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Extended affine root system IV

(Simply-laced elliptic Lie algebras)

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Abstract. Let (R, G) be a pair consisting of an elliptic root system R with a marking G. Assume that the attached elliptic Dynkin diagram $\Gamma(R, G)$ is simply-laced (see Sect. 2). We associate three Lie algebras, explained in 1), 2) and 3) below, to the elliptic root system, and show that all three are isomorphic. The isomorphism class is called the elliptic algebra.

1) The first one is the subalgebra $\widetilde{\mathfrak{g}}(R)$ generated by the vacuum e^{α} for $\alpha \in R$ of the quotient Lie algebra $V_{Q(R)}/DV_{Q(R)}$ of the lattice vertex algebra (studied by Borcherds) attached to the elliptic root lattice $Q(R)$. This algebra is isomorphic to the 2-toroidal algebra and to the intersection matrix algebra proposed by Slodowy.

2) The second algebra $\widetilde{\mathfrak{e}}(\Gamma(R, G))$ is presented by Chevalley generators and generalized Serre relations attached to the elliptic Dynkin diagram $\Gamma(R, G)$.

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Since the Cartan matrix for the elliptic diagram has some positive off diagonal entries, the algebra is defined not only by Kac-Moody type relations but some others.

3) The third algebra $\widetilde{\mathfrak{h}}^{\mathbb{Z}}_{\mathrm{af}} * \mathfrak{g}_{\mathrm{af}}$ is defined as an amalgamation of a Heisenberg algebra and an affine Kac-Moody algebra, where the amalgamation relations between the two algebras are explicitly given. This algebra admits a sort of triangular decomposition in a generalized sense.

The first algebra $\widetilde{\mathfrak{g}}(R)$ does not depend on a choice of the marking G whereas the second $\widetilde{\mathfrak{e}}(\Gamma(R,G))$ and the third $\widetilde{\mathfrak{h}}^{\mathbb{Z}}_{\mathrm{af}} * \mathfrak{g}_{\mathrm{af}}$ do. This means the isomorphism depend on the choice of the marking i.e. on a choice of an element of $\mathrm{PSL}(2,\mathbb{Z})$.

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1. Introduction

We give a brief overview of the history, the motivation and the results of the present article. Some readers in a hurry may choose to skip to (1.3).

(1.1) Since the introduction, in the early eighties [Sa3-I], of the concept of a generalized root system and, in particular, of extended affine and elliptic (=

2-extended affine) root systems, there have been several attempts to construct Lie algebras realizing a given root system as the set of its "real roots". The answers were not unique, since there seemed no a priori constraint on the size of the set of "imaginary roots" and of the center of the algebra. Let us recall some of these works.

The first attempt was due to P. Slodowy ([Sl3]), who looked at the tensor of a simple algebra with the algebra of Laurent series of two variables for simply-laced elliptic root systems (cf. Sect. 2). Wakimoto [W] has constructed some of the representations of these algebras with trivial center action (see also [ISW] for further development). The idea was extended by U. Pollmann [P] to non simply-laced elliptic root systems, where she used the "twisted construction" using affine diagram automorphisms and called the algebra *biaffine algebra*.

The next attempt was due to H. Yamada [Y1], who constructed the tensor of the affine Kac-Moody algebra with the algebra of Laurent series of one variable by the use of vertex operators [F], [FK], which are, in fact, certain infinite dimensional central extensions of the two variables Laurent series extension of simple algebras. He put a constraint on the "size" of the center of the algebra (related to the marking on the elliptic root system, cf. Sect. 2) on the algebra in order to get the action of the central extension of the elliptic Weyl group. Still, the occurance of an infinite dimensional center in the algebra was a puzzle at the time.

The universal central extension of the tensor of a Lie algebra with a commutative algebra was explained by Kassel [Kas] in terms of Kähler differentials of the algebra. Moody-Eswara-Yokonuma [MEY] introduced the name *toroidal algebra* for the universal central extension of the tensor of a simple algebra with the algebra of Laurent series of several variables. They gave vertex algebra representations of 2-toroidal algebras (cf. also [BB], [EM]).

In [Sl2], [Sl4], Slodowy proposed to study a generalized intersection matrix (g.i.m. for short) algebra modifying the Kac-Moody algebra by admitting positive off-diagonal entries in the Cartan matrix, and an intersection matrix (i.m. for short) algebra as the quotient of the g.i.m. algebra by the ideal generated by the root spaces whose roots have norms ≥ 3. Then Berman-Moody

[BM] showed by the method of finite root system grading (cf. [BGK], [AABGP]) that the i.m. algebra for a suitable choice of root basis of an extended affine root system becomes the toroidal algebra.

The general idea of the construction of a Lie algebra $\mathfrak{g}(R)$ for any generalized root system R comes from vertex algebras, as we now explain. Generalizing the vertex representation of affine Kac-Moody algebras [FK], [F], Borcherds defined the vertex operator action for any element of the total Fock space V_Q of level 1 representations of the Heisenberg algebra attached to any given lattice Q [Bo1, 2]. He has also axiomatized the structure as the vertex algebra V and has shown that V/DV carries a Lie algebra structure for the derivation D of V. Applying the construction to the root lattice $Q(R)$ for a generalized root system R, we consider the Lie algebra $V_{Q(R)}/DV_{Q(R)}$. Then, we define $\mathfrak{g}(R)$ as its Lie subalgebra generated by all vacuum vectors e^{α} attached to (real) roots $\alpha \in R$. In order to get finite dimensional root space decomposition of $\mathfrak{g}(R)$, we extend the Cartan algebra $\mathfrak{h}$ to a non-degenerate space $\widetilde{\mathfrak{h}}$, and the algebra generated by $\widetilde{\mathfrak{h}}$ and $\mathfrak{g}(R)$ will be denoted by $\widetilde{\mathfrak{g}}(R)$.

The correspondence $R \mapsto \widetilde{\mathfrak{g}}(R)$ works as follows. If R is a finite or affine root system, then $\widetilde{\mathfrak{g}}(R)$ is the corresponding finite or affine Kac-Moody algebra, respectively. If the root lattice is hyperbolic (Lorentzian), the algebra becomes a quotient of the Kac-Moody algebra. If R is an extended affine root system, then $\mathfrak{g}(R)$ becomes a toroidal algebra. In general, the algebra $\widetilde{\mathfrak{g}}(R)$ has a root space decomposition (containing $R \subset Q(R)$ as its real roots) such that any root α has norm $q_R(\alpha) \leq 1$. Thus, the algebra is a quotient of the i.m. algebra for a suitable choice of a root basis.

Let us call, in particular, the algebra $\widetilde{\mathfrak{g}}(R)$ for a (simply-laced) elliptic root system R a (simply-laced) *elliptic Lie algebra*. The justification for this renaming of the 2-toroidal algebras comes from the study of elliptic singularities (see (1.2) and its remark). The algebras considered by Yamada and Pollman are quotients of the elliptic algebras.

(1.2) We turn to the question of presentations of the algebra $\widetilde{\mathfrak{g}}(R)$. If the Witt index of the root lattice $Q(R)$ is less than or equal to 1, then the root system

is either finite, affine or hyperbolic, so that it admits Weyl chambers and hence has some particular (so called simple) root basis. Accordingly, the algebra $\widetilde{\mathfrak{g}}(R)$ admits some particular (so called Chevalley) generator system and is presented by Serre relations. It is therefore natural to seek presentations of the algebra $\widetilde{\mathfrak{g}}(R)$ for a root system R with Witt index ≥ 2. In fact, for the 2-toroidal (= elliptic) case, Moody-Eswara-Yokonuma [MEY] have already given an infinite presentation of the algebra.

One of the main goals of the present article is to give another, finite, presentation of the elliptic algebra by generalizing the Serre relations (see (4.1) Definition 2) in terms of the elliptic Dynkin diagram explained below.

The elliptic root system R has a two dimensional radical and any one dimensional subspace G of the radical is called a marking (cf. Sect. 2). The pair (R,G) consisting of an elliptic root system with a marking admits some particular root basis $\Gamma(R,G)$, called the *elliptic Dynkin diagram* (Sect. 2, [Sa3-I]), even though there is no longer a good analogue of the Weyl chambers. The intersection matrix attached to the elliptic diagram, which we call an *elliptic Cartan matrix*, contains some positive entries in its off diagonal part, so that it is not a generalized Cartan matrix in the sense of Kac-Moody algebras [K]. Hence, the "classical" Serre relations which describe Kac-Moody-algebras are not sufficient to describe the elliptic algebra. Thus it is necessary to seek some generalizations of the Serre relations attached to the elliptic diagram.

A new impetus for the problem of reducing relations to the elliptic diagram came from a description of the elliptic Weyl group. In [SaT3-III], the elliptic Weyl group was presented via a generalization of a Coxeter system. Namely, the elliptic Weyl group is generated by involutive elements attached at vertices of the diagram and is determined by a system of relations which consist partly of the classical Coxeter relations involving 2 vertices of the elliptic diagram, together with some new relations involving 3 and 4 vertices of the diagram. The quetion was raised whether one could find presentations of the elliptic Lie algebras, the elliptic Artin groups and the elliptic Hecke algebras whose their defining relations involve only the same 2, 3 and 4 vertices of the elliptic diagram.

In the present article, we answer this problem affirmatively (see Remark 2): the elliptic algebra $\widetilde{\mathfrak{e}}(\Gamma(R,G))$ is generated by a system of Chevalley basis (i.e. $\mathfrak{sl}_2$-triplets attached at vertices of the elliptic diagram $\Gamma(R,G)$) and is defined by a generalization of the Serre relations involving three or four vertices of the diagram (Sect. 4 Definition (4.1), (4.1) Theorem 1). In the course of the identification of the algebra $\widetilde{\mathfrak{e}}(\Gamma(R,G))$ with $\widetilde{\mathfrak{g}}(R)$, we are necessarily lead to consider an amalgamation $\widetilde{\mathfrak{h}}_{\mathrm{af}}^{\mathbb{Z}} * \mathfrak{g}_{\mathrm{af}}$ of a Heisenberg algebra $\widetilde{\mathfrak{h}}_{\mathrm{af}}^{\mathbb{Z}}$ and an affine Kac-Moody algebra $\mathfrak{g}_{\mathrm{af}}$ which is again isomorphic to the elliptic algebra. This gives the third description of the elliptic algebra. As a by-product of this third presentation, the elliptic algebra obtains a sort of triangular decomposition (5.2.2) (in [BB], Berman and Bilig use a similar triangular decomposition of toroidal algebras to study of their representations).

Remark 1. The 2-extended affine root systems [Sa3-I] were introduced in order to describe the (transcendental) lattices generated by vanishing cycles for *simply elliptic singularities* [Sa1], which is why we call them elliptic root systems [SaT3-III]. In fact, *the rank 2 radical of the root system corresponds to the lattice of an elliptic curve, and a rank 1 subspace of the radical, called a marking, corresponds to a choice of a primitive form* ([Sa2]). The motivation for introducing the elliptic algebra is to reconstruct the primitive form and the period mapping for the elliptic singularities in terms of the elliptic algebra (cf. the simple singularity case [Br], [Sl1, 2], [Ya2]). From the marked elliptic root system, one has already reconstructed the *flat structure on the invariants of elliptic Weyl groups* [Sa3-II], [Sat1, 2] and the *elliptic L-functions* [Sa3-V]. But we are still far from the goal.

Remark 2. The problems raised in [SaT3-III] on the description of the elliptic Artin groups and Hecke algebras were affirmatively solved by H. Yamada [Y3], where he rewrote in terms of elliptic diagrams the presentation of the fundamental group of the complement of the discriminant for simply elliptic singularities, given originally by van der Lek. The relations for an elliptic Artin group naturally "cover" the relations for an elliptic Weyl group. Still, the relationship between the presentations of the elliptic Weyl group and the Artin

group and the presentation of the elliptic algebra given in the present article is not yet clear.

(1.3) Let us give an overview of the contents of the present article.

Section 2 reviews the material from [Sa3-I] on generalized root systems and elliptic root systems R and their diagrams $\Gamma(R,G)$ which is necessary in the present article. Section 3 reviews Borcherds' description of a lattice vertex algebra, and then introduces the Lie algebra $\widetilde{\mathfrak{g}}(R)$ for any homogeneous generalized root system R. In (4.1), we give a finite and locally nilpotent presentation of an algebra $\widetilde{\mathfrak{e}}(\Gamma(R,G))$ attached to a simply-laced elliptic diagram $\Gamma(R,G)$. There is a natural surjective homomorphism : $\widetilde{\mathfrak{e}}(\Gamma(R,G)) \to \widetilde{\mathfrak{g}}(R)$ preserving the root space decomposition.

The elements of $\widetilde{\mathfrak{e}}(\Gamma(R,G))$ whose weights belong to the marking G span a Heisenberg subalgebra $\widetilde{\mathfrak{h}}^{\mathbb{Z}}_{\mathrm{af}}$. It is a slight surprise that the Heisenberg subalgebra and the affine Kac-Moody subalgebra $\mathfrak{g}_{\mathrm{af}}$ (attached to an affine subdiagram Γ_{af} of $\Gamma(R,G)$) in $\widetilde{\mathfrak{e}}(\Gamma(R,G))$ satisfy some quite simple relations (4.3.6). We consider, in Sect. 5, the abstract amalgamation $\widetilde{\mathfrak{h}}^{\mathbb{Z}}_{\mathrm{af}} * \mathfrak{g}_{\mathrm{af}}$ of the two algebras obeying those relations. The amalgamation is surjectively mapped onto $\widetilde{\mathfrak{e}}(\Gamma(R,G))$ preserving the root space decomposition. We determine the set of roots and the multiplicities of the amalgamation algebra in (5.2)–(5.5). It turns out that the set of roots of the amalgamation algebra and that of $\widetilde{\mathfrak{g}}(R)$ coincides and the multiplicities for the amalgamation algebra do not exceed those of the algebra $\widetilde{\mathfrak{g}}(R)$. This implies the isomorphism:

$$\widetilde{\mathfrak{g}}(R) \ \simeq \ \widetilde{\mathfrak{e}}(\Gamma(R,G)) \ \simeq \ \widetilde{\mathfrak{h}}^{\mathbb{Z}}_{\mathrm{af}} * \mathfrak{g}_{\mathrm{af}}.$$

Here we recall that the algebra $\widetilde{\mathfrak{g}}(R)$ does not depend on a choice of the marking G, but the other algebras $\widetilde{\mathfrak{e}}(\Gamma(R,G))$ and $\widetilde{\mathfrak{h}}^{\mathbb{Z}}_{\mathrm{af}} * \mathfrak{g}_{\mathrm{af}}$ do depend on the marking, i.e. on a choice of an element of $\mathrm{PSL}(2,\mathbb{Z})$. This should be understood by the following fact. Through the amalgamation, the elliptic algebra obtains a triangular decomposition:

$$\widetilde{\mathfrak{h}}^{\mathbb{Z}}_{\mathrm{af}} \ \oplus \ \mathfrak{n}^{+}_{\mathrm{ell}} \ \oplus \ \mathfrak{n}^{-}_{\mathrm{ell}}.$$

So, the ambiguity of the triangular decomposition of an elliptic algebra (more exactly, the ambiguity of the subalgebra $\widetilde{\mathfrak{h}}^{\mathbb{Z}}_{\mathrm{af}} \oplus \mathfrak{n}^{+}_{\mathrm{ell}}$) depends on an element of $\mathrm{PSL}(2,\mathbb{Z})$. Full study of this fact (i.e. $\mathrm{PSL}(2,\mathbb{Z})$ action on the elliptic flag variety) may require another work, and is beyond the scope of the present article.

For the convenience of the reader, in Appendix C, we list the reference numbers for the relations of the algebras $\widetilde{\mathfrak{g}}(R)$, $\widetilde{\mathfrak{e}}(\mathit{\Gamma}(R,G))$, $\mathfrak{g}(A_{\triangle})$, $\mathfrak{g}_{\mathrm{af}} = \mathfrak{e}(\mathit{\Gamma}_{\mathrm{af}})$, $\mathfrak{h}^{\mathbb{Z}'}_{\mathrm{af}}$, $\widetilde{\mathfrak{h}}^{\mathbb{Z}}_{\mathrm{af}}$ and $\widetilde{\mathfrak{h}}^{\mathbb{Z}}_{\mathrm{af}} * \mathfrak{g}_{\mathrm{af}}$ studied in the present article.

Notation. (1) For a sequence of elements $x_1, x_2, x_3, \cdots, x_n$ of a Lie algebra, put:

$$[x_1, x_2, x_3, \cdots, x_n] \ := \ [[\cdots[[x_1, x_2], x_3], \cdots, x_{n-1}], x_n],$$

and call it a *multi-bracket of length* n. The element x_i is called the *i-th entry* of the bracket. For any s with $1 < s \leq n$, by successive applications of the Jacobi identity, one gets an identity:

$$\begin{aligned}
[x_1, x_2, x_3, \cdots, x_n, y] \ = \ & [x_1, \cdots, x_{s-1}, y, x_s, x_{s+1}, \cdots, x_n] \\
& + [x_1, \cdots, x_{s-1}, [x_s, y], x_{s+1}, \cdots, x_n] \\
& + [x_1, \cdots, x_{s-1}, x_s, [x_{s+1}, y], \cdots, x_n] \\
& \qquad \cdots \\
& + [x_1, \cdots, x_{s-1}, x_s, x_{s+1}, \cdots, [x_n, y]]
\end{aligned}$$

We shall refer to this transformation as *"delivering y to the left"*.

(2) For a subset S of a root system R in F, we put

$$\pm S := S \cup (-S) = S \cup \{-s \mid s \in S\}.$$

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2. Generalized root systems and elliptic root systems

We recall the notions from [Sa3-I] on a generalized root system and an elliptic root system and their classification by elliptic diagrams together with some facts about them which are necessary in the present article.

(2.1) Let (F, q) be a pair consisting of an $\mathbb{R}$-vector space F and a quadratic form q on it with bounded rank (i.e. $max\{$rank of non-singular susbspace of $F\} < \infty$). The bilinear form $I : F \times F \to \mathbb{R}$ is attached to q by $I(x, y) := q(x + y) - q(x) - q(y)$ and $q(x) = I(x, x)/2$. Let $rad(q) := F^{\perp}$ be the *radical* of q. The signature $sig(q) = (\mu_+, \mu_0, \mu_-)$ is the triplet with $\mu_{\pm} := max\{$rank of positive (resp. negative) subspaces of $F\}$ and $\mu_0 := rank(rad(q))$. A vector $\alpha \in F$ is called *isotropic* if $q(\alpha) = 0$ and *non-isotropic* if $q(\alpha) \neq 0$. Take a non-isotropic vector α. Put $\alpha^{\vee} := \alpha/q(\alpha)$, then we have $\alpha^{\vee\vee} = \alpha$. Define the reflection w_α on F by $w_\alpha(u) := u - I(\alpha^{\vee}, u)\alpha$. We have $w_\alpha^2 = id_F$ and $I \circ w_\alpha = I$.

Definition. *A set R of non-isotropic vectors of F is called a generalized root system belonging to (F, q), if it satisfies:*

1) the additive subgroup $Q(R)$ in F generated by R is a full lattice of F (i.e. $Q(R) \otimes_{\mathbb{Z}} \mathbb{R} \simeq F$),
2) for all α and $\beta \in R$, one has $I(\alpha^{\vee}, \beta) \in \mathbb{Z}$,
3) for all $\alpha \in R$, the reflection w_α preserves the set R,
4) if $R = R_1 \cup R_2$ and $R_1 \perp R_2$ with respect to q, then either R_1 or R_2 is empty.

A root system R is called *reduced* if $\mathbb{Q}\alpha \cap R = \pm\alpha$ for any $\alpha \in R$. The subgroup $Q(R) \subset F$ is called the *root lattice* for the root system R. The group $W(R)$ generated by the reflections w_α for all $\alpha \in R$ is called the *Weyl group* of the root system R. Two root systems are *isomorphic* if there is an isomorphism between the ambient vector spaces which induces a bijection between the sets of roots. For any subspace H of $rad(q)$ which is defined over $\mathbb{Q}$, the image set of R in the quotient space F/H is a root system, called the *quotient root system*

modulo H, and is denoted by R/H. In particular, $R/rad(q)$ is called the *radical quotient* of R. (A subspace H is called *defined over* $\mathbb{Q}$ if $H \cap Q(R)$ is a full lattice of H; for example, $rad\,Q(R)$ is defined over $\mathbb{Q}$, i.e. $rad\,Q(R) := rad(q) \cap Q(R)$ is a free abelian group of rank μ_0.)

Definition. *We shall call a subset $\Pi \subset R$ a root basis of R, if*

$$i)\ Q(R) = \mathbb{Z}\Pi, \quad ii)\ W(R) = W(\Pi) \quad and \quad iii)\ R = W(R)\Pi, \tag{2.1.1}$$

where $\mathbb{Z}\Pi := \sum_{\alpha\in\Pi} \mathbb{Z}\alpha$ and $W(\Pi) := \langle w_\alpha \mid \alpha \in \Pi \rangle$.

Note that $R = W(\Pi)\Pi$ alone implies the conditions i), ii) and iii).

One can show that the cardinality of the set of norms of roots $\{q(\alpha) \mid \alpha \in R\}$ is finite and that the proportion $q(\alpha)/q(\beta)$ for any $\alpha, \beta \in R$ is a rational number so that the integer:

$$t(R) := lcm\{q(\alpha) \mid \alpha \in R\}/gcd\{q(\alpha) \mid \alpha \in R\}, \tag{2.1.2}$$

called the *total tier number* of the root system, is a well defined positive integer. So, up to a constant factor, the bilinear form I may be assumed to take rational values on $Q(R)$. In particular, by choosing a constant factor c such that $gcd\{c \cdot q(\alpha) \mid \alpha \in R\} = 1$, we define the normalized forms:

$$q_R := c \cdot q, \qquad I_R := c \cdot I \tag{2.1.3}$$

(the constant c will be referred to as $I_R : I$). Then $Q(R)$ becomes an even lattice with respect to the form q_R (and I_R). We shall always consider the vector space F to be equipped with the integral lattice structure $Q(R)$ and the rational structure $F_{\mathbb{Q}} := \mathbb{Q} \otimes_{\mathbb{Z}} Q(R)$. A root system R is called *homogeneous* if $t(R) = 1$, and, hence, $q_R(\alpha) = \pm 1$ for $\alpha \in R$.

Let R be a root system. The set $R^\vee := \{\alpha^\vee \mid \alpha \in R\}$, called the *dual* of R, is also a root system with respect to the same quadratic form q. It satisfies $q(\alpha^\vee) = 1/q(\alpha)$ for $\alpha \in R$, with $t(R^\vee) = t(R)$. The sets R and $R^\vee$ span the same vector space F, but the lattice $Q(R^\vee) := \mathbb{Z}R^\vee$ (equipped with the normalized quadratic form $q_{R^\vee} = (I_{R^\vee} : I) \cdot q$ satisfying $q_R(\alpha) \cdot q_{R^\vee}(\alpha^\vee) = t(R)$) may define a different $\mathbb{Z}$ structure on F.

(2.2) We call R a *k-extended affine root system of rank l*, if q is positive semi-definite with $rank(rad(q)) = k$ and $rank(Q(R)) = l + k$. Let us recall briefly the classification of k-extended affine root systems when k (=the Witt index of q) is less than or equal to 2.

One has the equivalence: $\#R < \infty \iff \#W(R) < \infty \iff q$ is definite $\iff k = 0$. This is the case studied in the classical literature. The root systems are classified into types A_l $(l \geq 1)$, B_l $(l \geq 2)$, C_l $(l \geq 3)$, D_l $(l \geq 4)$, E_l $(l = 6, 7, 8)$, F_4 and G_2.

If $k = 1$, then R turns out to be an affine root system in the sense of Macdonald [Mc] (cf. [K], [Mo]). These are classified by the types $P^{(t_1)}$, where P is the type of the finite root system $R_{\mathrm{f}} := R/rad(q)$ and t_1 is called the first tier number (an integer satisfying $t_1 | t(R)$).

If $k = 2$, then R is called an elliptic root system. In most cases these are classified by the types $P^{(t_1,t_2)}$, where P is the type of the finite root system $R_{\mathrm{f}} = R/rad(q)$ and t_1, t_2 are called the first and the second tier number (integers satisfying $t_1 | t(R)$, $t_2 | t(R)$).

In general, let R be a homogeneous k-extended affine root system. Then by a suitable choice of the basis $a_1, \cdots, a_k$ of the radical $rad(q)$, one has an expression of the set of roots

$$\begin{aligned} R &= R_{\mathrm{f}} \oplus \mathbb{Z}a_1 \oplus \cdots \oplus \mathbb{Z}a_k \\ &= \{\alpha + n_1 a_1 + \cdots + n_k a_k \mid \alpha \in R_{\mathrm{f}},\ n_i \in \mathbb{Z} \text{ for } i = 1, \cdots, k\} \end{aligned} \tag{2.2.1}$$

where R_{f} is a "splitting sub-root system of R" defined in a positive subspace F_f of F and is isomorphic to the radical quotient finite root system $R/rad(q)$ of R.

If $k \leq 1$, then the Weyl group $W(R)$ acts properly discontinuously on a domain in F^*. The fundamental domain of the action bounded by the reflection hyperplanes is called a *Weyl chamber*. Then the set $\Gamma(R)$ of those roots which are normal to the walls of a fixed Weyl chamber gives a *simple root basis* of R and defines the classical or affine Dynkin diagram according as R is a finite or an affine root system. If $k \geq 2$ then the Weyl group acts nowhere properly discontinuously on F or F^* so that there is no concept of a Weyl chamber.

Nevertheless, for the elliptic root system R (i.e. $k = 2$), by a use of a marking G (see (2.3)), we introduce some particular root basis $\Gamma(R, G)$ of R which leads to a definition of the elliptic Dynkin diagrams for (R, G). We recall some more details on this in the following (2.3)–(2.6).

(2.3) Let R be an elliptic root system of rank l. That is: R is a generalized root system whose quadratic form q is positive semi-definite with a 2-dimensional radical. We have $l = rank(F/rad(q)) = rank(F) - 2$. If R is elliptic, so is $R^\vee$.

Definition. *A marking of an elliptic root system R is a rank 1 subspace G of $rad(q)$ defined over $\mathbb{Q}$. The pair (R, G) is called a marked elliptic root system.*

The dual $R^\vee$ is also marked by the same space G. They define different integral structures

$$\mathbb{Z}a = G_{\mathbb{Z}} := G \cap Q(R) \quad \text{and} \quad \mathbb{Z}a^\vee = G^\vee_{\mathbb{Z}} := G \cap Q(R^\vee), \tag{2.3.1}$$

where a and $a^\vee$ are fixed, once and for all, integral generators of the marking G satisfying $a^\vee : a > 0$ (here $a^\vee$ is not $(a)^\vee$ for the operation $(\,)^\vee$ defined in (2.1)). The marking G induces a projection

$$\pi_G : \ Q(R) \to Q_{\mathrm{af}} := Q(R)/G_{\mathbb{Z}}. \tag{2.3.2}$$

The image $R_{\mathrm{af}} := R/G := \pi_G(R)$ is an affine root system belonging to the affine root lattice Q_{af}. We assume R_{af} to be *reduced*. Once and for all, for the rest of the present paper, we choose and fix a simple root basis $\Gamma(R_{\mathrm{af}})$ of the affine root system R_{af} and a set $\Gamma_{\mathrm{af}} \subset R$ such that the projection π_G induces a bijection $\pi_G|\Gamma_{\mathrm{af}} \xrightarrow{\sim} \Gamma(R_{\mathrm{af}})$. In particular, the intersection matrix $(I(\alpha^\vee, \beta))_{\alpha,\beta\in\Gamma_{\mathrm{af}}}$ equals the affine Cartan matrix $(I(\alpha^\vee, \beta))_{\alpha,\beta\in\Gamma(R_{\mathrm{af}})}$. The set Γ_{af} is unique up to an isomorphism of (R, G). We identify Γ_{af} with the affine Dynkin diagram $\Gamma(R_{\mathrm{af}})$, the lattice $\oplus_{\alpha\in\Gamma_{\mathrm{af}}}\mathbb{Z}\alpha$ with Q_{af}, and the set of roots $R \cap \oplus_{\alpha\in\Gamma_{\mathrm{af}}}\mathbb{Z}\alpha$ with R_{af}, respectively. The following three properties are well known: i) Γ_{af} forms a basis of Q_{af} such that the affine roots R_{af} are contained in $Q^+_{\mathrm{af}} \cup Q^-_{\mathrm{af}}$ where $Q^\pm_{\mathrm{af}} := \big(\pm\sum_{\alpha\in\Gamma_{\mathrm{af}}}\mathbb{Z}_{\geq 0}\alpha\big) \setminus \{0\}$, ii) there exist positive integers $\{n_\alpha \in \mathbb{Z}_{>0}\}_{\alpha\in\Gamma_{\mathrm{af}}}$ such that $\sum_{\alpha\in\Gamma_{\mathrm{af}}} n_\alpha\alpha$ is a generator of the radical

of $Q_{\rm af}$ (null roots of $Q_{\rm af}$). iii) There exists $\alpha_0 \in \Gamma_{\rm af}$ with $n_{\alpha_0} = 1$ such that $\Gamma_{\rm f} := \Gamma_{\rm af} \setminus \{\alpha_0\}$ is a Dynkin diagram of the finite root system $R_{\rm f} = R/rad(q)$ with the root lattice $Q_f := \oplus_{\alpha\in\Gamma_{\rm f}}\mathbb{Z}\alpha$. So, the root lattice $Q(R)$ and the radical $rad\,Q(R)$ split over $\mathbb{Z}$ as:

$$Q(R) \;=\; Q_{\rm af} \oplus \mathbb{Z}a = Q_{\rm f} \oplus \mathbb{Z}b \oplus \mathbb{Z}a, \tag{2.3.3}$$

$$rad\,Q(R) \;=\; \mathbb{Z}b \oplus \mathbb{Z}a, \tag{2.3.4}$$

where b is a lifting of the generator of the null roots in $Q_{\rm af}$.

$$b := \sum_{\alpha\in\Gamma_{\rm af}} n_\alpha \alpha. \tag{2.3.5}$$

Similarly, $\Gamma^\vee_{\rm af} := \{\alpha^\vee \mid \alpha \in \Gamma_{\rm af}\} \subset R^\vee$ is bijective to a simple root basis of the affine root system $R^\vee_{\rm af}$. So, we get a generator $b^\vee := \sum_{\alpha^\vee\in\Gamma^\vee_{\rm af}} n_{\alpha^\vee}\alpha^\vee$ of the radical of $Q(R^\vee_{\rm af})$.

(2.4) We introduce the first and the second tier numbers t_1, t_2 of a marked elliptic root system (R,G) and its dual $(R^\vee,G)$ as follows. These will describe some subtle relations between the two integral structures $Q(R)$ and $Q(R^\vee)$, and will be used to define the type of a marked elliptic root system.

$$\begin{array}{ll} t_1(R,G) := (b^\vee : b)\cdot(I_{R^\vee} : I), & t_1(R^\vee,G) := (b : b^\vee)\cdot(I_R : I), \\ t_2(R,G) := (a^\vee : a)\cdot(I_{R^\vee} : I), & t_2(R^\vee,G) := (a : a^\vee)\cdot(I_R : I). \end{array} \tag{2.4.1}$$

These are positive integers satisfying the relation

$$t(R) \;=\; t_1(R,G)\cdot t_1(R^\vee,G) \;=\; t_2(R,G)\cdot t_2(R^\vee,G). \tag{2.4.2}$$

The isomorphism class of (R,G), in most cases, is determined by the triplet (P,t_1,t_2) where P is the type of the finite root system $R/rad(q)$ and t_1 and t_2 are the first and the second tier numbers of (R,G), and so, the symbol $P^{(t_1,t_2)}$ is called the *type* of (R,G).

Let us define some further numerical invariants which lead to the diagram $\Gamma(R,G)$. For $\alpha \in R$, put

$$k(\alpha) := inf\{k \in \mathbf{Z_{>0}} \mid \alpha + k\cdot a \in R\} \;\text{ and }\; \alpha^* := \alpha + k(\alpha)\cdot a \tag{2.4.3}$$

(and define $k^\vee(\alpha^\vee)$ similarly for $\alpha^\vee \in R^\vee$). Then, one has the following proportionalities: for $\alpha \in \Gamma_{\rm af}$,

$$\frac{n_{\alpha^\vee}}{n_\alpha} = \frac{t_1(R,G)}{q_{R^\vee}(\alpha^\vee)}, \quad \frac{k^\vee(\alpha^\vee)}{k(\alpha)} = \frac{q_{R^\vee}(\alpha^\vee)}{t_2(R,G)}. \tag{2.4.4}$$

Definition. *The set of exponents (resp. dual exponents) of (R,G) is the union of $\{0\}$ and*

$$m_\alpha := \frac{q(\alpha)}{k(\alpha)} \cdot n_\alpha \quad (\textit{resp.} \quad m_{\alpha^\vee} := \frac{q(\alpha^\vee)}{k^\vee(\alpha^\vee)} \cdot n_{\alpha^\vee}) \tag{2.4.5}$$

for $\alpha \in \Gamma_{\rm af}$. Let $m_{\max} := max\{m_\alpha \mid \alpha \in \Gamma_{\rm af}\}$ be the largest exponent. Put

$$\Gamma_{\max} := \{\alpha \in \Gamma_{\rm af} \mid m_\alpha = m_{\max}\}, \quad \Gamma^*_{\max} := \{\alpha^* \mid \alpha \in \Gamma_{\max}\}. \tag{2.4.6}$$

Finally, we introduce the finite set of roots:

$$\Gamma(R,G) := \Gamma_{\rm af} \cup \Gamma^*_{\max}. \tag{2.4.7}$$

Fact 1. *The set $\Gamma(R,G)$ forms a root basis of the elliptic root system R.*

Note that the proportionalities in (2.4.4) imply the proportionalities: $m_{\alpha^\vee}/m_\alpha = t_1(R,G) \cdot t_2(R,G)/t(R)$ and therefore $\Gamma(R^\vee,G) = (\Gamma(R,G))^\vee$. The matrix $(a_{\alpha,\beta} := I(\alpha^\vee,\beta))_{\alpha,\beta\in\Gamma(R,G)}$ is called the *elliptic Cartan matrix* (which is not a generalized Cartan matrix in the sense of Kac-Moody theory [K] because of the positive off-diagonal entry $a_{\alpha,\alpha^*} = 2$ for $\alpha \in \Gamma_{\max}$).

(2.5) The *elliptic diagram* (which we shall identify with the root basis $\Gamma(R,G)$ because of the following theorem) is defined by the following rule:

i) vertices are in one-to-one correspondence with $\Gamma(R,G)$,
ii) the type of the bond between the vertices $\alpha,\ \beta \in \Gamma(R,G)$ is defined according to the value $a_{\alpha,\beta}/a_{\beta,\alpha}$ by the usual convention (e.g. [B Chap.VI, §4 n°4.2]), except for the new additional convention: a double dotted bond $\circ ===\circ$ if $a_{\alpha,\beta} = a_{\beta,\alpha} = 2$ (i.e. between vertices α and α^* for $\alpha \in \Gamma_{\max}$).

Fact 2 ([Sa3-I] Theorem (9.6)). *The elliptic diagram is uniquely determined by the isomorphism class of (R,G). Conversely, the elliptic diagram $\Gamma(R,G)$ determines uniquely the isomorphism class of the marked elliptic root system (R,G) together with a root basis which is identified with the vertices of $\Gamma(R,G)$.*

The theorem allows us to identify the elliptic diagram with the root basis $\Gamma(R,G)$. Let us review the reconstruction of the root system (R,G) from the diagram $\Gamma(R,G)$.

a) Consider the vector space $\hat{F} := \oplus_{\alpha\in\Gamma(R,G)}\mathbb{R}\alpha$ spanned by the vertices of the diagram.
b) A symmetric bilinear form $\hat{I} : \hat{F}\times\hat{F}\to\mathbb{R}$ satisfying the relations $\frac{2\hat{I}(\alpha,\beta)}{\hat{I}(\alpha,\alpha)} = a_{\alpha,\beta}$ and $\hat{I}(\alpha,\alpha) > 0$ for $\alpha,\beta\in\Gamma(R,G)$ exists uniquely up to a positive constant factor.
c) The reflection of $\hat{F}$ with respect to $\alpha\in\Gamma(R,G)$ is denoted by $\hat{w}_\alpha$. The pre-Weyl group $\hat{W}$ is the group generated by $\hat{w}_\alpha$ for $\alpha\in\Gamma$.
d) The pre-Coxeter element $\hat{c}\in\hat{W}$ is defined as the product of $\hat{w}_\alpha$, ordered so that $\hat{w}_{\alpha^*}$ comes next to $\hat{w}_\alpha$ for $\alpha\in\Gamma_{\max}$.

Then one has:

(i) The eigenvalues of $\hat{c}$ are given by 1 *and* $\exp(2\pi\sqrt{-1}m_\alpha/m_{\max})$ *for* $\alpha\in\Gamma_{\mathrm{af}}$.
(ii) The image of $1-\hat{c}^{m'_{\max}}$ *is contained in rad*$(\hat{I})$ *and is spanned by* $\frac{\alpha^*-\alpha}{k(\alpha)} - \frac{\beta^*-\beta}{k(\beta)}$ *for* $\alpha,\ \beta\in\Gamma_{\max}$, *where* $m'_{\max}$:= *the least common denominator of the* $m_\alpha/m_{\max}$ *for* $\alpha\in\Gamma_{\mathrm{af}}$.

Fact 3. *Put* $F := \hat{F}/(1-\hat{c}^{m'_{\max}})\hat{F}$, I :=*the form on* F *induced from* $\hat{I}$, G := *the subspace in* F *spanned by* $\alpha^*-\alpha$ *for* $\alpha\in\Gamma_{\max}$ *and* R :=*the image set of* $\hat{W}\cdot\Gamma(R,G)$. *Then* R *is an elliptic root system belonging to* (F,I) *with the marking* G. *The image set in* F *of the vertices of* $\Gamma(R,G)$ *forms a root basis of the elliptic root system. The root lattice in* F *generated by* $\Gamma(R,G)$ *is given by*

$$Q(R) = \mathbb{Z}\Gamma(R,G)/\Big\langle \frac{\alpha^*-\alpha}{k(\alpha)} - \frac{\beta^*-\beta}{k(\beta)} \mid \alpha,\ \beta\in\Gamma_{\max}\Big\rangle, \tag{2.5.1}$$

$$Q(R^\vee) = \mathbb{Z}\Gamma(R,G)^\vee/\Big\langle \frac{\alpha^{*\vee}-\alpha^\vee}{k^\vee(\alpha^\vee)} - \frac{\beta^{*\vee}-\beta^\vee}{k^\vee(\beta^\vee)} \mid \alpha,\ \beta\in\Gamma_{\max}\Big\rangle. \tag{2.5.2}$$

(2.6) A marked elliptic root system (R, G) is called *simply-laced* if its diagram $\Gamma(R, G)$ consists only of simply-laced bonds $\circ$——$\circ$ and of doubly dotted bonds $\circ$===$\circ$.

Fact 4. *A simply-laced elliptic root system (R, G) is homogeneous and hence $t(R) = t_1(R) = t_2(R) = 1$, $m'_{max} = m_{max}$ (see Appendix A for explicit values), and $k(\alpha) = 1$ for $\alpha \in R$. The set of roots decomposes (cf. (2.3.3)):*

$$R \;=\; R_{\mathrm{af}} + \mathbb{Z}a \;=\; R_{\mathrm{f}} + \mathbb{Z}b + \mathbb{Z}a. \tag{2.6.1}$$

The simply-laced elliptic root systems are of types $A_l^{(1,1)}$ for $l \geq 2$, $D_l^{(1,1)}$ for $l \geq 4$ and $E_l^{(1,1)}$ for $l = 6, 7, 8$, whose diagrams are exhibited in Appendix A.

3. The Lie algebra $\widetilde{\mathfrak{g}}(R)$ associated to a generalized root system

The goal of this section is to introduce the Lie algebras $\mathfrak{g}(R)$ and $\widetilde{\mathfrak{g}}(R)$ attached to a generalized root system R as certain subalgebras of $V_{Q(R)}/DV_{Q(R)}$ for the root lattice $Q(R)$ ((3.2) Definition 1). Here V_Q is the lattice vertex algebra attached to a lattice Q and D is its derivation, studied by Borcherds [Bo1], which we recall in the first half of the section. For vertex algebras, see also [K2], [MN].

(3.1) First, we review the construction of the lattice vertex algebra in our context. Let Q be an even lattice with an integral symmetric bilinear form I attached to a quadratic form q such that $q(x) = \frac{I(x,x)}{2}$. There is a canonical central extension: $0 \to \mathbb{Z}/2\mathbb{Z} \to \hat{Q} \to Q \to 0$ defined by the skew symmetric form I *mod* 2. Fixing a section $e : Q \mapsto \hat{Q}$, $\alpha \mapsto e^\alpha$, we have the product rule: $e^\alpha e^\beta = \kappa^{I(\alpha,\beta)} e^\beta e^\alpha$, where κ is the multiplicative generator of the center $\mathbb{Z}/2\mathbb{Z}$. Giving an additive cocycle $\varepsilon : Q \times Q \to \mathbb{Z}/2\mathbb{Z}$ such that $\varepsilon(\alpha, \beta) + \varepsilon(\beta, \alpha) \equiv I(\alpha, \beta)$ *mod* 2 is equivalent to giving a product rule: $e^\alpha e^\beta = \kappa^{\varepsilon(\alpha,\beta)} e^{\alpha+\beta}$. We shall often use a $\mathbb{Z}$-bilinear cocycle ε defined from an ordered basis $\alpha_1, \cdots, \alpha_n$ of the lattice Q as follows: $\varepsilon(\alpha_i, \alpha_j) := I(\alpha_i, \alpha_j)$ *mod* 2 if $i \geq j$, otherwise 0. Then, we have $\varepsilon(\alpha, 0) = \varepsilon(0, 0) = \varepsilon(0, \alpha) = 0$ for any $\alpha \in Q$ and $e^0 = 1 \in \hat{Q}$.

Let $\mathbb{Q}\{Q\}$ be the quotient of the group ring $\mathbb{Q}[\hat{Q}]$ divided by the ideal generated by $1+\kappa$. The image of the section $\{e^\alpha \mid \alpha \in Q\}$ (denoted by the same symbol) gives a basis of $\mathbb{Q}\{Q\}$ with the product rule $e^\alpha e^\beta = (-1)^{\varepsilon(\alpha,\beta)} e^{\alpha+\beta}$.

Put $F_{\mathbb{Q}} := \mathbb{Q} \otimes_{\mathbb{Z}} Q$ and let $\widetilde{F}_{\mathbb{Q}}$ be a $\mathbb{Q}$ vector space equipped with a non-degenerate symmetric bilinear form $\widetilde{I}$ such that i) $\widetilde{F}_{\mathbb{Q}}$ contains $F_{\mathbb{Q}}$ as a subspace, ii) the restriction of $\widetilde{I}$ on $F_{\mathbb{Q}}$ coincides with I. Such an $\widetilde{F}_{\mathbb{Q}}$ having the lowest rank $(=rank(Q) + rank(rad(Q)))$ is unique, and shall be called the *non-degenerate hull* of $F_{\mathbb{Q}}$. We identify $\widetilde{F}_{\mathbb{Q}}$ with its dual space $\widetilde{F}^*_{\mathbb{Q}}$ by $\widetilde{I} : \widetilde{F}_{\mathbb{Q}} \to \widetilde{F}^*_{\mathbb{Q}}$; $x \mapsto \widetilde{I}(x)(y) = \langle \widetilde{I}(x), y\rangle := \widetilde{I}(x,y)$. Put $\widetilde{\mathfrak{h}} := \widetilde{F}^*_{\mathbb{Q}} = \widetilde{I}(\widetilde{F}_{\mathbb{Q}})$, $\mathfrak{h} := \widetilde{I}(F_{\mathbb{Q}})$ (note that $\mathfrak{h}$ is not the dual space $F^*_{\mathbb{Q}}$ of $F_{\mathbb{Q}}$ if I is degenerate) and $h_x := \widetilde{I}(x) \in \widetilde{\mathfrak{h}}$ for any $x \in \widetilde{F}_{\mathbb{Q}}$. We introduce the form $\widetilde{I}^*$ on $\widetilde{\mathfrak{h}}$ by $\widetilde{I}^*(\widetilde{I}(x), \widetilde{I}(y)) := \widetilde{I}(x,y)$ for any $x, y \in \widetilde{F}_{\mathbb{Q}}$.

Let us define the space

$$V_Q := S\left(\bigoplus_{n\in\mathbb{Z}_{>0}} \widetilde{\mathfrak{h}}(-n)\right) \otimes \mathbb{Q}\{Q\}, \tag{3.1.1}$$

where $S(\oplus_{n\in\mathbb{Z}_{>0}}\widetilde{\mathfrak{h}}(-n))$ is the symmetric tensor algebra of the direct sum of an infinite sequence $\{\widetilde{\mathfrak{h}}(-n)\}_{n\in\mathbb{Z}_{>0}}$ of copies of $\widetilde{\mathfrak{h}}$ (copies of an element $h \in \widetilde{\mathfrak{h}}$ are denoted by $h(-1), h(-2), \cdots$). Then V_Q has the following structures.

i) As the tensor product of algebras, V_Q is an algebra.

ii) For any $h \in \widetilde{\mathfrak{h}}$ and $n \in \mathbb{Z}$, we define the left-operator $h(n)$ on V_Q as follows: if $n < 0$ then $h(n)$ is multiplication by $h(n)$. If $n = 0$ then $h(0)e^\alpha = \langle h, \alpha\rangle e^\alpha$ for any $\alpha \in Q$. If $n > 0$ then $h(n)e^\alpha = 0$ for any $\alpha \in Q$. One has the rule: $[h(m), g(n)] = m\delta_{m+n,0}\widetilde{I}^*(h,g)$ for any $h, g \in \widetilde{\mathfrak{h}}$ and $m, n \in \mathbb{Z}$.

iii) The algebra V_Q has the Cartan involution $\omega : \omega(e^\alpha) = e^{-\alpha}$, $\omega(h) = -h$ for any $\alpha \in Q$ and $h \in \widetilde{\mathfrak{h}}$.

iv) There is the linear map $deg : V_Q \to V_Q$ such that $deg\, e^\alpha = q(\alpha)e^\alpha$ and $deg\, h(-n)v = h(-n)(nv + deg\, v)$ for any $\alpha \in Q$, $h \in \widetilde{\mathfrak{h}}$ and $v \in V_Q$. We say $u \in V_Q$ is of *degree* n if $deg\, u = nu\,(n \in \mathbb{Z})$.

v) The algebra V_Q is Q-graded. That is: $V_Q = \oplus_{\alpha\in Q}(V_Q)_\alpha$, where $(V_Q)_\alpha := S(\oplus_{n\in\mathbb{Z}_{>0}}\widetilde{\mathfrak{h}}(-n)) \otimes e^\alpha$. An element $u \in (V_Q)_\alpha$ is said to have *grade* α.

For any $n \in \mathbb{Z}$ we define the n-th product, denoted by (n), of $u = h_1(-n_1) \cdots h_k(-n_k)e^\alpha \in V_Q$ ($h_1, \cdots, h_k \in \widetilde{\mathfrak{h}}$, $\alpha \in Q$ and $k \geq 0$) and $v \in V_Q$ by

$$\begin{aligned} u_{(n)}v \quad &:= \text{the coefficient of } z^{-n-1} \text{ in} \\ &\left({}^\circ_\circ Q(h_1(-n_1), z) \cdots Q(h_k(-n_k), z) \exp\left(Q(h_\alpha(0), z)\right) e^\alpha {}^\circ_\circ\right) v, \end{aligned} \tag{3.1.2}$$

where for $h \in \widetilde{\mathfrak{h}}$ and $n \geq 0$, we put

$$Q(h(-n), z) := \frac{1}{n!}\left(\frac{d}{dz}\right)^n \left(\sum_{i \neq 0} \frac{h(-i)}{i} z^i + h(0)\log(z)\right). \tag{3.1.3}$$

Here "${}^\circ_\circ X {}^\circ_\circ$" is the "normal ordering of X", where one rearranges the ordering of products in the formal expression of X in such way that the creation operators $h(-i)$ ($i \geq 1$) occur to the left of all annihilation operators $h(i)$ ($i \geq 0$) and e^α occur to the left of all operators $h(i)$ for $i \in \mathbb{Z}$, and $\exp(h_\alpha(0)\log(z))e^\beta = e^\beta z^{I(\alpha,\beta)}$. Extending (3.1.2) linearly in u, the vector space V_Q is equipped with countably many bilinear operations (n). Then, the system of these operations defines the vertex algebra structure on V_Q ([Bo1], [FLM]).

For any $h \in \widetilde{\mathfrak{h}}$, $u \in V_Q$ and $n \in \mathbb{Z}$, we have $h(-n)_{(-1)}u = h(-n)u$ (see (3.1.2)). So, V_Q is generated as (-1)-th product algebra by $h(-n)$ and e^α for $h \in \widetilde{\mathfrak{h}}$, $n \in \mathbb{Z}_{>0}$ and $\alpha \in Q$. Define an operator

$$D: \ V_Q \to V_Q \ , \quad a \mapsto \ a_{(-2)}1. \tag{3.1.4}$$

Then, we have

$$D(u_{(n)}v) = (Du)_{(n)}v + u_{(n)}Dv. \tag{3.1.5}$$

In addition, for any $h \in \widetilde{\mathfrak{h}}$, $\alpha \in Q$ and $n \in \mathbb{Z}_{>0}$,

$$De^\alpha = h_\alpha(-1)e^\alpha, \qquad Dh(-n) = nh(-n-1), \tag{3.1.6}$$

so D is a homogeneous operator of degree 1.

Fact 5 (Borcherds[Bo1]). *The product $u_{(0)}v$ for $u, v \in V_Q$ induces a Lie algebra structure on the quotient space V_Q/DV_Q and a left V_Q/DV_Q-module structure on V_Q.*

In this section, let us tentatively denote the algebra V_Q/DV_Q by $\widetilde{\mathfrak{g}}(Q)$.

We shall use the same symbols to express an element in $\widetilde{\mathfrak{g}}(Q)$ as an element in V_Q. The Lie bracket of $\widetilde{\mathfrak{g}}(Q)$ is given by $[u,v] = u_{(0)}v$. The algebra $\widetilde{\mathfrak{g}}(Q)$ inherits $\mathbb{Z}$- and Q-grading structures since D is homogeneous and preserves the Q-grading.

If $u, v \in V_Q$ have degrees $l, m \in \mathbb{Z}$ respectively then $u_{(n)}v$ has degree $l + m - n - 1$. So, the subspace of degree 1 is closed under the 0-th product.

Fact 6. $\widetilde{\mathfrak{g}}(Q)_1 = (V_Q/DV_Q)_1 \simeq (V_Q)_1/D(V_Q)_0$ *is a Lie subalgebra of* $\widetilde{\mathfrak{g}}(Q)$.

Here, we recall some terminologies from [K], [MP]. Given a Lie algebra $\mathfrak{g}$ and its abelian subalgebra $\mathfrak{h}$, we say an element $x \in \mathfrak{g}$ *has weight* $f \in \mathfrak{h}^*$ if $[h,x] = \langle h, f\rangle x$ for any $h \in \mathfrak{h}$, and we say x is a *weight vector of weight* f. Let $\mathfrak{g}_{\mathrm{f}}$ be the set of all elements which have weight f. Its dimension $dim_{\mathbb{Q}}\,\mathfrak{g}_{\mathrm{f}}$ is called the *multiplicity* of f. If $\mathfrak{g}_{\mathrm{f}} \neq \{0\}$, f is called a *root* and $\mathfrak{g}_{\mathrm{f}}$ is called its *root space*. If $\mathfrak{g}$ is spanned (as a vector space) by root spaces, then we say that $\mathfrak{g}$ *has a root space decomposition* with respect to $\mathfrak{h}$.

Let $u \in V_Q$ be an element of grade $\alpha \in Q$. Then $h(-1)_{(0)}u = \langle h, \alpha\rangle u$ for any $h \in \widetilde{\mathfrak{h}}$ (use (3.1.2)). This means that the two concepts: weight and grade coincide. Therefore $\widetilde{\mathfrak{g}}(Q)$ has a root space decomposition with respect to $\widetilde{\mathfrak{h}}(-1)$ and the root space $\widetilde{\mathfrak{g}}(Q)_\alpha$ of $\alpha \in Q$ is $\widetilde{\mathfrak{g}}(Q)_\alpha = \big(\mathcal{S}/(D + h_\alpha(-1))\mathcal{S}\big) \otimes e^\alpha$, where we put $\mathcal{S} := S(\oplus_{n\in\mathbb{Z}_{>0}}\widetilde{\mathfrak{h}}(-n))$. The subalgebra $\widetilde{\mathfrak{g}}(Q)_1$ also has a root space decomposition with respect to $\widetilde{\mathfrak{h}}(-1)$. Note that $\widetilde{\mathfrak{g}}(Q)_1$ has no root $\alpha \in Q$ such that $q(\alpha) > 1$ (recall that the degree of $h_1(-n_1)\cdots h_k(-n_k)e^\alpha$ is $n_1 + \cdots n_k + q(\alpha) \geq q(\alpha)$). Some of the root spaces are described explicitly as follows.

$$\begin{array}{lcll} (\widetilde{\mathfrak{g}}(Q)_1)_\alpha & = & \mathbb{Q}\, e^\alpha & \text{for} \quad q(\alpha) = 1, \\ (\widetilde{\mathfrak{g}}(Q)_1)_\mu & = & \big(\widetilde{\mathfrak{h}}(-1)/\mathbb{Q}\mu(-1)\big)\, e^\mu & \text{for} \quad q(\mu) = 0. \end{array} \tag{3.1.7}$$

A root $\alpha \in Q$ such that $q(\alpha) = 1$ (resp. $q(\alpha) \leq 0$) is called a *real root* (resp. an *imaginary root*).

Let us list some bracket rules in $\widetilde{\mathfrak{g}}(Q)$, which can be calculated from the table of (0)-th products in Appendix B. We shall use them in the present paper without mentioning it explicitly. Put $Q_1 := \{\alpha \in Q \mid q(\alpha) = 1\}$. Let $\alpha, \beta \in Q$, $\bar{h}, \bar{g} \in \mathfrak{h}$, $h, g \in \widetilde{\mathfrak{h}}$ and $\mu, \lambda \in rad\, Q$.

0. $[h(-1), g(-1)] = 0,$
I. $[h(-1), e^{\alpha}] = \langle h, \alpha \rangle e^{\alpha},$
$[h(-1), \bar{g}(-1)e^{\mu}] = \langle h, \mu \rangle \bar{g}(-1)e^{\mu},$
II.1. $[e^{\alpha}, e^{\beta}] =$

$$\begin{cases} 0 & \text{if} \quad I(\alpha,\beta) \geq 0 \\ (-1)^{\varepsilon(\alpha,\beta)} e^{\alpha+\beta} & \text{if} \quad I(\alpha,\beta) = -1 \\ (-1)^{\varepsilon(\alpha,\beta)} h_{\alpha^\vee}(-1) e^{\alpha+\beta} & \text{if} \quad I(\alpha,\beta) = -2 \end{cases} \tag{3.1.8}$$

for $\alpha, \beta \in Q_1,$
II.2. $ad(e^{\alpha})^{1-\langle h_{\alpha^\vee}, \beta \rangle} e^{\beta} = 0 \quad \text{if} \quad I(\alpha,\beta) \leq 0 \text{ and } \alpha \in Q_1,$
III. $[\bar{h}(-1)e^{\mu}, e^{\alpha}] = (-1)^{\varepsilon(\mu,\alpha)} \langle \bar{h}, \alpha \rangle e^{\alpha+\mu},$
IV. $[\bar{h}(-1)e^{\mu}, \bar{g}(-1)e^{\lambda}] = (-1)^{\varepsilon(\mu,\lambda)} \widetilde{I}^{*}(\bar{h}, \bar{g}) h_{\mu}(-1) e^{\mu+\lambda}.$

(3.2) In this subsection, we attach a Lie algebra $\widetilde{\mathfrak{g}}(R)$ to any homogeneous generalized root system R belonging to the lattice $Q(R)$. Recall the even lattice structure I_R on $Q(R)$ (2.1.3). In the rest of this section, we use the normalized bilinear form I_R, but will denote it by I for short.

Definition 1. *Define subalgebras* $\widetilde{\mathfrak{g}}(R)$ *and* $\mathfrak{g}(R)$ *of* $\widetilde{\mathfrak{g}}(Q(R))$ *by*

$$\begin{array}{lll} \widetilde{\mathfrak{g}}(R) & := & \langle\, h(-1), e^{\alpha} \mid h \in \widetilde{\mathfrak{h}},\, \alpha \in R \,\rangle, \\ \mathfrak{g}(R) & := & \langle\, e^{\alpha} \mid \alpha \in R \,\rangle. \end{array} \tag{3.2.1}$$

The algebra $\widetilde{\mathfrak{g}}(R)$ is a subalgebra of $\widetilde{\mathfrak{g}}(Q(R))_1$ (since all the generators are in $\widetilde{\mathfrak{g}}(Q(R))_1$ and (3.1) Fact 6) and inherits the $Q(R)$-grading structure of $\widetilde{\mathfrak{g}}(Q(R))_1$. Since all the generators are weight vectors, the algebra $\widetilde{\mathfrak{g}}(R)$ has a root space decomposition with respect to $\widetilde{\mathfrak{h}}$. If α is a root of $\widetilde{\mathfrak{g}}(R)$, then $I(\alpha,\alpha) \leq 2$. Note that $\mathfrak{g}(R) = [\widetilde{\mathfrak{g}}(R), \widetilde{\mathfrak{g}}(R)]$, $\mathfrak{g}(R) = [\mathfrak{g}(R), \mathfrak{g}(R)]$ (use the bracket table (3.1.8)).

For $S \subset R$, we consider the subalgebra $\mathfrak{g}(S)$ of $\mathfrak{g}(R)$:

$$\mathfrak{g}(S) := \langle\, e^{\alpha} \mid \alpha \in \pm S \,\rangle. \tag{3.2.2}$$

Assertion 1. *For any subset* $S \subset R$, *one has* $\mathfrak{g}(S) = \mathfrak{g}(W(S)S)$. *In particular, if* Π *is a root basis of* R *(in the sense of (2.1.1)), then* $\{e^{\alpha} \mid \alpha \in \pm\Pi\}$ *generates* $\mathfrak{g}(R)$.

Proof. First we claim $e^{w_\alpha(\beta)} = const \cdot ad(e^\alpha)^{-I(\alpha^\vee,\beta)}e^\beta$. It is enough to show $ad(e^\alpha)^{-I(\alpha^\vee,\beta)}e^\beta \neq 0$ since the real root space is one dimensional (3.1.7). If it were zero, we would get $e^\beta = 0$ by applying $(ad\, e^{-\alpha})^{-I(\alpha^\vee,\beta)}$. It is impossible, and claim is proved. We return to the proof of (4). Any element of $W(S)S$ is an image of S by successive applications of reflections with respect to elements of S. Applying the claim above successively, we obtain $\mathfrak{g}(W(S)S) = \mathfrak{g}(S)$. □

Let us consider the case when R is a homogeneous k-extended affine root system of rank l. We shall call the algebra $\widetilde{\mathfrak{g}}(R)$ *k-extended affine Lie algebra*. Notice that in this case the bracket table (3.1.8) restores the structure of the Lie algebra $\widetilde{\mathfrak{g}}(R)$.

Lemma 1. *The algebra $\widetilde{\mathfrak{g}}(R)$ decomposes to root spaces as:*

$$\widetilde{\mathfrak{g}}(R) = \widetilde{\mathfrak{h}}(-1) \oplus \underset{\alpha\in R}{\oplus} \mathbb{Q}e^\alpha \oplus \underset{\mu\in rad\, Q(R)\setminus\{0\}}{\oplus} (\mathfrak{h}(-1)/\mathbb{Q}\mu(-1))e^\mu. \qquad (3.2.3)$$

If we replace $\widetilde{\mathfrak{h}}$ by $\mathfrak{h}$ in (3.2.3), then we get an explicit description of $\mathfrak{g}(R)$.

The ranks of $\widetilde{\mathfrak{h}}$ and $\mathfrak{h}$ are $l+2k$ and $l+k$, respectively. So, we know the multiplicity for any root of $\widetilde{\mathfrak{g}}(R)$: the multiplicity of $\alpha \in R$ is equal to 1 and the multiplicity of $\mu \in rad\, Q(R) \setminus \{0\}$ is equal to $l+k-1$.

According to the explicit expression of $\mathfrak{g}(R)$ above and the bracket table (3.1.8), it is easy to see that $\mathfrak{g}(R)$ is isomorphic to the universal central extension of the algebra $\mathfrak{g}_{\mathrm{f}} \otimes \mathbb{Q}[e^{\pm a_1}, \cdots, e^{\pm a_k}]$ for a simple Lie algebra $\mathfrak{g}_{\mathrm{f}}$ (use (2.6.1)). That is: $\mathfrak{g}(R)$ is the k-toroidal Lie algebra in the sense [MEY]. See Appendix D for more explicit description of $\widetilde{\mathfrak{g}}(R)$.

Assertion 2. *If R is a simply-laced finite or affine root system, then the algebra $\mathfrak{g}(R)$ is isomorphic to a finite or affine Kac-Moody algebra, respectively.*

Proof. Take a simple root basis Γ of R giving the finite, or affine Dynkin diagram. Take a proper cocycle ε (as in (3.1)). Then the Serre relations are satisfied by the Chevalley generator system $\{\alpha^\vee := h_{\alpha^\vee}(-1), e_\alpha := e^\alpha, f_\alpha := e^{-\alpha}\}_{\alpha\in\Gamma}$ and $h_{\alpha^\vee}(-1)$ for $\alpha \in \Gamma$ are linearly independent. Apply Gabber-Kac theorem. □

4. The elliptic Lie algebra $\widetilde{\mathfrak{e}}(R,G)$ presented by generators and relations

In the previous section, we have introduced the Lie algebra $\widetilde{\mathfrak{g}}(R)$ attached to a homogeneous elliptic root system R. In this section, we introduce the second Lie algebra $\widetilde{\mathfrak{e}}(\Gamma(R,G)) = \widetilde{\mathfrak{e}}(\Gamma)$ attached to a simply-laced *marked* elliptic root system (R,G). The algebra $\widetilde{\mathfrak{e}}(\Gamma)$ is presented by generators and relations determined by its elliptic diagram $\Gamma = \Gamma(R,G)$. In (4.1) Theorem 1, we state the main result of the present article: the isomorphism of the two algebras, where the surjectivity $\varphi : \widetilde{\mathfrak{e}}(\Gamma) \to \widetilde{\mathfrak{g}}(R)$ follows immediately from the definition.

The rest of this article is devoted to the proof of the theorem. This section gives a preparation by studying some subalgebras of $\widetilde{\mathfrak{e}}(\Gamma)$. In (4.2), we study the subalgebras $\mathfrak{e}(A)$ of $\widetilde{\mathfrak{e}}(\Gamma)$ attached to the A-parts A of the diagram $\Gamma(R,G)$, which turn out to be the affine Kac-Moody algebras $\mathfrak{g}(A_\triangle)$ by a choice of a generator system $A_\triangle$. In (4.3) we consider the subalgebra $\mathfrak{h}_{\mathrm{af}}^{\mathbb{Z}'}$ of $\widetilde{\mathfrak{e}}(\Gamma)$, whose weight belongs to the marking $\mathbb{Z}a$ and which turns out to be a Heisenberg algebra.

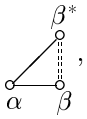
(4.1) Let $\Gamma(R,G)$ be the elliptic diagram of a simply-laced marked elliptic root system (R,G) (2.6). We construct the Lie algebra from the diagram as follows. As in (2.5) a)–d), we reconstruct the root lattice $Q(R)$ (2.5.1) from $\Gamma(R,G)$, where $\Gamma(R,G)$ is identified with a root basis of R as a subset of $Q(R)$. As in (3.1) put $F_{\mathbb{Q}} := \mathbb{Q} \otimes_{\mathbb{Z}} Q(R)$ and let $(\widetilde{F}_{\mathbb{Q}}, \widetilde{I})$ be its non-degenerate hull. The space $\widetilde{F}_{\mathbb{Q}}$ is identified with its dual space $\widetilde{\mathfrak{h}} := Hom_{\mathbb{Q}}(\widetilde{F}_{\mathbb{Q}}, \mathbb{Q})$ by $\widetilde{I} : x \mapsto h_x$, where $h_x \in \widetilde{\mathfrak{h}}$ is defined by $h_x(y) := \widetilde{I}(x,y)$ for any $y \in \widetilde{F}_{\mathbb{Q}}$. Recall $\alpha^* - \alpha = a$ (see (2.4.3) and (2.6) Fact 4), and hence we have $h_{\alpha^{*\vee}} - h_{\alpha^\vee} = h_{a^\vee}$ for any $\alpha \in \Gamma_{\max}$.

Definition 2. *The Lie algebra $\widetilde{\mathfrak{e}}(\Gamma(R,G))$ is the algebra presented by the following generators and relations:*

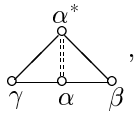
Generators: $\widetilde{\mathfrak{h}}$ *and* $\{E^\alpha \mid \alpha \in \pm\Gamma(R,G)\}$

Relations:

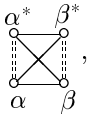
0. $\widetilde{\mathfrak{h}}$ *is abelian,*
I. $[h, E^\alpha] = \langle h, \alpha\rangle E^\alpha$,
II.1. $[E^\alpha, E^{-\alpha}] = h_{\alpha^\vee}$,
$[E^\alpha, E^\beta] = 0 \quad$ *for* $\quad I(\alpha,\beta) \geq 0$,
II.2. $(ad\, E^\alpha)^{1-\langle h_{\alpha^\vee},\beta\rangle} E^\beta = 0 \quad$ *for* $\quad I(\alpha,\beta) \leq 0$,

III. $[E^\alpha, E^\beta, E^{\beta^*}] = 0$
$[E^{-\alpha}, E^{-\beta}, E^{-\beta^*}] = 0$ *for* 

,

IV. **1.** $[E^{\alpha^*}, E^{-\alpha}, E^\beta, E^\gamma] = 0$
$[E^{-\alpha^*}, E^\alpha, E^{-\beta}, E^{-\gamma}] = 0$
2. $[E^\beta, E^\alpha, E^\gamma, E^{\alpha^*}] = 0$
$[E^{-\beta}, E^{-\alpha}, E^{-\gamma}, E^{-\alpha^*}] = 0$ *for* 

,

V. $[E^{-\alpha}, E^{\alpha^*}, E^\beta] = E^{\beta^*}$
$[E^\alpha, E^{-\alpha^*}, E^{-\beta}] = E^{-\beta^*}$ *for* 

, (4.1.1)

where h *runs over* $\widetilde{\mathfrak{h}}$ *in* **I**, α, β *run over* $\pm\Gamma(R,G)$ *in* **I**, **II**, *and* α, β, γ *run over* $\pm\Gamma_{\rm af}$ *in* **III**, **IV** *and* **V**.

Remark 3. The definition of a root system is invariant under the scalar multiplication cI of the form I for $c \in \mathbb{Q} \setminus \{0\}$. We shall use $h_{\alpha^\vee}$ instead of h_α for $\alpha \in R$ so that the result does not depend on the choice of c (recall from (3.2) that $\widetilde{\mathfrak{g}}(R)$ is constructed by the use of I_R). Then $\langle h_{\alpha^\vee}, \alpha\rangle = 2$. Similarly, $h_{a^\vee}$ does not move after rescaling of I (see (2.4.1)). See also the proof of (4.3) Lemma 4 **H-II**.

Remark 4. The relations **0**–**II** are the well known Kac-Moody type relations. The relations **III**–**V** are new relations caused by the new type bonds $\circ{=}{=}{=}\circ$ in $\Gamma(R,G)$. Note that the relations **III**–**V** reduce to the classical relations if we identify $E^{\pm\alpha^*}$ with $E^{\pm\alpha}$ for $\alpha \in \Gamma_{\rm max}$.

For a root lattice $Q(R) = Q_{\rm af} \oplus \mathbb{Z}a$ of a simply-laced elliptic root system R, we always use $\mathbb{Z}$-bilinear cocycle ε satisfying: $\varepsilon(\alpha,-\alpha) = \varepsilon(\alpha,a) = \varepsilon(a,\alpha) = 0$ for any $\alpha \in \pm\Gamma(R,G)$. The following is a construction of such a cocycle ε. Recall from the splitting (2.3.3) that the set $\Gamma_{\rm af} \cup \{a\}$ forms a $\mathbb{Z}$-basis of $Q(R)$. Put $\Gamma_{\rm af} \cup \{a\} = \{\alpha_{-1}, \alpha_0, \cdots, \alpha_l\}$ tentatively, and define

$$\varepsilon(\alpha_i,\alpha_j) := \begin{cases} I(\alpha_i,\alpha_j) \ mod\ 2 & \text{if} \quad i \geq j, \\ 0 & \text{otherwise.} \end{cases} \tag{4.1.2}$$

Extend this $\mathbb{Z}$-linearly to $Q(R)$.

The following is the main theorem of the present article.

Theorem 1. *The following correspondence extends to an isomorphism from* $\widetilde{\mathfrak{e}}(\Gamma(R,G))$ *to* $\widetilde{\mathfrak{g}}(R)$*:*

$$\begin{array}{ccccc} h & \mapsto & h(-1) & \quad for \quad & h \in \widetilde{\mathfrak{h}}, \\ E^\alpha & \mapsto & e^\alpha & \quad for \quad & \alpha \in \pm\Gamma(R,G). \end{array} \tag{4.1.3}$$

Here, the cocycle ε *is so chosen to satisfy* $\varepsilon(\alpha,-\alpha) = \varepsilon(\alpha,a) = \varepsilon(a,\alpha) = 0$ *for any* $\alpha \in \pm\Gamma(R,G)$*.*

The rest of this article is devoted to the proof of the theorem. In the sequel, for short, we shall denote $\Gamma(R,G)$ by Γ, and so, $\widetilde{\mathfrak{e}}(\Gamma(R,G))$ by $\widetilde{\mathfrak{e}}(\Gamma)$.

First, let us see that the correspondence (4.1.3) induces a homomorphism.

Assertion 3. *The map defined in (4.1.3) extends to a surjective Lie homomorphism* φ

$$\varphi : \widetilde{\mathfrak{e}}(\Gamma(R,G)) \to \widetilde{\mathfrak{g}}(R). \tag{4.1.4}$$

Proof. We can check the vanishing of the φ-images of the defining relations (4.1.1) of $\widetilde{\mathfrak{e}}(\Gamma)$ using (3.1.8). The surjectivity follows from (2.4) Fact 1 and (3.2) Assertion 1. See also Remark 3. □

The algebra $\widetilde{\mathfrak{e}}(\Gamma)$ has the root space decomposition with respect to $\widetilde{\mathfrak{h}}$ and the set of roots is contained in $Q(R)$, since all the generators are weight vectors (see the relations **0** and **I**) and their weights are in $Q(R)$. By definition of φ (4.1.3), we know a root space $\widetilde{\mathfrak{e}}(\Gamma)_\alpha$ is sent to the root space $\widetilde{\mathfrak{g}}(R)_\alpha$ for any root α. There exists the involution ω, called the Cartan involution, on $\widetilde{\mathfrak{e}}(\Gamma)$ defined by $h \mapsto -h$ for $h \in \widetilde{\mathfrak{h}}$ and $E^\alpha \mapsto E^{-\alpha}$ for $\alpha \in \pm\Gamma$. This follows from the fact that the system of the relations are invariant under the transformation.

As in (3.2.2), for a subset $S \subset \Gamma$, we consider a subalgebra $\mathfrak{e}(S)$ of $\widetilde{\mathfrak{e}}(\Gamma)$:

$$\mathfrak{e}(S) := \langle E^\alpha \,|\, \alpha \in \pm S \rangle. \tag{4.1.5}$$

Note that $\mathfrak{e}(S)$ contains $h_{\alpha^\vee}$ for any $\alpha \in S$ because of the relation **II.1**.

Assertion 4. *If a subset $S \subset \Gamma$ is linearly independent in $F_{\mathbb{Q}}$ and its intersection matrix $(I(\alpha^\vee, \beta))_{\alpha,\beta\in S}$ forms a generalized Cartan matrix in the sense of [K], then $\mathfrak{e}(S)$ is isomorphic to an affine Kac-Moody algebra constructed from the Cartan matrix, in other words, to $\mathfrak{g}(S)$.*

Proof. Under the assumption of linear independence of $\{h_{\alpha^\vee}\}_{\alpha\in S}$, it is enough to show $\{E^\alpha, h_{\alpha^\vee}, E^{-\alpha}\}_{\alpha\in S}$ satisfy the Kac-Moody relations (after suitable changes of sign). But they do satisfy the relations **0**–**II**. (cf. (3.2) Assertion 2). □

Recall the projection π_G in (2.3.2) and let π be its restriction to $\Gamma(R,G)$: $\pi : \Gamma(R,G) \to \pi_G(\Gamma_{\mathrm{af}})$. For any section $\iota : \pi_G(\Gamma_{\mathrm{af}}) \to \Gamma$ of π, $\iota(\pi_G(\Gamma_{\mathrm{af}}))$ satisfies the condition of Assertion 4. So, $\mathfrak{e}(\iota(\pi_G(\Gamma_{\mathrm{af}})))$ is a Kac-Moody algebra of simply-laced type.

(4.2) At the end of (4.1), we have found affine Kac-Moody subalgebras of $\widetilde{\mathfrak{e}}(\Gamma)$ whose null roots are $\mathbb{Z}\cdot(b+ma)$ for some $m \in \mathbb{Z}$. In this subsection, we find affine Kac-Moody subalgebras of $\widetilde{\mathfrak{e}}(\Gamma)$ whose null roots are the integral marking $G_{\mathbb{Z}} = \mathbb{Z}a$ (2.3.1). This subsection is aimed to show Lemma 2, which plays the crucial role in the sequel of the proof of Theorem 1.

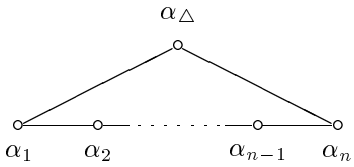
Definition 3. *An A-part A of $\Gamma(R,G)$ is a union of a maximal linear subdiagram A_{af} of Γ_{af} and $\{\alpha^* \mid \alpha \in A_{\mathrm{af}} \cap \Gamma_{\max}\}$. That is: $A = A_{\mathrm{af}} \cup (A_{\mathrm{af}} \cap \Gamma_{\max})^*$.*

Explicit list of A-parts is given in Appendix A. Notice that for any $\alpha, \beta \in \Gamma(R,G)$, there exists an A-part A which contains α and β (check case by case, note that $A_1^{(1,1)}$ is not simply-laced).

For any $\beta \in A \cap \Gamma_{\max}$, the difference $\beta^* - \beta$ is the generator a of the integral marking $G_{\mathbb{Z}}$ (2.3.1). We see that A generates an affine root lattice of type $A_n^{(1)}$ (whose null roots are $G_{\mathbb{Z}}$) by introducing the extension node

$$\alpha_\triangle := -a - \sum_{\alpha\in A_{\mathrm{af}}} \alpha, \tag{4.2.1}$$

and introducing an affine root basis $A_\triangle := A_{\mathrm{af}} \cup \{\alpha_\triangle\}$, whose diagram is given below.



We have $W(A_\triangle)A_\triangle = W(A)A$ (since $-\alpha_\triangle = \beta^* + \sum_{\alpha\in A_{\rm af}\setminus\{\beta\}}\alpha$ is the highest root with respect to the root basis $(A_{\rm af}\setminus\{\beta\})\cup\{\beta^*\}$, one has $\alpha_\triangle \in W(A)A$. Conversely, $-\beta^*$ is the highest root with respect to the root basis $A_\triangle\setminus\{\beta\}$, etc.). The set $\{e^\alpha, h_{\alpha^\vee}, e^{-\alpha}\}_{\alpha\in A_{\rm af}}\cup\{e^{\alpha_\triangle}, h_{\alpha_\triangle^\vee}, (-1)^{\varepsilon(\alpha_\triangle,\alpha_\triangle)}e^{-\alpha_\triangle}\}$ forms a standard generator of the Kac-Moody algebra $\mathfrak{g}(A_\triangle)$.

Lemma 2. *For any given A-part A, there exists an isomorphism* $\Upsilon : \mathfrak{g}(A) \to \mathfrak{e}(A)$ *which satisfies*

$$\begin{array}{lrcll} \Upsilon: & h_{\alpha^\vee}(-1) & \mapsto & h_{\alpha^\vee} & \text{for} \quad \alpha\in A, \\ \Upsilon: & e^\alpha & \mapsto & E^\alpha & \text{for} \quad \alpha\in \pm A. \end{array} \tag{4.2.2}$$

Its inverse is the restriction of φ *(4.1.4) on* $\mathfrak{e}(A)$.

Proof. Put $A_{\rm af} = A\cap\Gamma_{\rm af} =: \{\alpha_1,\cdots,\alpha_n\}$ such that $I(\alpha_i^\vee,\alpha_{i+1}) = -1$ for $i = 1,\cdots,n-1$. We fix an element $\alpha_s \in A\cap\Gamma_{\max}$ and put $A_{s^*} := A_{\rm af}\cup\{\alpha_s^*\}$. Then, $W(A_{s^*})A_{s^*} = W(A_\triangle)A_\triangle$ and hence $\mathfrak{g}(A_{s^*}) = \mathfrak{g}(A_\triangle)$ ((3.2) Assertion 1). Let us define a map Υ_s: $\{h_{\alpha^\vee}(-1),\ e^\alpha,\ e^{-\alpha} \mid \alpha\in A_\triangle\} \to \widetilde{\mathfrak{e}}(\Gamma)$ by

$$\begin{array}{lrcll} \Upsilon_s: & h_{\alpha^\vee}(-1) & \mapsto & h_{\alpha^\vee} & \text{for} \quad \alpha\in A_\triangle, \\ \Upsilon_s: & e^\alpha & \mapsto & E^\alpha & \text{for} \quad \alpha\in \pm A_\triangle. \end{array} \tag{4.2.3}$$

where $E^{\pm\alpha_\triangle}$ are defined as follows:

$$\begin{array}{rcl} E^{\alpha_\triangle} & := & v_+[E^{-\alpha_1},\cdots,E^{-\alpha_{s-1}},E^{-\alpha_s^*},E^{-\alpha_{s+1}},\cdots,E^{-\alpha_n}], \\ E^{-\alpha_\triangle} & := & v_-[E^{\alpha_1},\cdots,E^{\alpha_{s-1}},E^{\alpha_s^*},E^{\alpha_{s+1}},\cdots,E^{\alpha_n}]. \end{array} \tag{4.2.4}$$

Here $v_\pm$ are the constants in $\{\pm 1\}$ defined by the relations:

$$\begin{array}{rcl} e^{\alpha_\triangle} & = & v_+[e^{-\alpha_1},\cdots,e^{-\alpha_{s-1}},e^{-\alpha_s^*},e^{-\alpha_{s+1}},\cdots,e^{-\alpha_n}], \\ e^{-\alpha_\triangle} & = & v_-[e^{\alpha_1},\cdots,e^{\alpha_{s-1}},e^{\alpha_s^*},e^{\alpha_{s+1}},\cdots,e^{\alpha_n}]. \end{array} \tag{4.2.5}$$

The proof of Lemma 2 consists of three steps. The first step shows the map defined in (4.2.3) and (4.2.4) induces a homomorphism Υ_s. The second step shows the homomorphism Υ_s satisfies (4.2.2). So, Υ_s is independent of a choice

of $\alpha_s \in A \cap \Gamma_{\max}$ and let us call it Υ. We remark that, as a consequence, $E^{\pm\alpha_\triangle}$ are defined independent of a choice of α_s. The third step shows $\varphi \circ \Upsilon = id_{\mathfrak{g}(A)}$, but this is clear after the first and second step.

First step: Let us check the vanishing of the Υ_s images in $\widetilde{\mathfrak{e}}(\Gamma)$ of the defining relations of the affine Kac-Moody algebra $\mathfrak{g}(A_\triangle)$:

$$\begin{array}{lll} \textbf{A-0.} & [h_{\alpha^\vee}(-1), h_{\beta^\vee}(-1)] = 0, & \\ \textbf{A-I.} & [h_{\alpha^\vee}(-1), e^\beta] = \langle h_{\alpha^\vee}, \beta\rangle e^\beta, & \\ \textbf{A-II.1.} & [e^\alpha, e^{-\alpha}] = (-1)^{\varepsilon(\alpha,-\alpha)} h_{\alpha^\vee}(-1), & \\ \textbf{A-II.2.} & (ad\, e^\alpha)^{1-\langle h_{\alpha^\vee},\beta\rangle} e^\beta = 0 & \text{for} \quad I(\alpha,\beta) \le 0, \\ & [e^\alpha, e^\beta] = 0 & \text{for} \quad I(\alpha,\beta) \ge 0, \end{array} \tag{4.2.6}$$

where $\alpha, \beta \in \pm A_\triangle$. To do this, we prepare the following assertion.

Assertion 5. *(1) Let $\{\beta_1, \cdots, \beta_k\}$ be a sequence of $\pm A_{s*}$ satisfying that $I(\beta_i, \beta_j) = 0$ for all $1 \le i, j \le k$ with $|i-j| \ge 2$ and $I(\beta_i^\vee, \beta_{i+1}) = -1$. Then,*

$$[E^{\beta_1}, \cdots, E^{\beta_k}, [E^{-\beta_1}, \cdots, E^{-\beta_k}]] = (-1)^{k-1} (h_{\beta_1^\vee} + \cdots + h_{\beta_k^\vee}), \tag{4.2.7}$$

$$[E^{-\beta_i}, [E^{\beta_1}, \cdots, E^{\beta_k}]] = \begin{cases} 0 & \text{for} \quad 1 < i < k, \\ [E^{\beta_2}, \cdots, E^{\beta_k}] & \text{for} \quad i = 1, \\ -[E^{\beta_1}, \cdots, E^{\beta_{k-1}}] & \text{for} \quad i = k. \end{cases} \tag{4.2.8}$$

(2) If $1 < s < n$, we have following relations in $\widetilde{\mathfrak{e}}(\Gamma)$:

$$\begin{array}{ll} \textbf{B.1.} & [E^{\alpha_1}, \cdots, E^{\alpha_{s-1}}, E^{\alpha_s^*}, E^{\alpha_{s+1}}, \cdots, E^{\alpha_n}, E^{\alpha_s}] = 0, \\ & [E^{-\alpha_1}, \cdots, E^{-\alpha_{s-1}}, E^{-\alpha_s^*}, E^{-\alpha_{s+1}}, \cdots, E^{-\alpha_n}, E^{-\alpha_s}] = 0, \\ \textbf{B.2.} & [E^{\alpha_1}, \cdots, E^{\alpha_{s-1}}, E^{\alpha_s^*}, E^{\alpha_{s+1}}, \cdots, E^{\alpha_n}, E^{-\alpha_s}] = 0, \\ & [E^{-\alpha_1}, \cdots, E^{-\alpha_{s-1}}, E^{-\alpha_s^*}, E^{-\alpha_{s+1}}, \cdots, E^{-\alpha_n}, E^{\alpha_s}] = 0. \end{array} \tag{4.2.9}$$

(3) If $\alpha, \beta \in \Gamma_{\max}$ satisfy $I(\alpha^\vee, \beta) = -1$, then we have

$$[E^\alpha, E^{\beta^*}] = [E^{\alpha^*}, E^\beta], \quad [E^{-\alpha}, E^{-\beta^*}] = [E^{-\alpha^*}, E^{-\beta}].$$

Proof. (1) These relations are inside the affine Kac-Moody subalgebra of $\mathfrak{e}(S) \simeq \mathfrak{g}(S)$ for a set $S \subset R$ (see (4.1) Assertion 4). So they hold up to sign. Here, we calculate the sign carefully in $\mathfrak{g}(S)$.

(4.2.7): Using (3.1.8) **II.1** recursively, we have

$$\begin{aligned}[[e^{\beta_1},\cdots,e^{\beta_k}],[e^{-\beta_1},\cdots,e^{-\beta_k}]] &= [(-1)^{\sigma_1}e^{\beta_1+\cdots+\beta_k},(-1)^{\sigma_1}e^{-(\beta_1+\cdots+\beta_k)}]\\ &= [e^{\beta_1+\cdots+\beta_k},e^{-(\beta_1+\cdots+\beta_k)}]\\ &= (-1)^{\sigma_2}(\beta_1+\cdots+\beta_k)^\vee(-1),\end{aligned}$$

where, σ_1, $\sigma_2 \in \mathbb{Z}/2\mathbb{Z}$ are determined by ε (4.1.2) and, further, σ_2 is calculated as follows:

$$\begin{aligned}\sigma_2 &= \varepsilon(\beta_1+\cdots+\beta_k,-\beta_1-\cdots-\beta_k)\\ &= \varepsilon(\beta_1,\beta_2)+\varepsilon(\beta_2,\beta_1)+\cdots+\varepsilon(\beta_{k-1},\beta_k)+\varepsilon(\beta_k,\beta_{k-1})\\ &= I_R(\beta_1,\beta_2)+\cdots+I_R(\beta_{k-1},\beta_k) = -k+1.\end{aligned}$$

(4.2.8): Similar to the proof of (4.2.7).

(2) **B.1**: Consider the first one. Delivering E^{α_s} to the left, we have

$$[\cdots,E^{\alpha_{s-2}},[E^{\alpha_{s-1}},E^{\alpha_s}],E^{\alpha_s^*},\cdots]+[\cdots,E^{\alpha_{s-1}},E^{\alpha_s^*},[E^{\alpha_{s+1}},E^{\alpha_s}],E^{\alpha_{s+2}},\cdots].$$

For the first term, delivering $E^{\alpha_s^*}$ to the left, we have

$$[\cdots,E^{\alpha_{s-2}},[E^{\alpha_{s-1}},E^{\alpha_s},E^{\alpha_s^*}],\cdots] \overset{(\mathbf{III})}{=} 0.$$

For the second term, delivering $[E^{\alpha_{s+1}},E^{\alpha_s}]$ to the left, using the fact that $[E^{\alpha_i},[E^{\alpha_{s+1}},E^{\alpha_s}]]=0$ for $i<s-1$ (the relation **II-1**), we have

$$[\cdots,E^{\alpha_{s-2}},[E^{\alpha_{s-1}},[E^{\alpha_{s+1}},E^{\alpha_s}]],E^{\alpha_s^*},\cdots]+[\cdots,[E^{\alpha_s^*},[E^{\alpha_{s+1}},E^{\alpha_s}]],\cdots].$$

The second term is 0 by the relation **III**. For the first term, delivering $E^{\alpha_s^*}$ to the left, we have

$$[\cdots,[E^{\alpha_{s-1}},[E^{\alpha_{s+1}},E^{\alpha_s}],E^{\alpha_s^*}],\cdots] \overset{(\mathbf{IV.2})}{=} 0.$$

The second one is similarly calculated.

B.2: The first one: Delivering $E^{-\alpha_s}$ to the left, we have

$$\begin{aligned}&[\cdots,E^{\alpha_{s-1}},[E^{\alpha_s^*},E^{-\alpha_s}],E^{\alpha_{s+1}},\cdots]\\ =\ &[\cdots,E^{\alpha_{s-2}},[E^{\alpha_{s-1}},[E^{\alpha_s^*},E^{-\alpha_s}]],E^{\alpha_{s+1}},\cdots]\\ =\ &[\cdots,[E^{\alpha_{s-1}},[E^{\alpha_s^*},E^{-\alpha_s}],E^{\alpha_{s+1}}],\cdots]\\ \overset{(\mathbf{IV.1})}{=}\ &0.\end{aligned}$$

The second one is similarly calculated to the first one.

(3) The first equality:

$$\begin{array}{rcl}
[E^{\beta^*}, E^{\alpha}] & \overset{(\mathbf{V})}{=} & [[E^{-\alpha}, E^{\alpha^*}, E^{\beta}], E^{\alpha}] \\
& = & [-h_{\alpha^\vee}, E^{\alpha^*}, E^{\beta}] + [E^{-\alpha}, E^{\alpha^*}, [E^{\beta}, E^{\alpha}]] \\
& \overset{(\mathbf{I} \text{ and } \mathbf{III})}{=} & -2[E^{\alpha^*}, E^{\beta}] + [E^{-\alpha}, [E^{\beta}, E^{\alpha}], E^{\alpha^*}] \\
& = & -[E^{\alpha^*}, E^{\beta}].
\end{array}$$

The second equality is calculated similarly. □

We return to the proof of the first step. Let us check vanishing of the Υ_s-images of the relations (4.2.6) in $\widetilde{\mathfrak{e}}(\Gamma)$. This is clear for the relations which does not contain $\alpha_\triangle$. So we check only the relations which contain $\alpha_\triangle$.

A-0 and **A-I**: These are direct consequences of the relations **0** and **I** in (4.1.1).

A-II.1: Let us denote the sequence $\alpha_1, \cdots, \alpha_{s-1}, \alpha_s^*, \alpha_{s+1}, \cdots, \alpha_n$ by $\alpha_1, \cdots, \alpha_n$ in the calculation below.

$$\begin{array}{rcl}
[E^{\alpha_\triangle}, E^{-\alpha_\triangle}] & = & -v_+v_-[E^{\alpha_1}, \cdots, E^{\alpha_n}, [E^{-\alpha_1}, \cdots, E^{-\alpha_n}]] \\
& \overset{(4.2.7)}{=} & -(-1)^{n-1}v_+v_-\,(h_{\alpha_1^\vee} + \cdots + h_{\alpha_n^\vee}) \\
& = & (-1)^{n-1}v_+v_-\,h_{\alpha_\triangle^\vee}.
\end{array}$$

Let us prove $(-1)^{n-1}v_+v_- = (-1)^{\varepsilon(\alpha_\triangle, \alpha_\triangle)}$. From the definitions of $v_\pm$ (4.2.5) and the proof of (4.2.7), one has

$$\begin{array}{rcl}
[v_+e^{\alpha_\triangle}, v_-e^{-\alpha_\triangle}] & = & [e^{\alpha_1}, \cdots, e^{\alpha_n}, [e^{-\alpha_1}, \cdots, e^{-\alpha_n}]] \\
& = & (-1)^{n-1}h_{\alpha_\triangle^\vee}(-1).
\end{array}$$

On the other hand, recall (3.1.8) **I-2**:

$$[v_+e^{\alpha_\triangle}, v_-e^{-\alpha_\triangle}] = (-1)^{\varepsilon(\alpha_\triangle, \alpha_\triangle)}v_+v_-h_{\alpha_\triangle^\vee}(-1).$$

A-II.2: First we assume the type of R is not $A_2^{(1,1)}$. In this case, we can find $\alpha_s \in A \cap \Gamma_{\max}$ such that $1 < s < n$. If $i \neq s$, then the relation is inside a finite Kac-Moody subalgebra $\mathfrak{e}(A_{s^*} \setminus \{\alpha_s\})$ (see (4.1) Assertion 4). The norms of weights $(1 - \langle h_{\alpha^\vee}, \beta\rangle)\alpha + \beta = \alpha + w_\alpha\beta$ or $\alpha + \beta$ of the relations **A-II.2** are $q_R(\alpha + w_\alpha\beta) = 2 - I_R(\alpha, \beta) \geq 2$ or $q_R(\alpha + \beta) = 2 + I_R(\alpha, \beta) \geq 2$, respectively. Such roots do not exist in the finite Kac-Moody algebra, and Υ_s-images of **A-II.2** vanish. If $i = s$, then this case drops to **B.1** or **B.2** in Assertion 5 (2).

Let us consider the case $A_2^{(1,1)}$. In this case, $\Gamma_{\rm af} = \Gamma_{\rm max}$. By the use of Assertion 5 (3), we have $v_\pm E^{\pm\alpha_\triangle} = [E^{\mp\alpha_1^*}, E^{\mp\alpha_2}] = [E^{\mp\alpha_1}, E^{\mp\alpha_2^*}]$. These mean that this case can be reduced to the case $i \neq s$.

These complete the proof of the first step.

Second step: Let us see that the homomorphism Υ_s satisfies (4.2.2). It is enough to show $\Upsilon_s(e^{\pm\alpha_s^*}) = E^{\pm\alpha_s^*}$ only for the fixed $\alpha_s \in A \cap \Gamma_{\rm max}$. (*Proof.* Let $\alpha, \beta \in A \cap \Gamma_{\rm max}$ and suppose $I(\alpha^\vee, \beta) = -1$, then using the facts $[e^{\mp\beta}, e^{\pm\beta^*}, e^{\pm\alpha}] = e^{\pm\alpha^*}$ in $\mathfrak{g}(A)$ (by (3.1.8)) and the relation **V**: $[E^{\mp\beta}, E^{\pm\beta^*}, E^{\pm\alpha}] = E^{\pm\alpha^*}$ in $\widetilde{\mathfrak{e}}(\Gamma)$, we see that $\Upsilon_s(e^{\pm\beta^*}) = E^{\pm\beta^*}$ implies $\Upsilon_s(e^{\pm\alpha^*}) = E^{\pm\alpha^*}$. Apply this to the fact that the set $A \cap \Gamma_{\rm max}$ is connected by simple bonds. □.) So, the next assertion finishes the proof of Lemma 2.

Assertion 6. *$E^{\alpha_s^*} = \Upsilon_s(e^{\alpha_s^*})$ and $E^{-\alpha_s^*} = \Upsilon_s(e^{-\alpha_s^*})$. That is,*

$$\begin{array}{rcl} E^{\alpha_s^*} & = & \tau_+[E^{-\alpha_{s-1}}, \cdots, E^{-\alpha_1}, E^{-\alpha_\triangle}, E^{-\alpha_n}, \cdots, E^{-\alpha_{s+1}}], \\ E^{-\alpha_s^*} & = & \tau_-[E^{\alpha_{s-1}}, \cdots, E^{\alpha_1}, E^{\alpha_\triangle}, E^{\alpha_n}, \cdots, E^{\alpha_{s+1}}]. \end{array} \tag{4.2.10}$$

where $\tau_\pm \in \{\pm 1\}$ are the constants in $\{\pm 1\}$ defined by

$$\begin{array}{rcl} e^{\alpha_s^*} & = & \tau_+[e^{-\alpha_{s-1}}, \cdots, e^{-\alpha_1}, e^{-\alpha_\triangle}, e^{-\alpha_n}, \cdots, e^{-\alpha_{s+1}}], \\ e^{-\alpha_s^*} & = & \tau_-[e^{\alpha_{s-1}}, \cdots, e^{\alpha_1}, e^{\alpha_\triangle}, e^{\alpha_n}, \cdots, e^{\alpha_{s+1}}]. \end{array} \tag{4.2.11}$$

Proof. Let us show the first one.

$$\begin{array}{rl} & \tau_+ v_-[E^{-\alpha_{s-1}}, \cdots, E^{-\alpha_1}, \\ & \qquad [E^{\alpha_1}, \cdots, E^{\alpha_{s-1}}, E^{\alpha_s^*}, E^{\alpha_{s+1}}, \cdots, E^{\alpha_n}], E^{-\alpha_n}, \cdots, E^{-\alpha_{s+1}}] \\ \overset{\text{(a)}}{=} & \tau_+ v_-[E^{-\alpha_{s-1}}, \cdots, E^{-\alpha_2}, [E^{-\alpha_1}, [E^{\alpha_1}, \cdots, E^{\alpha_n}]], E^{-\alpha_n}, \cdots, E^{-\alpha_{s+1}}] \\ \overset{(4.2.8)}{=} & \tau_+ v_-[E^{-\alpha_{s-1}}, \cdots, E^{-\alpha_2}, [E^{\alpha_2}, \cdots, E^{\alpha_n}], E^{-\alpha_n}, \cdots, E^{-\alpha_{s+1}}] \\ = & \cdots \\ = & \tau_+ v_-[[E^{\alpha_s^*}, E^{\alpha_{s+1}}, \cdots, E^{\alpha_n}], E^{-\alpha_n}, \cdots, E^{-\alpha_{s+1}}] \\ \overset{(4.2.8)}{=} & \tau_+ v_-[E^{\alpha_s^*}, E^{\alpha_{s+1}}, \cdots, E^{\alpha_{n-1}}, E^{-\alpha_{n-1}}, \cdots, E^{-\alpha_{s+1}}] \\ = & \cdots \\ = & \tau_+ v_- E^{\alpha_s^*}, \end{array}$$

where in (a), deliver $[E^{\alpha_1}, \cdots, E^{\alpha_n}]$ to the left and use the relation (4.2.8).

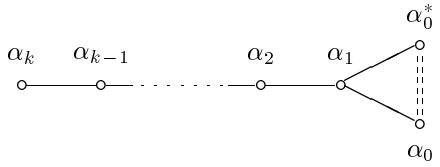
Let us show $\tau_+ v_- = 1$. From the definitions of τ_+ and v_- and proceeding similar calculations as above, one has

$$\begin{aligned}\tau_+ e^{\alpha_s^*} &= [e^{-\alpha_{s-1}}, \cdots, e^{-\alpha_1}, e^{-\alpha_\triangle}, e^{-\alpha_n}, \cdots, e^{-\alpha_{s+1}}] \\ &= [e^{-\alpha_{s-1}}, \cdots, e^{-\alpha_1}, v_-[e^{\alpha_1}, \cdots, e^{\alpha_{s-1}}, e^{\alpha_s^*}, e^{\alpha_{s+1}}, \cdots, e^{\alpha_n}], \\ &\qquad\qquad e^{-\alpha_n}, \cdots, e^{-\alpha_{s+1}}] \\ &= v_- e^{\alpha_s^*}.\end{aligned}$$

This shows $\tau_+ v_- = 1$, and finish the proof of Lemma 2. □

Remark 5. In [Sl2], Slodowy has shown a weaker statement than that of Lemma 2 that the diagram $A_{\mathrm{af}} \cup \{\beta^*\}$ is braid equivalent to the diagram $A_\triangle$, and hence the i.m. algebra for $A_{\mathrm{af}} \cup \{\beta^*\}$ is isomorphic to the Kac-Moody algebra for $A_\triangle$.

(4.3) Let us construct elements of $\widetilde{\mathfrak{e}}(\Gamma)$ whose weight belongs to the integral marking $G_{\mathbb{Z}} = \mathbb{Z}a$ (2.3.1). For the purpose, let us define $E^{\pm\alpha^*}$ not only for $\alpha \in \Gamma_{\max}$ but also $\alpha \in \Gamma_{\mathrm{af}} \setminus \Gamma_{\max}$. First we recall $\alpha^* := \alpha + a$ ((2.4.3), (2.6) Fact 4) and $\Gamma = \Gamma_{\mathrm{af}} \cup \Gamma^*_{\max}$ (2.4.7), where $\Gamma_{\max}$ is a connected subdiagram of the affine diagram Γ_{af} such that the compliment $\Gamma_{\mathrm{af}} \setminus \Gamma_{\max}$ is a union $\cup_{j=1}^r \Gamma_j$ of A_{l_j}-type diagrams (Appendix A). Let $\alpha_0 \in \Gamma_{\max}$ be an element connected to a connected component Γ_j. Let the elements of Γ_j be ordered from the side α_0 as $\alpha_1, \alpha_2, \cdots, \alpha_k$, as in the diagram below.



Inductively, we define $E^{\alpha_i^*} := [E^{-\alpha_{i-1}}, E^{\alpha_{i-1}^*}, E^{\alpha_i}]$ and $E^{-\alpha_i^*} := [E^{\alpha_{i-1}}, E^{-\alpha_{i-1}^*}, E^{-\alpha_i}]$ for $i = 1, \cdots, k$. Then we have $[h, E^{\pm\alpha^*}] = \langle h, \pm\alpha^* \rangle E^{\pm\alpha^*}$ for any $h \in \widetilde{\mathfrak{h}}$ and $\alpha \in \Gamma_{\mathrm{af}}$. For any A-part to which α belongs contains $E^{\pm\alpha^*} \in \mathfrak{e}(A)$. One easily check (using the argument in the second step of the proof of (4.2) Lemma 2) the correspondence:

$$\varphi : E^{\pm\alpha^*} \mapsto e^{\pm\alpha^*} \qquad \text{for } \alpha \in \Gamma_{\mathrm{af}}. \tag{4.3.1}$$

Assertion 7. *Let n be a positive integer and $\alpha \in \Gamma_{\mathrm{af}}$, then,*

$$\begin{aligned}
[\underbrace{e^{\alpha^*}, e^{-\alpha}}_{1}, \cdots, \underbrace{e^{\alpha^*}, e^{-\alpha}}_{n}] &= 2^{n-1} h_{\alpha^\vee}(-1) e^{ka}, \\
[\underbrace{e^{\alpha}, e^{-\alpha^*}}_{1}, \cdots, \underbrace{e^{\alpha}, e^{-\alpha^*}}_{n}] &= 2^{n-1} h_{\alpha^\vee}(-1) e^{-ka}.
\end{aligned} \tag{4.3.2}$$

Proof. One can show this by induction on k using (3.1.8). □

In view of Assertion 7 and (4.3.1), we make the next definition.

Definition 4. *Define $H_{\alpha^\vee}^{(n)}$ for any $\alpha \in \Gamma_{\mathrm{af}}$ and for any integer $n \in \mathbb{Z}$ by*

$$\begin{cases}
H_{\alpha^\vee}^{(m)} & := 2^{-(m-1)}[E^{\alpha^*}, E^{-\alpha}, \cdots, E^{\alpha^*}, E^{-\alpha}], \\
H_{\alpha^\vee}^{(-m)} & := 2^{-(m-1)}[E^{\alpha}, E^{-\alpha^*}, \cdots, E^{\alpha}, E^{-\alpha^*}], \\
H_{\alpha^\vee}^{(0)} & := h_{\alpha^\vee},
\end{cases} \tag{4.3.3}$$

where $m \in \mathbb{Z}_{>0}$, and for the first (resp. second cases), E^{α^*} and $E^{-\alpha}$ (resp. E^{α} and $E^{-\alpha^*}$) appear m times. Put $H_{-\alpha^\vee}^{(n)} := -H_{\alpha^\vee}^{(n)}$ for any $\alpha \in \Gamma_{\mathrm{af}}$ and any $n \in \mathbb{Z}$.

Note that $\varphi(H_{\alpha^\vee}^{(n)}) = h_{\alpha^\vee}(-1)e^{na}$ for any $\alpha \in \pm\Gamma_{\mathrm{af}}$ and $n \in \mathbb{Z} \setminus \{0\}$ (see (4.3.1)).

Lemma 3. *For any given A-part $A \subset \Gamma$, Υ in (4.2.2) satisfies:*

$$\Upsilon : \begin{array}{ccc}
h_{\alpha^\vee}(-1) & \mapsto & h_{\alpha^\vee}, \\
e^{\alpha} & \mapsto & E^{\alpha}, \\
h_{\alpha^\vee}(-1)e^{na} & \mapsto & H_{\alpha^\vee}^{(n)},
\end{array} \tag{4.3.4}$$

where α runs over $\{\pm\alpha, \pm\alpha^* \mid \alpha \in A\} = \pm A \cup \pm A^*$.

Proof. It is clear from (4.2) Lemma 2, (4.3.1) and the definitions of $H_{\alpha^\vee}^{(n)}$. □

The algebra $\widetilde{\mathfrak{e}}(\Gamma)$ inherits all the relations in $\mathfrak{g}(A)$ ((4.2) Lemma 2). First we should list up some calculations in $\widetilde{\mathfrak{g}}(R)$ by a use of (3.1.8).

Assertion 8. *Let $n, m \in \mathbb{Z}$, $\alpha, \beta \in R$ and $h, g \in \mathfrak{h}$, then one has:*

H-II. $[h(-1)e^{ma}, g(-1)e^{na}] = \widetilde{I}^*(h,g) m \delta_{m+n,0} h_a(-1) e^{(m+n)a}$,

I*. $[e^{\alpha}, h(-1)e^{ma}, g(-1)e^{na}] =$
$-\frac{1}{2}\langle h, \alpha\rangle\langle g, \alpha\rangle [e^{\alpha}, h_{\alpha^\vee}(-1)e^{(n+m)a}]$,

II*.1. $[h(-1)e^{na}, e^{\alpha}, e^{-\alpha}] =$
$[h(-1)e^{na}, e^{-\alpha}, e^{\alpha}] = \langle h, \alpha\rangle h_{\alpha^\vee}(-1)e^{na}$,

II*.2. $[h(-1)e^{na}, e^{\alpha}, e^{\beta}] = 0 \quad for \quad I(\alpha, \beta) \geq 0$.

(4.3.5)

Proof. Direct calculation by a use of (3.1.8). See (4.1) Remark 3. □

Lemma 4. *The following relations hold in* $\widetilde{\mathfrak{e}}(\Gamma)$*:*

$$\begin{array}{ll}
\textbf{H-I.} & [h, H^{(n)}_{\alpha^\vee}] = \langle h, na\rangle H^{(n)}_{\alpha^\vee}, \\
\textbf{H-II.} & [H^{(m)}_{\alpha^\vee}, H^{(n)}_{\beta^\vee}] = I(\alpha^\vee, \beta) m\delta_{m+n,0} h_{a^\vee}, \\
\textbf{I*.} & [E^\alpha, H^{(m)}_{\beta^\vee}, H^{(n)}_{\gamma^\vee}] = -\frac{1}{2} I(\beta^\vee, \alpha) I(\gamma^\vee, \alpha) [E^\alpha, H^{(n+m)}_{\alpha^\vee}], \\
\textbf{II*.1.} & [H^{(n)}_{\alpha^\vee}, E^\beta, E^{-\beta}] = [H^{(n)}_{\alpha^\vee}, E^{-\beta}, E^\beta] = I(\alpha^\vee, \beta) H^{(n)}_{\beta^\vee}, \\
\textbf{II*.2.} & [H^{(n)}_{\alpha^\vee}, E^\beta, E^\gamma] = 0 \quad \textit{for} \quad I(\beta, \gamma) \geq 0.
\end{array} \tag{4.3.6}$$

where $h \in \widetilde{\mathfrak{h}}$, $n, m \in \mathbb{Z}$ *and* $\alpha, \beta, \gamma \in \pm\Gamma_{\mathrm{af}}$.

Proof. Since the formula **H-II** in (4.3.6) contains a basis $h_{a^\vee}$, we treat this case separately. The relation **H-II** in (4.3.5) is a specialization of the relation **IV** in (3.1.8):

$$\begin{aligned}
[h(-1)e^{ma}, g(-1)e^{na}] &= \widetilde{I}^*_R(h,g) m\delta_{m+n,0} \widetilde{I}_R(a,\cdot)(-1)e^{(m+n)a} \\
&= \frac{1}{c}\widetilde{I}^*(h,g) m\delta_{m+n,0} h_{a^\vee}(-1)e^{(m+n)a},
\end{aligned}$$

where, in the second equality we replace $\widetilde{I}_R$ by $\widetilde{I}$ ($I_R = c \cdot I$ (2.1.3)) and use the fact that $h_a(-1) = h_{a^\vee}(-1)$ (we use the normalized form I_R in $\widetilde{\mathfrak{g}}(R)$) and that $c \cdot \widetilde{I}^*_R = \widetilde{I}^*$ on $\widetilde{\mathfrak{h}}$ (use $\alpha^\vee = c \cdot \alpha$ for $\alpha \in R$). One has $\frac{1}{c} I(\alpha^\vee, \beta^\vee) = I(\alpha^\vee, \beta) = I(\beta^\vee, \alpha) \in \mathbb{Z}$ for any $\alpha, \beta \in R$. See also (4.1) Remark 3.

H-I: Clear from the definition of $H^{(n)}_{\alpha^\vee}$. **II*.1**: Take an A-part A containing $\{\alpha, \beta\}$ (4.2). And apply (4.3) Lemma 3 for A. These are Υ-images of the relations **II*.1** in (4.3.5), respectively. Similar argument shows that we get the following relations:

$$[H^{(n)}_{\alpha^\vee}, E^\beta] = \frac{1}{2} I(\alpha^\vee, \beta)[H^{(n)}_{\beta^\vee}, E^\beta] \tag{4.3.7}$$

(note that the relations can be induced from the relation **I***). **I*** and **II*.2**: Use (4.3.7) for the first bracket and after that take an A-part A of $\{\beta, \gamma\}$. These are Υ-images of the relations **I*** and **II*.2** in (4.3.5), respectively. □

(4.4) We introduce three Lie subalgebras of $\widetilde{\mathfrak{e}}(\Gamma)$: $\mathfrak{h}_{\mathrm{af}}^{\mathbb{Z}'}$, $\widetilde{\mathfrak{h}}_{\mathrm{af}}^{\mathbb{Z}}$ and $\mathfrak{g}_{\mathrm{af}}$. They are the subalgebras generated by $\mathfrak{B}^0 \cup \{h_{a^\vee}\}$, $\mathfrak{B}^\circ \cup \widetilde{\mathfrak{h}}$ and $\mathfrak{B}^+ \cup \mathfrak{B}^-$, respectively, where

$$\begin{array}{lcl} \mathbb{Z}' & := & \mathbb{Z} \setminus \{0\}, \\ \mathfrak{B}^0 & := & \{H_{\alpha^\vee}^{(n)} \mid \alpha \in \Gamma_{\mathrm{af}}, n \in \mathbb{Z}'\}, \\ \mathfrak{B}^+ & := & \{E^\alpha \mid \alpha \in \Gamma_{\mathrm{af}}\}, \\ \mathfrak{B}^- & := & \{E^{-\alpha} \mid \alpha \in \Gamma_{\mathrm{af}}\}. \end{array} \tag{4.4.1}$$

Lemma 5. *(1)* $\mathfrak{h}_{\mathrm{af}}^{\mathbb{Z}'}$ *is a Heisenberg algebra:*

$$\mathfrak{h}_{\mathrm{af}}^{\mathbb{Z}'} = \mathbb{Q}h_{a^\vee} \oplus \bigoplus_{n \in \mathbb{Z}'} \mathfrak{h}_{\mathrm{af}}^{(n)} \quad \textit{where} \quad \mathfrak{h}_{\mathrm{af}}^{(n)} := \oplus_{\alpha \in \Gamma_{\mathrm{af}}} \mathbb{Q} H_{\alpha^\vee}^{(n)}. \tag{4.4.2}$$

(2) $\widetilde{\mathfrak{h}}_{\mathrm{af}}^{\mathbb{Z}}$ *is an extension of* $\mathfrak{h}_{\mathrm{af}}^{\mathbb{Z}'}$*:* $\widetilde{\mathfrak{h}}_{\mathrm{af}}^{\mathbb{Z}} = \widetilde{\mathfrak{h}} \oplus \bigoplus_{n \in \mathbb{Z}'} \mathfrak{h}_{\mathrm{af}}^{(n)}$.

(3) $\mathfrak{g}_{\mathrm{af}} = \mathfrak{e}(\Gamma_{\mathrm{af}})$ *is isomorphic to the affine Kac-Moody algebra* $\mathfrak{g}(\Gamma_{\mathrm{af}})$.

(4) $\widetilde{\mathfrak{e}}(\Gamma)$ *is generated by* $\widetilde{\mathfrak{h}}_{\mathrm{af}}^{\mathbb{Z}}$ *and* $\mathfrak{g}_{\mathrm{af}}$.

Proof. (1), (2): The relations **H-I** and **H-II** in (4.3.6) are the defining relations for the Heisenberg algebra and its extension. Linear independence of components of direct sum follows from linear independence of their φ-images. (3): See (4.1) Assertion 4. (4): It follows the fact that for $\alpha \in \Gamma_{\mathrm{af}}$, one has $[H_{\alpha^\vee}^{(1)}, E^\alpha] = 2E^{\alpha^*}$ and $[H_{\alpha^\vee}^{(-1)}, E^{-\alpha}] = -2E^{-\alpha^*}$. Let us show the first relation. By definition, we have $[H_{\alpha^\vee}^{(1)}, E^\alpha] = [E^{\alpha^*}, E^{-\alpha}, E^\alpha]$. Delivering E^α to the left, we get $[E^{\alpha^*}, -h_{\alpha^\vee}]$ ($[E^{\alpha^*}, E^\alpha] = 0$ since $q_R(\alpha + \alpha^*) > 1$ and (4.3) Lemma 3) and it is $2E^{\alpha^*}$. The second relation is shown similarly. □

5. The amalgamation $\widetilde{\mathfrak{h}}_{\mathrm{af}}^{\mathbb{Z}} * \mathfrak{g}_{\mathrm{af}}$ of affine Kac-Moody and Heisenberg algebras

We introduce the third Lie algebra $\widetilde{\mathfrak{h}}_{\mathrm{af}}^{\mathbb{Z}} * \mathfrak{g}_{\mathrm{af}}$ attached to a simply-laced *marked* elliptic root system as an amalgamation of a Heisenberg algebra and an affine Kac-Moody algebra. The algebra admits a triangular decomposition in a generalized sense. By definition, there is a natural surjective homomorphism ϱ : $\widetilde{\mathfrak{h}}_{\mathrm{af}}^{\mathbb{Z}} * \mathfrak{g}_{\mathrm{af}} \to \widetilde{\mathfrak{e}}(\Gamma)$.

In this section, we prove that the three Lie algebras $\widetilde{\mathfrak{g}}(R)$, $\widetilde{\mathfrak{e}}(\Gamma)$ and $\widetilde{\mathfrak{h}}^{\mathbb{Z}}_{\mathrm{af}} * \mathfrak{g}_{\mathrm{af}}$ are isomorphic. The proof is achieved by showing the multiplicity of the root spaces of $\widetilde{\mathfrak{h}}^{\mathbb{Z}}_{\mathrm{af}} * \mathfrak{g}_{\mathrm{af}}$ does not exceed that of $\widetilde{\mathfrak{g}}(\Gamma)$.

(5.1) Let (R, G) be a simply-laced marked elliptic root system. Recall the algebras $\widetilde{\mathfrak{h}}^{\mathbb{Z}}_{\mathrm{af}}$ and $\mathfrak{g}_{\mathrm{af}} = \mathfrak{e}(\Gamma_{\mathrm{af}}) \simeq \mathfrak{g}(\Gamma_{\mathrm{af}})$ in (4.4). They have a common abelian subalgebra $\mathfrak{h}_{\mathrm{af}} := \oplus_{\alpha \in \Gamma_{\mathrm{af}}} \mathbb{Q} h_{\alpha^\vee}$.

Notation. By the *amalgamation* $\mathfrak{g}_1 * \mathfrak{g}_2$ of two Lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$, we mean the Lie subalgebra generated by $\mathfrak{g}_i$ $(i = 1, 2)$ in $T(\mathfrak{g}_1 \oplus \mathfrak{g}_2)/I$, where $T(V)$ means the tensor algebra of a vector space V and I is the both-side ideal generated by the elements $g_i \otimes h_i - h_i \otimes g_i - [\, g_i, h_i]$ for all $g_i, h_i \in \mathfrak{g}_i$. If, further, there are Lie algebra homomorphisms $\varphi_i : \mathfrak{g} \to \mathfrak{g}_i \, (i = 1, 2)$, we denote by $\mathfrak{g}_1 *_{\mathfrak{g}} \mathfrak{g}_2$ the Lie algebra defined similarly but adding more relations $\varphi_1(g) - \varphi_2(g)$ for $g \in \mathfrak{g}$ to the generators of the ideal I. For an abuse of notation, we sometimes call a quotient algebra of $\mathfrak{g}_1 * \mathfrak{g}_2$ also an amalgamation of $\mathfrak{g}_1$ and $\mathfrak{g}_2$ and denote it by $\mathfrak{g}_1 * \mathfrak{g}_2$.

Definition 5. *We define the Lie algebra* $\widetilde{\mathfrak{h}}^{\mathbb{Z}}_{\mathrm{af}} * \mathfrak{g}_{\mathrm{af}}$ *as the quotient algebra of the amalgamation* $\widetilde{\mathfrak{h}}^{\mathbb{Z}}_{\mathrm{af}} *_{\mathfrak{h}_{\mathrm{af}}} \mathfrak{g}_{\mathrm{af}}$ *of* $\widetilde{\mathfrak{h}}^{\mathbb{Z}}_{\mathrm{af}}$ *and* $\mathfrak{g}_{\mathrm{af}}$ *divided by the ideal defined by the following relations* **I*** *and* **II****:*

$$
\begin{array}{ll}
\textbf{I*.} & [E^\alpha, H^{(m)}_{\beta^\vee}, H^{(n)}_{\gamma^\vee}] = -\frac{1}{2} I(\beta^\vee, \alpha) I(\gamma^\vee, \alpha) [E^\alpha, H^{(n+m)}_{\alpha^\vee}], \\
\textbf{II*.1.} & [H^{(n)}_{\alpha^\vee}, E^\beta, E^{-\beta}] = [H^{(n)}_{\alpha^\vee}, E^{-\beta}, E^\beta] = I(\alpha^\vee, \beta) H^{(n)}_{\beta^\vee}, \\
\textbf{II*.2.} & [H^{(n)}_{\alpha^\vee}, E^\beta, E^\gamma] = 0 \quad \textit{for} \quad I(\beta, \gamma) \geq 0.
\end{array}
\tag{5.1.1}
$$

where $\alpha, \beta, \gamma \in \pm\Gamma_{\mathrm{af}}$, $m, n \in \mathbb{Z}$ *and* $H^{(0)}_{\alpha^\vee} := h_{\alpha^\vee}$ *for any* $\alpha \in \pm\Gamma_{\mathrm{af}}$.

All the relations in (5.1.1) are satisfied in $\widetilde{\mathfrak{e}}(\Gamma)$ (**0**, **I** and **II** are in (4.1.1) and the rests are in (4.3.6)). So, due to (4.4) Lemma 5 (4), the next fact follows.

Assertion 9. *The natural homomorphisms in (4.4) from* $\widetilde{\mathfrak{h}}^{\mathbb{Z}}_{\mathrm{af}}$ *and* $\mathfrak{g}_{\mathrm{af}}$ *to* $\widetilde{\mathfrak{e}}(\Gamma)$ *induce a surjective homomorphism*

$$
\varrho \ : \ \widetilde{\mathfrak{h}}^{\mathbb{Z}}_{\mathrm{af}} * \mathfrak{g}_{\mathrm{af}} \ \to \ \widetilde{\mathfrak{e}}(\Gamma). \tag{5.1.2}
$$

As a consequence of the above assertion, the three algebras $\widetilde{\mathfrak{h}}_{\mathrm{af}}^{\mathbb{Z}}$, $\mathfrak{h}_{\mathrm{af}}^{\mathbb{Z}'}$ and $\mathfrak{g}_{\mathrm{af}}$ are considered as subalgebras of $\widetilde{\mathfrak{h}}_{\mathrm{af}}^{\mathbb{Z}} * \mathfrak{g}_{\mathrm{af}}$. We can consider the root space decomposition of $\widetilde{\mathfrak{h}}_{\mathrm{af}}^{\mathbb{Z}} * \mathfrak{g}_{\mathrm{af}}$ with respect to $\widetilde{\mathfrak{h}}$ since all the generators of $\widetilde{\mathfrak{h}}_{\mathrm{af}}^{\mathbb{Z}} * \mathfrak{g}_{\mathrm{af}}$ are weight vectors (**I** in (4.1.2) and **H-I** in (4.3.6)) and their weights are in $Q_{\mathrm{af}} \oplus \mathbb{Z}a = Q(R)$ (2.3.3). The set of all roots is denoted by Δ. By definition, ϱ maps root spaces of $\widetilde{\mathfrak{h}}_{\mathrm{af}}^{\mathbb{Z}} * \mathfrak{g}_{\mathrm{af}}$ to those of $\widetilde{\mathfrak{e}}(\Gamma)$.

Due to the symmetry of the defining relations, there exists an involution ω, called the Cartan involution, on $\widetilde{\mathfrak{h}}_{\mathrm{af}}^{\mathbb{Z}} * \mathfrak{g}_{\mathrm{af}}$ defined by $h \mapsto -h$ for $h \in \widetilde{\mathfrak{h}}$, $H_{\alpha^\vee}^{(n)} \mapsto -H_{\alpha^\vee}^{(-n)}$ for $\alpha \in \pm\Gamma_{\mathrm{af}}$, $n \in \mathbb{Z}$ and $E^\alpha \mapsto E^{-\alpha}$ for $\alpha \in \pm\Gamma_{\mathrm{af}}$. The Cartan involution brings a root space of α to a root space of $-\alpha$.

(5.2) We study the root space decomposition of $\widetilde{\mathfrak{h}}_{\mathrm{af}}^{\mathbb{Z}} * \mathfrak{g}_{\mathrm{af}}$ with respect to $\widetilde{\mathfrak{h}}$. Recall the cones Q_{af}^+ and Q_{af}^- in the affine lattice Q_{af} (2.3). Put

$$Q^+ := Q_{\mathrm{af}}^+ \oplus \mathbb{Z}a, \quad Q^0 := \mathbb{Z}a = G_{\mathbb{Z}}, \quad Q^- := Q_{\mathrm{af}}^- \oplus \mathbb{Z}a. \tag{5.2.1}$$

Lemma 6. *The set of roots Δ of $\widetilde{\mathfrak{h}}_{\mathrm{af}}^{\mathbb{Z}} * \mathfrak{g}_{\mathrm{af}}$ is contained in $Q^+ \cup Q^0 \cup Q^-$. So we have the decomposition*

$$\widetilde{\mathfrak{h}}_{\mathrm{af}}^{\mathbb{Z}} * \mathfrak{g}_{\mathrm{af}} = \widetilde{\mathfrak{h}}_{\mathrm{af}}^{\mathbb{Z}} \oplus \mathfrak{n}_{\mathrm{ell}}^+ \oplus \mathfrak{n}_{\mathrm{ell}}^-, \tag{5.2.2}$$

where, $\mathfrak{n}_{\mathrm{ell}}^\sigma := \oplus_{\alpha \in \Delta \cap Q_{\mathrm{af}}^\sigma} (\widetilde{\mathfrak{h}}_{\mathrm{af}}^{\mathbb{Z}} * \mathfrak{g}_{\mathrm{af}})_\alpha$ for $\sigma \in \{\pm 1\}$. The $\mathfrak{n}_{\mathrm{ell}}^\sigma$ is an ideal of the algebra $\mathfrak{u}_{\mathrm{ell}}^\sigma := \langle \widetilde{\mathfrak{h}}_{\mathrm{af}}^{\mathbb{Z}}, \mathfrak{n}_{\mathrm{af}}^\sigma \rangle = \widetilde{\mathfrak{h}}_{\mathrm{af}}^{\mathbb{Z}} \oplus \mathfrak{n}_{\mathrm{ell}}^\sigma$ generated by $\mathfrak{n}_{\mathrm{af}}^\sigma := \langle E^\alpha \mid \alpha \in \sigma\Gamma_{\mathrm{af}} \rangle$ and is nilpotent in the sense:

$$\bigcap_{m=1}^{\infty} [\underbrace{\mathfrak{n}_{\mathrm{ell}}^\sigma, \cdots, \mathfrak{n}_{\mathrm{ell}}^\sigma}_{m\text{-times}}] = \{0\} \quad \textit{for} \quad \sigma \in \{\pm 1\}. \tag{5.2.3}$$

Proof. To prove this, we add notation to (4.1.1):

$$\begin{array}{lcl} \mathfrak{B} & := & \mathfrak{B}^+ \cup \mathfrak{B}^0 \cup \mathfrak{B}^-, \\ G & := & \widetilde{\mathfrak{h}} \cup \{[B_1, \cdots, B_k] \mid B_i \in \mathfrak{B}^+ \cup \mathfrak{B}^0,\ k \in \mathbb{Z}_{>0}\} \\ & & \qquad \cup \{[B_1, \cdots, B_k] \mid B_i \in \mathfrak{B}^- \cup \mathfrak{B}^0,\ k \in \mathbb{Z}_{>0}\}, \\ V & := & \text{the subspace of } \widetilde{\mathfrak{h}}_{\mathrm{af}}^{\mathbb{Z}} * \mathfrak{g}_{\mathrm{af}} \text{ spaned by } G. \end{array} \tag{5.2.4}$$

It is enough to show $[B_1, \cdots, B_k] \in V$ where $B_i \in \mathfrak{B}$. We show this by induction on k. Case $k = 1$ is clear. Assume case k, then $[B_1, \cdots, B_k, B_{k+1}] =$

$[[B_1,\cdots,B_k],B_{k+1}]$. Expressing $[B_1,\cdots,B_k]\in V$ as a linear combination of elements in G, it is enough to show following three cases: (i) $[C_1^+,\cdots,C_m^+,B_{k+1}]\in V$ for any $C_i^+\in\mathfrak{B}^+\cup\mathfrak{B}^\circ$ and any $m\in\mathbb{Z}_{>0}$, (ii) $[C_1^-,\cdots,C_m^-,B_{k+1}]\in V$ for any $C_i^-\in\mathfrak{B}^-\cup\mathfrak{B}^\circ$ and any $m\in\mathbb{Z}_{>0}$, and (iii) $[h,B_{k+1}]\in V$ for any $h\in\widetilde{\mathfrak{h}}$. Case (iii) is clear by **I** and **H-I**. Let us consider case (i). Case (ii) can be shown similarly. If $B_{k+1}\in\mathfrak{B}^+\cup\mathfrak{B}^0$, it is clearly in V. So we assume $B_{k+1}^-:=B_{k+1}\in\mathfrak{B}^-$. Delivering B_{k+1}^- to the left, we have $\sum_{i=1}^k[\cdots,C_{i-1}^+,[C_i^+,B_{k+1}^-],C_{i+1}^+,\cdots]$. If $C_i^+\in\mathfrak{B}^+$, $[C_i^+,B_{k+1}^-]$ is in $\widetilde{\mathfrak{h}}$ by **II** and the i-th term is in V. If $H_i:=C_i^+\in\mathfrak{B}^0$, the i-th term is $[\cdots,C_{i-1}^+,[H_i,B_{k+1}^-],C_{i+1}^+,\cdots]$. If $i=1$, then using relations either **I***, **II*.1** or **II*.2** in (5.1.1), we can reduce the 1st-term to the case $\le k$. If $i>1$, deliver $[H_i,B_{k+1}^-]$ further to the left. Again, using relations in (5.1.1), we can reduce the ith-term to the case$\le k$. □

As a consequence of Lemma 6, we can already determine some of root spaces.

Lemma 7. *(1)* $(\widetilde{\mathfrak{h}}_{\mathrm{af}}^{\mathbb{Z}}*\mathfrak{g}_{\mathrm{af}})_0=\widetilde{\mathfrak{h}}$,

(2) $(\widetilde{\mathfrak{h}}_{\mathrm{af}}^{\mathbb{Z}}*\mathfrak{g}_{\mathrm{af}})_{\alpha+na}=\mathbb{Q}[E^\alpha,H_{\alpha^\vee}^{(n)}]$ *for any* $\alpha\in\Gamma_{\mathrm{af}}$ *and* $n\in\mathbb{Z}$,

(3) $(\widetilde{\mathfrak{h}}_{\mathrm{af}}^{\mathbb{Z}}*\mathfrak{g}_{\mathrm{af}})_{m\alpha+na}=0$ *for any* $\alpha\in\pm\Gamma_{\mathrm{af}}$, $n\in\mathbb{Z}$ *such that* $|m|\ge 2$, *and for any* $n\in\mathbb{Z}$.

(4) $(\widetilde{\mathfrak{h}}_{\mathrm{af}}^{\mathbb{Z}}*\mathfrak{g}_{\mathrm{af}})_{na}=\oplus_{\alpha\in\Gamma_{\mathrm{af}}}\mathbb{Q}H_\alpha^{(n)}$ *for any* $n\in\mathbb{Z}\setminus\{0\}$.

Proof. Recall the relations in (4.3.6). (1) According to Lemma 6, $(\widetilde{\mathfrak{h}}_{\mathrm{af}}^{\mathbb{Z}}*\mathfrak{g}_{\mathrm{af}})_0$ is spanned by $\widetilde{\mathfrak{h}}$ and elements $[H_{\beta_1^\vee}^{(n_1)},\cdots,H_{\beta_k^\vee}^{(n_k)}]$ for $\beta_i\in\Gamma_{\mathrm{af}}$ satisfying $n_1+\cdots+n_k=0$. If $k\ge 2$, first bracket is in $\mathbb{Q}h_{\alpha^\vee}$ by **H-II**. So if $k>2$, it is 0 by **H-I**. (2) For $\alpha\in\Gamma_{\mathrm{af}}$, $(\widetilde{\mathfrak{h}}_{\mathrm{af}}^{\mathbb{Z}}*\mathfrak{g}_{\mathrm{af}})_{\alpha+na}$ is spanned by elements $[\cdots,H_{\beta_{s-1}^\vee}^{(n_{s-1})},E^\alpha,H_{\beta_s^\vee}^{(n_s)},\cdots]$. That is: one entry is $E^\alpha\in\mathfrak{B}^+$ and the others are in $\mathfrak{B}^0$ satisfying $n_1+\cdots+n_k=n$. Using **H-II** and **I*** repeatedly, we finally find it is either in $\mathbb{Q}E^\alpha$ (if $n=0$) or in $\mathbb{Q}[E^\alpha,H_{\alpha^\vee}^{(n)}]$ (if $n\ne 0$). (3) $(\widetilde{\mathfrak{h}}_{\mathrm{af}}^{\mathbb{Z}}*\mathfrak{g}_{\mathrm{af}})_{m\alpha+na}$ for $m\ge 2$ is spanned by elements $[A,E^\alpha,B,E^\alpha,\cdots]$, where A and B are sequences of $\mathfrak{B}^0$. Similarly to the proof of (2) above, we know this becomes $[E^\alpha,H_{\alpha^\vee}^{(n)},E^\alpha,\cdots]$ and it is 0 by **II*.2**. (4) $(\widetilde{\mathfrak{h}}_{\mathrm{af}}^{\mathbb{Z}}*\mathfrak{g}_{\mathrm{af}})_{na}$ is spanned by elements $[H_{\beta_1^\vee}^{(n_1)},\cdots,H_{\beta_k^\vee}^{(n_k)}]$, where $\beta_1,\cdots,\beta_k\in\Gamma_{\mathrm{af}}$ and $n_1+\cdots+n_k=n$. Use **H-II**. □

(5.3) Let us study the root space decomposition in more detail by lifting the actions of reflections on $Q(R)$. Recall the reflection w_α on $Q(R)$ with respect to $\alpha \in R$ and the Weyl group $W(R) = \langle w_\alpha \mid \alpha \in R \rangle$ (2.1). We define the action $\widetilde{w}_\alpha$ on $\widetilde{\mathfrak{h}}$ by $\widetilde{w}_\alpha(h) := h - \langle h, \alpha \rangle h_{\alpha^\vee}$ for any $h \in \widetilde{\mathfrak{h}}$ so that $\langle \widetilde{w}_\alpha(h), \beta \rangle = \langle h, w_\alpha(\beta) \rangle$ for any $\beta \in Q(R)$. Here, we lift the action of the subgroup $W_{\mathrm{af}} := \langle w_{\mathrm{af}} \mid \alpha \in \Gamma_{\mathrm{af}} \rangle$ of $W(R)$ generated by reflections of elements in Γ_{af}.

Fisrt note that the action of $ad\, E^\alpha$ on $\widetilde{\mathfrak{h}}^{\mathbb{Z}}_{\mathrm{af}} * \mathfrak{g}_{\mathrm{af}}$ is locally nilpotent for any $\alpha \in \pm\Gamma_{\mathrm{af}}$. (*Proof.* It is enough to show its nilpotency for any generators. For $\widetilde{\mathfrak{h}}$, see **I**, for $\mathfrak{B}^\pm$, see **II**, and for $\mathfrak{B}^\circ$, see **II*.2**.$\square$) Thanks to this fact, we can define the exponential of $ad\, E^\alpha$: $exp(ad\, E^\alpha) := \sum_{i=0}^{\infty} \frac{1}{i!} (ad\, E^\alpha)^i$ for any $\alpha \in \pm\Gamma_{\mathrm{af}}$. This is an automorphism of the algebra $\widetilde{\mathfrak{h}}^{\mathbb{Z}}_{\mathrm{af}} * \mathfrak{g}_{\mathrm{af}}$. Composing these automorphisms, we define

$$n_\alpha := exp(ad\, E^\alpha) exp(ad\, (-E^{-\alpha})) exp(ad\, E^\alpha) \tag{5.3.1}$$

for any $\alpha \in \Gamma_{\mathrm{af}}$. The following facts are more or less standard: i) n_α is an automorphism of $\widetilde{\mathfrak{h}}^{\mathbb{Z}}_{\mathrm{af}} * \mathfrak{g}_{\mathrm{af}}$ whose restriction on $\widetilde{\mathfrak{h}}$ is equal to $\widetilde{w}_\alpha$. ii) n_α maps $(\widetilde{\mathfrak{h}}^{\mathbb{Z}}_{\mathrm{af}} * \mathfrak{g}_{\mathrm{af}})_\beta$ to $(\widetilde{\mathfrak{h}}^{\mathbb{Z}}_{\mathrm{af}} * \mathfrak{g}_{\mathrm{af}})_{w_\alpha \beta}$ isomorphically for any $\alpha \in \Gamma_{\mathrm{af}}$ and $\beta \in Q(R)$. (*Proof.* i) This can be shown by a direct calculation (4.1) **I**. ii) Let $x \in (\widetilde{\mathfrak{h}}^{\mathbb{Z}}_{\mathrm{af}} * \mathfrak{g}_{\mathrm{af}})_\beta$ for $\beta \in Q(R)$. Then, for $h \in \widetilde{\mathfrak{h}}$, we have $[h, n_\alpha(x)] = n_\alpha[n_\alpha^{-1}(h), x] = n_\alpha[\widetilde{w}_\alpha^{-1}(h), x] = \langle \widetilde{w}_\alpha(h), \beta \rangle n_\alpha(x) = \langle h, w_\alpha(\beta) \rangle n_\alpha(x)$. Similarly we can show n_α^{-1} maps $(\widetilde{\mathfrak{h}}^{\mathbb{Z}}_{\mathrm{af}} * \mathfrak{g}_{\mathrm{af}})_{w_\alpha \beta}$ to $(\widetilde{\mathfrak{h}}^{\mathbb{Z}}_{\mathrm{af}} * \mathfrak{g}_{\mathrm{af}})_\beta$. $\square$)

As a consequence, we obtain the following fact.

Fact 7. *The set of roots Δ of $\widetilde{\mathfrak{h}}^{\mathbb{Z}}_{\mathrm{af}} * \mathfrak{g}_{\mathrm{af}}$ is W_{af}-invariant.*

Define a *height* of an element $x = \sum_{\alpha \in \Gamma_{\mathrm{af}}} m_\alpha \alpha + na \in Q_{\mathrm{af}} \oplus \mathbb{Z}a = Q(R)$ by

$$h(x) := \sum_{\alpha \in \Gamma_{\mathrm{af}}} m_\alpha \in \mathbb{Z}. \tag{5.3.2}$$

Let us call the element x *positive* (resp. *negative*) if $h(x) > 0$ (resp. $h(x) < 0$).

Lemma 8. *The set of roots Δ is equal to the union $R \cup rad\, Q(R)$. The multiplicity of any root in R is equal to 1.*

Proof. Consider the W_{af} orbit of any root $x \in \Delta$. If the orbit contains an element y with $h(y) = 0$. Then $y = na = x$ for some $n \in \mathbb{Z}$ by (5.2) Lemma 6.

If the orbit contains both positive and negative elements, then there exist an element y in the orbit and $\alpha \in \Gamma_{\mathrm{af}}$ such that $h(y) > 0 > h(w_\alpha(y))$. Express $y = \sum_{\beta\in\Gamma_{\mathrm{af}}} m_\beta\beta + na$, where all $m_\beta \in \mathbb{Z}$ are non-negative and $h(y) = \sum_{\beta\in\Gamma_{\mathrm{af}}} m_\beta$. Then the coefficient m_α only in the expression of y can be non-zero, since all coefficients of $w_\alpha(y) := y - I(\alpha^\vee, y)\alpha = \sum_{\beta\in\Gamma_{\mathrm{af}}\setminus\{\alpha\}} m_\beta\beta + (m_\alpha - I(\alpha^\vee, y)\ )\alpha + na$ are simultaneously non-positive. So, $y = m_\alpha\alpha + na$ and hence $m_\alpha = 1$ by (5.2) Lemma 7 (3). Thus, $y \in R$ (recall (2.6.1)) and hence $x \in W_{\mathrm{af}} \cdot R = R$. The multiplicity of $(\widetilde{\mathfrak{h}}^{\mathbb{Z}}_{\mathrm{af}} * \mathfrak{g}_{\mathrm{af}})_y$ is equal to 1 because of (5.2) Lemma 7 (2).

Assume that all elements of the orbit $W_{\mathrm{af}} \cdot x$ have positive heights and $y \in W_{\mathrm{af}} \cdot x$ attains the minimal height. Then, for any $\beta \in \Gamma_{\mathrm{af}}$, one has $h(y) \le h(w_\beta(y)) = h(y - I(y, \beta)\beta) = h(y) - I(y, \beta)$. So, $I(y, \beta) \le 0$. Using the expression: $y = \sum_{\beta\in\Gamma_{\mathrm{af}}} m_\beta\beta + na$, one has $I(y, y) = \sum_{\beta\in\Gamma_{\mathrm{af}}} m_\beta I(y, \beta) \le 0$, and so, $y \in rad\, Q(R)$. The case when all elements of $W_{\mathrm{af}} \cdot x$ have negative weight is reduced to the positive case by the Cartan involution (5.1). □

A root in R is called a *real root* and a root in $rad\, Q(R)$ is called an *imaginary root* (see (3.1.7)).

(5.4) Let us denote the composition map $\varphi \circ \varrho$ by ξ:

$$\xi := \varphi \circ \varrho : \ \widetilde{\mathfrak{h}}^{\mathbb{Z}}_{\mathrm{af}} * \mathfrak{g}_{\mathrm{af}} \to \widetilde{\mathfrak{g}}(R). \tag{5.4.1}$$

The map ξ preserves root spaces of $\widetilde{\mathfrak{h}}^{\mathbb{Z}}_{\mathrm{af}} * \mathfrak{g}_{\mathrm{af}}$ and $\widetilde{\mathfrak{g}}(R)$.

For any $\alpha \in R$ and $\mu \in rad\, Q(R)$, we define elements in $\widetilde{\mathfrak{h}}^{\mathbb{Z}}_{\mathrm{af}} * \mathfrak{g}_{\mathrm{af}}$ as follows:

$$\begin{aligned} E^\alpha \quad &:= \quad \text{the element of the root space } (\widetilde{\mathfrak{h}}^{\mathbb{Z}}_{\mathrm{af}} * \mathfrak{g}_{\mathrm{af}})_\alpha \\ &\qquad\qquad \text{which is mapped to } e^\alpha \in \widetilde{\mathfrak{g}}(R)_\alpha \text{ by } \xi, \\ H^\mu_{\alpha^\vee} \quad &:= \quad (-1)^{\varepsilon(\alpha,\mu-\alpha)}[E^\alpha, E^{\mu-\alpha}], \end{aligned} \tag{5.4.2}$$

These are well-defined since the restriction of ξ on a real root space $(\widetilde{\mathfrak{h}}^{\mathbb{Z}}_{\mathrm{af}} * \mathfrak{g}_{\mathrm{af}})_\alpha$ is isomorphic by (5.2) Lemma 8 (for definition of ε, see (4.1.2)). This definition is consistent with the generators E^α for $\alpha \in \pm\Gamma_{\mathrm{af}}$ by definition of φ. Note that $H^\mu_{\alpha^\vee}$ is a weight vector of weight μ and $\xi(H^\mu_{\alpha^\vee}) = h_{\alpha^\vee}(-1)e^\mu$ by (3.1.8) **II.1**.

Assertion 10. *Inside the algebra* $\widetilde{\mathfrak{h}}_{\mathrm{af}}^{\mathbb{Z}} * \mathfrak{g}_{\mathrm{af}}$*, one has the following formulae for any* $\alpha, \beta \in R$*,* $h \in \widetilde{\mathfrak{h}}$ *and* $\mu \in rad\, Q(R)$*.*

$$\begin{array}{ll}
\textbf{R-0} & H^{na}_{\alpha^\vee} = H^{(n)}_{\alpha^\vee} \quad \textit{for} \quad n \in \mathbb{Z}, \\
\textbf{R-I.} & (1)\,[h, E^\alpha] = \langle h, \alpha \rangle E^\alpha, \quad (2)\,[h, H^\mu_{\alpha^\vee}] = \langle h, \mu \rangle H^\mu_{\alpha^\vee}, \\
\textbf{R-II.} & [E^\alpha, E^\beta] = \begin{cases} 0 & \textit{if} \quad I(\alpha^\vee, \beta) \geq 0, \\ (-1)^{\varepsilon(\alpha,\beta)} E^{\alpha+\beta} & \textit{if} \quad I(\alpha^\vee, \beta) = -1, \\ (-1)^{\varepsilon(\alpha,\beta)} H^{\alpha+\beta}_{\alpha^\vee} & \textit{if} \quad I(\alpha^\vee, \beta) = -2, \end{cases} \\
\textbf{R-III.} & [H^\mu_{\alpha^\vee}, E^\beta] = (-1)^{\varepsilon(\mu,\beta)} I(\alpha^\vee, \beta) E^{\beta+\mu}, \\
\textbf{R-H.} & -H^\mu_{\alpha^\vee} = H^\mu_{\alpha^\vee}.
\end{array} \tag{5.4.3}$$

Proof. **R-0**: Recall the note after the definition (4.3.3) and (5.2) Lemma 7 (4).

R-I. (1), **R-II**. Case $I(\alpha^\vee, \beta) = -1$ and **R-III**: Recall (5.3) Lemma 8.

R-I. (2) and **R-II**. Case $I(\alpha^\vee, \beta) = -2$: Recall the definition (5.4.2).

R-II. Case $I(\alpha^\vee, \beta) \geq 0$: Recall (5.3) Lemma 8.

R-H: We have $[H^\mu_{\alpha^\vee}, E^\alpha, E^{-\alpha}] = [[H^\mu_{\alpha^\vee}, E^\alpha], E^{-\alpha}] - [H^\mu_{\alpha^\vee}, [E^\alpha, E^{-\alpha}]]$ where the second term is 0, since $[E^\alpha, E^{-\alpha}] = \pm H^0_{\alpha^\vee} = \pm h_{\alpha^\vee}$ (**R-0**) and $[h_{\alpha^\vee}, H^\mu_{\alpha^\vee}] = I(\alpha^\vee, \mu) H^\mu_{\alpha^\vee}$ (**R-I** (2)) $= 0$. Applying **R-III** in both hand sides, this implies the equality: $-[E^{-\alpha+\mu}, E^\alpha] = [E^{\alpha+\mu}, E^{-\alpha}]$. By definition (5.4.2), this implies **R-H**. □

Remark 6. A careful calculation using relations (5.4.3) shows a formula:

$$\textbf{R-H-II.} \qquad [H^\mu_{\alpha^\vee}, H^\nu_{\beta^\vee}] = I(\alpha^\vee, \beta)(-1)^{\varepsilon(\mu,\nu)}(-H^{\mu+\nu}_{(\alpha+\nu)^\vee} + H^{\mu+\nu}_{\alpha^\vee}). \tag{5.4.4}$$

Put $\Gamma^\delta_{\mathrm{af}} := \Gamma_{\mathrm{af}} \cup \{\delta^*\}$ for a fixed element $\delta \in \Gamma_{\mathrm{af}}$. We have $\#\Gamma^\delta_{\mathrm{af}} = l + 2$.

Assertion 11. *The algebra* $\widetilde{\mathfrak{h}}_{\mathrm{af}}^{\mathbb{Z}} * \mathfrak{g}_{\mathrm{af}}$ *is generated by* $\widetilde{\mathfrak{h}}$ *and* $\{E^\alpha \mid \alpha \in \pm\Gamma^\delta_{\mathrm{af}}\}$*.*

Proof. We have to show that $H^{(n)}_{\alpha^\vee}$ for $\alpha \in \Gamma_{\mathrm{af}}$ and $n \in \mathbb{Z}' = \mathbb{Z} \setminus \{0\}$ are generated by $\{E^\alpha \mid \alpha \in \pm\Gamma^\delta_{\mathrm{af}}\}$. The elements $H^{(\pm 1)}_{\delta^\vee}$ are generated, since one has $[E^{-\delta}, E^{\delta^*}] = H^a_{-\delta^\vee} = -H^{(1)}_{\delta^\vee}$ (**R-0** and **R-H**) and $[E^\delta, E^{-\delta^*}] = H^{-a}_{\delta^\vee} = H^{(-1)}_{\delta^\vee}$ (**R-0**). Using the fact that Γ_{af} is connected by simple bonds and using the relations **II*.1** in (5.1.1), we see all the $H^{(\pm 1)}_{\alpha^\vee}$'s ($\alpha \in \Gamma_{\mathrm{af}}$) are generated. We have $[E^\alpha, H^{(n)}_{\alpha^\vee}, H^{(\pm 1)}_{\alpha^\vee}] = -2[E^\alpha, H^{(n\pm 1)}_{\alpha^\vee}]$ by **I*** and $[E^\alpha, H^{(n\pm 1)}_{\alpha^\vee}, E^{-\alpha}] = -2H^{(n\pm 1)}_{\alpha^\vee}$ by **II*.1**. Hence, by induction, we see $H^{(n)}_{\alpha^\vee}$ is generated for all $\alpha \in \Gamma_{\mathrm{af}}$ and $n \in \mathbb{Z}$. □

Assertion 12. *The set* $\{H^{\mu}_{\alpha} \mid \alpha \in \Gamma^{\delta}_{\mathrm{af}}\}$ *spans* $(\widetilde{\mathfrak{h}}^{\mathbb{Z}}_{\mathrm{af}} * \mathfrak{g}_{\mathrm{af}})_{\mu}$ *for* $\mu \in rad\, Q(R) \backslash \{0\}$.

Proof. Take an element $[E^{\beta_1}, \cdots, E^{\beta_k}]$ such that $\beta_1 + \cdots + \beta_k = \mu$ where $\beta_1, \cdots, \beta_k \in \pm\Gamma^{\delta}_{\mathrm{af}}$. One has $k > 1$ since μ is not a real root. Then, $\beta_1 + \cdots + \beta_{k-1} = \mu - \beta_k$ and $[E^{\beta_1}, \cdots, E^{\beta_{k-1}}] \in (\widetilde{\mathfrak{h}}^{\mathbb{Z}}_{\mathrm{af}} * \mathfrak{g}_{\mathrm{af}})_{\mu-\beta_k}$. Since $\mu - \beta_k$ is a real root, $[E^{\beta_1}, \cdots, E^{\beta_{k-1}}] \in \mathbb{Q}E^{\mu-\beta_k}$, the multi-bracket belongs to $\mathbb{Q}[E^{\mu-\beta_k}, E^{\beta_k}] = \mathbb{Q}\, H^{\mu}_{\beta^{\vee}_k}$. Use **R-H** if necessary. □

(5.5) We consider the multiplicities of imaginary roots. We already know the multiplicity of an imaginary root μ of height 0 (i.e. $\mu \in G_{\mathbb{Z}} = \mathbb{Z}a$) is equal to $l + 4$ or $l + 1$ according as $\mu = 0$ or not (see (5.2) Lemma 7 (1), (4)). So, let us consider an imaginary root $\mu = mb + na$, for $m \in \mathbb{Z}$, $m \neq 0$ and $n \in \mathbb{Z}$. We have lower band of the multiplicity of μ, which is $l + 1$ by (3.2) Lemma 1. On the other hand, the root space is spanned by $l + 2$ generators $\{H^{\mu}_{\alpha} \mid \alpha \in \Gamma^{\delta}_{\mathrm{af}}\}$. So the next lemma determines the multiplicity of μ.

Lemma 9. *The set* $\{H^{\mu}_{\alpha} \mid \alpha \in \Gamma^{\delta}_{\mathrm{af}}\}$ *are linearly dependent for* $\mu \in rad\, Q(R) \setminus \mathbb{Z}a$.

Proof. We prepare two assertions for the proof.

Assertion 13. *Let* $m \in \mathbb{Z}_{>0}$ *and* $\alpha \in \Gamma_{\mathrm{af}}$. *Then,* $H^{mb}_{\alpha^{\vee}} \in \mathfrak{g}_{\mathrm{af}}$. *Furthermore, there exists a sequence* $\beta_1, \cdots, \beta_k$ *in* Γ_{af} *such that* $H^{mb}_{\alpha^{\vee}} = const.[E^{\beta_1}, \cdots, E^{\beta_k}]$.

Proof. Note that $mb - \alpha$ is a real root of the affine root system $R(\Gamma_{\mathrm{af}})$. So E^{α} and $E^{mb-\alpha}$ are the elements of $\mathfrak{g}_{\mathrm{af}}$. Additionally, $E^{mb-\alpha}$ has the expression: $E^{mb-\alpha} = const.[E^{\beta_1}, \cdots, E^{\beta_{k-1}}]$ for some $\beta_i \in \Gamma_{\mathrm{af}}$, since $mb - \alpha$ is positive with respect to the root basis Γ_{af}. □

Assertion 14. *Let* $A := \{y, x_1, \cdots, x_n\}$ *be a sequence of elements in a Lie algebra* $\mathfrak{g}$. *Then,*

$$[\, y, [x_1, \cdots, x_n]] = \sum_{i=1}^{n} \sum_{\sigma \in \mathfrak{S}_n} m^{(i)}_{\sigma} [\, z^{(i)}_{\sigma(1)}, \cdots, z^{(i)}_{\sigma(n)}, x_i],$$

where $m_\sigma^{(i)}$*'s are some integers independent of* y *and* x_i*'s and* $\{z_1^{(i)}, \cdots, z_n^{(i)}\} := A \setminus \{x_i\}$ *for* $i = 1, \cdots, n$.

Proof. Induction on n. It is clear for case $n = 1$. Assuming case n, let us show case $n+1$:

$$[y, [x_1, \cdots, x_n, x_{n+1}]] = [y, [x_1, \cdots, x_n], x_{n+1}] - [y, x_{n+1}, [x_1, \cdots, x_n]]$$

$$= \sum_{i=1}^{n} \sum_{\sigma \in \mathfrak{S}_n} m_\sigma^{(i)} \big([[z_{\sigma(1)}^{(i)}, \cdots, z_{\sigma(n)}^{(i)}, x_i], x_{n+1}] - [w_{\sigma(1)}^{(i)}, \cdots, w_{\sigma(n)}^{(i)}, x_i] \big),$$

where $\{z_1^{(i)}, \cdots, z_n^{(i)}\} := \{y, x_1, \cdots, x_n\} \setminus \{x_i\}$, $\{w_1^{(i)}, \cdots, w_n^{(i)}\} := \{[y, x_{n+1}], x_1, \cdots, x_n\} \setminus \{x_i\}$ and some integers $m_\sigma^{(i)}$ for $i = 1, \cdots, n$ and $\sigma \in \mathfrak{S}_n$. And use the fact that $[P, [y, x_{n+1}], Q] = [P, y, x_{n+1}, Q] - [P, x_{n+1}, y, Q]$. □

We return to a proof of Lemma 9.

Put $\mu = mb + na$ for $m \in \mathbb{Z} \setminus \{0\}$ and $n \in \mathbb{Z}$. We may assume that $m > 0$ due to the Cartan involution ω (5.1). First we consider the case $n > 0$ (the case $n < 0$ can be shown similarly). According to Assertion 13, take a sequence $\beta_1, \cdots, \beta_k$ in Γ_{af} such that $H_{\delta^\vee}^{mb} = const.[E^{\beta_1}, \cdots, E^{\beta_k}]$. Its ξ-image is $h_{\delta^\vee}(-1)e^{mb}$. We can assume the constant $= 1$ in our following proof. The ξ-image of $[E^{\delta^*}, E^{-\delta}, \cdots, E^{\delta^*}, E^{-\delta}]$ (E^{δ^*} and $E^{-\delta}$ appear n times) is $2^{n-1}h_{\delta^\vee}(-1)e^{na}$. The fact that $[h_{\delta^\vee}(-1)e^{mb}, h_{\delta^\vee}(-1)e^{na}] = 2nh_a(-1)e^{mb+na} \neq 0$ gives us a non-zero element which has the form

$$\begin{aligned} \big[[E^{\delta^*}, E^{-\delta}, \cdots, E^{\delta^*}, E^{-\delta}], [E^{\beta_1}, \cdots, E^{\beta_k}]\big] = \\ -\big[[E^{\beta_1}, \cdots, E^{\beta_k}], [E^{\delta^*}, E^{-\delta}, \cdots, E^{\delta^*}, E^{-\delta}]\big], \end{aligned} \tag{5.5.1}$$

whose ξ-image is $2^n n h_{a^\vee}(-1)e^{mb+na}$. Apply Assertion 14 to LHS of (5.5.1), and from the definition of $H_{\alpha^\vee}^{\mu}$ in (5.4.2), we conclude that this element is in $\oplus_{\alpha \in \Gamma_{\mathrm{af}}} \mathbb{Q}\, H_{\alpha^\vee}^{mb+na}$. On the other hand, apply Assertion 14 to RHS of (5.5.1), then, similarly, we conclude that this element can be written as $c_{\delta^*} H_{\delta^{*\vee}}^{mb+na} + c_\delta H_{\delta^\vee}^{mb+na}$. We have $c_{\delta^*} = 2^n n \neq 0$ and $c_\delta = -2^n n$. So, we find a non-trivial linear relation among the basis $H_{\alpha^\vee}^{\mu}$ for $\alpha \in \Gamma_{\mathrm{af}}^{\delta}$.

Secondly, we consider the case $n = 0$. Recall that the null vector b can be expressed by $b = \sum_{\alpha \in \Gamma_{\mathrm{af}}} n_\alpha \alpha$ (2.3.5). We have $\sum_{\alpha \in \Gamma_{\mathrm{af}}} n_\alpha h_\alpha(-1)e^{mb} = 0$ (see

(3.1.7)). The LHS is inside $\mathfrak{g}(\Gamma_{\mathrm{af}})$. Recall that the subalgebra $\mathfrak{g}_{\mathrm{af}} = \mathfrak{e}(\Gamma_{\mathrm{af}})$ is isomorphic to the affine Kac-Moody algebra $\mathfrak{g}(\Gamma_{\mathrm{af}})$ by the restriction of ξ, and that $\xi(H^{mb}_{\alpha^\vee}) = h_{\alpha^\vee}(-1)e^{\mu}$. So, we have $\sum_{\alpha\in\Gamma_{\mathrm{af}}} n_\alpha H^{mb}_{\alpha^\vee} = 0$. We find again a non-trivial linear relation among the basis. This completes the proof of Lemma 9. □

Lemma 9 together with (5.1) Assertion 9, (4.1) Assertion 3, and (3.2) Lemma 1 proves the isomorphisms of all the three root spaces

$$\widetilde{\mathfrak{g}}(R)_\alpha \underset{\varphi}{\simeq} \widetilde{\mathfrak{e}}(\Gamma(R,G))_\alpha \underset{\varrho}{\simeq} (\widetilde{\mathfrak{h}}^{\mathbb{Z}}_{\mathrm{af}} * \mathfrak{g}_{\mathrm{af}})_\alpha \tag{5.5.2}$$

for $\alpha \in \Delta = R \cup rad\, Q(R)$. In fact, in vierw of (3.2.3), we have the relation:

R-H*. $$\sum_{\alpha\in R} c_\alpha H^{\mu}_{\alpha^\vee} = 0 \iff \sum_{\alpha\in R} c_\alpha \alpha^\vee \in \mathbb{Q}\mu \quad \text{for} \quad c_\alpha \in \mathbb{Q}. \tag{5.5.3}$$

These prove our main result (4.1) Theorem 1 and

Theorem 2. *Three Lie algebras* $\widetilde{\mathfrak{g}}(R)$, $\widetilde{\mathfrak{e}}(\Gamma(R,G))$ *and* $\widetilde{\mathfrak{h}}^{\mathbb{Z}}_{\mathrm{af}} * \mathfrak{g}_{\mathrm{af}}$ *attached to the marked elliptic root system* (R,G) *are isomorphic.*

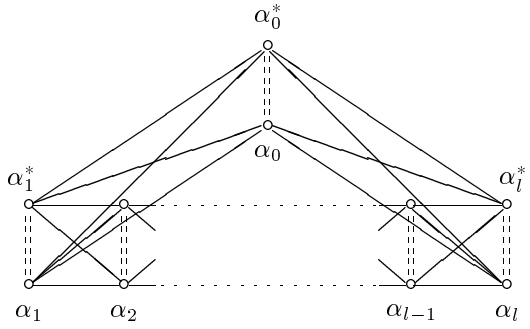
Appendix

A. Table of diagrams of simply-laced marked elliptic root system

1) $m_i := m_{\alpha_i}$: the exponent of α_i.
2) Explicit description of Γ_{af} and $\Gamma_{\max}$.
3) The A-parts.

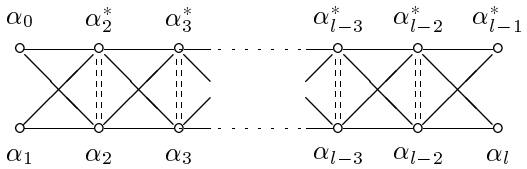
Type $A^{(1,1)}_l$ $(l \geq 2)$**.**

1) $m_i = 1\,(0 \geq i \geq l)$; $m'_{\max} = m_{\max} = 1$.
2) $\Gamma_{\mathrm{af}} = \{\alpha_0, \cdots, \alpha_l\}$, $\Gamma_{\max} = \{\alpha_0, \cdots, \alpha_l\}$.
3) $\Gamma(R,G) \setminus \{\alpha_j, \alpha^*_j\}$ for $j = 0, \cdots, l$



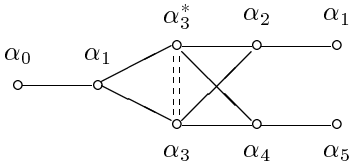
Type $D_l^{(1,1)}(l \geq 4)$.

1) $m_0 = 1$, $m_1 = 1$, $m_i = 2\,(2 \leq i \leq l-1)$, $m_{l-1} = 1$, $m_l = 1$;
   $m'_{\max} = m_{\max} = 2$.
2) $\Gamma_{\mathrm{af}} = \{\alpha_0, \cdots, \alpha_l\}$, $\Gamma_{\max} = \{\alpha_2, \cdots, \alpha_{l-2}\}$.
3) $\{\alpha_0, \alpha_1, \alpha_2, \alpha_2^*\}$, $\{\alpha_{l-2}, \alpha_{l-2}^*, \alpha_{l-1}, \alpha_l\}$,
   $\{\alpha_p, \alpha_q\} \cup \{\alpha_i, \alpha_i^* \mid i = 2, \cdots, l-2\}$ for $p \in \{0,1\}, q \in \{l-1, l\}$.



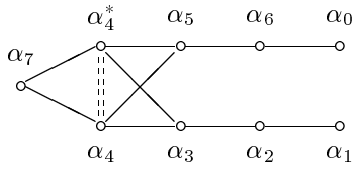
Type $E_6^{(1,1)}$.

1) $m_0 = 1$, $m_1 = 1$, $m_2 = 2$, $m_3 = 3$, $m_4 = 2$, $m_5 = 1$, $m_6 = 1$;
   $m'_{\max} = m_{\max} = 3$.
2) $\Gamma_{\mathrm{af}} = \{\alpha_0, \cdots, \alpha_6\}$, $\Gamma_{\max} = \{\alpha_3\}$.
3) $\{\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_3^*, \alpha_6\}$, $\{\alpha_0, \alpha_3, \alpha_3^*, \alpha_3, \alpha_4, \alpha_5, \alpha_6\}$,
   $\{\alpha_1, \alpha_2, \alpha_3, \alpha_3^* \alpha_4, \alpha_5, \alpha_6\}$.



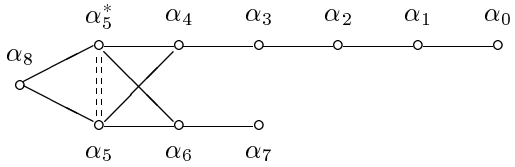
Type $E_7^{(1,1)}$.

1) $m_0 = 1$, $m_1 = 1$, $m_2 = 2$, $m_3 = 3$, $m_4 = 4$, $m_5 = 3$, $m_6 = 2$, $m_7 = 1$;
   $m'_{\max} = m_{\max} = 4$.
2) $\Gamma_{\mathrm{af}} = \{\alpha_0, \cdots, \alpha_7\}$, $\Gamma_{\max} = \{\alpha_4\}$.
3) $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_4^*, \alpha_7\}$, $\{\alpha_0, \alpha_6, \alpha_5, \alpha_4, \alpha_4^*, \alpha_7\}$,
   $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_4^*, \alpha_5, \alpha_6, \alpha_0\}$.



Type $E_8^{(1,1)}$.

1) $m_0 = 1$, $m_1 = 2$, $m_2 = 3$, $m_3 = 4$, $m_4 = 5$, $m_5 = 6$, $m_6 = 4$, $m_7 = 2$,
   $m_8 = 3$;
   $m'_{\max} = m_{\max} = 6$.
2) $\Gamma_{\mathrm{af}} = \{\alpha_0, \cdots, \alpha_8\}$, $\Gamma_{\max} = \{\alpha_5\}$.
3) $\{\alpha_7, \alpha_6, \alpha_5, \alpha_5^*, \alpha_8\}$, $\{\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_5^*, \alpha_8\}$,
   $\{\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_5^*, \alpha_6, \alpha_7\}$.



B. Table of 0-th products

Let $\alpha, \beta \in Q$ and $h, g \in \widetilde{\mathfrak{h}}$.

i) $$(e^\alpha)_{(0)}e^\beta = \begin{cases} 0 & \text{if} \quad I(\alpha,\beta) \geq 0, \\ (-1)^{\varepsilon(\alpha,\beta)}e^{\alpha+\beta} & \text{if} \quad I(\alpha,\beta) = -1, \\ (-1)^{\varepsilon(\alpha,\beta)}h_\alpha(-1)e^{\alpha+\beta} & \text{if} \quad I(\alpha,\beta) = -2, \end{cases}$$

ii) $$(e^\alpha)_{(0)}(h(-1)e^\beta) = \begin{cases} 0 & \text{if} \quad I(\alpha,\beta) \geq 1, \\ -(-1)^{\varepsilon(\alpha,\beta)}\langle h,\alpha\rangle e^{\alpha+\beta} & \text{if} \quad I(\alpha,\beta) = 0, \\ (-1)^{\varepsilon(\alpha,\beta)}(h(-1) - \langle h,\alpha\rangle h_\alpha(-1))e^{\alpha+\beta} & \text{if} \quad I(\alpha,\beta) = -1, \end{cases}$$

iii) $$(h(-1)e^\alpha)_{(0)}(g(-1)e^\beta) = \begin{cases} 0 & \text{if} \quad I(\alpha,\beta) \geq 2 \\ (-1)^{\varepsilon(\alpha,\beta)}\big(-\langle h,\alpha+\beta\rangle + \widetilde{I}^*(h,g)\big)e^{\alpha+\beta} & \text{if} \quad I(\alpha,\beta) = 1, \\ (-1)^{\varepsilon(\alpha,\beta)}\Big(-\langle g,\alpha\rangle h + \langle h,\alpha+\beta\rangle g + \\ \quad \big(\langle h,\alpha+\beta\rangle\langle g,\alpha\rangle + \widetilde{I}^*(h,g)\big)h_\alpha\Big)(-1)e^{\alpha+\beta} & \text{if} \quad I(\alpha,\beta) = 0. \end{cases}$$

iv) $$((e^\alpha)_{(0)})^{1-I(\alpha,\beta)}e^\beta = 0 \quad \text{for} \quad q(\alpha) = 1 \text{ and } I(\alpha,\beta) \leq 0.$$

C. List of Relations

- Relations in the lattice vertex Lie algebra $\widetilde{\mathfrak{g}}(Q) = V_Q/DV_Q$:
  (3.1.8) **0**, **I**, **II.1**, **II.2**, **III** and **IV**.
- Relations for the elliptic Lie algebra $\widetilde{\mathfrak{e}}(\mathit{\Gamma}(R,G))$:
  (4.1.1) **0**, **I**, **II.1**, **II.2**, **III**, **IV.1**, **IV.2** and **V**.
- Relations for the affine Kac-Moody algebra $\mathfrak{g}(A_\triangle)$:
  (4.2.6) **A-0**, **A-I**, **A-II.1** and **A-II.2**.
- Relations for the affine Kac-Moody algebra $\mathfrak{g}_{\mathrm{af}} = \mathfrak{e}(\mathit{\Gamma}_{\mathrm{af}}) \simeq \mathfrak{g}(\mathit{\Gamma}_{\mathrm{af}})$:
  (4.1.1) **0**, **I**, **II.1** and **II.2**.
- Relations for the Heisenberg algebras $\mathfrak{h}_{\mathrm{af}}^{\mathbb{Z}'}$ and $\widetilde{\mathfrak{h}}_{\mathrm{af}}^{\mathbb{Z}}$:
  (4.3.6) **H-I** and **H-II**.
- Amalgamation relations among $\mathfrak{h}_{\mathrm{af}}^{\mathbb{Z}'}$ and $\mathfrak{g}_{\mathrm{af}}$:
  (4.3.6), (5.1.1) **I***, **II*.1** and **II*.2**.
- Relations among root spaces of $\widetilde{\mathfrak{h}}_{\mathrm{af}}^{\mathbb{Z}} * \mathfrak{g}_{\mathrm{af}}$:
  (5.4.3), (5.4.4), (5.5.3) **R-I**, **R-II**, **R-III**, **R-H**, **R-H*** and **R-H-II**.

D. An explicit description of $\widetilde{\mathfrak{g}}(R)$

Recall from (2.2) that there exists a sub-diagram Γ_{f} of Γ_{af} for a finite root system R_{f} so that the root lattice has the splitting $Q(R) = Q_{\mathrm{f}} \oplus \mathbb{Z}b \oplus \mathbb{Z}a$ (2.3.3) and the set of roots decomposes as $R = R_{\mathrm{f}} \oplus \mathbb{Z}b \oplus \mathbb{Z}a$ (2.6.1). Then, one can find basis Λ_a and Λ_b of the non-degenerate hull $\widetilde{F}_{\mathbb{Q}}$ of $F_{\mathbb{Q}}$ (3.1) such that

$$\widetilde{F}_{\mathbb{Q}} = F_{\mathbb{Q}} \oplus \mathbb{Q}\Lambda_b \oplus \mathbb{Q}\Lambda_a,$$

where

$$\widetilde{I}(\Lambda_a, a) = \widetilde{I}(\Lambda_b, b) = 1,\ \widetilde{I}(\Lambda_a, b) = \widetilde{I}(\Lambda_b, a) = 0 \text{ and } \widetilde{I}(\Lambda_a, \Gamma_{\mathrm{f}}) = \widetilde{I}(\Lambda_b, \Gamma_{\mathrm{f}}) = 0.$$

Recall the identification: $h\colon \widetilde{F}_{\mathbb{Q}} \to^{\sim} \widetilde{\mathfrak{h}};\ x \mapsto h_x$ such that $\langle h_x, y\rangle = \widetilde{I}(x, y)$ for $x, y \in \widetilde{F}_{\mathbb{Q}}$. Then we put:

$$da^\vee := h_{a^\vee}, \quad db^\vee := h_{b^\vee}, \quad , \frac{\partial}{\partial a} := h_{\Lambda_a}, \quad \frac{\partial}{\partial b} := h_{\Lambda_b} \quad \text{and}$$
$$\mathfrak{h}_{\mathrm{f}} := \bigoplus_{\alpha \in \Gamma_{\mathrm{f}}} \mathbb{Q}h_{\alpha^\vee}, \quad \mathfrak{g}_\alpha := \mathbb{Q}e^\alpha \quad \text{for} \quad \alpha \in R_{\mathrm{f}}.$$

Then the elliptic algebra is given by

$$\widetilde{\mathfrak{g}}(R) = \mathbb{Q}\frac{\partial}{\partial a} \oplus \mathbb{Q}\frac{\partial}{\partial b} \oplus \bigoplus_{\alpha \in R_{\mathrm{f}}} \left(\mathfrak{g}_\alpha \otimes \mathbb{Q}[e^{\pm a}, e^{\pm b}]\right) \oplus$$
$$\bigoplus_{m,n \in \mathbb{Z}} \left(\mathfrak{h}_{\mathrm{f}} \oplus \frac{\mathbb{Q}da^\vee + \mathbb{Q}db^\vee}{\mathbb{Q}(mda^\vee + ndb^\vee)}\right) \otimes \mathbb{Q}e^{ma+nb}.$$

The non-degenerate full $\widetilde{\mathfrak{h}}$ of $\mathfrak{h}$, the Heisenberg subalgebra $\mathfrak{h}_{\mathrm{af}}^{\mathbb{Z}'}$, the affine Kac-Moody subalgebra $\mathfrak{g}_{\mathrm{af}}$ and "nilpotent" subalgebras $\mathfrak{n}_{\mathrm{ell}}^{\pm}$ are given by the followings.

$$\begin{aligned}
\widetilde{\mathfrak{h}} &= \mathbb{Q}\frac{\partial}{\partial a} \oplus \mathbb{Q}\frac{\partial}{\partial b} \oplus \mathfrak{h}_{\mathrm{f}} \oplus \mathbb{Q}da^\vee \oplus \mathbb{Q}db^\vee,\\
\mathfrak{h}_{\mathrm{af}}^{\mathbb{Z}'} &= \mathbb{Q}da^\vee \oplus \bigoplus_{n \in \mathbb{Z}\setminus\{0\}} \left(\mathfrak{h}_{\mathrm{f}} \oplus \mathbb{Q}db^\vee\right) \otimes e^{na},\\
\mathfrak{g}_{\mathrm{af}} &= \bigoplus_{\alpha \in R_{\mathrm{f}}} \left(\mathfrak{g}_\alpha \otimes \mathbb{Q}[e^{\pm b}]\right) \oplus \bigoplus_{m \in \mathbb{Z}\setminus\{0\}} \left(\mathfrak{h}_{\mathrm{f}} \otimes e^{mb}\right) \oplus \mathfrak{h}_{\mathrm{f}} \oplus \mathbb{Q}db^\vee,\\
\mathfrak{n}_{\mathrm{ell}}^{\pm} &= \bigoplus_{\alpha \in R_{\mathrm{f}}} \bigoplus_{m \lesseqgtr 0} \left(\mathfrak{g}_\alpha \otimes e^{mb} \otimes \mathbb{Q}[e^{\pm a}]\right) \oplus \bigoplus_{\alpha \in R_{\mathrm{f}}^{\pm}} \left(\mathfrak{g}_\alpha \otimes \mathbb{Q}[e^{\pm a}]\right) \oplus\\
&\qquad \bigoplus_{m \lesseqgtr 0} \left((\mathfrak{h}_{\mathrm{f}} \oplus \mathbb{Q}da^\vee) \otimes \mathbb{Q}e^{mb} \otimes \mathbb{Q}[e^{\pm a}]\right).
\end{aligned}$$

References

[AABGP] Allison, Bruce N., Azam Saeid, Berman Stephan, Gao Yun, and Pianzola Arturo, *Extended Affine Lie Algebras and their Root systems*, Memoirs of AMS, vol 126, number **603**, (1997).

[Az] Azam, Saed, *Construction of extended affine Lie algebras by the twisting process*, preprint (1998).

[B] Bourbaki, N., *Lie alg. chap.3,4 et 5*, Hermann, Paris, (1968).

[BB] Berman, Stephen and Billig, Yuly, *Irreducible Representations for Toroidal Lie Algebras*, preprint (1998)

[BGK] Berman, Stephen, Y. Gao and Y. Krylyuk, *Quantum tori and the structure of elliptic quasi-simple Lie algebras*, J. Funct. Anal **135** (1996), 339–389.

[BJT] Berman, Stephen, Jurisich, Elizabeth and Tan, Shaobin, *Beyond Borcherds Lie Algebras and Inside*, preprint (1998).

[BM] Berman, Stephen and Moody, R.V., *Lie algebras graded by finite root systems and the intersection matrix algebras of Slodowy*, Inventiones Math., **108**, (1992), 323–347.

[Bo1] Borcherds, Richard, *Vertex algebras, Kac-Moody algebras and the Monster*, Proc. Natl. Acad. Sci. USA **83**, 3068–3071 (1986).

[Bo2] Borcherds, Richard, *Monstrous moonshine and monstrous Lie superalgebras*, Inventiones Math. **109**, 405–444 (1992).

[EM] S. Eswara Rao, R.V. Moody, *Vertex representations for n-toroidal Lie algebras and a generalization of the Virasoro algebra*, Comm. Math. Phys. **159** (1994), 239–264

[F] Frenkel, Igor B, *Representations of Kac-Moody Algebras and Dual Resonance Models*, Lect. Appl. Math. **21**,(1985) 325–353.

[FK] Frenkel, Igor B and Kac, Victor, *Basic Representations of Affine Lie Algebras and Dual Resonance Models*, Inventiones Math. **62**, (1980) 23–66.

[FLM] Frenkel, Igor B, Lepowsky and Meurman, *Vertex operator algebras and the Monster*, Academic Press, c1988.

[ISW] Iohara, Kenji, Saito, Yoshihisa and Wakimoto, Minoru, *Hirota Bilinear Forms with 2-Toroidal Symmetry*, Phys. Lett. A **254** (1999), no. 1-2, 37–46.

[K] Kac, Victor, *Infinite Dimensional Lie Algebras*, Birkhauser (third edition).

[K2] Kac, Victor, *Vertex Algebras for Beginners*, AMS, University lecture series **10**, (1998)

[Kas] Kassel, C., *Kaehler Differentials and coverings of complex simple Lie algebras extended over a commutative ring,* J. Pure Appl. Algebra **34**,(1984), 265–275.

[Ma] Macdonald, I.g. Ivan, *Affine root Systems and Dedekind's η-Function*, Inventiones Math. **15**, (1972), 91–143.

[MEY] Moody, R.V., Eswara Rao, S. and Yokonuma, T., *Toroidal Lie algebras and vertex representation*, Geom. Dedicata **35**, (1990), 283–307.

[MN] Matsuo, Atsushi and Nagatomo, Kiyokazu, *On axioms for a vertex algebra and the locality of quantum fields,* MSJ Memoirs Vol. **4**, 1999.

[MP] Moody, R.V. and Pianzola, A., *Lie algebras with triangular decomposition*, John Wiley, New York, 1995.

[P] Pollmann, Ulrike, *Realization der biaffinen Wurzelsystem von Saito in Lie-Algebren*, Hamburger Beitrage zur Mathematik, Heft **29**, (1994).

[PS] Pressley, A. and Segal, G., *Loop groups*, Oxford: Oxford University Press 1986.

[Sa1] Saito, Kyoji, *Einfach Elliptische Singularitaten*, Inventiones Math., **23** (1974), 289–325.

[Sa2] Saito, Kyoji, *Period mapping associated to a primitive form*, Publ. RIMS. **19** (1983),1231–1264.

[Sa3-I] Saito, Kyoji, *Extended Affine Root systems I (Coxeter Transformations)*, Publ. RIMS, Kyoto Univ., **21**, (1985) 75–179,

[Sa3-II] Saito, Kyoji, *Extended Affine Root systems II (Flat Invariants)*, Publ. RIMS, Kyoto Univ., **26**, (1990) 15–78,

[SaT3-III] Saito, Kyoji and Takebayashi, T., *Extended Affine Root systems III (Elliptic Weyl Groups)*, Publ. RIMS, Kyoto Univ., **33**, (1997) 301–329,

[Sa3-V] Saito, Kyoji, *Extended Affine Root systems V (Elliptic eta-product and Elliptic L-function)*, preprint (1998).

[Sa4] Saito, Kyoji, *Around the Theory of the Generalized Weight System: Relations with Singularity Theory, the Generalized Weyl Group and Its Invariant Theory, Etc.*, Amer. Math. Soc. Transl. (2) **183**, (1998) 101–143.

[Sas] Sasaki, Shoichi: Master thesis, Saitama University, 1996.

[Sat1] Satake, Ikuo, *Automorphism of the Extended Affine Root System and Modular Property for the Flat Theta Invariants*, Publ. RIMS, Kyoto Univ., **31**, (1995), 1–32.

[Sat2] Satake, Ikuo, *Flat Structure for the simple Elliptic Singularities of type* $\tilde{E}^6$, Proc. Japan Acad. Ser. A, **69**, (1993), 247–251.

[Sat3] Satake, Ikuo, *Flat Theta Invariants and Jacobi form of type* $\tilde{D}_4$, Proceedings of 37th Taniguchi Symposium, Birkhauser (1998).

[Se] Serre, Jean Pierre, *Algebres de Lie semi-simple complexes*, Benjamin, New York and Amsterdam, (1966).

[Sl1] Slodowy, Peter, *Simple singularities and simple algebraic group*, LNM **815**, (1980)

[Sl2] Slodowy, Peter, *Singularitaeten, Kac-Moody-Lie algebren, assozierte Gruppen und Verallgemeinerungen*, Habilitationsshrift (1984).

[Sl3] Slodowy, Peter, *Another new class of Lie algebras*, (1984).

[Sl4] Slodowy, Peter, *Beyond Kac-Moody algebras and inside*, Can. Math. Soc. Proc., **5** (1986), 361–371.

[VL] Van der Lek, H., *Extended Artin Groups*, Proc. Symposia in Pure Math., **40**, Part 2 (1983), 117–122.

[W] Wakimoto, Minoru, *Extended Affine Lie Algebras and a Certain Series of Hermitian Representations*, Department of Mathematics, Faculty of Science, Hiroshima University, 1985.

[Y1] Yamada, Hiroshi, *Extended affine Lie Algebras and their Vertex representations*, Publ. RIMS, Kyoto Univ., **25**, (1989), 587–603.

[Y2] Yamada, Hiroshi, *Lie group theoretical construction of period mapping*, Math. Z. **220**, 231–255 (1995)

[Y3] Yamada, Hiroshi, *Elliptic Root Systems and Elliptic Artin Group*, Preprint (1998).

[Yo] Yoshii, Daigo, *Elliptic Lie algebra of type* $E_8^{(1,1)}$, Master thesis, 1997.

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