2 again. We obtain

$$\mathrm{ch}_{q,z} L^N_{k,l}(0) = \sum_{\substack{0 \leqslant a \leqslant m \\ 0 \leqslant a' \leqslant m'}} \left(\tilde{\mathcal{P}}(q^{N-1},z) \cdot \tilde{\mathcal{P}}(q^{N-2},z) \cdots \tilde{\mathcal{P}}(q,z)\right)^{a,m}_{a',m'} \cdot \tilde{p}^{a',m'}(k,l,1;q,z),$$

with

$$\tilde{p}^{a,m}(k,l,1;q,z) = \begin{cases} z^{l-2a}, & m = l; \\ 0 & \text{otherwise.} \end{cases}$$

Finally, noting that $\tilde{\mathcal{P}}^{0,0}_{a,m}(q,z) = 1$ for all a and m, we can replace the summation with respect to a and m with the multiplication of the row vector $\tilde{\mathcal{P}}^{(0,0)}(q^N, z)$. □

When $k = 1$, the matrix $\tilde{\mathcal{P}}$ is

$$\begin{pmatrix} 1 & 1 & 1 \\ q(1+z^2) & q & 0 \\ qz^{-2}+q^2 & qz^{-2} & 0 \end{pmatrix}. \tag{17}$$

Let us write the characters for $k = 1$. The proof will be supplied elsewhere. Let D be the matrix

$$D = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$$

and $\mathbf{e} \in \mathbb{Z}^3$. Define

$$c(N, \mathbf{e}) = \sum_{\substack{\mathbf{n}=(n_1,n_2,n_3) \\ n_1,n_2,n_3 \geqslant 0}} z^{2(n_3-n_1)} q^{\frac{1}{2}\mathbf{n}D\mathbf{n}+\mathbf{e}\mathbf{n}} \prod_{i=1}^{3} \begin{bmatrix} N+1-(D\mathbf{n}-n+\mathbf{e})_i \\ n_i \end{bmatrix}.$$

Then

$$\begin{aligned} \left(\mathcal{P}^{(N)}\right)^{0,0}_{0,0}(q,z) &= c(N,(0,0,0)), \\ \left(\mathcal{P}^{(N)}\right)^{0,0}_{0,1}(q,z) &= c(N,(0,0,1)), \\ \left(\mathcal{P}^{(N)}\right)^{0,0}_{1,1}(q,z) &= c(N,(1,1,1)). \end{aligned}$$

The characters of the coinvariant $L^N_{k,l}(0)$ in the case $k = 1$ are

$$\begin{aligned} \mathrm{ch}_{q,z} L^{(N)}_{1,0}(0) &= c(N,(0,0,0)), \\ \mathrm{ch}_{q,z} L^{(N)}_{1,1}(0) &= z \cdot c(N,(0,0,1)) + z^{-1} \cdot c(N,(1,1,1)). \end{aligned}$$

4. The space of coinvariants $L^{(N)}_{k,l}(\mathfrak{n})$: Monomial basis and dual space

In this section, following [FS], we introduce a scaled limit of $\widehat{sl}_2$, and consider $L^{(N)}_{1,0}(\mathfrak{n})$ in this limit. Then, we compute a monomial basis and the character of $L^{(N)}_{1,l}(\mathfrak{n})$ $(l = 0, 1)$ by using the functional description of the dual space to the scaled coinvariants.

4.1. Embedding into $\widehat{sl}_3$.

Consider the scaled algebra $sl_2^{(\varepsilon)}$ with generators

$$f^\varepsilon = \varepsilon f,\ e^\varepsilon = \varepsilon e,\ h^\varepsilon = \varepsilon^2 h,$$

and its scaled limit $\overline{sl}_2$ where $\varepsilon \to 0$. Let $\overline{sl}_2 \otimes \mathbb{C}[t, t^{-1}]$ be the associated loop algebra, with generators $\bar{e}_i, \bar{f}_i, \bar{h}_i$ and commutation relations

$$[\bar{e}_i, \bar{f}_j] = \bar{h}_{i+j}, \qquad [\bar{e}_i, \bar{h}_j] = [\bar{f}_i, \bar{h}_j] = [\bar{f}_i, \bar{f}_j] = [\bar{e}_i, \bar{e}_j] = 0. \tag{18}$$

As explained in the introduction, there is an embedding of the family of Lie algebras $\{sl_2^{(\varepsilon)} \otimes \mathbb{C}[t, t^{-1}]\}_{0 \leqslant \varepsilon \leqslant 1}$ into $\widehat{sl}_3$. In this embedding $\overline{sl}_2$ is mapped to the maximal nilpotent subalgebra of sl_3, with the identification $\bar{e} = E^{12}, \bar{f} = E^{23}, \bar{h} = E^{13}$. We use this fact below to identify a certain subspace of an irreducible level 1 $\widehat{sl}_3$-module with $L_{1,0}$ in the scaled limit.

Let $(H_1, H_2) = (E^{11} - E^{22}, E^{22} - E^{33})$ be the basis for the sl_3 Cartan subalgebra and d the homogeneous degree operator. Let $\bar{v}$ be the highest weight vector with the highest weight $(d, H_1, H_2, c) = (0, 0, 0, 1)$. We have

$$\bar{e}_i \bar{v} = \bar{f}_i \bar{v} = \bar{h}_i \bar{v} = 0 \text{ for } i \leqslant 0.$$

Define the subspace $W = U(\overline{sl}_2 \otimes \mathbb{C}[t, t^{-1}])\bar{v}$. As shown in [FS], we have

$$\bar{e}(z)^2 = \bar{f}(z)^2 = \bar{h}(z)^2 = \bar{e}(z)\bar{h}(z) = \bar{f}(z)\bar{h}(z) = 0 \tag{19}$$

on W.

The space W is graded by the action of (H_1, H_2). We define the character of a graded subspace $V \subset W$ by

$$\mathrm{ch}_{q,z_1,z_2} V \quad = \quad \mathrm{trace} q^d z_1^{H_1} z_2^{H_2}.$$

Let $W_{m,n}$ be the graded component of degree (m, n) with respect to (H_1, H_2).

Theorem 4.1.1. *The following elements span $W_{m,n}$:*

$$v_{\alpha,\beta} = \prod_{j=1}^{m} \bar{e}_{\alpha_j + 2(m-j)+1-n} \prod_{j=1}^{n} \bar{f}_{\beta_j + 2(n-j)+1} \ \bar{v}, \tag{20}$$

where α, β are partitions (i.e., $\alpha_1 \geqslant \alpha_2 \geqslant \cdots \geqslant \alpha_m \geqslant 0$).

First we prove a technical lemma. We use the notation $F_i^{(n)}$ for a sum of terms of the form $\bar{f}_{i_1} \cdots \bar{f}_{i_n}$, with degree $\sum i_j = i$.

Lemma 4.1.2.

$$\bar{e}_{-n-a} F_i^{(n)} \bar{v} = \sum_{b \geqslant 0} \bar{e}_{1-n+b} F_{i-b-a-1}^{(n)} \bar{v} \text{ for } a \geqslant 0.$$

Proof. Consider the case $n = 1$. We have

$$\bar{e}_{-1-a} \bar{f}_i \bar{v} = [\bar{e}_{-1-a}, \bar{f}_i]\bar{v} = [\bar{e}_0, \bar{f}_{i-a-1}]\bar{v} = \bar{e}_0 \bar{f}_{i-a-1} \bar{v}.$$

Then use induction on n. Consider

$$\begin{aligned} \bar{e}_{-n-a} \bar{f}_b F_i^{(n-1)} \bar{v} \quad &= \quad (\bar{f}_b \bar{e}_{-n-a} + [\bar{e}_{1-n}, \bar{f}_{b-1-a}]) F_i^{(n-1)} \bar{v} \\ &= \bar{f}_b \sum_{c \geqslant 0} \bar{e}_{2-n+c} F_{i-2-c-a}^{(n-1)} \bar{v} + \bar{f}_{b-1-a} \sum_{c \geqslant 0} \bar{e}_{2-n+c} F_{i-c-1}^{(n-1)} \bar{v} + \bar{e}_{1-n} F_{b+i-1-a}^{(n)} \bar{v}. \end{aligned}$$

In the second line, we assumed the lemma is proved for $n-1$. The last term is of the desired form, and the other terms can be reduced to this form by using the following procedure:

$$\begin{aligned} \bar{f}_a \bar{e}_{2-n+b} F_c^{(n-1)} \bar{v} \quad &= \quad (\bar{e}_{2-n+b} \bar{f}_a + [\bar{f}_{a+1+b}, \bar{e}_{1-n}]) F_c^{(n-1)} \bar{v} \\ &= \bar{f}_{a+1+b} \sum_{d \geqslant 0} \bar{e}_{2-n+d} F_{c-1-d}^{(n-1)} \bar{v} + (\bar{e}_{2-n+b} \bar{f}_a - \bar{e}_{1-n} \bar{f}_{a+1+b}) F_c^{(n-1)} \bar{v}. \end{aligned}$$

In the second step, the last two terms are again of the desired form, and the degree of $F_{c-d-1}^{(n-1)}$ in the first term is less than c, the degree on the left hand side of the equation. Therefore if we repeat this step a finite number of times, the process will terminate when the degree is low enough so that $F_{c-d-1}^{(n-1)} \bar{v} = 0$. We are left with a sum of terms with $\bar{e}_i$ acting on the left, with degree $i \geqslant 1 - n$. Therefore the lemma follows. □

Proof of Theorem 4.1.1. Clearly, $W_{m,n}$ is spanned by elements of the form

$$\bar{e}_{i_1} \cdots \bar{e}_{i_{m-l}} \bar{h}_{j_1} \cdots \bar{h}_{j_l} \bar{f}_{k_1} \cdots \bar{f}_{k_{n-l}} \bar{v}, \tag{21}$$

where all the indices are positive.

Also, any element of the form (21) can be expressed as a sum of terms of the form

$$\bar{e}_{i_1}\cdots\bar{e}_{i_m}\bar{f}_{j_1}\cdots\bar{f}_{j_n}\bar{v}, \text{ with } j_a > 0, \tag{22}$$

rewriting $\bar{h}_i = [\bar{e}_{-a}, \bar{f}_{i+a}]$ with a sufficiently large, and using the fact that $\bar{h}_i$ commute with all other generators.

The relations $\bar{e}(z)^2 = \bar{f}(z)^2 = 0$ in W mean that (22) can be rewritten as a sum of similar elements, with indices which satisfy

$$|i_k - i_{k+1}| \geqslant 2 \quad \text{and} \quad |j_k - j_{k+1}| \geqslant 2. \tag{23}$$

Now, using Lemma 4.1.2, any element of the form $\bar{e}_a \prod_{i=1}^n \bar{f}_{j_i}\bar{v}$ with $a < 1-n$ can be rewritten as a sum of such terms with $a \geqslant 1-n$. The result is a sum of terms of the form

$$\prod_{k=1}^m \bar{e}_{i_k} \prod_{k=1}^n \bar{f}_{j_k}\bar{v}, \quad i_\alpha \geqslant 1-n.$$

We again use $\bar{e}(z)^2 = \bar{f}(z)^2 = 0$ so that the indices again satisfy (23). Note that this step preserves the degree with respect to the grading operator d of $\prod \bar{e}_j$ and $\prod \bar{f}_j$, separately, whereas using the lemma, the degree of the product of $\bar{e}$'s is raised and $\bar{f}$'s lowered. Therefore, repeating the process a finite number of times, we obtain a sum of terms of the form (20). □

We next show that in fact, (20) form a basis for $W_{m,n}$. In order to show linear independence, we introduce the dual space.

4.2. **Dual space.** Define the space $S_{m,n}$ to be the space of rational functions in two sets of variables, $\mathbf{x} = (x_1, \ldots, x_m)$ and $\mathbf{y} = (y_1, \ldots, y_n)$, of the form

$$F(\mathbf{x},\mathbf{y}) = \prod_{i=1}^m x_i \prod_{i=1}^n y_i \frac{\prod_{i<j}(x_i - x_j)^2 \prod_{i<j}(y_i - y_j)^2}{\prod_{i,j}(x_i - y_j)} f(\mathbf{x},\mathbf{y}), \tag{24}$$

where $f(\mathbf{x},\mathbf{y})$ is a symmetric polynomial in the variables $\mathbf{x}$ and $\mathbf{y}$ separately.

Consider the variables $\{z_1, \ldots, z_{m-l}\}, \{u_1, \ldots, u_l\}, \{w_1, \ldots, w_{n-l}\}$ in some order σ. Let $\iota : (\mathbf{x},\mathbf{y}) \to (\mathbf{z},\mathbf{u},\mathbf{w})$ be defined as follows: Arrange the letters z_i, u_j in the order in which they appear in σ. Then $\iota(\mathbf{x})$ is this sequence. Similarly, $\mathbf{y}$ is mapped to the ordering in σ of w_i, u_j. For example, if the sequence of variables is $(u_1 w_1 z_1 u_2)$, then $\iota(\mathbf{x}) = (u_1, z_1, u_2)$ and $\iota(\mathbf{y}) = (u_1, w_1, u_2)$.

We define a coupling of an element in $S_{m,n}$ to an element in $W_{m,n}$ by using generating functions for elements in $W_{m,n}$,

$$\left\langle F(\mathbf{x},\mathbf{y}), \sigma\left(\prod_{i=1}^{m-l}\bar{e}(z_i)\prod_{i=1}^{l}\bar{h}(u_i)\prod_{i=1}^{n-l}\bar{f}(w_i)\right)\bar{v}\right\rangle = \left(\prod_{\{i,j;\ \iota(x_i)=\iota(y_j)\}} x_i^{-1}(x_i - y_j)\right) F(\mathbf{x},\mathbf{y})\Big|_{\substack{\mathbf{x}\mapsto\iota(\mathbf{x})\\ \mathbf{y}\mapsto\iota(\mathbf{y})}}. \tag{25}$$

Namely, we first remove the pole in (24) at $x_i = y_j$, for x_i and y_j that are mapped by ι to the same variable of type u, then we make the substitution. After this substitution, the remaining pole factors are only of the form $(z_i - w_j)^{-1}$. They are understood as follows. If the variable z_i appears to the right of the variable w_j, then we expand the rational function

in positive powers of z_i. We expand in negative powers if it appears to the left of w_j. We will use the notation

$$\frac{1}{w-z} = w^{-1}\sum_{n\geqslant 0}(z/w)^n, \quad \frac{1}{z-w} = z^{-1}\sum_{n\geqslant 0}(w/z)^n.$$

Note that

$$\frac{zw}{z-w} + \frac{zw}{w-z} = z\delta(z/w), \tag{26}$$

where $\delta(z) = \sum_{i\in\mathbb{Z}} z^i$ is the multiplicative delta function.

For example, the coupling (25) means

$$\begin{aligned} \left\langle F(x_1,y_1), \bar{e}(z)\bar{f}(w)\bar{v}\right\rangle &= \frac{zw}{z-w}f(z,w), \\ \left\langle F(x_1,y_1), \bar{f}(w)\bar{e}(z)\bar{v}\right\rangle &= -\frac{zw}{w-z}f(z,w), \\ \left\langle F(x_1;y_1), \bar{h}(u)\bar{v}\right\rangle &= uf(u;u). \end{aligned} \tag{27}$$

Lemma 4.2.1. *The coupling (25) is well defined.*

Proof. The fact that $\langle F, g_i\bar{v}\rangle = 0$ $(g = e, f, h \in sl_2; i \leqslant 0)$ follows from the fact that the right hand side of (25) contains only positive powers of the rightmost variable in the product of currents acting on $\bar{v}$.

The fact that $F(\mathbf{x},\mathbf{y})$ is a symmetric function in x_i and y_i implies that

$$\langle F, \cdots[\bar{e}(z),\bar{e}(w)]\cdots\bar{v}\rangle = \langle F, \cdots[\bar{f}(z),\bar{f}(w)]\cdots\bar{v}\rangle = 0.$$

The same argument holds for $\langle F, [\bar{e}(z),\bar{h}(w)]\bar{v}\rangle = \langle F, [\bar{f}(z),\bar{h}(w)]\bar{v}\rangle = 0$.

Since $F(\mathbf{x},\mathbf{y}) = 0$ if $x_1 = x_2$ or $y_1 = y_2$, we have

$$\langle F, \cdots e(z)^2\cdots\bar{v}\rangle = \langle F, \cdots \bar{f}(z)^2\cdots\bar{v}\rangle = 0.$$

From (26) and (27) it is clear that $\langle F, \cdots([\bar{e}(z),\bar{f}(w)] - \delta(z/w)\bar{h}(z))\cdots\bar{v}\rangle = 0$.

Therefore, the lemma follows. □

We define a complete (lexicographic) ordering among all partitions,

$$\lambda > \mu \text{ if } \lambda_i = \mu_i,\ i = 1,\dots,j-1 \text{ and } \lambda_j > \mu_j \tag{28}$$

for some j. We extend the ordering to pairs of partitions $\{(\alpha,\beta)\}$ by setting $(\alpha,\beta) > (\alpha',\beta')$ if $\alpha > \alpha'$, or $\alpha = \alpha'$ and $\beta > \beta'$.

We pick the basis for the symmetric polynomials $f(\mathbf{x},\mathbf{y})$ to be

$$f_{\alpha,\beta}(\mathbf{x},\mathbf{y}) = (\mathbf{x}^\alpha\mathbf{y}^\beta)^{\mathrm{Symm}},$$

where the symmetrization is with respect to $\mathbf{x}$ and $\mathbf{y}$ separately, and we use the standard notation

$$\mathbf{x}^\alpha \stackrel{\mathrm{def}}{=} x_1^{\alpha_1}\cdots x_m^{\alpha_m}.$$

Note that the separately symmetric polynomials $f_{\alpha,\beta}$ are ordered by the ordering given above. We extend this ordering even for the separately symmetric Laurent polynomials since we will use it in that way in the proof of Lemma 4.2.2.

We define $F_{\alpha,\beta}(\mathbf{x},\mathbf{y})$ to be a function of the form (24) with f replaced by $f_{\alpha,\beta}$. These functions form a basis of $S_{m,n}$.

Lemma 4.2.2. *The coupling (25) between $F_{\alpha,\beta}$ and $v_{\alpha,\beta}$ of (20) is triangular, with nonzero diagonal elements.*

Proof. We define an ordering among all separately symmetric Laurent polynomials of the form $\left(\mathbf{x}^{\alpha}\mathbf{y}^{\beta}\right)^{\mathrm{Symm}}$ in a similar manner as for pairs of partitions (α,β). Then, we have

$$F_{\alpha,\beta}(\mathbf{x},\mathbf{y}) = \left(\prod_{i=1}^{m} x_i^{\alpha_i+2(m-i)+1-n} \prod_{j=1}^{n} y_j^{\beta_j+2(n-j)+1}\right)^{\mathrm{Symm}} + \text{ lower terms.}$$

The coupling (25) is equivalent to

$$\langle (\mathbf{x}^{\mathbf{i}'}\mathbf{y}^{\mathbf{j}'})^{\mathrm{Symm}}, \bar{e}_{i_1}\cdots\bar{e}_{i_m}\bar{f}_{j_1}\cdots\bar{f}_{j_n}\bar{v}\rangle = \delta_{\mathbf{i},\mathbf{i}'}\delta_{\mathbf{j},\mathbf{j}'}. \tag{29}$$

Therefore the coupling of the basis $F_{\alpha,\beta}$ to the vectors $v_{\alpha,\beta}$ is triangular. □

Theorem 4.2.3. *The coupling (25) is nondegenerate, and the space $S_{m,n}$ is dual to $W_{m,n}$.*

Proof. This follows from Theorem 4.1.1 and Lemma 4.2.2. □

Corollary 4.2.4. *The vectors $\{v_{\alpha,\beta}\ ;\alpha,\beta$ partitions$\}$ of (20) form a basis of W.*

Let $S_{m,n}(d)$ denote the subspace of $S_{m,n}$ of functions $F(\mathbf{x},\mathbf{y})$ with homogeneous degree d. Using Theorem 4.2.3, we have

$$\mathrm{ch}_{q,z_1,z_2} W = \sum_{m,n,d} \dim S_{m,n}(d) q^d z_1^m z_2^n = \sum_{m,n\geqslant 0} \frac{q^{n^2+m^2-mn}}{(q)_m (q)_n} z_1^m z_2^n.$$

When specialized to $z_1 = z_2^{-1} = z^2$, this is the character of $L_{1,0}$:

$$\mathrm{ch}_{q,z} L_{1,0} = \sum_{n\in\mathbb{Z}} \frac{q^{n^2} z^{2n}}{(q)_\infty}.$$

as noted in [FS]. Since W is the scaled limit of $L_{1,0}$, by general arguments we can say only that $\mathrm{ch}_{q,z,z^{-1}} W \geqslant \mathrm{ch}_{q,z} L_{1,0}$, but the calculation above allows us to see that in fact, the equality holds.

4.3. **The dual to the coinvariants.** Let $\mathfrak{g}(M,N) \subset \overline{sl}_2 \otimes \mathbb{C}[t,t^{-1}]$ be the subalgebra generated by $\{\bar{e}_M, \bar{e}_{M+1}, \ldots, \bar{f}_N, \bar{f}_{N+1}, \ldots\}$, and define

$$\overline{W}^{M,N} = W/\mathfrak{g}(M,N)W \quad \text{for } M,N \geqslant -1, (M,N) \neq (-1,-1). \tag{30}$$

We further define $W^{M,N} = \overline{W}^{M,N}$ if $M,N \geqslant 0$, and $W^{M,N} = \overline{W}^{M,N}/\mathbb{C}\bar{v}$ if $M = -1$ or $N = -1$. The quotient $W^{M,N}$ inherits an sl_3 Cartan grading (m,n) and a grading by the degree operator d.

Lemma 4.3.1. *The orthogonal complement to the space $\mathfrak{g}(M,N)W$ in $S_{m,n}$ is the subspace $\overline{S}^{M,N}_{m,n}$, consisting of F (see (24)) with the degree restrictions*

$$\deg_{x_1} F(\mathbf{x},\mathbf{y}) < M, \qquad \deg_{y_1} F(\mathbf{x},\mathbf{y}) < N. \tag{31}$$

If $m = 0$ there is no restriction on $\deg_{x_1} F(\mathbf{x},\mathbf{y})$, and the same is true for n. The degree of F is counted in such a way that

$$\deg_x \frac{1}{x-y} = \deg_y \frac{1}{x-y} = -1. \tag{32}$$

Proof. We only note that we expand

$$\frac{1}{z-w} = \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{w}{z}\right)^n \tag{33}$$

when we compute the coupling of F with an element of the form $\bar{e}(z)\cdots\bar{f}(w)\cdots\bar{v}$. □

In terms of f (see (24)) the restrictions read as

$$\deg_{x_1} f(\mathbf{x},\mathbf{y}) \leqslant M-2m+n, \qquad \deg_{y_1} f(\mathbf{x},\mathbf{y}) \leqslant N-2n+m. \tag{34}$$

Note that $\overline{S}^{M,N}_{0,0} = \mathbb{C}$ for all M, N.

We define $S^{M,N}_{m,n}$ to be the dual space of $W^{M,N}_{m,n}$. For $M, N \geqslant 0$, we have $S^{M,N}_{m,n} = \overline{S}^{M,N}_{m,n}$. For $M = -1$ or $N = -1$, $S^{M,N}_{m,n}$ is the subspace of $\overline{S}^{M,N}_{m,n}$, which is codimension 1 and $S^{M,N}_{0,0} = 0$.

The reason for the above definition of the space $W^{M,N}_{m,n}$ and its dual at $(m,n) = (0,0)$ will be clear in Section 3.

We have

Theorem 4.3.2. *The following vectors form a basis for* $W^{M,N}_{m,n}$ *for* $(m,n) \neq (0,0)$*:*

$$\bar{e}_{i_1}\cdots\bar{e}_{i_m}\bar{f}_{j_1}\cdots\bar{f}_{j_n}\bar{v} \tag{35}$$

with $i_k \geqslant i_{k+1}+2$, $j_k \geqslant j_{k+1}+2$, $M > i_k \geqslant 1-n$ *and* $N+m > j_k > 0$.

This space is finite dimensional and the character is given by

$$\mathrm{ch}_{q,z_1,z_2} W^{M,N} = \sum_{m,n\geqslant 0} q^{n^2+m^2-mn}\begin{bmatrix} M-m+n \\ m \end{bmatrix}\begin{bmatrix} N-n+m \\ n \end{bmatrix} z_1^m z_2^n. \tag{36}$$

Proof. Since we have Corollary 4.2.4, it is enough to show, first that a vector of the form (35) belongs to the orthogonal complement to the space $\bar{S}^{M,N}_{m,n}$ if $i_1 \geqslant M$ or $j_1 \geqslant N+m$, and second that the characters of the space of vectors of the form (35) satisfying the conditions in the statement of the theorem, is equal to that of the space $S^{M,N}_{m,n}$. We only note that N is replaced by $N+m$ because $\bar{f}_{j_1}$ is placed to the right of $\bar{e}_{i_1}\cdots\bar{e}_{i_m}$. The rest of the proof is straightforward. □

Theorem 4.3.3. *The character of the coinvariants* $L^{(N)}_{1,0}(\mathfrak{n})$ *is given by*

$$\mathrm{ch}_{q,z} L^{(N)}_{1,0}(\mathfrak{n}) = \sum_{s\geqslant 0} q^{s^2}\begin{bmatrix} N \\ 2s \end{bmatrix} z^{-2s}. \tag{37}$$

We have, in particular,

$$\dim L^{(N)}_{1,0}(\mathfrak{n}) = 2^{N-1}.$$

Proof. By expanding the identity $(a+b)^N = (a+b)^{N-s}(a+b)^s$ with $ba = qab$, we have

$$\begin{aligned}\mathrm{ch}_{q,z_1,z_2} W^{0,N}|_{z_1=z_2^{-1}=z^2} &= \sum_{n\geqslant s\geqslant 0} q^{s^2+n(n-s)}\begin{bmatrix} N-s \\ n \end{bmatrix}\begin{bmatrix} s \\ n-s \end{bmatrix} z^{-2s} \\ &= \sum_{s\geqslant 0} q^{s^2}\begin{bmatrix} N \\ 2s \end{bmatrix} z^{-2s}.\end{aligned}$$

We have, in particular, that

$$\dim W^{0,N} \quad = \quad 2^{N-1}.$$

This is equal to the dimensions of the coinvariants $L^{(N)}_{1,0}(\mathcal{Z};\mathfrak{n})$ given by Verlinde's formula.

By general arguments we have

$$\dim L^{(N)}_{1,0}(\mathfrak{n}) \geq \dim L^{(N)}_{1,0}(\mathcal{Z};\mathfrak{n}).$$

Note that $W^{0,N}$ is the scaled limit of $L^{(N)}_{1,0}(\mathfrak{n})$. We showed in the previous subsection that $\mathrm{ch}_{q,z,z^{-1}} W = \mathrm{ch}_{q,z} L_{1,0}$. Therefore, again, by general arguments, we have

$$\dim W^{0,N} \geq \dim L^{(N)}_{1,0}(\mathfrak{n}).$$

Thus, we have shown the equality of their dimensions, and thereby the equality of their characters. □

Corollary 4.3.4. *The vectors of the form*

$$e_{i_1} \cdots e_{i_m} f_{j_1} \cdots f_{j_n} \bar{v}_{1,0}$$

with the same conditions as in Theorem 4.3.2 form a monomial basis for the space of coinvariants $L^{(N)}_{1,0}(\mathfrak{n})$.

5. Recursion relations for the spaces $W^{M,N}$ and the dual spaces

5.1. **Short exact sequences.** In this section we describe the spaces of coinvariants $W^{0,N}$ using induction on N. This will allow us to produce another monomial basis for the space $L^{(N)}_{1,0}(\mathfrak{n})$. We also write down the recurrence relations for the corresponding characters. To establish the induction we use the spaces $W^{M,N}$ with M general.

Introduce the maps

$$\begin{aligned} T^* &: S^{M,N}_{m,n} \to S^{M,N+1}_{m,n}, \\ U^* &: S^{M,N}_{m,n} \to S^{M+1,N}_{m,n}, \\ \varphi^* &: S^{M,N}_{m,n} \to S^{M+1,N-2}_{m,n-1}, \\ \varepsilon^* &: S^{M,N}_{m,n} \to S^{M-2,N+1}_{m-1,n} \end{aligned}$$

as follows. For $F(\mathbf{x},\mathbf{y}) \in S^{M,N}_{m,n}$, define

$$\begin{aligned} T^*(F(\mathbf{x},\mathbf{y})) &= y_1 y_2 \ldots y_n F(\mathbf{x},\mathbf{y}), \\ U^*(F(\mathbf{x},\mathbf{y})) &= x_1 x_2 \ldots x_m F(\mathbf{x},\mathbf{y}), \\ \varphi^*(F(\mathbf{x},\mathbf{y})) &= \frac{1}{y_1} \frac{x_1 x_2 \ldots x_m}{y_2^2 y_3^2 \ldots y_n^2} F(\mathbf{x},\mathbf{y})\Big|_{y_1=0}, \\ \varepsilon^*(F(\mathbf{x},\mathbf{y})) &= \frac{1}{x_1} \frac{y_1 y_2 \ldots y_n}{x_2^2 x_3^2 \ldots x_m^2} F(\mathbf{x},\mathbf{y})\Big|_{x_1=0}. \end{aligned}$$

Recall that the spaces $S^{M,N}_{m,n}$ are defined only for $M, N \geqslant -1$, with $(M,N) \neq (-1,-1)$, and $m, n \geqslant 0$. We define $S^{M,N}_{m,n} = 0$ if $m < 0$ or $n < 0$. The maps $T^*, U^*, \varphi^*, \varepsilon^*$ are defined only when both source and target spaces are defined.

Lemma 5.1.1. *The maps $T^*, U^*, \varphi^*, \varepsilon^*$ are well defined. Moreover, the sequences*

$$0 \to S^{M,N-1}_{m,n} \xrightarrow{T^*} S^{M,N}_{m,n} \xrightarrow{\varphi^*} S^{M+1,N-2}_{m,n-1} \to 0, \qquad M \geqslant -1, N \geqslant 1 \tag{38}$$

and

$$0 \to S^{M-1,N}_{m,n} \xrightarrow{U^*} S^{M,N}_{m,n} \xrightarrow{\varepsilon^*} S^{M-2,N+1}_{m-1,n} \to 0, \qquad M \geqslant 1, N \geqslant -1 \tag{39}$$

are exact.

Proof. By symmetry reason it is enough to show one of these, say (39).

It is clear that U^* is well-defined and injective.

If $m \geqslant 1$, the map ε^* sends F of (24) to

$$\varepsilon^*(F) \;=\; \prod_{i=2}^{m} x_i \prod_{i=1}^{n} y_i \frac{\prod_{2\leqslant i<j}(x_i-x_j)^2 \prod_{i<j}(y_i-y_j)^2}{\prod_{i\geqslant 2,j}(x_i-y_j)} f(0,x_2,\dots,x_m;\mathbf{y}).$$

If $m \geqslant 2$, it is obvious that this is well-defined and surjective. If $m = 1$, it is less obvious because in the space $S^{M-2,N+1}_{0,n}$, the first condition of (31) is void and also because $S^{-1,N}_{0,0} \neq \overline{S}^{-1,N}_{0,0}$.

The first condition of (24) for $S^{M,N}_{1,n}$ becomes

$$\deg_{x_1} x_1 \prod_{i=1}^{n} y_i \frac{\prod (y_i-y_j)^2}{\prod_{i=1}^{n}(x_1-y_i)} f(x_1;\mathbf{y}) < M.$$

This is equivalent to $\deg_{x_1} f(x_1;\mathbf{y}) < M+n-1$. Therefore, if $n \geqslant 1$ or $M \geqslant 2$, then

$$\deg_{x_1} f(x_1;\mathbf{y}) \;=\; 0$$

is allowed and the map ε^* is surjective.

If $n = 0$ and $M = 1$, we have $S^{1,N}_{1,0} = 0$, and therefore, we must use the definition $S^{-1,N+1}_{0,0} = 0$.

The rest of the proof is straightforward. □

Next, define the maps

$$\begin{aligned} T &: W^{M,N}_{m,n} \to W^{M,N-1}_{m,n}, \\ U &: W^{M,N}_{m,n} \to W^{M-1,N}_{m,n}, \\ \varphi &: W^{M,N}_{m,n} \to W^{M-1,N+2}_{m,n+1}, \\ \varepsilon &: W^{M,N}_{m,n} \to W^{M+2,N-1}_{m+1,n} \end{aligned}$$

as follows. Let $A = \sigma(\bar{e}_{i_1}\dots\bar{e}_{i_a}\bar{h}_{s_1}\dots\bar{h}_{s_b}\bar{f}_{j_1}\dots\bar{f}_{j_c})\bar{v} \in W^{M,N}_{m,n}$, where σ denotes the order in which $\bar{e}_i, \bar{h}_i$ and $\bar{f}_i$ are written, with $a+b=m,\ b+c=n$. Define

$$\begin{aligned} T(A) &= \sigma(\bar{e}_{i_1}\dots\bar{e}_{i_a}\bar{h}_{s_1-1}\dots\bar{h}_{s_b-1}\bar{f}_{j_1-1}\dots\bar{f}_{j_c-1})\bar{v}, \\ U(A) &= \sigma(\bar{e}_{i_1-1}\dots\bar{e}_{i_a-1}\bar{h}_{s_1-1}\dots\bar{h}_{s_b-1}\bar{f}_{j_1}\dots\bar{f}_{j_c})\bar{v}, \\ \varphi(A) &= \sigma(\bar{e}_{i_1-1}\dots\bar{e}_{i_a-1}\bar{h}_{s_1+1}\dots\bar{h}_{s_b+1}\bar{f}_{j_1+2}\dots\bar{f}_{j_c+2})\bar{f}_1\bar{v}, \\ \varepsilon(A) &= \sigma(\bar{e}_{i_1+2}\dots\bar{e}_{i_a+2}\bar{h}_{s_1+1}\dots\bar{h}_{s_b+1}\bar{f}_{j_1-1}\dots\bar{f}_{j_c-1})\bar{e}_1\bar{v}. \end{aligned}$$

Lemma 5.1.2. *The maps $T, U, \varphi, \varepsilon$ are well defined.*

Proof. Consider, for example, the map φ. Clearly, φ preserves the relations (18),(19) and the highest weight condition, $\bar{g}_i\bar{v}=0$, where $g=e,f,h$ and $i\leqslant 0$. For example, the images of the zero vectors $\bar{e}_0\bar{v}$, $\bar{f}_0\bar{v}$,

$$\varphi(\bar{e}_0\bar{v})=\bar{e}_{-1}\bar{f}_1\bar{v}, \qquad \varphi(\bar{f}_0\bar{v})=\bar{f}_2\bar{f}_1\bar{v},$$

are also zero vectors.

Finally, φ maps the subalgebra generated by $\{\bar{e}_i,\bar{f}_j, i\geqslant M, j\geqslant N\}$ to the the subalgebra generated by $\{\bar{e}_i,\bar{f}_j, i\geqslant M-1, j\geqslant N+2\}$. $\square$

We have another easy

Lemma 5.1.3. *The maps T,U,φ,ε are dual to maps $T^*,U^*,\varphi^*,\varepsilon^*$ with respect to pairing defined by (25).*

Corollary 5.1.4. *The sequences*

$$0\to W^{M+1,N-2}_{m,n-1}\xrightarrow{\varphi} W^{M,N}_{m,n}\xrightarrow{T} W^{M,N-1}_{m,n}\to 0, \qquad M\geqslant -1, N\geqslant 1, \tag{40}$$

and

$$0\to W^{M-2,N+1}_{m-1,n}\xrightarrow{\varepsilon} W^{M,N}_{m,n}\xrightarrow{U} W^{M-1,N}_{m,n}\to 0, \qquad M\geqslant 1, N\geqslant -1, \tag{41}$$

are exact.

By Corollary 5.1.4 we have the following recursion relations for characters:

$$\begin{aligned}
&\mathrm{ch}W^{M,N}(q,z_1,z_2)=qz_2\ \mathrm{ch}W^{M+1,N-2}(q,q^{-1}z_1,q^2z_2)+\mathrm{ch}W^{M,N-1}(q,z_1,qz_2),\\
&\mathrm{ch}W^{M,N}(q,z_1,z_2)=qz_1\ \mathrm{ch}W^{M-2,N+1}(q,q^2z_1,q^{-1}z_1)+\mathrm{ch}W^{M-1,N}(q,qz_1,z_2).
\end{aligned}\tag{42}$$

It can be verified, using the q-Pascal identities

$$\begin{bmatrix} m\\ n\end{bmatrix}=q^n\begin{bmatrix} m-1\\ n\end{bmatrix}+\begin{bmatrix} m-1\\ n-1\end{bmatrix}, \qquad \begin{bmatrix} m\\ n\end{bmatrix}=\begin{bmatrix} m-1\\ n\end{bmatrix}+q^{m-n}\begin{bmatrix} m-1\\ n-1\end{bmatrix},$$

that the character (36) satisfies the recursion relations (42).

5.2. **The fh-basis.** Recall that the spaces $L^{(N)}_{1,0}(\mathfrak{n})$ have ef bases given by (21). However, we take a quotient with respect to action of e_i, h_j, f_j, $i\geqslant 0$ $j\geqslant N$. So, it is more natural to have a basis in terms of the remaining generators, $h_1,\dots h_{N-1}, f_1,\dots f_{N-1}$. In this section we describe such a basis, which we call the fh-basis.

We are interested in $W^{0,N}$, and $W^{1,N}$ (see Section 5.3). We construct the fh-bases in these spaces inductively, by using the short exact sequences (40), (41). In fact, we have to consider spaces $W^{-1,N}$, too, in order to close the induction steps.

Let $\mathcal{C}^{0,N}$ be the set of all $(\mathbf{a},\mathbf{b})=(a_{N-1},\dots,a_1,a_0;b_{N-1},\dots,b_1)\in\{0,1\}^{2N-1}$ such that

$$b_{i+1}+a_{i+1}+a_i\leqslant 1, \qquad b_{i+1}+b_i+a_i\leqslant 1, \tag{43}$$

$$\begin{aligned}
&\text{for each } i \text{ such that } b_i\neq 0 \text{ there exists } j>i\\
&\text{such that } a_j\neq 0 \text{ and } b_s=0 \text{ for } i+1\leqslant s\leqslant j,
\end{aligned}\tag{44}$$

and

$$a_0=0.$$

For $(\mathbf{a},\mathbf{b})\in\mathcal{C}^{0,N}$, define $\tau^{(0,N)}(\mathbf{a},\mathbf{b})\in W$ to be

$$\tau^{(0,N)}(\mathbf{a},\mathbf{b})=\bar{f}_{N-1}^{a_{N-1}}\bar{h}_{N-1}^{b_{N-1}}\dots\bar{f}_2^{a_2}\bar{h}_2^{b_2}\bar{f}_1^{a_1}\bar{h}_1^{b_1}\bar{v}.$$

Note that $\{\bar{f}_i,\bar{h}_j\}$ form a commutative algebra, so we can write them in any order.

It is convenient to write elements $(\mathbf{a},\mathbf{b}) \in \mathcal{C}^{0,N}$ as follows.

$$\begin{pmatrix} & b_{N-1} & \cdots & b_2 & & b_1 & \\ a_{N-1} & & \cdots & & a_1 & & a_0 = 0 \end{pmatrix}.$$

Then the map τ can be read off and relations (43) become "triangle relations".

Similarly, let $\mathcal{C}^{1,N}$ be the set of all $(\mathbf{a},\mathbf{b})$ satisfying the conditions (43), (44) and that

$$\text{there exists } j \text{ such that } a_j \neq 0 \text{ and } b_s = 0 \text{ for } s \leqslant j. \tag{45}$$

For any $(\mathbf{a},\mathbf{b}) \in \mathcal{C}^{1,N}$, define $\tau^{(1,N)}(\mathbf{a},\mathbf{b}) \in W$ to be

$$\tau^{(1,N)}(\mathbf{a},\mathbf{b}) = \bar{h}_N^{b_N} \bar{f}_{N-1}^{a_N} \bar{h}_{N-1}^{b_{N-1}} \dots \bar{f}_2^{a_3} \bar{h}_2^{b_2} \bar{f}_1^{a_2} \bar{h}_1^{b_1} \bar{f}_0^{a_1} \bar{e}_1^{1-a_0} \bar{v}.$$

We write $(\mathbf{a},\mathbf{b}) \in \mathcal{C}^{1,N}$ as

$$\begin{pmatrix} b_N & & b_{N-1} & \cdots & b_2 & & b_1 & & \\ & a_N & & \cdots & & a_2 & & a_1 & a_0 \end{pmatrix},$$

Finally, set $\mathcal{C}^{-1,N} = \mathcal{C}^{1,N-2}$.

For $(\mathbf{a},\mathbf{b}) \in \mathcal{C}^{-1,N}$, define $\tau^{(-1,N)}(\mathbf{a},\mathbf{b}) \in W$ to be

$$\tau^{(-1,N)}(\mathbf{a},\mathbf{b}) = \bar{f}_{N-1}^{a_{N-2}} \bar{f}_{N-2}^{a_{N-3}} \bar{h}_{N-2}^{b_{N-2}} \dots \bar{f}_2^{a_1} \bar{h}_2^{b_2} \bar{f}_1^{a_0} \bar{h}_1^{b_1} \bar{v}.$$

We write $(\mathbf{a},\mathbf{b}) \in \mathcal{C}^{-1,N}$ as

$$\begin{pmatrix} & & b_{N-2} & \cdots & & b_3 & & b_2 & & b_1 \\ a_{N-2} & a_{N-3} & & \cdots & a_2 & & a_1 & & a_0 & \end{pmatrix},$$

Example: We have

$$W^{-1,1} = W^{1,-1} = W^{-1,0} = W^{0,-1} = 0, \qquad W^{0,0} = W^{0,1} = W^{1,0} = \mathbb{C}\bar{v},$$

$$W^{-1,2} = \mathbb{C}\bar{f}_1\bar{v}, \qquad W^{1,1} = \langle \bar{v}, \bar{f}_0\bar{e}_1\bar{v} \rangle, \qquad W^{0,2} = \langle \bar{v}, \bar{f}_1\bar{v} \rangle.$$

Theorem 5.2.1. *For $M = -1,0,1$, $N = -1,0,1,2,\dots$, $M+N \geqslant 1$, the vectors*

$$\{\tau^{(M,N)}(\mathbf{a},\mathbf{b}) \mid (\mathbf{a},\mathbf{b}) \in \mathcal{C}^{M,N}\}$$

form a basis in $W^{M,N}$, and the dimension of $W^{M,N}$ is 2^{M+N-1}.

Proof. The proof is by induction on N. First, by inspection we check that $W^{-1,1} = W^{1,-1} = 0$, and $W^{0,0} = v$, is consistent with the statement of the Theorem.

Next, assume that

$$\{\tau^{(-1,N_0)}(\mathbf{a},\mathbf{b}) \mid (\mathbf{a},\mathbf{b}) \in \mathcal{C}^{-1,N_0}\} \text{ is a basis of } W^{-1,N_0},$$
$$\{\tau^{(0,N_0-1)}(\mathbf{a},\mathbf{b}) \mid (\mathbf{a},\mathbf{b}) \in \mathcal{C}^{0,N_0-1}\} \text{ is a basis of } W^{0,N_0-1},$$
$$\{\tau^{(1,N_0-2)}(\mathbf{a},\mathbf{b}) \mid (\mathbf{a},\mathbf{b}) \in \mathcal{C}^{1,N_0-2}\} \text{ is a basis of } W^{1,N_0-2}.$$

Let us prove that $\{\tau^{(-1,N_0+1)}(\mathbf{a},\mathbf{b}) \mid (\mathbf{a},\mathbf{b}) \in \mathcal{C}^{-1,N_0+1}\}$ is a basis of W^{-1,N_0+1}. We use the sequences (40) with $M = -1$, $N = N_0 + 1$.

We show the following:

(i) there exists an injection $\tilde{\varphi} : \mathcal{C}^{0,N_0-1} \to \mathcal{C}^{-1,N_0+1}$ satisfying

$$\varphi(\tau^{(0,N_0-1)}(\mathbf{a},\mathbf{b})) = \tau^{(-1,N_0+1)}(\tilde{\varphi}(\mathbf{a},\mathbf{b}));$$

(ii) there exists a bijection $\tilde{T} : \mathcal{C}^{-1,N_0+1}/\operatorname{Im}\tilde{\varphi} \to \mathcal{C}^{-1,N_0}$ satisfying

$$T(\tau^{(-1,N_0+1)}(\mathbf{a},\mathbf{b})) = \tau^{(-1,N_0)}(\tilde{T}(\mathbf{a},\mathbf{b})).$$

In fact, the map (i) is given by

$$\tilde{\varphi}(a_{N_0-2}\dots,a_0;b_{N_0-2}\dots,b_1)=(a_{N_0-2}\dots,a_0,1;b_{N_0-2}\dots,b_1,0).$$

Then we have,

$$\mathcal{C}^{-1,N_0+1}/\mathrm{Im}=\{(\mathbf{a},\mathbf{b})\in\mathcal{C}^{-1,N_0+1};a_0=b_1=0\}.$$

The map (ii) is given by

$$\tilde{T}(a_{N_0-1},\dots,a_1,0;b_{N_0-1},\dots,b_2,0)=(a_{N_0-1},\dots,a_1;b_{N_0-1},\dots,b_2).$$

Similarly, we prove that $\{\tau^{(0,N_0)}(\mathbf{a},\mathbf{b})\mid(\mathbf{a},\mathbf{b})\in\mathcal{C}^{0,N_0}\}$ is a basis of W^{0,N_0} using the sequences (40) with $M=0,N=N_0$. Finally, we prove that $\{\tau^{(1,N_0-1)}(\mathbf{a},\mathbf{b})\mid(\mathbf{a},\mathbf{b})\in\mathcal{C}^{1,N_0-1}\}$ is a basis of W^{1,N_0-1} using the sequences (41) with $M=1,N=N_0-1$. □

Corollary 5.2.2. *The vectors*

$$\{f_{N-1}^{a_{N-1}}h_{N-1}^{b_{N-1}}\dots f_2^{a_2}h_2^{b_2}f_1^{a_1}h_1^{b_1}v\mid(\mathbf{a},\mathbf{b})\in\mathcal{C}^{0,N}\}$$

form a basis in $L_{1,0}^{(N)}(\mathfrak{n})$.

Proof. The proof is similar to Theorem 4.3.2. By Theorem 5.2.1, the vectors described by Corollary 5.2.2 are linearly independent in the scaling limit. By general arguments, these vectors are independent before the scaling limit. By Theorem 5.2.1, there are 2^{N-1} of them. By Theorem 4.3.3 the dimension of $L_{1,0}^{(N)}(\mathfrak{n})$ is also 2^{N-1}, so Corollary 5.2.2 follows. □

5.3. **The case** $l=1$. In this section we describe ef- and fh- bases for $L_{1,1}^{(N)}(\mathfrak{n})$, and compute the corresponding character.

As shown in [FM], there exists an isomorphism of linear spaces

$$D^{1/2}:L_{1,0}\to L_{1,1},$$

defined by

$$D^{1/2}(\sigma(f_{i_1}\dots f_{i_a}h_{j_1}\dots h_{j_b}e_{s_1}\dots e_{s_c}v_{1,0}))=$$
$$\sigma(f_{i_1+1}\dots f_{i_a+1}h_{j_1}\dots h_{j_b}e_{s_1-1}\dots e_{s_c-1})f_0v_{1,1}.$$

Let us define the space of coinvariants

$$L_{k,l}^{M,N}(\mathfrak{n})=L_{k,l}/sl_2^{(M,N)}(\mathfrak{n})L_{k,l},$$

where we used the notation $sl_2^{(M,N)}(\mathfrak{n})$ for the subalgebra of $sl_2\otimes\mathbb{C}[t]$ generated by e_i $(i\geqslant M)$ and f_i $(i\geqslant N)$. Note that $L_{k,l}^{(N)}(\mathfrak{n})=L_{k,l}^{(0,N)}(\mathfrak{n})$.

The map $D^{1/2}$ maps $L_{1,0}^{1,N-1}(\mathfrak{n})$ to $L_{1,1}^{(N)}(\mathfrak{n})$.

Theorem 5.3.1. *The vectors*

$$\{f_{N-1}^{a_{N-1}}h_{N-1}^{b_{N-1}}\dots f_2^{a_2}h_2^{b_2}f_1^{a_1}h_1^{b_1}f_0^{a_0}v\mid(\mathbf{a},\mathbf{b})\in\mathcal{C}^{1,N}\}$$

form a basis in $L_{1,1}^{(N)}(\mathfrak{n})$.

Proof. The fh-basis of $W^{1,N-1}$ is described by Theorem 5.2.1. By the argument similar to the proof of Corollary 5.2.2, the same vectors form a basis in $L_{1,0}^{1,N-1}(\mathfrak{n})$. Theorem 5.3.1 is proved by application of the map $D^{1/2}$ to this basis. □

Similarly, applying the map $D^{1/2}$ to the basis of Theorem 4.3.2, we obtain

Theorem 5.3.2. *The vectors*

$$\{e_{2m-2-n+\alpha_1}\dots e_{2-n+\alpha_{m-1}}e_{-n+\alpha_m}f_{2n+\beta_1}\dots f_{4+\beta_{n-1}}f_{2+\beta_n}f_0v_1\},$$

$$n+2-2m>\alpha_1\geqslant\alpha_2\geqslant\dots\geqslant\alpha_m\geqslant 0,$$
$$N+m-2n>\beta_1\geqslant\beta_2\geqslant\dots\geqslant\beta_n\geqslant 0,$$

form a basis in $L^{(N)}_{1,1}(\mathfrak{n})$.

Now, we compute the character of $L^{(N)}_{1,1}(\mathfrak{n})$. Using the map $D^{1/2}$ once again, we obtain

Lemma 5.3.3.

$$\mathrm{ch}_{q,z}L^{(N)}_{1,1}(\mathfrak{n})=\mathrm{ch}_{q,z_1,z_2}W^{1,N-1}|_{z_1=q^{-1}z^2,z_2=qz^{-2}}. \tag{46}$$

Theorem 5.3.4.

$$\mathrm{ch}_{q,z}L^{(N)}_{1,1}(\mathfrak{n})=\sum_{s\geqslant 0}q^{s(s-1)}\begin{bmatrix}N\\2s-1\end{bmatrix}z^{-2s-1}.$$

Remark: Taking the limit $N\to\infty$ of the characters $\mathrm{ch}_{q,z}L^{(N)}_{1,l}(\mathfrak{n})$ $(l=0,1)$, given by Theorems 4.3.3 and 5.3.4, we obtain

$$\lim_{N\to\infty}\mathrm{ch}_{q,z}L^{(N)}_{1,l}(\mathfrak{n})=\sum_s q^{s(s-l)}\frac{1}{(q)_{2s-l}}z^{-2s-l},$$

where $(q)_m=(1-q)(1-q^2)\dots(1-q^m)$.

At $z=1$ the above characters coincide with the characters of Virasoro modules in the minimal CFT $\mathcal{M}(3,5)$. Namely, they are characters of the vacuum representation and of the representation with highest weight $-1/20$, cf. [KKMM].

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