On a complete description of the Stokes geometry for higher order ordinary differential equations with a large parameter via integral representations.*

Dedicated to Professor K. Aomoto on his sixtieth birthday

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*Supported in part by Japan Society for the Promotion of Science under the Japan-U.K. Scientific Cooperation Project and through Grant-in-Aid for Scientific Research (B) (11440042), Japanese Ministry of Education, Science, Sports and Culture.

†Supported in part by Grant-in-Aid for Encouragement of Young Scientists (11740087), Japanese Ministry of Education, Science, Sports and Culture.

Abstract

We make a detailed study of particular third order linear ordinary differential equations with a large parameter of Laplace type through integral representations of their solutions, hoping that it will give us a better understanding of the redundant new turning points. In the particular examples we discuss, the redundancy is tied up with a simple function-theoretic property of a real one-dimensional curve determined by a Hamiltonian system associated with the third order differential equations.

As is now commonly accepted (cf. [AKT1], [AKT2] and references cited there), exact WKB analysis for higher order equations is incredibly difficult; the first difficulty is the existence of crossing points of Stokes curves ([BNR]), and if new Stokes curves and new turning points are introduced to resolve the apparent contradiction caused by such crossing points ([BNR], [AKT1], etc.), we next encounter (in general, infinitely many) "redund new turning points" and "new Stokes curves" ([AKT1, p.77]). At the same time, we expect only a very limited number of new turning points are necessary to obtain a complete Stokes geometry ([AKT1, p.77]). In this article we examine the third order equations of the form (1) below; our observation is that the existence of the redundant new Stokes curves is tied up with the multiple-valued nature of the integrand of an integral representation of solutions. This indicates an interesting problem in the theory of hyperasymptotics of integrals, as integrals with singular integrands have not yet been seriously investigated. (Cf. [BH, p.673].) This observation also gives us a hope that exact WKB analysis of subholonomic systems with a large parameter may be done without much trouble if the system originates from an integral whose integrand is non-singular. (Cf. [A], [S].)

To obtain a clear-cut picture of the problem we concentrate our attention to equations of the following form:

$$(\frac{d^3}{dx^3} + (c_1 x + d_1)\eta^2 \frac{d}{dx} + (c_0 x + d_0)\eta^3)\psi = 0, \tag{1}$$

where η is a large parameter and c_j and d_k $(j,k = 0,1)$ are fixed complex numbers. Note that both the equation discussed in Example 2.5 of [AKT1] and that discussed in Example 2.2 of [AKT2] are of this form. Since the

Laplace method applies to an equation of the form (1), we can find an integral representation of a solution of (1):

$$\psi(x) = \int_\gamma C(\zeta)^{-1} \exp(\eta(x\zeta + g(\zeta)))d\zeta, \tag{2}$$

where $g(\zeta)$ is a multi-valued analytic function that satisfies

$$\frac{dg}{d\zeta} = \frac{D(\zeta)}{C(\zeta)} \tag{3}$$

with $C(\zeta) = c_1\zeta + c_0$ and $D(\zeta) = \zeta^3 + d_1\zeta + d_0$. Let us now fix g and set

$$f(x,\zeta) = x\zeta + g(\zeta). \tag{4}$$

If we let σ, τ, u and v respectively denote $\mathrm{Re}\,\zeta, \mathrm{Im}\,\zeta, \mathrm{Re}\,f$ and $\mathrm{Im}\,f$ and let γ denote a steepest descent path of u passing through a saddle point of the integral, then γ is a (real 1-dimensional) solution curve of the following Hamiltonian system:

$$\begin{cases} \dfrac{d\sigma}{dt} &= \dfrac{\partial v}{\partial \tau} \\ \dfrac{d\tau}{dt} &= -\dfrac{\partial v}{\partial \sigma}. \end{cases} \tag{5}$$

We note that a point $\alpha = \alpha(x)$ is, by definition, a saddle point if it satisfies

$$\frac{df}{d\zeta}(\alpha) = x + \frac{dg}{d\zeta}(\alpha) = 0, \tag{6}$$

and hence it follows from (3) that $(x,\zeta) = (x, \alpha(x))$ is a characteristic point of equation (1) with $\eta = 1$ (except for the case $C(\alpha) \equiv 0$), i.e.,

$$(xC(\alpha/\eta) + D(\alpha/\eta))\eta^3|_{\eta=1} = 0. \tag{7}$$

We also note that $C(\zeta) = c_1\zeta + c_0$ and hence the singularity of $g(\zeta)$ is a logarithmic one (if $c_1 \neq 0$). Therefore a solution curve of (5) is, in general, a logarithmic spiral near $\zeta = -c_0/c_1$. However, our computer-assisted experiments strongly indicate that the following conjecture should be true:

Conjecture. *Let $\alpha(x)$ and $\beta(x)$ be saddle points of the integral* (2) *and suppose that a steepest descent path γ connects $\alpha(x)$ and $\beta(x)$. Then the variation of* $\arg(\zeta + c_0/c_1)$ *along γ is smaller than 2π, i.e., there exists a constant $\theta_0(x)$ for which*

$$|\arg(\zeta + \frac{c_0}{c_1}) - \theta_0(x)| < \pi \tag{8}$$

holds for ζ in γ.

This conjecture, if true, would mean that for each fixed x a steepest descent segment connecting two saddle points should lie on one preferred branch of the logarithmic singularity contained in f. The existence of such a segment should indicate that the point x lies on a Stokes curve, an ordinary one or a new one. In fact, the Riemann surface $\mathcal{R}$ of the characteristic root of equation (1) (with $\eta = 1$, cf. [AKT1, p. 77]) is given by

$$\{(x, \zeta); xC(\zeta) + D(\zeta) = 0\} \tag{9}$$

in this case, and hence we can choose ζ as a coordinate of $\mathcal{R}$. Letting $x(\zeta) = -D(\zeta)/C(\zeta)$ denote the projection $\mathcal{R} \to \mathbb{C}_x$, we then find

$$\frac{dg}{d\zeta}d\zeta = \frac{D(\zeta)}{C(\zeta)}d\zeta = -x(\zeta)d\zeta = -\frac{d}{d\zeta}(x(\zeta)\zeta)d\zeta + \zeta\frac{dx(\zeta)}{d\zeta}d\zeta. \tag{10}$$

Since a point in $\mathcal{R}$ that projects to x is expressed by a characteristic root at x in this coordinate, we integrate (10) to obtain

$$\begin{aligned} f(x, \alpha(x)) - f(x, \beta(x)) &= x(\alpha(x) - \beta(x)) + \int_{\beta(x)}^{\alpha(x)} \frac{dg}{d\zeta}d\zeta \qquad (11) \\ &= \int_{\beta(x)}^{\alpha(x)} \zeta\frac{dx(\zeta)}{d\zeta}d\zeta \\ &= \int_a^x (\alpha(x) - \beta(x))dx. \end{aligned}$$

Here a is an ordinary turning point where $\alpha(x)$ and $\beta(x)$ merge. Hence

$$\mathrm{Im}(f(x, \alpha(x)) - f(x, \beta(x))) = 0 \tag{12}$$

is equivalent to

$$\mathrm{Im}\int_a^x (\alpha(x) - \beta(x))dx = 0, \tag{13}$$

which is the ordinary definition of a Stokes curve.

Remark. Not only an ordinary Stokes curve but a new Stokes curve can also be defined by the relation (13). For example, the new Stokes curve (i.e., the imaginary axis) of the example of [BNR] (cf. [AKT1, Example 2.1]) is expressed as

$$\mathrm{Im} \int_{-1}^{x} (\xi_1(x) - \xi_2(x))dx = 0$$

where $\xi_1(x)$ and $\xi_2(x)$ denote the characteristic roots that merge at a turning point $x = -1$ and the integration is done along a path starting from $x = -1$, turning around another turning point $x = 1$ once and reaching a point x near the imaginary axis. We note that either $\xi_1(x)$ or $\xi_2(x)$ moves onto another branch $\xi_0(x)$ of the characteristic roots after turning around $x = 1$.

In this way the ζ-space of integral representation of a solution of (1) is canonically identified with the Riemann surface $\mathcal{R}$. In particular, the residue of $(\partial f/\partial\zeta)d\zeta = (x + dg/d\zeta)d\zeta$ at $\zeta = -c_0/c_1$ coincides with that of the characteristic root $\zeta(x)dx$ at $x = \infty$ thanks to (9). (The points $\zeta = -c_0/c_1$ and $\zeta = \infty$ correspond to $x = \infty$ by the projection $x = x(\zeta)$.) Furthermore, in view of the connection between the integral (2) and the Borel transform of the WKB solution of (1) (cf. [U], [AKT2], etc.), that is, noting that a mapping $y = -f(x, \zeta) = -(x\zeta + g(\zeta))$ from the ζ-space to the Borel plane transforms a saddle point of the integral (2) into a singular point of the Borel transform of the WKB solution of (1), we conclude that the existence of a steepest descent segment of $u = \mathrm{Re}\, f$ connecting two saddle points should imply that the point x lies on an effective (ordinary or new) Stokes curve.

Thus an important consequence of the validity of the conjecture is the following: since $g(\zeta)$ is a rational function, we find only finitely many (actually 3 in our case) saddle points, and hence finitely many types of equations given by (12). Among them finitely many equations with the same preferred branch of the logarithmic singularity for both $f(x, \alpha(x))$ and $f(x, \beta(x))$ describe all the possible curves across which the configuration of steepest descent paths may change. Otherwise stated, the conjecture would entail that no Stokes phenomena should occur on a Stokes curve defined by (13) with "complicated" path of integration (i.e., a path making more than one turn around the singular point $\zeta = -c_0/c_1$ on $\mathcal{R}$). We thus obtain a complete Stokes geometry via the integral representation (2) without being bothered by redundant new Stokes curves. In particular, the vague expression "small with respect to periods" used in [AKT1, §2] acquires a definite meaning in this

case: a steepest descent segment joining two saddle points does not make more than one turn around the singular point of $g(\zeta)$.

At the end of this article we call the attention of the reader to the fact that the clear-cut description given above is highly dependent on the fact that $C(\zeta)$ is a linear form; in general the conjecture is not true, and the whole scope is still far beyond our understanding.

References

[A] Aoki,T.: Exact WKB analysis of linear ordinary differential equations with a large parameter of the third order.

[AKT1] Aoki,T., T. Kawai and Y. Takei: New turning points in the exact WKB analysis for higher-order ordinary differential equations. Analyse algébrique des perturbations singulières. I. Hermann, 1994, pp. 69-84.

[AKT2] ———: On the exact WKB analysis for the third order ordinary differential equations with a large parameter. Asian J. Math., **2** (1998),

[BNR] Berk, H. L., W. M. Nevins and K. V. Roberts: New Stokes lines in WKB theory. J. Math. Phys., **23** (1982), 988-1002.

[BH] Berry, M. V. and C. J. Howls: Hyperasymptotics for integrals with saddles: Proc. Roy. Soc. London, **A 434** (1991), 657-675.

[S] Shudo, A.: Complex semi-classical theory and tunneling phenomena. Bussei-Kenkyû, **71** (199) 581-591. (In Japanese.)

[U] Uchiyama, K.: On examples of Voros analysis of complex WKB theory. Analyse algébrique des perturbations singulières. I. Hermann, 1994, pp. 104-109.