Minimal Models of Integrable Lattice Theory and Truncated Functional Equations

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Abstract

We consider the integrable XXZ model with the special open boundary conditions. We perform Quantum Group reduction of this model in roots of unity and use it for the definition Minimal Models of Interable lattice theory. It is shown that after this Quantum Group reduction Sklyanin's transfer-matrices satisfy the closed system of the truncated functional relations. We solve these equations for the simplest case.

I. Introduction

One of the most interesting open problem in Conformal Field Theory is the connection between its Integrable Structure [1] and the Fermi–Bose correspondence. This correspondence manifests itself in the generalizations of Roger–Ramanujian Identities for Characters of Minimal Models of CFT [3] discovered by Kedem, Klassen, McCoy and Melzer [2]. The

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simplest example of such identity is the following equation for a character of $M(3,4)$ minimal model(Ising model):

$$\frac{1}{\prod_{n=1}^{\infty}(1-t^n)}\sum_{k=-\infty}^{\infty}[t^{k(12k+1)}-t^{(4k+1)(3k+1)}]=\sum_{\substack{m-\text{even}\\ m>0}}^{\infty}\frac{t^{m^2/2}}{(1-t)\dots(1-t^m)} \tag{1}$$

The l. h. s. of the identity has a natural explanation as Feigin–Fuks–Felder representation of the space of states of the model in terms of free bosonic field. The r. h. s. can be interpreted [2] as a partition function of right moving (chiral) fermions of one kind.

The similar identities were found in [2] for generic Minimal Model $M(p,p+1)$ with $p\geq 3$. In this case the r. h. s. can be considered as partition functions for $p-2$ species of right moving fermions. It was shown by R.Kedem and B.McCoy the fermionic states arise as low lying excitations in Bethe Ansatz approach for the 3-Potts model which corresponds to $M(5,6)$ Minimal Model. It would be interesting to generalize these results for general $M(p,p+1)$ case.

In [1] the continuous field theory version of the commuting transfer-matrices of Integrable lattice theory have been constructed and the conjecture that eigenvectors of these transfer-matrices form the fermionic basis of Virasoro characters has been claimed.

So the investigation this connection proceeds be interesting both in the continuous field theory and in the Integrable lattice theory and is one of the main motivations of this work.

The plan of the paper is the following. In part II we consider the integrable XXZ model with special open boundary conditions which was investigated firstly by Alcaraz *et al.* [4]. According to Pasqier and Saleur [5] we use the Quantum Group reduction of this model for the definition Minimal Models of Interable lattice theory ($LM(p,p+1)$) which coincide with Minimal Models of CFT $M(p,p+1)$ in the termodinamic limit. In part III we show that after this Quantum Group reduction Sklyanin's transfer-matrix $T(u)$ satisfies a closed system of the Truncated Functional relations. In part IV we solve these equations for the case $LM(3,4)$.

II. Minimal Models of Integrable Lattice Theory

Let us consider one dimensional XXZ chain with free boundary conditions [4]

$$\begin{aligned}
H_{xxz} &= \sum_{n=1}^{N-1}[\sigma_n^+\sigma_{n+1}^-+\sigma_n^-\sigma_{n+1}^+ +\frac{q+q^{-1}}{4}\sigma_n^z\sigma_{n+1}^z \\
&\qquad +\frac{q-q^{-1}}{4}(\sigma_n^z-\sigma_{n+1}^z)] \\
\sigma_n^{\pm} &= 1\otimes\ldots\otimes\sigma^{\pm}\otimes\ldots\otimes 1 \\
\sigma_n^{z} &= 1\otimes\ldots\otimes\sigma^{z}\otimes\ldots\otimes 1 \\
\sigma^+ &= \begin{pmatrix}0&1\\0&0\end{pmatrix},\quad \sigma^-=\begin{pmatrix}0&0\\1&0\end{pmatrix},\quad \sigma^z=\begin{pmatrix}1&0\\0&-1\end{pmatrix}
\end{aligned} \tag{2}$$

It was observed in [4, 6] that the eigenvalues of $2L$-site XXZ Hamiltonian (2) with the coupling $\eta=\pi/4$ and $\eta=\pi/6$ (we denote $q\equiv e^{\imath\eta}$) are exactly coincide with some of the eigenenergies of L-site self dual quantum Ising and 3-Potts models with free ends (see also [7]).

The remarkable properties of the model are connected with its $U_q(sl(2))$ symmetry found by Pasqier and Saleur [5]. Namely Hamiltonian H_{xxz} commutes with the generators X, Y, H of this quantum algebra defined as follows

$$\begin{aligned} X &= \sum_{n=1}^{N} q^{\frac{1}{2}(\sigma_1^z+\ldots+\sigma_{n-1}^z)}\sigma_n^+ q^{-\frac{1}{2}(\sigma_{n+1}^z+\ldots+\sigma_N^z)} \\ X &\to Y, \quad \sigma^+ \to \sigma^- \\ H &= \sum_{n=1}^{N} \frac{\sigma_n^z}{2}; \end{aligned}$$

and satisfy the following relations

$$[H,X] = X, \quad [H,Y] = -Y, \quad [X,Y] = \frac{q^{2H} - q^{-2H}}{q - q^{-1}} \tag{3}$$

Due to the Quantum Group symmetry the spectrum of the Hamiltonian can be classified according to the Represenation Theory of the algebra. For generic q its representations are equivalent to the ones of the ordinary $U(sl(2))$ algebra and the configuration space $(\mathbf{C}^2)^N$ of the spin chain can be splited into a direct sum of irreducible highest weight representations ρ_j (j is the highest weight) which are one-to-one correspondence to the ordinary $sl(2)$ representations. For example in $N=4$ case $(\mathbf{C}^2)^4$ can be decomposed as $\rho_2 + 3\rho_1 + 2\rho_0$.

We will concentrate on the case $q^{p+1} = -1$ [9, 5]. In this case the generators X and Y are nilpotent on the space of states of the model

$$X^{p+1} = 0, \quad Y^{p+1} = 0 \tag{4}$$

As the consequence we obtain the very different picture of decomposition the configuration space $(\mathbf{C}^2)^N$.

For example if $q^4 = -1$ and we try to decompose $(\mathbf{C}^2)^4$ we get the following [5]. The $(\mathbf{C}^2)^N$ decomposes now into sum of one "bad" 8-dimensional representation (ρ_2, ρ_1) of type I and four another "good" representations $2\rho_2 + 2\rho_1$ of type II [5]. The type II representations are isomorphic to the ordinary $U(sl(2))$ ones. What about the type I representation (ρ_2, ρ_1) it can be considered as a result of gluing of two representations ρ_2 and ρ_1. This (ρ_2, ρ_1) representation is indecomposable, but is not irreducible for it contains 3-dimensional invariant subspace.

In general $q^{p+1} = -1$ case [9, 5] the configuration space splits into sum of "bad" type I representations with highest weights $S_z \geq p/2$ and "good" type II repr-s with highest weights $S_z < p/2$ which are simultaneously not subspaces of some "bad" ones. The highest weights vectors v_j of the good repr-s can be characterized [5] by the following condition

$$v_j \in V_p \equiv \mathrm{Ker}\, X / \mathrm{Im}\, X^p \tag{5}$$

Because of $U_q(sl(2))$ invariance of H_{xxz} we can restrict its action on the space V_p. We will call the result of this Quantum Group reduction as the Minimal Model of Integrable Lattice Theory ($LM(p,p+1)$) because its termodinamic limit of this theory is $M(p,p+1)$, the ordinary Minimal Model of CFT with the Virasoro central charge $c = 1 - \frac{6}{p(p+1)}$ [4, 5, 8].

III. The Quantum Group reduction and the truncation of the fusion functional relations

The XXZ model(2) was solved by Alcaraz et al [4] using the coordinate Bethe ansatz method. Sklyanin constructed the family of transfer-matrices $T(u)$ [10, 11, 13] commuting between themselves and with H_{xxz}. It was shown in [12] that the Sklyanin's transfer-matrix $T(u)$ commutes with the Quantum Group $U_q(sl(2))$. So the action of $T(u)$ as well as H_{xxz} can be restricted on the space V_p if $q^{p+1} = -1$. To perform this Quantum Group reduction for $T(u)$ we will use Baxter's T-Q equation[8]

$$t(u)Q(u) = \phi(u+\eta/2)Q(u-\eta) + \phi(u-\eta/2)Q(u+\eta) \tag{6}$$

where $\phi(u) = \sin 2u \sin^{2N} u$ and $t(u) = \sin 2u\, T(u)$ and $Q(u)$ are eigenvalues of Baxter's auxilary matrix $\hat{Q}(u)$ commuting with $\hat{T}(u)$.

The eigenvalue $Q(u)$ for the eigenvector with $M = N/2 - S_z$ reversed spins has the following form

$$Q(u) = \prod_{m=1}^{M} \sin(u-u_m)\sin(u+u_m) \tag{7}$$

The equation(6) is equivalent to Bethe ansatz equations[13]

$$\left[\frac{\sin(u_k+\eta/2)}{\sin(u_k-\eta/2)}\right]^{2N} = \prod_{m\neq k}^{M} \frac{\sin(u_k-u_m+\eta)\sin(u_k+u_m+\eta)}{\sin(u_k-u_m-\eta)\sin(u_k+u_m-\eta)} \tag{8}$$

for the model (2) provided that $T(u)$ has no poles.

The Baxter's equation(6) can be thought as a discrete version of a second order differential equation[1]. So we can look for its second (linearly independent) solution $P(u)$ with the same eigenvalue $t(u)$ as in (6)

$$t(u)P(u) = \phi(u+\eta/2)P(u-\eta) + \phi(u-\eta/2)P(u+\eta) \tag{9}$$

It follows from (6) and (9) that

$$\frac{Q(u+\eta)P(u) - P(u+\eta)Q(u)}{\phi(u+\eta/2)} = \frac{Q(u)P(u-\eta) - P(u)Q(u-\eta)}{\phi(u-\eta/2)} \tag{10}$$

If η/π is irrational it means that both parts of (10) are equal const and we can choose it be 1. It means just a normalization of $P(u)$. Thus we obtain the "quantum Wronskian" condition[1]

$$Q(u-\eta/2)P(u+\eta/2) - P(u-\eta/2)Q(u+\eta/2) = \phi(u) \tag{11}$$

If η is a rational part of π the expresions in (10) could be equal a periodic function $f(u)$ such that $f(u+\eta) = f(u)$, but we assume that in this case $f(u)$ is also equal to 1.

Inserting (11) to (6) or (9) we obtain

$$t(u) = Q(u-\eta)P(u+\eta) - P(u-\eta)Q(u+\eta) \tag{12}$$

Let us divide (11) by $Q(u+\eta/2)Q(u-\eta/2)$, we get

$$\frac{P(u+\eta/2)}{Q(u+\eta/2)} - \frac{P(u-\eta/2)}{Q(u-\eta/2)} = \frac{\phi(u)}{Q(u+\eta/2)Q(u-\eta/2)} \tag{13}$$

The r. h. s. of (13) a fraction of two trigonometric polynomials. So there exist the single way to express it as the following sum

$$\frac{\phi(u)}{Q(u+\eta/2)Q(u-\eta/2)} = R(u) + \frac{A(u+\eta/2)}{Q(u+\eta/2)} - \frac{B(u-\eta/2)}{Q(u-\eta/2)} \tag{14}$$

where $R(u+\pi) = R(u)$ is a trigonometric polynomial whose degree $degR(u) = 2N+2-4M$ and $A(u+\pi) = A(u)$ and $B(u+\pi) = B(u)$ are some trigonometric polynomials whose degree are less than $degQ(u) = 2M$. Then (12) can be rewritten as follows

$$\begin{aligned} \frac{t(u)}{Q(u+\eta)Q(u-\eta)} &= R(u+\eta/2) + R(u-\eta/2) + \\ &+ \frac{A(u+\eta)}{Q(u+\eta)} - \frac{B(u)}{Q(u)} - \frac{A(u)}{Q(u)} - \frac{B(u-\eta)}{Q(u-\eta)} \end{aligned} \tag{15}$$

The term $\frac{A(u)-B(u)}{Q(u)}$ in the r. h. s. must vanish for it has extra poles which are absent in the l. h. s. of (15). So we obtain that $A(u) = B(u)$ and hence

$$\frac{\phi(u)}{Q(u+\eta/2)Q(u-\eta/2)} = R(u) + \frac{A(u+\eta/2)}{Q(u+\eta/2)} - \frac{A(u-\eta/2)}{Q(u-\eta/2)} \tag{16}$$

Provided that $F(u)$ is such function that

$$R(u) = F(u+\eta/2) - F(u-\eta/2) \tag{17}$$

we can rewrite (13)

$$\begin{aligned} &\frac{P(u+\eta/2)}{Q(u+\eta/2)} - \frac{P(u-\eta/2)}{Q(u-\eta)} = \\ &= F(u+\eta/2) + \frac{A(u+\eta/2)}{Q(u+\eta/2)} - F(u-\eta/2) - \frac{A(u-\eta/2)}{Q(u-\eta/2)} \end{aligned} \tag{18}$$

It means that we can choose the second solution of Baxter's equation (9) in the following form [15]

$$P(u) = F(u)Q(u) + A(u) \tag{19}$$

The new constructed solution of Baxter's equation is not a periodic function in the general case. It is a periodic function if and only if there exists a periodic solution of (17) $F(u)$.

We delay the discussion of the question when such solution does exist and get now the consequences of its existence provided it takes place.

Let us define the following function

$$t_k(u) = Q\left(u - \tfrac{(k+1)\eta}{2}\right) P\left(u + \tfrac{(k+1)\eta}{2}\right) - P\left(u - \tfrac{(k+1)\eta}{2}\right) Q\left(u + \tfrac{(k+1)\eta}{2}\right) \tag{20}$$

where k is any nonnegative integer.

Comparing with (11) and (12) we see that $t_0(u) = \phi(u)$ and $t_1(u) = t(u)$. It is easily to obtain [15] from (20) the following functional equations for $t_k(u)$

$$\begin{aligned} t_k\left(u+\tfrac{\eta}{2}\right)t_k\left(u-\tfrac{\eta}{2}\right) - t_{k+1}(u)t_{k-1}(u) = \phi\left(u+\tfrac{(k+1)\eta}{2}\right)\phi\left(u-\tfrac{(k+1)\eta}{2}\right) \\ t_k(u)t_1\left(u-\tfrac{(k+1)\eta}{2}\right) = t_{k+1}\left(u-\tfrac{\eta}{2}\right)\phi\left(u-\tfrac{k\eta}{2}\right) + t_{k-1}\left(u+\tfrac{\eta}{2}\right)\phi\left(u-\tfrac{(k+2)\eta}{2}\right) \end{aligned} \tag{21}$$

These functional relations [16] coincide with the relations for eigenvalues of the fused transfer-matrices received by Zhou in [8].

Let us return now to the question about existence of the periodic solution of (17). Recalling that $\phi(-u) = -\phi(u)$ and the definition (14) of $R(u)$ we conclude that $R(u)$ also is an odd trygonometric polynomial whose degree $degR(u) = 2N + 2 - 4M \geq 2$ provided the number of the reversed spins M is not more than $N/2$. So such polynomial can be written as follows

$$R(u) = \sum_{m=1}^{2N+2-4M} R_m \sin mu \tag{22}$$

where R_m are some coefficients which are defined from (14).

If η/π is irrational the periodic solution of (17) does exist and has the following form

$$F(u) = -\frac{1}{2}\sum_{m=1}^{2N+2-4M} \frac{R_m}{\sin\frac{m\eta}{2}}\cos mu \tag{23}$$

But if η/π is rational number the situation is more sophisticated. Namely we will concentrate on the case $\eta = \frac{\pi}{p+1}$ corresponding to $LM(p, p+1)$ model. In this case we get from (23)

$$F(u) = -\frac{1}{2}\sum_{m=1}^{2N+2-4M} \frac{R_m}{\sin\frac{m\pi}{2(p+1)}}\cos mu \tag{24}$$

So if we try to find the solution in sector when $2N + 2 - 4M \geq 2(p+1)$ or equivalently $S_z \geq p/2$ we see from (24) that the periodic solution does not exist. The reason is just a resonance.

Thus we conclude that a periodic $F(u)$ and the second periodic solution of Baxter's equation exist if we apply it to subspaces of the configuration space with $S_z < p/2$! We see that this inequality coinsides with the condition for the Quantum Group reduction from the part II . So we arrive to the main statement of this section:

The second periodic solution of Baxter equation for $\eta = \pi/(p+1)$ case exists if and only if the configuration space of the model is undergone Quatum Group reduction.

And for this case we have the additional relations for eigenvalues of the transfer-matrices $t_k(u)$. The equation (20) takes now the following form

$$t_k(u) = Q\left(u-\tfrac{(k+1)\pi}{2(p+1)}\right)P\left(u+\tfrac{(k+1)\pi}{2(p+1)}\right) - Q\left(u+\tfrac{(k+1)\pi}{2(p+1)}\right)P\left(u-\tfrac{(k+1)\pi}{2(p+1)}\right) \tag{25}$$

So we see that

$$t_p(u) = 0 \tag{26}$$

and

$$t_{p-1-k}(u) = -t_k(u + \pi/2) \tag{27}$$

So we obtain the second main statement of this section:
Quantum Group reduction of the model is equivalent to the truncation of functional relations (21). [1]

IV. The solution of the truncated functional equatios. Case $\eta = \pi/4$.

In the case $\eta = \pi/4$ (or $q^4 = -1$) the truncated fuctional equations take the following form

$$t(u + \pi/8)t(u - \pi/8) = \phi(u + \pi/4)\phi(u - \pi/4) - \phi(u)\phi(u + \pi/2) \tag{28}$$

After substitution $\phi(u) = \sin 2u \sin^{2N} u$ to (28) we obtain

$$t(u + \pi/8)t(u - \pi/8) = 2^{2N}[\sin^{2N+2} 2u - \cos^{2N+2} 2u] \tag{29}$$

For $T(u) = \frac{t(u)}{\sin 2u}$ we can rewrite the last equation as

$$2^{2N-1}T(u + \pi/8)T(u - \pi/8) = \prod_{k=1}^{N/2} |\varphi_k(u + \pi/8)|^2 |\varphi_k(u - \pi/8)|^2 \tag{30}$$

where we suggest N be even and

$$\varphi_k(u) = \sin(2u + \pi/4) - \omega^k \sin(2u - \pi/4) \tag{31}$$

and $\omega = \exp \frac{\imath math \pi}{N+1}$.

Using that $\varphi_k(u + \pi/8) = -\omega^k \varphi_k(u - \pi/8)$ we find $2^{N/2}$ real solutions of (30)

$$T(u) = T_{n_1 \ldots n_{N/2}} \equiv 2^{\frac{1-2N}{2}} \prod_{k=1}^{N/2} |\varphi_k\left(u + \frac{\pi n_k}{4}\right)|^2 \tag{32}$$

where $n_1 \ldots n_{N/2} = 0, 1$.

Another form the last expression is

$$T_{n_1 \ldots n_{N/2}} = 2^{\frac{1-2N}{2}} \prod_{k=1}^{N/2} \left[1 + (-1)^{n_k} \cos \frac{\pi k}{N+1} \cos 4u\right] \tag{33}$$

We recognize in integers $\{n_k\}$ the occupation numbers of the fermions of one dimensional Ising model. It is not suprising because $LM(3,4)$ exactly coincides with Ising model as was mentioned above [4, 6, 7].

[1] This statement was also checked numerically by J. Suzuki [17].

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