THE INDICES OF LOG CANONICAL SINGULARITIES

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Abstract. Let $(P \in X, \Delta)$ be a three dimensional log canonical pair such that Δ has only standard coefficients and P is a center of log canonical singularities for (X, Δ). Then we get an effective bound of the indices of these pairs and actually determine all the possible indices. Furthermore, under certain assumptions including the log Minimal Model Program, an effective bound is also obtained in dimension $n \geq 4$.

0. Introduction

The main purpose of this paper is to investigate the indices of log canonical pairs. Let $(P \in X)$ be a log canonical singularity which is not log terminal. If $\dim X = 2$, then the index is $1, 2, 3, 4$, or 6. This fact is well-known to specialists. Shihoko Ishii generalized this result to three dimensional isolated log canonical singularities which are not log terminal. More precisely, she proved that a positive integer r is the index of such a singularity if and only if $\varphi(r) \leq 20$ and $r \neq 60$, where φ is the Euler function (for related topics, see [Sh2]). In this paper, we generalize it to higher dimensional (not necessarily isolated) log canonical singularities which are not log terminal. We note that if $(P \in X)$ is a log canonical singularity such that P is not a center of log canonical singularities, then the index is not bounded (see Example (5.1)). So, we shall prove the following (for the precise statement, see Corollary (4.21) and Remark (4.22)).

Theorem 0.1. *Let $(P \in X, \Delta)$ be a three dimensional log canonical pair such that Δ has only standard coefficients and P is a center of log canonical singularities for the pair (X, Δ). Then the index of (X, Δ) at P is bounded. More precisely, the positive integer r is the index of such a pair if and only if $\varphi(r) \leq 20$ and $r \neq 60$. In particular, if there exists another center of log canonical singularities W such that $P \subsetneq W$, then the index is $1, 2, 3, 4$, or 6.*

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This is related to (birational) automorphisms on $K3$ surfaces, Abelian surfaces and elliptic curves. Unfortunately, (birational) automorphisms on higher dimensional "Calabi-Yau" varieties are not well understood. If we can prove the conjectures about such automorphism groups (see Conjecture (3.2)), then Theorem (0.1) is generalized to the following (for the precise and effective statement, see Theorem (4.20)). Precisely, we prove Theorem (0.2) and get Theorem (0.1) as a corollary.

Theorem 0.2. *Assume the log Minimal Model Program for dimension $\leq n$. Let $(P \in X, \Delta)$ be an n-dimensional log canonical pair such that Δ has only standard coefficients and P is a center of log canonical singularities for the pair (X, Δ). If the conjectures (F'_{n-1}) and (F_l) hold true for $l \leq n-2$ (see Conjecture (3.2)), then the index of (X, Δ) at P is bounded.*

This theorem is an answer to [Is3, 4.16]. We should mention that the idea of this paper is due to [Is2] and [Is3], and the proof relies on [Fj1] (see also [Sh2]).

We explain the contents of this paper. In Section 1, we fix our notation and recall some definitions used in this paper, some of which were introduced in [Fj1]. In Section 2, we generalize Shokurov's connectedness lemma. This section is a continuation of [Fj1, Section 2]. Section 3 deals with birational automorphism groups and we collect some known results for low dimensional varieties. Section 4 is devoted to the proof of the main result, Theorem (4.20). In Section 5, we collect some examples of log canonical singularities. Finally, in Section 6, we explain how to translate statements on algebraic varieties into those on analytic spaces.

Notation. (1) We will make use of the standard notation and definitions as in [KoM].

(2) The log Minimal Model Program (log MMP, for short) means the log MMP for $\mathbb{Q}$-factorial dlt pairs.

(3) A *variety* means an algebraic variety over $\mathbb{C}$ and an *analytic space* a reduced complex analytic space.

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1. Preliminaries

In this section, we fix our notation and recall some definitions. For analytic spaces, we have to modify Definitions (1.6), (1.8) and Lemma (1.7) (see Section 6).

Notation 1.1. Let X be a normal variety over $\mathbb{C}$. The *canonical divisor* K_X is defined so that its restriction to the regular part of X is a divisor of a regular n-form. The reflexive sheaf of rank one $\omega_X := \mathcal{O}(K_X)$ corresponding to K_X is called the *canonical sheaf*. Let $(P \in X, \Delta)$ be a germ of a normal variety with $\mathbb{Q}$-divisor such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier. The *index* of (X, Δ) at P, denoted by $\mathrm{I}(P \in X, \Delta)$, is the smallest positive integer r such that $r(K_X + \Delta)$ is Cartier at P.

The following is the definition of singularities of pairs. Note that the definitions in [KMM] or [KoM] are slightly different from ours.

Definition 1.2. Let X be a normal variety and $D = \sum d_i D_i$ an effective $\mathbb{Q}$-divisor such that $K_X + D$ is $\mathbb{Q}$-Cartier. Let $f : Y \to X$ be a proper birational morphism. Then we can write

$$K_Y = f^*(K_X + D) + \sum a(E, X, D)E,$$

where the sum runs over all the distinct prime divisors $E \subset Y$, and $a(E, X, D) \in \mathbb{Q}$. This $a(E, X, D)$ is called the *discrepancy* of E with respect to (X, D). We define

$$\mathrm{discrep}(X, D) := \inf_E \{a(E, X, D) \mid E \text{ is exceptional over } X\}.$$

On the assumption that $0 \le d_i \le 1$ for every i, we say that (X, D) is

$$\begin{cases} \text{terminal} \\ \text{canonical} \\ \text{klt} \\ \text{plt} \\ \text{lc} \end{cases} \quad \text{if} \quad \mathrm{discrep}(X, D) \quad \begin{cases} > 0, \\ \ge 0, \\ > -1 \quad \text{and} \quad \llcorner D \lrcorner = 0, \\ > -1, \\ \ge -1. \end{cases}$$

Moreover, (X, D) is *divisorial log terminal* (dlt, for short) if there exists a log resolution (see [KoM, Notation 0.4 (10)]) with $a(E, X, D) > -1$ for every exceptional divisor E. Here klt (resp. plt, lc) is short for *Kawamata log terminal* (resp. *purely log terminal*, *log canonical*). If $D = 0$, then the notions klt, plt and dlt coincide and in this case we say that X has *log terminal* (lt, for short) singularities.

Let $S := \{1 - 1/m \,|\, m \in \mathbb{N} \cup \{\infty\}\}$. We say that the divisor $D = \sum d_i D_i$ has *only standard coefficients* if $d_i \in S$ for every i (cf. [Sh2, 1.3]).

In the following definition, we define the *compact center of log canonical singularities*.

Definition 1.3 (cf. [Ka2, Definition 1.3]). A subvariety W of X is said to be a *center of log canonical singularities* for the pair (X, D), if there exists a proper birational morphism from a normal variety $\mu : Y \to X$ and a prime divisor E on Y with the discrepancy coefficient $a(E, X, D) \leq -1$ such that $\mu(E) = W$.

The set of all centers of log canonical singularities is denoted by $\mathrm{CLC}(X, D)$. The union of all the subvarieties in $\mathrm{CLC}(X, D)$ is denoted by $\mathrm{LLC}(X, D)$ and called the *locus of log canonical singularities* for (X, D). $\mathrm{LLC}(X, D)$ is a closed subset of X.

We denote the set of *compact* (with respect to the classical topology) elements in $\mathrm{CLC}(X, D)$ by $\mathrm{CLC}^{\mathrm{c}}(X, D)$. If $W \in \mathrm{CLC}^{\mathrm{c}}(X, D)$, then W is said to be a *compact center of log canonical singularities* for the pair (X, D). It can be checked easily that $W \in \mathrm{CLC}^{\mathrm{c}}(X, D)$ if and only if there exists a proper birational morphism from a normal variety $\mu : Y \to X$ and a compact prime divisor E on Y with the discrepancy coefficient $a(E, X, D) \leq -1$ such that $\mu(E) = W$. We define $\mathrm{LLC}^{\mathrm{c}}(X, D)$ as the union of subvarieties in $\mathrm{CLC}^{\mathrm{c}}(X, D)$.

Remark 1.4. Let (X, D) be a dlt pair. Then there exists a log resolution $f : Y \to X$ such that f induces an isomorphism over every generic point of center of log canonical singularities for the pair (X, D) and $a(E, X, D) > -1$ for every f-exceptional divisor E. This is obvious by the original definition of dlt (see [Sh1, 1.1]). See also [Sz, Divisorial Log Terminal Theorem].

Definitions (1.5), (1.6), (1.8), (1.9), and (1.10) are reformulations of the definitions of [Fj1].

Definition 1.5 (cf. [Fj1, Definition 4.6]). Assume that X is nonsingular and $\mathrm{Supp} D$ is a simple normal crossing divisor and $D = \sum_i d_i D_i$ is a $\mathbb{Q}$-divisor such that $d_i \leq 1$ (d_i may be negative) for every i. In this case we say that (X, D) is *B-smooth*.

Let (X, D) be dlt or B-smooth. We write $D = \sum_i d_i D_i$ such that D_i's are distinct prime divisors. Then the *B-part* of D is defined by $D^B := \sum_{d_i=1} D_i$. We define the compact and non-compact B-part of D as follows:

$$D^{c} := \sum_{\substack{d_i = 1 \\ D_i \text{ is compact.}}} D_i \ , \quad D^{nc} := \sum_{\substack{d_i = 1 \\ D_i \text{ is non-compact.}}} D_i \, .$$

If (X, D) is dlt or B-smooth, then a center of log canonical singularities is an irreducible component of an intersection of some B-part

divisors. (See the Divisorial Log Terminal Theorem of [Sz] and [KoM, Section 2.3].) When we consider a center of log canonical singularities W, we always consider the pair (W, Ξ) such that $K_W + \Xi = (K_X + D)|_W$, where Ξ is defined by repeatedly using the adjunction. Note that if (X, D) is dlt (resp. B-smooth), then (W, Ξ) is dlt (resp. B-smooth) by the adjunction.

If (X, D) is dlt or B-smooth and W is a center of log canonical singularities for the pair (X, D), then we write $(W, \Xi) \Subset (X, D)$. If there is no confusion, we write $W \Subset X$.

Definition 1.6 (cf. [Fj1, Definition 1.5])**.** Let (X, D) and (X', D') be normal varieties with $\mathbb{Q}$-divisors such that $K_X + D$ and $K_{X'} + D'$ are $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisors.

We say that $f : (X, D) \dashrightarrow (X', D')$ is a B-birational map (resp. morphism) if $f : X \dashrightarrow X'$ is a proper birational map (resp. morphism) and there exists a common resolution $\alpha : Y \to X$, $\beta : Y \to X'$ of $f : X \dashrightarrow X'$ such that $\alpha^*(K_X + D) = \beta^*(K_{X'} + D')$.

The following lemma, which is a corollary of [Fj1, Claims (A_n), (B_n)], is very useful.

Lemma 1.7. *Let (X_i, D_i) be dlt or B-smooth for $i = 1, 2$. Let $f : (X_1, D_1) \dashrightarrow (X_2, D_2)$ be a B-birational map and $(W_1, \Xi_1) \Subset (X_1, D_1)$ a minimal (with respect to $\Subset$) center of log canonical singularities. Then there exists a minimal center of log canonical singularities $(W_2, \Xi_2) \Subset (X_2, D_2)$ such that (W_1, Ξ_1) and (W_2, Ξ_2) are B-birationally equivalent to each other.*

Proof. By Remark (1.4), we may assume that (X_i, D_i) is B-smooth for $i = 1, 2$. Let $g_i : (Y, E) \to (X_i, D_1)$ be a common resolution of f. We note that $g_1^*(K_{X_1} + D_1) = K_Y + E = g_2^*(K_{X_2} + D_2)$. By [Fj1, Claim (A_n)], we can take $(W, \Xi) \Subset (Y, E)$ such that $g_1|_W : (W, \Xi) \to (W_1, \Xi_1)$ is B-birational. It is obvious that $(W, \Xi) \Subset (Y, E)$ is a minimal center of log canonical singularities for the pair (Y, E) since (W, Ξ) and (W_1, Ξ_1) have the same discrepancies. By applying [Fj1, Claim (B_n)] (see also Section 6) to $g_2 : (Y, E) \to (X_2, D_2)$, $(W, \Xi) \to (W_2, \Xi_2)$ is B-birational, where $W_2 = g_2(W)$ and $(W_2, \Xi_2) \Subset (X_2, D_2)$. It is obvious that (W_2, Ξ_2) is a minimal center of log canonical singularities for the pair $(X_2.D_2)$ by [Fj1, Claim (A_n)]. □

Definition 1.8 (cf. [Fj1, Definition 3.1])**.** Let (X, D) be a pair of a normal variety and a $\mathbb{Q}$-divisor such that $K_X + D$ is $\mathbb{Q}$-Cartier. We define

$$\begin{aligned} \mathrm{Bir}(X, D) &:= \{\sigma : (X, D) \dashrightarrow (X, D) \mid \sigma \text{ is a B-birational map}\}, \\ \mathrm{Aut}(X, D) &:= \{\sigma : X \to X \mid \sigma \text{ is an automorphism and } \sigma^* D = D\}. \end{aligned}$$

Since $\mathrm{Bir}(X, D)$ acts on $H^0(X, \mathcal{O}_X(m(K_X + D)))$ for every integer m such that $m(K_X+D)$ is a Cartier divisor, we can define B-pluricanonical representations $\rho_m : \mathrm{Bir}(X, D) \to \mathrm{GL}(H^0(X, m(K_X + D)))$.

The following is the definition of semi divisorial log terminal pairs. The notion of semi divisorial log terminal is much better than that of semi log canonical for the inductive treatment (see [Fj1]).

Definition 1.9 (cf. [Fj1, Definition 1.1]). Let X be a reduced algebraic scheme, which satisfies S_2 condition. We assume that it is pure n-dimensional and normal crossing in codimension 1. Let Δ be an effective $\mathbb{Q}$-Weil divisor on X (cf. [FA, 16.2 Definition]) such that $K_X+\Delta$ is $\mathbb{Q}$-Cartier.

Let $X = \cup X_i$ be a decomposition into irreducible components, and $\mu : X' := \amalg X'_i \to X = \cup X_i$ the normalization. A $\mathbb{Q}$-divisor Θ on X' is defined by $K_{X'} + \Theta := \mu^*(K_X + \Delta)$ and a $\mathbb{Q}$-divisor Θ_i on X'_i by $\Theta_i := \Theta|_{X'_i}$.

We say that (X, Δ) is a *semi divisorial log terminal n-fold* (an sdlt n-fold, for short) if X_i is normal, that is, X'_i is isomorphic to X_i, and (X', Θ) is dlt.

Definition 1.10 (cf. [Fj1, Definition 4.1]). Let (X, Δ) be a proper sdlt n-fold and m a divisible integer. We define admissible and preadmissible sections inductively on dimension.

- $s \in H^0(X, m(K_X+\Delta))$ is *preadmissible* if the restriction $s|_{(\amalg_i \llcorner \Theta_i \lrcorner)} \in H^0(\amalg_i \llcorner \Theta_i \lrcorner, m(K_{X'} + \Theta)|_{(\amalg_i \llcorner \Theta_i \lrcorner)})$ is admissible.
- $s \in H^0(X, m(K_X + \Delta))$ is *admissible* if s is preadmissible and $g^*(s|_{X_j}) = s|_{X_i}$ for every B-birational map $g : (X_i, \Theta_i) \dashrightarrow (X_j, \Theta_j)$ for every i, j.

Note that if $s \in H^0(X, m(K_X + \Delta))$ is admissible, then the restriction $s|_{X_i}$ is $\mathrm{Bir}(X_i, \Theta_i)$-invariant for every i.

The next lemma-definition is frequently used in Section 4.

Lemma-Definition 1.11. *Let $(o \in X, \Theta)$ be a pointed variety with $\mathbb{Q}$-divisor Θ such that $K_X+\Theta$ is $\mathbb{Q}$-Cartier. Then there exists a resolution $f : (Y, \Xi) \to (o \in X, \Theta)$ such that*

(1) *f is projective,*
(2) *$K_Y + \Xi = f^*(K_X + \Theta)$,*
(3) *$f^{-1}(o)$ is a simple normal crossing divisor in Y,*
(4) *$f^{-1}(o) \cup \Xi$ is also a simple normal crossing divisor in Y.*

We say that $f : (Y, \Xi) \to (o \in X, \Theta)$ is a very good resolution of $(o \in X, \Theta)$.

Proof. Let $f_1 : Y_1 \to X$ be any log resolution such that f_1 is projective. Apply the embedded resolution to $f_1^{-1}(o)_{\mathrm{red}} \subset Y_1$. Then we get a sequence of blowing-ups $g : Y_2 \to Y_1$. By Hironaka's theorem (see [Hi] or [BM]), the proper transform of $f_1^{-1}(o)_{\mathrm{red}}$ by g in Y_2 is smooth and the exceptional locus of g is a simple normal crossing divisor, which intersects the proper transform of $f_1^{-1}(o)_{\mathrm{red}}$ transversally. We note that g is an isomorphism over $Y_1 \setminus f_1^{-1}(o)_{\mathrm{red}}$. Therefore, all the components of $(g \circ f_1)^{-1}(o)_{\mathrm{red}}$ of codimension ≥ 2 in Y_2 is smooth. By blowing up these components, we obtain $f_3 : Y_3 \to Y_2 \to Y_1 \to X$ and Ξ_3 such that $K_{Y_3} + \Xi_3 = {f_3}^*(K_X + \Theta)$ and $f_3^{-1}(o)$ is pure codimension 1 in Y_3. We note that Y_3 is smooth. By applying the embedded resolution to $(f_3^{-1}(o) \cup \Xi_3)_{\mathrm{red}}$, we get $f : Y \to Y_3 \to Y_2 \to Y_1 \to X$ and Ξ such that $K_Y + \Xi = f^*(K_X + \Theta)$. By the construction of f, f is projective and $f : (Y, \Xi) \to (o \in X, \Delta)$ satisfies the conditions (3) and (4). □

The following lemma-definition follows from [FA, 8.2.2 Lemma, 17.10 Theorem].

Lemma-Definition 1.12 ($\mathbb{Q}$-factorial dlt model)**.** *Assume that the log MMP holds in dimension n. Let (X, Θ) be an lc n-fold. Then there exists a projective birational morphism $f : (Y, \Xi) \to (X, \Theta)$ from a $\mathbb{Q}$-factorial dlt pair (Y, Ξ) such that $K_Y + \Xi = f^*(K_X + \Theta)$. Furthermore, if (X, Θ) is dlt, then we may take f a small projective morphism. We say that (Y, Ξ) is a $\mathbb{Q}$-factorial dlt model of (X, Θ).*

The following lemma is a special case of [FA, 17.10 Theorem]. We use this in Lemma (4.4).

Lemma 1.13. *Assume the log MMP in dimension n. Let $(o \in Y, D)$ be a germ of a log canonical singularity such that $o \in \mathrm{CLC}(Y, D)$. Without loss of generality, we may assume that $\mathrm{LLC}^{\mathrm{c}}(Y, D) = o$. Let $h : (V, F) \to (o \in Y, D)$ be a very good resolution. Let*

$$V = V^0 \overset{p_1}{\dashrightarrow} V^1 \overset{p_2}{\dashrightarrow} \cdots \overset{p_i}{\dashrightarrow} V^i \overset{p_{i+1}}{\dashrightarrow} V^{i+1} \overset{p_{i+2}}{\dashrightarrow} \cdots \overset{p_{l-1}}{\dashrightarrow} V^{l-1} \overset{p_l}{\dashrightarrow} V^l = Z$$

be the $(K_V + G)$-log MMP over Y, where $F := G - H$ such that G and H are both effective $\mathbb{Q}$-divisors without common irreducible components. We denote $F_0 = F, G_0 = G, H_0 = H$, and $F_i = p_{i*}F_{i-1}, G_i = p_{i*}G_{i-1}, H_i = p_{i*}H_{i-1}$, for every i and $F_l = E$. Then we obtain that $H_l = 0$ and $f : (Z, E) \to (Y, D)$ is a $\mathbb{Q}$-factorial dlt model, and $g := p_l \circ p_{l-1} \circ \cdots \circ p_1$ induces an isomorphism at every generic point of center of log canonical singularities for the pair (V, F). Furthermore, $\mathrm{LLC}^{\mathrm{c}}(Z, E) = E^{\mathrm{c}}$.

Proof. Since h is a very good resolution, $\mathrm{LLC}^{\mathrm{c}}(V, F) = F^{\mathrm{c}}$ and H contains no centers of log canonical singularities for the pair (V, F). We

note that $\mathrm{LLC}(V, F) = \mathrm{LLC}(V, G)$. By induction on i, we assume that $\mathrm{LLC}^{\mathrm{c}}(V^i, G_i) = F_i^c = G_i^c$ and $g_i := p_i \circ p_{i-1} \circ \cdots \circ p_1$ induces an isomorphism at every generic point of center of log canonical singularities for the pair (V, F), and H_i contains no centers of log canonical singularities for the pair (V^i, G_i).

If p_{i+1} is a divisorial contraction, then p_{i+1} contracts an irreducible component of H_i. Thus $\mathrm{LLC}^{\mathrm{c}}(V^{i+1}, G_{i+1}) = F_{i+1}^c = G_{i+1}^c$. It is obvious that H_{i+1} contains no centers of log canonical singularities for the pair (V^{i+1}, G_{i+1}) and g_{i+1} induces an isomorphism at every generic point of center of log canonical singularities for the pair (V, F).

If p_{i+1} is a flip, then the flipping locus is included in H_i. In particular, every divisor whose center is in the flipping locus has discrepancy > -1 with respect to $K_{V^i} + G_i$. After the flip p_{i+1}, the discrepancies do not decrease. Therefore, $\mathrm{LLC}^{\mathrm{c}}(V^{i+1}, G_{i+1}) = F_{i+1}^c = G_{i+1}^c$. Of course, p_{i+1} is an isomorphism at every generic point of center of log canonical singularities for the pair (V^i, G_i). In particular, H_{i+1} contains no centers of log canonical singularities for the pair (V^{i+1}, G_{i+1})

Thus we get the result by the induction on i. We note that $H_l = 0$ and $G_l = F_l = E$. □

Remark 1.14. Let $h : (V, F) \to (o \in Y, \Delta)$ be a log resolution such that $K_V + F = h^*(K_Y + D)$. Assume that $h^{-1}(o)$ is not pure codimension 1. Then $\mathrm{LLC}^{\mathrm{c}}(V, F)$ does not necessarily coincide with F^c. We note that if there exist two irreducible components F', F'' of F^{nc} such that $F' \cap F'' \in \mathrm{CLC}^{\mathrm{c}}(V, F)$, then $F^c \subsetneq \mathrm{LLC}^{\mathrm{c}}(V, F)$. This is why we need the notion of the very good resolution.

2. Connectedness Lemmas

In this section, we treat connectedness lemmas. They play important roles in Section 4. The results are stated for algebraic varieties. However, by the same argument, we can generalize them for analytic spaces which are projective over analytic germs (see Section 6). The following lemma is well-known (for the proof, see [FA, 17.4 Theorem] or [Ka2, Theorem 1.4]).

Lemma 2.1 (Connectedness Lemma, cf. [Sh1, 5.7], [FA, 17.4]). *Let X and Y be normal varieties and $f : X \to Y$ be a proper surjective morphism with connected fibers. Let D be a $\mathbb{Q}$-divisor on X such that $K_X + D$ is $\mathbb{Q}$-Cartier. Write $D = \sum d_i D_i$, where D_i is an irreducible component of D for every i. Assume the following conditions:*

(1) *if $d_i < 0$, then $f(D_i)$ has codimension at least two in Y, and*
(2) *$-(K_X + \Delta)$ is f-nef and f-big.*

Then $\mathrm{LLC}(X, D) \cap f^{-1}(y)$ *is connected for every point* $y \in Y$.

The next proposition is also well-known.

Proposition 2.2 (cf. [Sh1, 6.9] and [FA, 12.3.2]). *Let* (S, Θ) *be a dlt surface. Let* $f : S \to R$ *be a proper surjective morphism onto a smooth curve* R *with connected fibers. Assume that* $K_S + \Theta$ *is numerically* f*-trivial. Then* $\llcorner \Theta \lrcorner \cap f^{-1}(r)$ *has at most two connected components for every* $r \in R$. *Moreover, if* $\llcorner \Theta \lrcorner \cap f^{-1}(o)$ *has exactly two connected components for* $o \in R$, *then, in a neighborhood of* $f^{-1}(o)$, (S, Θ) *is plt and* $\llcorner \Theta \lrcorner$ *has no vertical component with respect to* f.

By using Proposition (2.2), we can prove the next corollary easily.

Corollary 2.3. *Let* (T, Ξ) *be a B-smooth surface and* (S, Θ) *be a dlt surface. Let* $g : (T, \Xi) \dashrightarrow (S, \Theta)$ *be a B-birational map and* $h : T \to R$ *and* $f : S \to R$ *be proper surjective morphisms onto a smooth curve* R *with connected fibers such that* $h = f \circ g$. *Assume that there exists an irreducible component* C *of* Ξ^B *such that* $C \subset h^{-1}(r)$ *for some* $r \in R$. *Then* $\Xi^B \cap h^{-1}(r) = \mathrm{LLC}(T, \Xi) \cap h^{-1}(r)$ *is connected.*

Proof. By applying the elimination of indeterminacy, we may assume that g is a B-birational morphism. Apply Proposition (2.2) to $f : S \to R$ and Lemma (2.1) to $g : T \to S$. Thus we get $\Xi^B \cap h^{-1}(r)$ is connected. □

On the assumption that the log MMP holds in dimension n, we get higher dimensional generalizations of Proposition (2.2) and Corollary (2.3). Proposition (2.4) is a special case of [Fj1, Proposition 2.1 (0)].

Proposition 2.4. *Assume that the log MMP holds in dimension* n. *Let* (X, Θ) *be a proper connected dlt* n*-fold and* $K_X + \Theta \equiv 0$. *Assume that* $K_X + \Theta$ *is Cartier. In particular,* Θ *is an integral divisor. Then one of the following holds:*

(1) Θ *is connected,*
(2) $\Theta = \Theta_1 \amalg \Theta_2$, *where* Θ_i *is connected and irreducible for* $i = 1, 2$. *In particular,* (X, Θ) *is plt. We note that* (X, Θ) *is canonical since* $K_X + \Theta$ *is Cartier and plt. Furthermore, there exists a B-birational map* $(\Theta_1, 0) \dashrightarrow (\Theta_2, 0)$.

Proof. By replacing (X, Θ) with its $\mathbb{Q}$-factorial dlt model, we may assume that X is $\mathbb{Q}$-factorial. By [Fj1, Proposition 2.1 (0) and Remark 2.2 (2)], $\llcorner \Theta \lrcorner = \Theta$ has at most two connected components. If Θ is connected, then (1) holds. So, we may assume that Θ has two connected components. By the proof of [Fj1, Proposition 2.1], there exists a sequence of $(K + \Theta - \varepsilon\Theta)$-flips and divisorial contractions $p : X \dashrightarrow Z$

and $(K_Z + p_*\Theta - \varepsilon p_*\Theta)$-Fano contraction to $(n-1)$-dimensional lc pair (V, P), denoted by $u : Z \to V$, where P is the divisor such that $K_Z + p_*\Theta = u^*(K_V + P)$ (see [Fj1, Lemma 2.3]). We denote $p_*\Theta = \Theta'_1 \amalg \Theta'_2$. Then $(\Theta'_i, \mathrm{Diff}(p_*\Theta - \Theta'_i)) \simeq (V, P)$ for $i = 1, 2$. It is because $\Theta'_i \simeq V$ by Zariski's Main Theorem. Since $K_Z + p_*\Theta$ is Cartier and $p_*\Theta = \llcorner p_*\Theta \lrcorner$, and $(Z, p_*\Theta)$ is lc, $\mathrm{Diff}(p_*\Theta - \Theta'_i)$ is an integral divisor. Therefore, the divisor P is also integral. Since $p_*\Theta$ has no vertical component with respect to u, we have $P = 0$. By [N2, Appendix] or [Fj2, Corollary 4.5], $(V, 0)$ is lt. Therefore, $(\Theta'_i, \mathrm{Diff}(p_*\Theta - \Theta'_i)) = (\Theta'_i, 0)$ is lt for $i = 1, 2$. Then $(Z, p_*\Theta)$ is plt in a neighborhood of $p_*\Theta$. Thus, we get (2) (see [Fj1, Proposition 2.1, Lemma 2.3, and Lemma 2.4]). □

The following proposition is a higher dimensional analogue of Corollary (2.3). The proof is similar to that of Proposition (2.4).

Proposition 2.5. *Assume that the log MMP holds in dimension n. Let (X, Θ) be a dlt n-fold and $f : X \to R$ be a proper surjective morphism onto a normal variety R with connected fibers. Assume that $\dim R \geq 1$, $K_X + \Theta$ is Cartier, and $K_X + \Theta$ is numerically f-trivial. Let $o \in R$ be a closed point. Assume that there exists an irreducible component Θ_o of Θ such that $f(\Theta_o) = o$. Then $f^{-1}(o) \cap \Theta$ is connected.*

Proof. As in the proof of Proposition (2.4), We may assume that X is $\mathbb{Q}$-factorial. By [Fj1, Proposition 2.1 (1)], it is enough to think about the case where there exists a sequence of $(K + \Theta - \varepsilon\Theta)$-flips and divisorial contractions $p : X \dashrightarrow Z$ over R and $(K_Z + p_*\Theta - \varepsilon p_*\Theta)$-Fano contraction to $(n-1)$-dimensional lc pair (V, P) over R, denoted by $u : Z \to V$. In this case, $\mathrm{Center}_Z\Theta_o$ is in $p_*\Theta \cap h^{-1}(o)$, where $h : Z \to V \to R$. If $p_*\Theta \cap h^{-1}(o)$ is connected, then we get the result (see [Fj1, Lemma 2.4]). So, we may assume that $p_*\Theta \cap h^{-1}(o)$ is not connected. By shrinking R to a small analytic neighborhood of $o \in R$, we may assume that $p_*\Theta = \Theta'_1 \amalg \Theta'_2$ in a neighborhood of $h^{-1}(o)$. Without loss of generality, we may assume that $\mathrm{Center}_Z\Theta_o \subset \Theta'_1$. In a neighborhood of $h^{-1}(o)$, $u|_{\Theta'_i} : \Theta'_i \to V$ is finite for $i = 1, 2$. It is because, if $u|_{\Theta'_i} : \Theta'_i \to V$ is not finite, then $\Theta'_1 \cap \Theta'_2 \neq \emptyset$ since Θ'_{3-i} is u-ample. By using Zariski's Main Theorem (see, for example, [Ue, Theorem 1.11]), we get $\Theta'_i \simeq V$ and $(\Theta'_i, \mathrm{Diff}(p_*\Theta - \Theta'_i)) \simeq (V, P)$ in a neighborhood of $h^{-1}(o)$ for $i = 1, 2$. Since $p_*\Theta$ has no u-vertical component in a neighborhood of $h^{-1}(o)$, we get $P = 0$. Note that P is integral (see the proof of Proposition (2.4)). Then $(\Theta'_i, \mathrm{Diff}(p_*\Theta - \Theta'_i)) \simeq (V, 0)$ in a neighborhood of $h^{-1}(o)$. Therefore, $(\Theta'_i, \mathrm{Diff}(p_*\Theta - \Theta'_i))$ is lt in a neighborhood of $h^{-1}(o)$. So, $(Z, p_*\Theta)$ is plt in a neighborhood of $h^{-1}(o)$.

This contradicts the assumption that $\mathrm{Center}_Z\Theta_o \subset \Theta'_1$. So, $p_*\Theta$ is connected in a neighborhood of $h^{-1}(o)$. Then we get the result. □

The next proposition plays an essential role in the proof of the main theorem (see Proposition (4.5)).

Proposition 2.6. *Assume that the log MMP holds in dimension* $\leq n$. *Let* (T,Ξ) *be a B-smooth* n*-fold and* (S,Θ) *a dlt* n*-fold. Let* $g : (T,\Xi) \dashrightarrow (S,\Theta)$ *be a B-birational map and* $h : (T,\Xi) \to (o \in R)$ *and* $f : (S,\Theta) \to (o \in R)$ *be proper surjective morphisms with connected fibers onto a germ* $(o \in R)$ *of a normal variety* R *with* $\dim R \geq 1$. *Assume the following conditions:*

(1) $K_S + \Theta$ *and* $K_T + \Xi$ *are Cartier divisors,*
(2) $K_S + \Theta \equiv_f 0$ *and* $K_T + \Xi \equiv_h 0$,
(3) $h^{-1}(o)$ *and* $h^{-1}(o) \cup \Xi$ *are simple normal crossing divisors,*
(4) g *induces an isomorphism at every generic point of center of log canonical singularities for the pair* (T,Ξ),
(5) *there exists an irreducible component* Ξ_o *of* Ξ^B *such that* $h(\Xi_o) = o \in R$.

Then $\mathrm{LLC}^c(T,\Xi) = \Xi^c$ *is connected. Furthermore, if* $\Xi' \subset \Xi^B$ *is an irreducible component such that* $h^{-1}(o) \cap \Xi' \neq \emptyset$, *then* $\Xi' \cap \Xi^c \neq \emptyset$.

Proof. First, if $\dim T = \dim S = 2$, then this proposition is true by Lemma (2.1) and Corollary (2.3).

Next, apply the elimination of indeterminacy. We may assume that g is a morphism. By using Proposition (2.5) and applying Lemma (2.1) to the morphism g, we get $\mathrm{LLC}(T,\Xi) \cap h^{-1}(o) = \Xi^B \cap h^{-1}(o)$ is connected.

Finally, we go back to the original B-birational map g. Let D be the maximum connected component of Ξ^c such that $\Xi_o \subset D$. For the proof of this proposition, it is enough to exclude the following situation;

(♣) there exist irreducible divisors Ξ_1 and Ξ_2 which satisfy the following conditions:
  (i) Ξ_1 and Ξ_2 are irreducible components of Ξ^B,
  (ii) there exists a connected component C of $\Xi_1 \cap h^{-1}(o)$ such that $C \cap D \neq \emptyset$ and $C \cap (\Xi_2 \cap h^{-1}(o)) \neq \emptyset$,
  (iii) $(\Xi_2 \cap h^{-1}(o)) \cap D = \emptyset$.

Note that $D \subset h^{-1}(o)$. We use the induction on n to exclude (♣). By the above argument, when $\dim T = \dim S = 2$, this proposition is true. Assume that this proposition is true in dimension $n-1$. If $h(\Xi_1) = o$, then $\Xi_1 \subset \Xi^c$. This contradicts the definition of D. So we get $\Xi_1 \subset \Xi^{nc}$. Let (S',Θ') be a proper transform of $(\Xi_1, (\Xi - \Xi_1)|_{\Xi_1})$, which can be taken by the condition (4), and $h' : (\Xi_1, (\Xi - \Xi_1)|_{\Xi_1}) \to R'$ be the

Stein factorization of $h : \Xi_1 \to R$. Since S' is normal, there exists $f' : S' \to R'$. We define $o' := h'(C)$. Apply the hypothesis of the induction to $(\Xi_1, (\Xi - \Xi_1)|_{\Xi_1}), (S', \Theta')$, and $(o' \in R')$. We note that the conditions (1), (2), and (4) are satisfied by the adjunction. The condition (3) is true since Ξ_1 is an irreducible component of Ξ and (5) is also true since $C \cap D \neq \emptyset$. Therefore, we obtain that, in the fiber $(h')^{-1}(o')$, $\Xi_1 \cap D$ and $\Xi_1 \cap \Xi_2$ are connected by $((\Xi - \Xi_1)|_{\Xi_1})^c$. Thus there exists $\Xi_3 \subset \Xi^c$ such that $\Xi_3 \cap D \neq \emptyset$, and $\Xi_3 \not\subset D$. This contradicts the definition of D. We note the condition (3) and $\mathrm{LLC}^c(T, \Xi) = \Xi^c$. □

3. Finiteness and Boundedness

In this section, we investigate the birational automorphism groups. First, we prove the following proposition, which is an easy consequence of [Ue, Theorem 14.10] and a special case of [Fj1, Conjecture 3.2].

Proposition 3.1. *Let $(S, 0)$ be a normal n-fold with only canonical singularities such that $K_S \sim 0$. Then the image $\rho_m(\mathrm{Bir}(S, 0))$ is finite, where $\rho_m : \mathrm{Bir}(S, 0) \to \mathrm{GL}(H^0(S, mK_S))$.*

Proof. Let $f : T \to S$ be a resolution. Then $K_T = f^*K_S + E$, where E is an effective Cartier divisor. It is obvious that $\sigma \in \mathrm{Bir}(S, 0)$ induces $f^{-1} \circ \sigma \circ f \in \mathrm{Bir}(T, -E)$. We denote

$$\mathrm{Bir}(T) = \{\text{all birational maps of } Y \text{ onto itself}\}.$$

Then $\mathrm{Bir}(T, -E) \subset \mathrm{Bir}(T)$. The image of $\mathrm{Bir}(T) \to \mathrm{GL}(H^0(T, mK_T))$ is finite by [Ue, Theorem 14.10], Thus the image of $\mathrm{Bir}(T, -E) \to \mathrm{GL}(H^0(T, m(K_T - E)))$ is also finite. Note that $H^0(T, m(K_T - E)) = H^0(T, mK_T)$. Therefore $\rho_m(\mathrm{Bir}(S, 0))$ is finite. □

By Proposition (3.1), we get the finiteness of B-pluricanonical representations. However, for the main theorem, we need the stronger results. So, we write down the required conjectures.

Conjecture 3.2 (Boundedness of B-canonical representations). *The following conjectures (F_l) and (F'_l) are used in the main results.*

- (F_l) *There exists a positive integer B_l such that $|\rho_1(\mathrm{Bir}(S, 0))| \leq B_l$ for every l-dimensional variety S with only canonical singularities such that $K_S \sim 0$.*
- (F'_l) *There exists a positive integer B'_l such that the order of $\rho_1(g)$ in $\mathrm{GL}(H^0(S, K_S))$ is bounded above by B'_l for every l-dimensional variety S with only canonical singularities such that $K_S \sim 0$ and for every $g \in \mathrm{Aut}(S, 0) = \mathrm{Aut}(S)$ such that g has a finite order.*

For low dimensional varieties, we know the details of canonical representations. We list the results needed in this paper for reader's convenience. This is [Is3, Proposition 4.8, Proposition 4.9].

Proposition 3.3. (1) *For an arbitrary elliptic curve C, denote the order $|\rho_1(\mathrm{Aut}(C))|$ by r, where $\rho_1 : \mathrm{Aut}(C) \to \mathrm{GL}(H^0(C, K_C))$. Then $\varphi(r) \leq 2$, which means $r = 1, 2, 3, 4, 6$ (see, for example,* [Ha, Chapter IV Corollary 4.7]). *Here φ is the Euler function.*

(2) ([Ni, 10.1.2]) *For an arbitrary $K3$ surface X, denote the order $|\rho_1(\mathrm{Aut}(X))|$ by r, where $\rho_1 : \mathrm{Aut}(X) \to \mathrm{GL}(H^0(X, K_X))$ is the induced representation. then $\varphi(r) \leq 20$, in particular $r \leq 66$. Machida and Oguiso proved that there are no $K3$ surfaces which satisfy $r = 60$ in* [MO].

(3) ([Fk, 3.2]) *For an arbitrary Abelian surface X, the order r of a finite automorphism on X satisfies $\varphi(r) \leq 4$, which means that $r = 1, 2, 3, 4, 5, 6, 8, 10$.*

Under Conjecture (3.2), we define the following constants.

Definition 3.4. Assuming that (F_l) holds true, we define

$$C_l := \{c \in \mathbb{N} \,|\, c = |\rho_1(\mathrm{Bir}(S, 0))|\},$$
$$D_l := \{d \in \mathbb{N} \,|\, d = \mathrm{l.c.m.}(2, c), \text{ where } c \in C_l\},$$
$$I_l := \{e \in \mathbb{N} \,|\, e \text{ is a divisor of } d \in D_l\}.$$

Assuming that (F'_l) holds true, we define

$$I'_l := \{\, c \in \mathbb{N} \,|\, c = |\rho_1(g)|\}.$$

Remark 3.5. The conjecture (F_l) implies (F'_l), and (F_l) holds true for $l \leq 1$ by Proposition (3.3) (1). In particular, we have that

$$I_0 = \{1, 2\}, \quad I'_0 = \{1\}, \quad I_1 = I'_1 = \{1, 2, 3, 4, 6\}.$$

Proposition 3.6. *The conjecture (F_2) holds true with $B_2 = 66$ and*

$$I'_2 = \{r \in \mathbb{N} \,|\, \varphi(r) \leq 20, r \neq 60\}.$$

Proof. Let $(S, 0)$ be a normal surface with only canonical singularities such that $K_S \sim 0$. Let $f : T \to S$ be the minimal resolution. Note that f is crepant, that is, $K_T = f^* K_S$. If $g \in \mathrm{Bir}(S, 0)$, then $g' := f^{-1} \circ g \circ f \in \mathrm{Bir}(T, 0)$. The discrepancy of every exceptional divisor over T is positive and that of another divisor is non-positive. Since B-birational map g' does not change discrepancies, we have that $g' \in \mathrm{Aut}(T, 0) = \mathrm{Aut}(T)$. By the classification of surfaces (see, for example, [Be, Theorem VIII.2]), T is Abelian or $K3$. If T is $K3$, then the second Betti number $b_2(T) = 22$. If T is Abelian, then the second Betti number $b_2(T) = 6$. Therefore, (F_2) holds true with $B_2 = 66$ by the proof of [Ue,

Proposition 14.4] and $I_2' = \{r \in \mathbb{N} \,|\, \varphi(r) \leq 20, r \neq 60\}$ by Proposition (3.3) (2). □

4. Indices of lc pairs with standard coefficients

In this section, we use the following notations and the log MMP in dimension n freely. All the results are stated for algebraic varieties. For analytic spaces, we recommend the reader to see Section 6.

Notation 4.1 (cf. [Sh1, §2]). Let $(P \in X, \Delta)$ be an n-dimensional log canonical pair such that $\Delta = \sum d_i \Delta_i$ has only standard coefficients. From now on, we may shrink X around P without mentioning it. If $\mathrm{I}(P \in X, \Delta) = a$, then $a(K_X + \Delta) \sim 0$. The corresponding finite cyclic cover

$$\pi : (Q \in Y, D) \to (P \in X, \Delta)$$

of degree a is ramified only over the components of Δ_i of Δ with $d_i < 1$ (see [Sh1, 2.3, 2.4]). Since Δ has only standard coefficients, D is reduced and $D = \pi^* \llcorner \Delta \lrcorner$. We say that this $\pi : (Q \in Y, D) \to (P \in X, \Delta)$ is the *index 1 cover* of the log divisor $K_X + \Delta$. By the construction, $K_Y + D = \pi^*(K_X + \Delta)$ has index 1 and the index 1 cover is unique up to étale equivalences. Let G be the cyclic group associated to the cyclic cover $\pi : Y \to X$. Then we have the followings:

$$(Q \in Y, D)/G \simeq (P \in X, \Delta),$$

$$(\diamondsuit) \qquad (\pi_* \mathcal{O}_Y(m(K_Y + D)))^G \simeq \mathcal{O}_X(mK_X + \llcorner m\Delta \lrcorner),$$

where $m \in \mathbb{Z}_{\geq 0}$. From now on, we assume that $P \in \mathrm{CLC}(X, \Delta)$. By [FA, Chapter 20], [Sh1, §2], or [KoM, Proposition 5.20], $(Q \in Y, D)$ is log canonical and $P \in \mathrm{CLC}(X, \Delta)$ is equivalent to $Q \in \mathrm{CLC}(Y, D)$. Therefore, we may assume that $\mathrm{LLC}^{\mathrm{c}}(Y, D) = Q$ without loss of generality.

Proposition 4.2. *Let m_0 be a non-negative integer. Let s be a G-invariant generator of $\mathcal{O}_Y(m_0(K_Y + D))$. Then $m_0(K_X + \Delta)$ is Cartier. In particular, $m_0\Delta$ is integral.*

Proof. By $(\diamondsuit)$, there exists a generator t of $\mathcal{O}_X(m_0K_X + \llcorner m_0\Delta \lrcorner)$ such that $\pi^* t = s$. In particular, $m_0K_X + \llcorner m_0\Delta \lrcorner$ is Cartier. Let l be a sufficiently divisible positive integer such that $l\Delta$ is integral. Since t is a generator of $\mathcal{O}_X(m_0K_X + \llcorner m_0\Delta \lrcorner)$, $t^{\otimes l}$ is that of $\mathcal{O}_X(lm_0K_X + l\llcorner m_0\Delta \lrcorner)$. On the other hand, $s^{\otimes l}$ is a generator of $\mathcal{O}_Y(lm_0(K_Y + D))$ and $(\pi_* \mathcal{O}_Y(lm_0(K_Y + D)))^G \simeq \mathcal{O}_X(lm_0(K_X + \Delta))$. Therefore, $t^{\otimes l}$ is also a generator of $\mathcal{O}_X(lm_0(K_X + \Delta))$. Thus $\llcorner m_0\Delta \lrcorner = m_0\Delta$, and $m_0(K_X + \Delta)$ is Cartier. □

Proposition 4.3 (cf. [Is3, Lemma 3.3]). *Let m_Q be the maximal ideal of Q and $\rho : G \to \mathrm{GL}(\omega_Y(D)/m_Q\omega_Y(D))$ the canonical representation. Let $m_0 := |\rho(G)|$. Then $\mathrm{I}(P \in X, \Delta) = m_0$.*

Proof. The cyclic group G acts on $\omega_Y(D)^{\otimes m_0}/m_Q\omega_Y(D)^{\otimes m_0}$ trivially. Let s' be a generator of $\omega_Y(D)^{\otimes m_0}/m_Q\omega_Y(D)^{\otimes m_0}$ and s'' a lift of s' in $\omega_Y(D)^{\otimes m_0}$. Put

$$s := \frac{1}{|G|}\sum_{\sigma\in G}\sigma^* s''.$$

Then s is a lift of s' and a G-invariant generator of $\mathcal{O}_Y(m_0(K_Y + D))$. By Proposition (4.2), we obtain that $r := \mathrm{I}(P \in X, \Delta)$ divides m_0. On the other hand, by considering the pull-back of the generator of $r(K_X + \Delta)$, we obtain that m_0 divides r. So we get $\mathrm{I}(P \in X, \Delta) = m_0$. □

The following lemma is a special case of Lemma (1.13).

Lemma 4.4. *Let $h : (V, F) \to (Q \in Y, D)$ be a very good resolution. Let*

$$V = V^0 \overset{p_1}{\dashrightarrow} V^1 \overset{p_2}{\dashrightarrow} \cdots \overset{p_i}{\dashrightarrow} V^i \overset{p_{i+1}}{\dashrightarrow} V^{i+1} \overset{p_{i+2}}{\dashrightarrow} \cdots \overset{p_{l-1}}{\dashrightarrow} V^{l-1} \overset{p_l}{\dashrightarrow} V^l = Z$$

be the $(K_V + F^B)$-log MMP over Y. We denote $F_0^B = F^B, F_0 = F$, and $F_i^B = p_{i*}F_{i-1}^B, F_i = p_{i*}F_{i-1}$, for every i and $F_l = E$. Then $f : (Z, E) \to (Y, D)$ is a $\mathbb{Q}$-factorial dlt model and $g := p_l \circ p_{l-1} \circ \cdots \circ p_1$ induces an isomorphism at every generic point of center of log canonical singularities for the pair (V, F). We note that $E = F_l = F_l^B$. Furthermore, $\mathrm{LLC}^{\mathrm{c}}(Z, E) = E^{\mathrm{c}}$.

Proof. Since $K_Y + D$ is Cartier, the effective part of F is F^B. Therefore, by Lemma (1.13), we get the result. □

The next proposition is very important. We prove it by using Proposition (2.6). Note that, if $(Q \in Y, 0)$ is an isolated singularity, then this proposition is obvious by Lemma (2.1).

Proposition 4.5. *Let $h : (V, F) \to (Q \in Y, D)$ be a very good resolution. Then $\mathrm{LLC}^{\mathrm{c}}(V, F) = F^{\mathrm{c}}$ is connected.*

Proof. Since h is a very good resolution, we have $\mathrm{LLC}^{\mathrm{c}}(V, F) = F^{\mathrm{c}}$. Run the $(K_V + F^B)$-log MMP over Y. We get a $\mathbb{Q}$-factorial dlt model $f : (Z, E) \to (Y, D)$ (see Lemma (4.4)). We put $(T, \Xi) := (V, F)$, $(S, \Theta) := (Z, E)$, and $(o \in R) := (Q \in Y)$ and apply Proposition (2.6). The conditions (1), (2) and (5) in Proposition (2.6) are satisfied since $K_Y + D$ is Cartier and $K_V + F = h^*(K_Y + D)$, $K_Z + E = f^*(K_Y + D)$ and $Q \in \mathrm{CLC}(Y, D)$. The condition (3) is in the definition of the

very good resolution and (4) has been already checked in Lemma (4.4). Therefore, we can apply Proposition (2.6). Thus we obtain that F^c is connected. □

The following is a corollary of Proposition (4.5). However, we don't use it for the proof of the main result.

Corollary 4.6. *Let $h' : (V', F') \to (P \in X, \Delta)$ be a very good resolution. Then $\mathrm{LLC}^{\mathrm{c}}(V', F') = F'^c$ is connected.*

Proof. Since h' is a very good resolution, we have that $\mathrm{LLC}^{\mathrm{c}}(V', F') = F'^c$. Let $h : (V, F) \to (Q \in Y, D)$ be a very good resolution which factors $Y \times_X V'$. By Proposition (4.5), $\mathrm{LLC}^{\mathrm{c}}(V, F) = F^c$ is connected. Since $h''(F^c) = F'^c$, where $h'' : V \to V'$, we obtain that F'^c is connected. □

Proposition 4.7. *There exists a $\mathbb{Q}$-factorial dlt model $f : (Z, E) \to (Y, D)$, that is, (Z, E) is dlt and*

$$K_Z + E = f^*(K_Y + D),$$

such that the following conditions are satisfied:

(1) $\mathrm{LLC}^{\mathrm{c}}(Z, E) = E^c$,
(2) *E^c is connected and $(E^c, (E - E^c)|_{E^c})$ is sdlt,*
(3) $K_Z + (E - E^c)|_{E^c} = (K_Z + E)|_{E^c} = f^*(K_Y + D)|_{E^c} \sim 0$.

Proof. Let $f : (Z, E) \to (Y, D)$ be a $\mathbb{Q}$-factorial dlt model constructed in Lemma (4.3). Let $h' : (U, H) \to (Z, E) \to (Q \in Y, D)$ be a very good resolution. Then H^c is connected by Proposition (4.5) and $g'(H^c) = E^c$, where $g' : U \to Z$. Therefore, E^c is connected. Since (Z, E) is $\mathbb{Q}$-factorial and dlt, E^c is Cohen-Macaulay and $(E^c, (E - E^c)|_{E^c})$ is sdlt by the adjunction. We note that $\mathrm{Diff}(E - E^c) = (E - E^c)|_{E^c}$ since (Z, E) is dlt and $K_Z + E$ is Cartier (see, for example, [FA, 16.6 Proposition]). □

Remark 4.8. By using Lemma (1.13) and Corollary (4.6), we get a similar result about $(P \in X, \Delta)$ by the same proof as that of Proposition (4.7). That is, there exists a $\mathbb{Q}$-factorial dlt model $f' : (Z', E') \to (X, \Delta)$, that is, (Z', E') is dlt and $K_{Z'} + E' = f'^*(K_X + \Delta)$, such that the following conditions are satisfied:

(1) $\mathrm{LLC}^{\mathrm{c}}(Z', E') = E'^c$,
(2) E'^c is connected and $(E'^c, \mathrm{Diff}(E' - E'^c))$ is sdlt,
(3) $K_{Z'} + \mathrm{Diff}(E' - E'^c) = (K_{Z'} + E')|_{E'^c} = f'^*(K_X + \Delta)|_{E'^c} \sim_{\mathbb{Q}} 0$.

See also Lemma (4.16).

Lemma 4.9. *We have the following isomorphisms:*

$$\begin{aligned}\omega_Y(D)^{\otimes m}/m_Q\omega_Y(D)^{\otimes m} &\simeq H^0(E^c, f^*m(K_Y+D)|_{E^c})\\ &\simeq H^0(E^c, m(K_{E^c}+(E-E^c)|_{E^c}))\\ &\simeq H^0(E^c, \mathcal{O}_{E^c}) \simeq \mathbb{C},\end{aligned}$$

where m_Q is the maximal ideal of Q.

Proof. We consider the following exact sequence,

$$(\spadesuit) \qquad 0 \to \mathcal{O}_Z(-E^c) \to \mathcal{O}_Z \to \mathcal{O}_{E^c} \to 0.$$

Thus we obtain

$$\mathcal{O}_Y/f_*\mathcal{O}_Z(-E^c) \simeq \mathbb{C}.$$

Note that E^c is connected and $f(E^c) = Q$. Since it is obvious that $m_Q \subset f_*\mathcal{O}_Z(-E^c)$, so we obtain $m_Q = f_*\mathcal{O}_Z(-E^c)$. By tensoring $\mathcal{O}_Z(f^*m(K_Y+D))$ to $(\spadesuit)$ and taking direct images, we get

$$\begin{aligned}0 \to m_Q \otimes \mathcal{O}_Y(m(K_Y+D)) &\to \mathcal{O}_Y(m(K_Y+D))\\ &\to H^0(E^c, f^*m(K_Y+D)|_{E^c}) \to \cdots.\end{aligned}$$

Therefore, we obtain the required isomorphisms. □

Proposition 4.10. *Let σ be an element of G. The B-birational automorphism $\sigma : (Y,D) \to (Y,D)$ induces a B-birational map $\sigma_Z := f^{-1}\circ\sigma\circ f : (Z,E) \dashrightarrow (Z,E)$. Let $\alpha, \beta : (V,F) \to (Z,E)$ be a common resolution of $\sigma_Z : (Z,E) \dashrightarrow (Z,E)$ such that $f\circ\alpha, f\circ\beta : (V,F) \to (Q \in Y, D)$ are very good resolutions. Then we get the following commutative diagram:*

$$\begin{array}{ccc} H^0(E^c, m(K_{E^c}+(E-E^c)|_{E^c})) & \xleftarrow[\sigma^*_{E^c}]{\sim} & H^0(E^c, m(K_{E^c}+(E-E^c)|_{E^c})) \\ {\scriptstyle f^*}\uparrow\simeq & & \simeq\uparrow{\scriptstyle f^*} \\ \omega_Y(D)^{\otimes m}/m_Q\omega_Y(D)^{\otimes m} & \xleftarrow[\sigma^*]{\sim} & \omega_Y(D)^{\otimes m}/m_Q\omega_Y(D)^{\otimes m}. \end{array}$$

Here $\sigma^*_{E^c} = (\alpha^*)^{-1}\circ\beta^*$, where $\alpha^*, \beta^* : H^0(E^c, m(K_{E^c}+(E-E^c)|_{E^c})) \simeq H^0(F^c, m(K_{F^c}+(F-F^c)|_{F^c}))$.

Proof. We note the following isomorphisms:

$$\begin{aligned}H^0(F^c, m(K_{F^c}+(F-F^c)|_{F^c})) &\simeq \mathbb{C},\\ H^0(E^c, m(K_{E^c}+(E-E^c)|_{E^c})) &\simeq \mathbb{C},\\ \omega_Y(D)^{\otimes m}/m_Q\omega_Y(D)^{\otimes m} &\simeq \mathbb{C}.\end{aligned}$$

Then we get the above commutative diagram by Lemma (4.9). □

Proposition 4.11. *In Proposition (4.10), if there exists a non-zero admissible section in $H^0(E^c, m_0(K_{E^c} + (E - E^c)|_{E^c}))$, then G acts on $\omega_Y(D)^{\otimes m_0}/m_Q\omega_Y(D)^{\otimes m_0}$ trivially.*

Proof. This can be checked by the same argument as that of [Fj1, Lemma 4.7]. Let $s \in H^0(E^c, m(K_{E^c} + (E - E^c)|_{E^c}))$ be a non-zero admissible section. It is sufficient to prove $\alpha^* s = \beta^* s$ in $H^0(F^c, m_0(K_{F^c} + (F - F^c)|_{F^c}))$. Let F^1 be any irreducible component of F^c. By applying [Fj1, Claim (B_n)] repeatedly, we get $F^2 \Subset F^1$ or $F^2 = F^1$ such that $\alpha : F^2 \to \alpha(F^2)$ and $\beta : F^2 \to \beta(F^2)$ are B-birational morphisms and $H^0(F^1, m_0(K_V + F)|_{F^1}) \simeq H^0(F^2, m_0(K_V + F)|_{F^2})$ (see the proof of [Fj1, Lemma 4.7]). Since $\alpha(F^2)$ is B-birationally equivalent to $\beta(F^2)$ and s is a non-zero admissible section, we obtain that $\alpha^*(s|_{\alpha(F^2)}) = \beta^*(s|_{\beta(F^2)})$ in $H^0(F^2, m_0(K_V + F)|_{F^2})$. Therefore, we have that $\alpha^* s = \beta^* s$ in $H^0(F^1, m_0(K_V + F)|_{F^1})$. Thus we obtain that $\alpha^* s = \beta^* s$ in $H^0(F^c, m_0(K_{F^c} + (F - F^c)|_{F^c}))$. □

Definition 4.12. Let $(P \in X, \Delta)$ and $(Q \in Y, D)$ be as in Notation (4.1). Let $h : (U, H) \to (Q \in Y, D)$ be a log resolution such that $K_U + H = h^*(K_Y + D)$. We define

$$\mu = \mu(P \in X, \Delta) := \min\{\dim W \mid W \in \mathrm{CLC}^{\mathrm{c}}(U, H)\}.$$

It is obvious that $0 \leq \mu \leq \dim X - 1$. We note that the index 1 cover $(Q \in Y, D)$ is defined uniquely up to étale equivalences. By Lemma (1.7) (see also Section 6), μ is independent of the choice of the resolution. Therefore, $\mu(P \in X, \Delta)$ is well-defined.

Remark 4.13. When $(P \in X, 0)$ is an n-dimensional isolated log canonical singularity, which is not log terminal, Shihoko Ishii defined the lc singularity of type $(0, i)$ by using the mixed Hodge structure of the simple normal crossing variety H^B (see [Is3, 2.7]), where $h : (U, H) \to (Q \in Y, 0)$ is a log resolution as in Definition (4.12). She also proved that

$$\dim \Gamma_{H^B} = n - 1 - i,$$

where Γ_{H^B} is the dual graph of H^B by [Is1, Theorem 2] and [Is4]. By the definition of $\mu(P \in X, 0)$, we have

$$\dim \Gamma_{H^B} = n - 1 - \mu.$$

Therefore, we get $\mu = i$ when $(P \in X, 0)$ is an isolated log canonical singularity. Furthermore, if the log MMP holds true in dimension $\leq n$, then the above dual graph Γ_{H^B} is pure $(n-1-\mu)$-dimension by Lemma (1.7) and Lemma (4.15) below.

Proposition 4.14. *Assume that $\mu(P \in X, \Delta) \leq n-2$. If (F_μ) holds true, then there is a non-zero admissible section $s \in H^0(E^c, m_0(K_{E^c} + (E - E^c)|_{E^c}))$ with $m_0 \in D_\mu$. In particular, s is G-invariant. Thus, $\mathrm{I}(P \in X, \Delta) \in I_\mu$.*

Proof. Let W be a compact minimal center of lc singularities for the pair (Z, E). By the definition of $\mu(P \in X, \Delta)$, $\dim W = \mu$. We note that $K_{E^c} + (E - E^c)|_{E^c} \sim 0$. By using the adjunction repeatedly, we have that $(K_{E^c} + (E - E^c)|_{E^c})|_W = K_W \sim 0$. So $(W, 0)$ has only canonical singularities. By Lemma (4.15) below, all the minimal compact centers of lc singularities are B-birationally equivalent to $(W, 0)$. Therefore, $H^0(\amalg W, m_1(K_Z + E)|_{\amalg W})$ has a non-zero admissible section with $m_1 := |\rho_1(\mathrm{Bir}(W, 0))| \in C_\mu$, where the sum runs over all the compact minimal centers of lc singularities for the pair (Z, E). By applying Proposition (4.17) below repeatedly, we get a non-zero admissible section with $m_0 \in D_\mu$, where $m_0 = \mathrm{l.c.m.}(2, m_1)$. Therefore, by Propositions (4.2), (4.10), (4.11) and the proof of Proposition (4.3), we get $\mathrm{I}(P \in X, \Delta)$ is a divisor of m_0. In particular, $\mathrm{I}(P \in X, \Delta) \in I_\mu$. □

We note that, in Lemmas (4.15) and (4.16) and Proposition (4.17), we use Proposition (2.4) and [Fj1, Proposition 2.1], which need the log MMP.

Lemma 4.15. *Let (S, Δ) be a proper connected sdlt n-fold with $K_S + \Delta \sim 0$. Let $\mu : (S', \Theta) = \amalg(S_i, \Theta_i) \to (S, \Delta)$ be the normalization. Then all the minimal centers of log canonical singularities for the pair (S', Θ) have the same dimension and are B-birationally equivalent to each other.*

Proof. We prove this lemma by induction on n. If $n = 1$, then this is trivial. Let

$$d := \min\{\dim W \mid W \in \mathrm{CLC}(S', \Theta)\}.$$

If $d \leq n-2$, then (S_i, Θ_i) is in Case (1) in Proposition (2.4) and Θ_i is not irreducible for every i. By applying the induction to $(n-1)$-dimensional sdlt pair $(\Theta_i, 0)$, we get the result.

If $d = n-1$, then (S_i, Θ_i) is plt for every i. Therefore, all the minimal elements in $\mathrm{CLC}(S', \Theta)$ have dimension $n-1$ and B-birationally equivalent to each other by Proposition (2.4). □

For the main result, the above lemma is sufficient. However, in the above lemma, the assumption that $K_S + \Delta \sim 0$ can be replaced by $K_S + \Delta \equiv 0$.

Lemma 4.16. *Let (S, Δ) be a proper connected sdlt n-fold with $K_S + \Delta \equiv 0$. Let $\mu : (S', \Theta) = \amalg(S_i, \Theta_i) \to (S, \Delta)$ be the normalization.*

Then all the minimal centers of log canonical singularities for the pair (S', Θ) have the same dimension and are B-birationally equivalent to each other.

Proof. We prove this lemma by induction on n. If $n = 1$, then this is trivial. Apply the inductive hyposesis to each connected component of $(n-1)$-dimensional sdlt pair $(\llcorner \Theta_i \lrcorner, \mathrm{Diff}(\Theta_i - \llcorner \Theta_i \lrcorner))$ for every i and use Lemma (1.7) and [Fj1, Proposition 2.1 (0) (0.2)]. Then we get the result. □

In the proof of Proposition (4.14), we used the following result, which is a special case of [Fj1, Proposition 4.3]. In our situation, all the dlt pairs have Kodaira dimension 0. So, [Fj1, Proposition 4.3] can be modified as follows:

Proposition 4.17. *Let (S, Θ) be a proper dlt n-fold such that $K_S + \Theta \sim 0$. Assume that there exists a non-zero admissible section $u \in H^0(\Theta, m(K_S + \Theta)|_\Theta)$. If m is even, then we can extend u to $v \in H^0(S, m(K_S + \Theta))$, that is, $v|_\Theta = u$. In particular, v is a non-zero admissible section.*

Proof. This is a special case of Cases (1) and (4) in the proof of [Fj1, Proposition 4.3] (for Case (4), see also [Fj3, Proposition 4.5]). For the latter part, see the case (3) in the proof of [Fj1, Theorem 3.5]. □

In the case where $\mu(P \in X, \Delta) = n - 1$, we can prove a slightly stronger result by using the canonical desingularization theorem.

Proposition 4.18. *Assume that $\mu(P \in X, \Delta) = n - 1$. Then there exists a projective birational morphism $f : (Z, E) \to (Y, D)$ from a dlt pair (Z, E), which satisfies the following conditions.*

(1) $K_Z + E = f^*(K_Y + D)$.
(2) *Let $\sigma \in G$. The B-birational automorphism $\sigma : (Y, D) \to (Y, D)$ induces the B-birational automorphism $\sigma_Z := f^{-1} \circ \sigma \circ f : (Z, E) \to (Z, E)$ over Y and the automorphism $\sigma_E : E \to E$.*
(3) *The morphism f is G-equivariant.*
(4) *There exists the following commutative diagram:*

$$\begin{array}{ccc} H^0(E, mK_E) & \xleftarrow[\sigma_E^*]{\sim} & H^0(E, mK_E) \\ {\scriptstyle f^*}\uparrow\simeq & & \simeq\uparrow{\scriptstyle f^*} \\ \omega_Y(D)^{\otimes m}/m_Q\omega_Y(D)^{\otimes m} & \xleftarrow[\sigma^*]{\sim} & \omega_Y(D)^{\otimes m}/m_Q\omega_Y(D)^{\otimes m}. \end{array}$$

Proof. We take the canonical desingularization $h : (V, F) \to (Q \in Y, D)$ (see [BM]). Since h is canonical, G acts on V. Since $\sigma_V :=$

$h^{-1} \circ \sigma \circ h$ is a B-birational automorphism, G also acts on F^B. Run the G-equivariant log MMP with respect to $K_V + F^B$ over Y (see [KoM, Example 2.21]). Then we get a G-equivariant dlt model $f : (Z, E) \to (Y, D)$, that is, G acts on Z and E. Since $\mu(P \in X, \Delta) = n-1$, $E = E^c$ is irreducible. By Lemma (4.9), we get (4). □

Proposition 4.19. *Let the notations be as in Proposition (4.18). If the conjecture (F'_{n-1}) holds true, then there exists a non-zero G-invariant section $s \in H^0(E, m_0 K_E)$ with $m_0 := |\rho_1(G)| \in I'_{n-1}$. In particular, G acts trivially on $\omega_Y(D)^{\otimes m_0}/m_Q \omega_Y(D)^{\otimes m_0}$. Thus we get $\mathrm{I}(P \in X, \Delta) = m_0 \in I'_{n-1}$.*

Proof. It is obvious by Propositions (4.18), (4.2), and (4.3). □

The following is the main theorem of this paper, which is a consequence of Propositions (4.10), (4.11), (4.14), (4.18), and (4.19).

Theorem 4.20. *Assume the log MMP for dimension $\leq n$. Let $(P \in X, \Delta)$ be an n-dimensional lc pair such that Δ has only standard coefficients and $P \in \mathrm{CLC}(X, \Delta)$. When $\mu(P \in X, \Delta) \leq n-2$ (resp. $\mu(P \in X, \Delta) = n-1$), we assume (F_μ) (resp. (F'_{n-1})) holds true. Then*

$$\begin{cases} \mathrm{I}(P \in X, \Delta) \in I'_{n-1} \\ \mathrm{I}(P \in X, \Delta) \in I_\mu \end{cases} \quad \textit{if} \quad \begin{cases} \mu(P \in X, \Delta) = n-1, \\ \mu(P \in X, \Delta) \leq n-2. \end{cases}$$

For three dimensional log canonical pairs, we obtain the following result as a corollary of Theorem (4.20) (for related results, see [Is3] and [Sh2, 1.10 Corollary]).

Corollary 4.21. *Let $(P \in X, \Delta)$ be a three dimensional lc pair such that Δ has only standard coefficients and $P \in \mathrm{CLC}(X, \Delta)$. Then*

$$\begin{cases} \mathrm{I}(P \in X, \Delta) \in \{1, 2\} \\ \mathrm{I}(P \in X, \Delta) \in \{1, 2, 3, 4, 6\} \\ \mathrm{I}(P \in X, \Delta) \in I'_2 \end{cases} \quad \textit{if} \quad \begin{cases} \mu(P \in X, \Delta) = 0, \\ \mu(P \in X, \Delta) = 1, \\ \mu(P \in X, \Delta) = 2, \end{cases}$$

where $I'_2 = \{r \in \mathbb{N} \,|\, \varphi(r) \leq 20, r \neq 60\}$. In particular, if there exists $W \in \mathrm{CLC}(X, \Delta)$ such that $P \subsetneq W$, then $\mathrm{I}(P \in X, \Delta) \in \{1, 2, 3, 4, 6\}$.

Remark 4.22. Shihoko Ishii proved that for every $r \in I'_2 = \{r \in \mathbb{N} \,|\, \varphi(r) \leq 20, r \neq 60\}$, there exist three dimensional isolated log canonical singularities such that $\mu(P \in X, 0) = 2$ and $\mathrm{I}(P \in X, 0) = r$ (see [Is3, Theorem 4.15]). For the singularities which satisfy $\mu \leq 1$, see Example (5.4).

For two dimensional log canonical pairs, the following corollary seems to be well-known to specialists.

Corollary 4.23. *Let $(P \in X, \Delta)$ be a two dimensional lc pair such that Δ has only standard coefficients and $P \in \mathrm{CLC}(X, \Delta)$. Then*

$$\begin{cases} \mathrm{I}(P \in X, \Delta) \in \{1, 2\} \\ \mathrm{I}(P \in X, \Delta) \in \{1, 2, 3, 4, 6\} \end{cases} \quad \text{if} \quad \begin{cases} \mu(P \in X, \Delta) = 0, \\ \mu(P \in X, \Delta) = 1. \end{cases}$$

5. Examples

In this section, we treat some examples of log canonical pairs.

Example 5.1. Let $X = (x^3 + y^3 + z^3 = 0) \subset \mathbb{C}^4 = \operatorname{Spec} \mathbb{C}[x, y, z, t]$. The cyclic group $\mathbb{Z}_m$ acts on X as follows:

$$(x, y, z, t) \mapsto (\varepsilon x, \varepsilon y, \varepsilon z, \varepsilon t),$$

where ε is a primitive m-th root of unity. Let $(o \in Y_m)$ be the quotient $X/\mathbb{Z}_m$, where o is the image of $(0, 0, 0, 0) \in X$. The cyclic group $\mathbb{Z}_m$ acts on $\omega_X = \mathcal{O}_X(K_X)$ as follows:

$$\omega \mapsto \varepsilon \omega,$$

where

$$\omega = \mathrm{Res} \frac{dx \wedge dy \wedge dz \wedge dt}{x^3 + y^3 + z^3},$$

which is a generator of ω_X. Therefore we obtain that $\mathrm{I}(o \in Y_m) = m$. This shows that the indices are not bounded if $o \notin \mathrm{CLC}(Y_m, 0)$.

Example 5.2. Let $X := (x^2 + y^3 + z^7 + t^6 z^6 = 0) \subset \mathbb{C}^4$ and $o = (0, 0, 0, 0) \subsetneq W := \{(x, y, z, t) \in X \,|\, x = y = z = 0\}$. Let $g : Z \to \mathbb{C}^3 = \operatorname{Spec} \mathbb{C}[x, y, z]$ be the weighted blowing up at $(0, 0, 0)$ with the weight $(\mathrm{wt}x, \mathrm{wt}y, \mathrm{wt}z) = (3, 2, 1)$. Let $h := g \times 1 : Z \times \mathbb{C} \to \mathbb{C}^4 = \operatorname{Spec} \mathbb{C}[x, y, z, t]$ and Y the strict transform of X by h. Let $f := h|_Y : Y \to X$ and E be the exceptional divisor of f. Then $K_Y = f^* K_X - E$ and Y is smooth, and $(E, 0)$ has only one lc singularity in $f^{-1}(o)$. So, by [FA, 17.2 Theorem], we obtain that X is lc and o and W are centers of log canonical singularities for the pair $(X, 0)$.

Example 5.3. Let $X = \mathbb{C}^4$ and

$$\Delta := \frac{1}{2}\Delta_1 + \frac{2}{3}\Delta_2 + \frac{7}{8}\Delta_3 + \frac{24}{25}\Delta_4 + \frac{599}{600}\Delta_5,$$

where Δ_i is a general hyperplane and $o := (0, 0, 0, 0) \in \Delta_i$ for every i. Then $(o \in X, \Delta)$ is a four dimensional lc pair such that $\mathrm{LLC}(X, \Delta) = o$ and $\mathrm{I}(o \in X, \Delta) = 600$.

Example 5.4. Let $(P \in Z, 0)$ be a two dimensional log canonical singularity which is not log terminal. Then, by Corollary (4.23), $\mu(P \in Z, 0) = 0$ or 1, and $\mathrm{I}(P \in Z, 0) = 1, 2, 3, 4$, or 6. Let $X := Z \times \mathbb{C}$ and $p : X \to \mathbb{C}$ be the second projection. Let $o \in \mathbb{C}$ and $H := p^*[o]$, and $Q := (P, o) \in X$. Then $(Q \in X, H)$ is log canonical and $Q \in \mathrm{CLC}(X, H)$ by [FA, 17.2 Theorem]. Furthermore, $\mu(P \in Z, 0) = \mu(Q \in X, H)$ and $\mathrm{I}(P \in Z, 0) = \mathrm{I}(Q \in X, H)$.

6. Appendix

In this appendix, we treat analytic germs of lc pairs. The main theorem holds true for analytic spaces.

Theorem 6.1 (Analytic version of the main results)**.** *Theorem (4.20) and Corollaries (4.21), (4.23) hold true even if X is an analytic space.*

In Section 1, we defined B-birational map (resp. morphism), $\mathrm{Bir}(X, D)$ and so on. For analytic spaces, B-bimeromorphic map (resp. morphism), $\mathrm{Bim}(X, D)$ etc. can be defined without difficulty. Details are left to the reader.

For the relative log MMP over analytic germs, we recommend the reader to see [Ka1, Section 1] or [N1]. By using this, Section 2 can be generalized to analytic spaces, which are projective over analytic germs. Note that in Propositions (4.7) and (4.18), E^c is projective. So we don't have to modify Section 3.

In Section 4 and Lemma (1.7), we often used [Fj1, Lemma 4.7]. In [Fj1, Lemma 4.7], we used Szabó's resolution lemma (see [Sz]). In [Sz], Szabó proved the resolution lemma for only algebraic varieties. Therefore, in the analytic case, we apply [BM, Theorem 12.4], which contains the analytic version of Szabó's resolution lemma.

In the proof of [Fj1, Lemma 4.7, Claim (B_n)], we cited [KoM, Lemma 2.45], which is an algebraic result. However, we can check that we didn't need it. In [Fj1, Claim (B_n)], we only treated B-smooth pairs. So, by using the following lemma, we can prove [Fj1, Claim (B_n)] without using [KoM, Lemma 2.45] (for related topics, see [Ku, 1.1], [Jh], and [Ko, Chapter VI 1.4.7]).

Lemma 6.2. *Let $(S, \Xi) \to (T, \Theta)$ be a B-bimeromorphic morphism between B-smooth pairs. Let W be an irreducible component of Ξ^B. Then there exists a finite sequence of blowing-ups:*

$$T^l \xrightarrow{p_l} T^{l-1} \xrightarrow{p_{l-1}} \cdots \xrightarrow{p_{k+1}} T^k \xrightarrow{p_k} T^{k-1} \xrightarrow{p_{k-1}} \cdots \xrightarrow{p_2} T^1 \xrightarrow{p_1} T^0 = T,$$

whose centers are $W^k \in \mathrm{CLC}(T^k, \Theta^k)$ and W^l is a divisor. Here $p_k^*(K_{T^{k-1}} + \Theta^{k-1}) = K_{T^k} + \Theta^k$ and $W^k \in \mathrm{CLC}(T^k, \Theta^k)$ is a center associated to the divisor W for every k.

Proof. This can be checked easily by computing discrepancies by the same way as in [Ko, Chapter VI 1.4.7]. □

Therefore, Theorem (6.1) can be proved without difficulty. Details are left to the reader.

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